— 1. INTRODUCTION — [K. Rosen]

Propositional logic

Propositional logic (and mathematics, in general) studies propositions: declarative sentences (a sentence that declares a fact) that is either true or false, but not both.

Ⓔ Toronto is the capital of Canada.

$1+1=2$

$2+2=3$

3 is a prime number

Ⓔ The following are not propositions.

What time is it? } not declarative
Read this carefully. }

$x+1=2$ } Neither true nor false.
$x+y=z$ }

We use letters to denote propositions: $p, q, r, \dots$
New propositions (called compound propositions) are constructed by combining one or more propositions using logical operators.

- Negation. If p is a proposition, its negation is denoted by $\neg p$.
  "It's not the case that p".

$\underbrace{\text{My PC runs Linux}}_{p} \rightsquigarrow$ It's not the case that my PC runs Linux.

$\underbrace{\text{My PC doesn't run Linux}}_{\neg p}$

$$\underbrace{1+1=2}_{p} \rightsquigarrow \underbrace{1+1\neq 2}_{\neg p}$$

$\neg p$ is true iff (if and only if) p is false.

- Conjunction $p, q \rightsquigarrow p \wedge q$ "p and q".
  Beware: sometimes the word "but" is used instead of "and": 2 is even but 3 is odd.

  $p \wedge q$ is true iff p and q are true.

- Disjunction. $p, q \rightsquigarrow p \vee q$ "p or q"
  $p \vee q$ is true iff p is true, q is true or both are.

  This corresponds to the "inclusive or" in English
  [Exclusive or: soup or salad comes with an entrée: most certainly means that the customer cannot have both soup and salad].

Conditional statement/

- Implication : $p, q \rightsquigarrow p \to q$ "if p, then q"

Because of its essential role in mathematical reasoning, a variety of terminology is used to express $p \to q$:

if p, q — p implies q
q if p — p only if q *
q when p

hypothesis/antecedent — consequence/conclusion

$p \to q$ is false when p is true and q is false; otherwise, it is true.

Useful way to understand its truth value: A pledge many politicians make when running for office:
"If I'm elected, I will lower taxes".
It is only when the politician is elected but does not lower taxes that he can be said to have broken his pledge.

Note that this definition is more general than the meaning attached to such statements in English: there needs to be no relationship between p and q:

If the Moon is made of cheese, then $2+3=4$.

- Biconditional statements / bi-implication. $p \leftrightarrow q$

It's equivalent to $(p \to q) \wedge (q \to p)$.
It's true iff the truth values of p and q are the same.

The main advantage of logical language over natural ones is that it removes ambiguity.

Predicate logic or first-order logic

Statements such as

$$x > 3 \quad \text{or} \quad x+1 = y$$

are often found in mathematical assertions. They are neither true nor false (when the values of the variables are not specified) and hence fall outside the scope of propositional logic.

In "$x>3$" there are two parts:

- the variable x
- the predicate "is greater than 3".

By denoting the predicate with P, the statement can be represented as $P(x)$.

E If $P(x)$ is "$x>3$" then
$P(4)$ is true, $P(2)$ is false, $P(y)$ is undefined.

E $Q(x,y)$ is "$x = y+2$".
$Q(3,1)$ is true.

Quantifiers

In addition to assigning values to variables, there is another way to get truth values from predicate statements. For that, it is necessary to assume a domain or universe of discourse

- Universal quantification.

$P(x) \rightsquigarrow \forall x\, P(x)$ "$P(x)$ for all values of x in the domain".

$\forall x P(x)$ is true iff $P(a)$ is true for all elements a in the universe.

E) $\forall x.\ x+1 > x$ is true in $\mathbb{R}$ and $\mathbb{N}$.

E) $\forall x.\ x \geq 0$ is true in $\mathbb{N}$ but not in $\mathbb{R}$.

(Alternative, mathematical notation:
$\forall x \in \mathbb{N}.\ x \geq 0$
$\forall x \in \mathbb{R}.\ x \geq 0$

E) $U = \{0, 1, 2, 3\}$ $\forall x.\ x^2 < 10$ is true

? E) $\forall x\, \forall y.\ x \cdot y > x+y$ is false in $\mathbb{N}$ and in $\mathbb{R}$.

E) $\forall x\, \forall y.\ x+y \geq x$ is true in $\mathbb{N}$, false in $\mathbb{R}$.

· Existential quantifier

$P(x) \rightsquigarrow \exists x\, P(x)$ "There exist an element x in the domain such that P(x)"

$\exists x P(x)$ is true iff $P(a)$ is true for some a in the universe.

(E) $\exists x.\, x<0$ is true in $\mathbb{R}$ but not in $\mathbb{N}$.

(E) $\forall x \exists y.\, x+y=0$ is $\begin{cases} \text{true in } \mathbb{R} \\ \text{false in } \mathbb{N} \end{cases}$

$\exists y \forall x.\, x+y=0$ is false in $\mathbb{R}$ and $\mathbb{N}$.

The order of the quantifiers matters!

Remark: What if the universe is empty (there are no elements)?

$\forall x P(x)$ is trivially true.

$\exists x P(x)$ is false.

Proof techniques

· **Direct proofs**: $p \to q$ can be proved by showing that if p is true then q must also be true.

(E) If n is even then n^2 is even. (Do it)

· Proof by contraposition To prove $p \to q$ is equivalent to showing its contrapositive: $\neg q \to \neg p$.

| false iff $\neg q$ is true, $\neg p$ is false
$\Leftrightarrow$ q is false, p is true
$\Leftrightarrow$ $p \to q$ is false.

(E) If $3n+2$ is odd, then n is odd.

· Vacuous and trivial proofs

$p \to q$ is true if the hypothesis is false.
$p \to q$ is true if so is q.

(E) If n^2 is odd then n is odd. (contrapositive: start with a direct proof)

· Proofs by contradiction: To show that p is true it is enough to show that $\neg p$ leads to a contradiction.

(E) Out of 22 (week) days, at least 4 must fall on the same day of the week.

· Counterexamples. To show that $\forall x P(x)$ is false, we need only find a value a such that $P(a)$ is false.

(E) Every positive integer is the sum of the squares of two integers.

$6 = 1^2 + 2^2 + 1^2$ but $7 = \ldots$!