Algorithmic Methods for Mathematical Models (AMMM)

Greedy Algorithms (for Combinatorial Optimization)

Luis Velasco

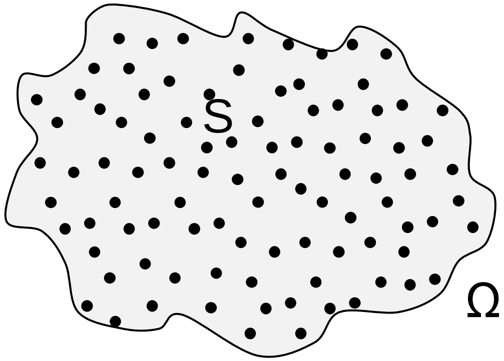
(luis.velasco@upc.edu)

Campus Nord D6-107

Combinatorial Optimization

- A combinatorial optimization problem is defined by:
  - N: finite **ground set** of elements, index *i*
  - Ω: set of **feasible solutions** of *N*
  - c_i: **cost** of the element *i*

$$\min_{\omega \subseteq N} \sum_{i \in \omega} c_i$$
$$\text{s.t.} \quad S \in \Omega$$



- Combinatorial problems can be modeled using binary variables $x_i \in \{0,1\}$, one per element.

The knapsack problem

- Given a set of items, each with a weight a_i and a value c_i, determine which items to include in a collection so that the total weight is less than a given capacity limit b and the total value is as large as possible.

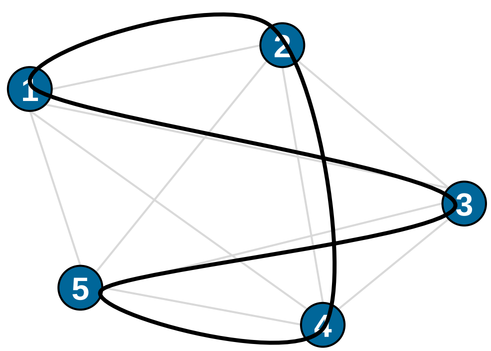
1. Parameters
   - $N = \{1, \ldots, n\}$: set of items (projects, objects, etc.)
   - c_i : benefit obtained choosing item i
   - a_i : weight of the item i
   - b: capacity
2. Variables
   - $x \in \{0, 1\}$: 1 the item is chosen, otherwise 0
3. Constraints $$\sum_{i=1}^{n} a_i x_i \leq b$$
4. Objective function $$\text{maximize} \sum_{i=1}^{n} c_i x_i$$

Steps for modeling

1. Parameters and sets
2. Decision Variables definition
3. Constraints
4. Objective function

Example: The Travelling Salesman Problem (TSP)

- Given a graph G(*V*,*E*) with a set of cities (*V*) and their pairwise distances
- The task is to find a **shortest** possible **tour** that visits each city exactly once (Hamiltonian cycle).



Example: The Travelling Salesman Problem (TSP)

