Fundamentals of Automatic Control

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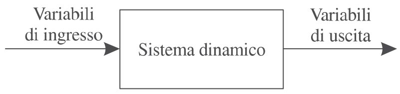
Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

2016/2017

Dynamical System



Input variables ($u \in R^m$) = represent the actions taken on the object (independent)
Output variables ($y \in R^p$) = represent which part of the objects' behaviour is, for any reasons, of interest
In most cases knowing the values of the input variables at an exact istant is not enough to determine the values of the output variables at the same istant.
State variables ($x \in R^n$) = represent the state inside the object that is going to be modelled: just what is needed to determine the output variables, once the values of the input variables and the istant of time are known.
The number n of the state variables is called *order* of the system.

Continuous time dynamanical system

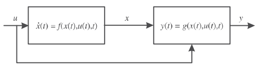
Output transformation

$$y(t) = g(x(t), u(t), t)$$

where g is a proper vector function.
State space equation

$$\dot{x}(t) = f(x(t), u(t), t)$$

where f is a vector function. This equation, under proper regularity hypothesis about f which are assumed to be verified, defines the state's evolution $x(t)$ for $t > t_0$, for any initial instant t_0, input function $u(t)$, $t \geq t_0$ and initial condition $x(t_0) = x_{t_0}$.
State or *input-state-output* or *internal representation* of a dynamical system.



It is also possible to have representations, called *input-output* or *external*, where the state variables are not explicit.

Classes of dynamical systems

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t), \quad x(t_0) = x_{t_0}, \\ y(t) &= g(x(t), u(t), t) \end{aligned}$$

- SISO systems (single variables) e MIMO systems (multiple variables)
- Strictly proper system when g does not depend from u otherwise proper.
- Not dynamical or static system if g does not depend from x
- Time-Invariant system, or even stationary, when f and g do not depend explicitly from t
- Linear system when f and g are linear with respect to x and u

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

- Linear Time Invariant (LTI) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

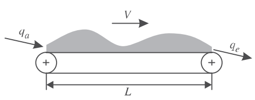
Classes of dynamical systems

- Finite dimensional system or *lumped element system*. To describe their internal situation at a time t you just need to know a finite number n of real scalars, that are n components of the state vector x. There are systems where the state at a time t is composed