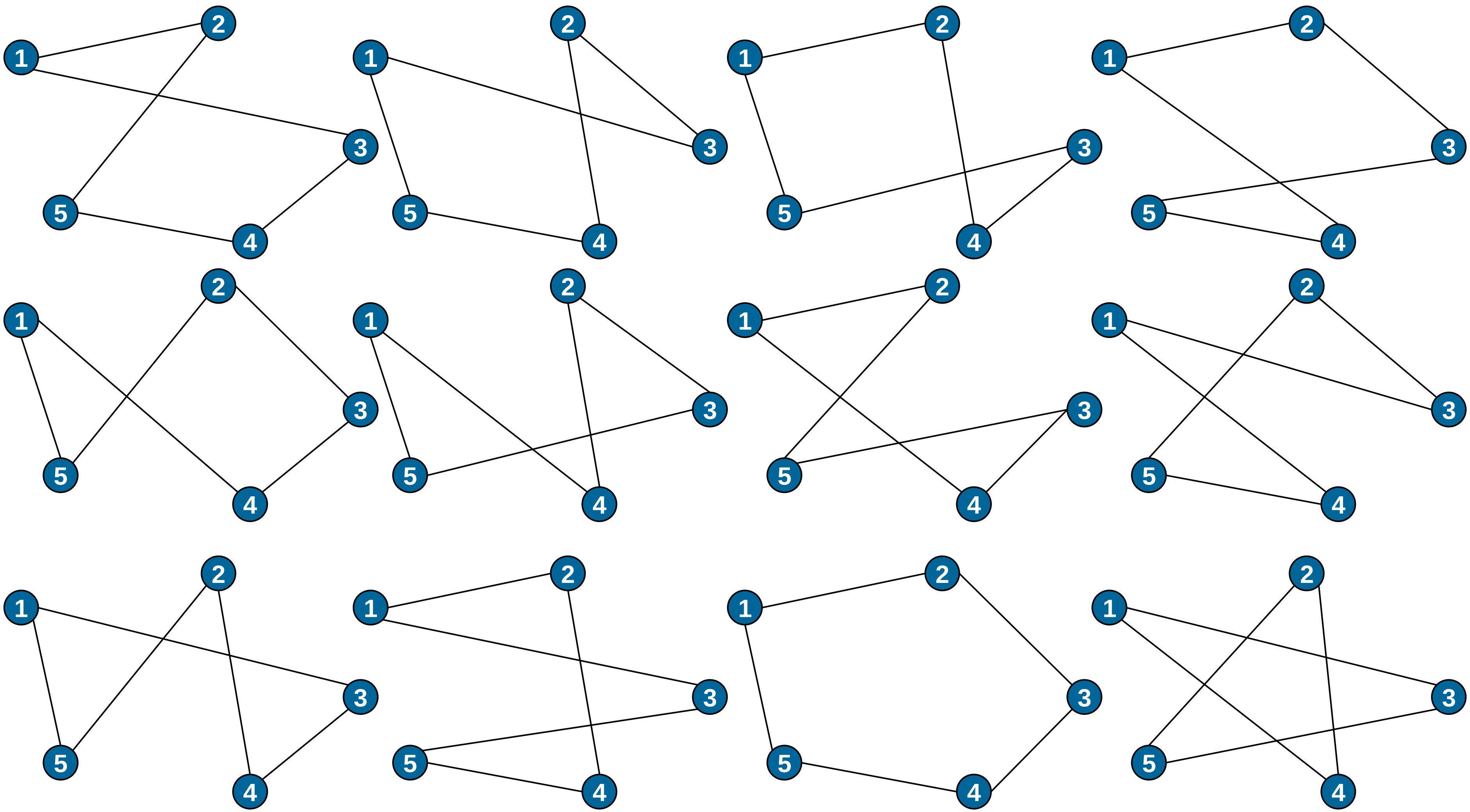
- TSP is a **combinatorial** problem. Its **search space** is fact(n-1)/2
  - the ground set is that of all edges connecting the cities to be visited,
  - *F* is formed by all edge subsets that determine a Hamiltonian cycle.
  - *f*(*S*) is the sum of the distances of all edges in each Hamiltonian cycle
- What **factorial** means?

n	search space
3	1
4	3
5	12
6	60
7	360
8	2,520
9	20,160
10	181,440

n	search space
20	6.08E+16
30	4.42E+30
40	1.02E+46
50	3.04E+62
60	6.93E+79
70	8.56E+97
80	4.47E+116
90	8.25E+135
100	4.67E+155

Information on the largest TSP instances solved to date can be found in:
http://www.math.uwaterloo.ca/tsp/optimal/index.html

The (n-1)!/2 combinations (n=5)



Dantzig-Fulkerson-Johnson formulation

$$\min \sum_{e \in E} c_e \cdot x_e$$

s.t.

$$\sum_{e=(i,j) \in E} x_e = 2 \quad \forall i \in N$$ Two edges per node

$$\sum_{e \in S} x_e = |S| - 1 \quad \forall S \subset E, |S| < |N|$$ Sub-tour elimination

$$x_e = \{0.1\} \quad \forall e \in E$$ Link e is in the tour

Greedy algorithm

- A greedy algorithm builds the solution in an iterative manner.
  - At each iteration, **the best element** from a **candidate list** is added to the **partial solution**
- In general they have **five pillars**:
  - A **candidate set** C, from which a solution is created
  - A **selection function**, which chooses the best candidate c to be added
  - A **feasibility function**, to determine if a candidate can be used
  - An **objective function** $f(S)$, which assigns a value to a (partial) solution
  - A **solution function**, which indicate when we have a complete solution

Greedy Algorithm for Combinatorial Problems

C: Candidate set, index c
$S \subseteq C$: (partial) solution
$q(c,S)$: quality of element c *given a partial solution* S (*greedy function*).

$$q(c,S) = \begin{cases} {}^{value}/_{cost} & (i.e., \boldsymbol{added\ value}) \\ \infty & if\ S \cup \{c\}\ is\ \boldsymbol{Infeasible} \end{cases}$$

Initialize C
$S \leftarrow \{\}$
while $\boxed{S \text{ is not a solution}}$ **do**
  evaluate $q(c, S)\ \forall\ c \in C$
  $c_{best} \leftarrow \boxed{\text{argmax}\{q(c, S) \mid c \in C}\}$
  $S \leftarrow S \cup \{c_{best}\}$
  update C, *e.g.*, $C \leftarrow C \setminus \{c_{best}\}$ ($\boxed{\text{you might want to exclude infeasible } c}$)
return <$f(S)$, S>

solution function
selection function
feasibility function

Example: TSP

The nearest neighbor (NN) algorithm

Given graph G(V,E)

Partial Solution

$S \leftarrow \{v\}$, where v is an arbitrary node (starting point) from V

while $S \neq$ V **do**

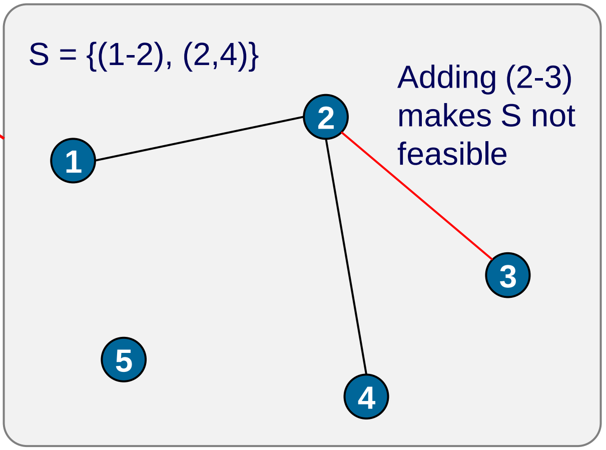
V is the candidate set

$C \leftarrow$ set of feasible links w.r.t. $S \subseteq E$

$e_{best} = (u \in S, v \notin S) \leftarrow \text{argmin}\{d(e) \mid e \text{ in } C\}$

$S \leftarrow S \cup \{v\}$

return Tour

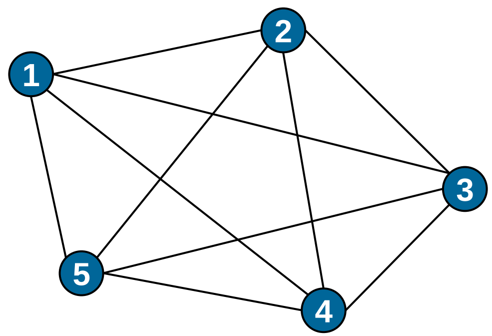


For |V| cities randomly distributed on a plane, the algorithm yields:

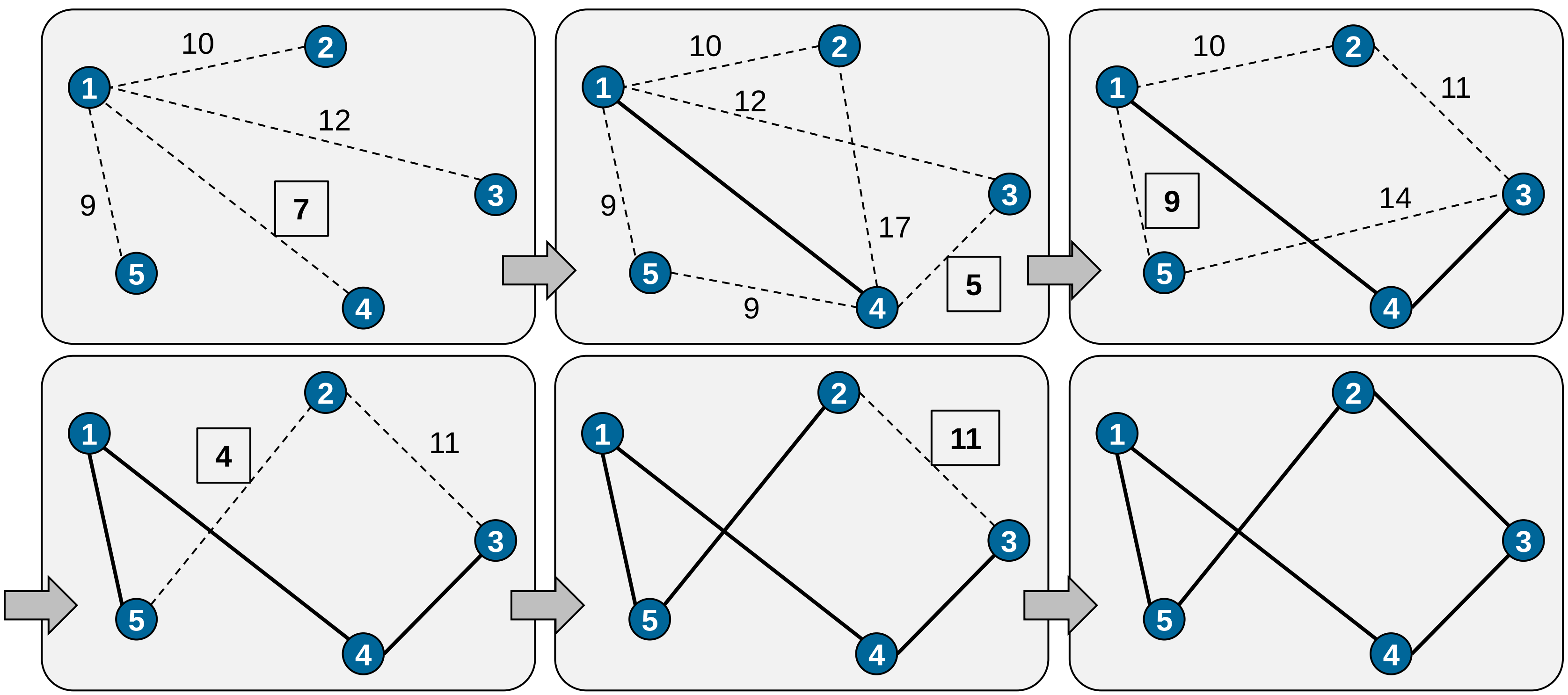
length = 1.25 * shortest (optimal) length

on average.

Example: TSP

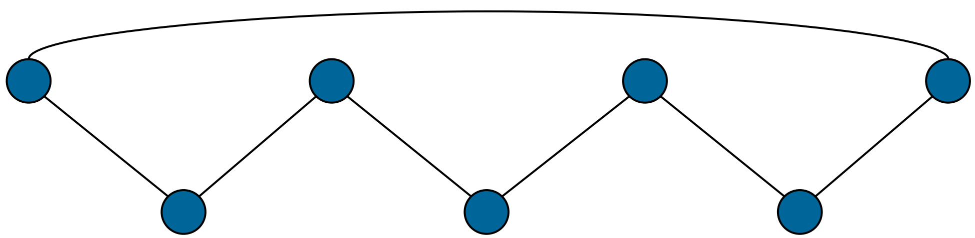


	1	2	3	4	5
1	-	10	12	7	9
2		-	11	17	4
3			-	5	14
4				-	9
5					-



Example: TSP

- A greedy algorithm suffers from myopia.
  - It looks for the best candidate at each iteration.

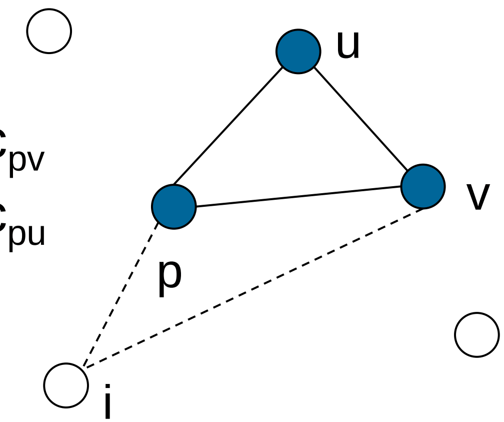


Other algorithms for the TSP

- **Nearest Insertion (greedy)**: From a small cycle, the algorithm expands the cycle by adding the nearest vertex.

$c_{ip}+c_{iv}-c_{pv}$

$c_{ip}+c_{iu}-c_{pu}$



- **Christofides algorithm**: Produces solutions within 3/2 of an optimal solution, i.e., $Z_{heuristic} \leq 3/2\, Z^*$.

- Create the minimum spanning tree MST *T* of G.
- Denote *O* the set of vertices with odd degree in *T*
- Find a perfect matching *M* with minimal weight in the complete graph over the vertices from *O*.
- Combine the edges of *M* and *T* to form a multigraph *H*.
- Form an Eulerian path in *H* (*H* is Eulerian because it is connected, with only even-degree vertices).
- Transform the path found in last step to be Hamiltonian by skipping visited nodes (*shortcutting*).

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Greedy Algorithms

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(lvelasco @ ac.upc.edu)

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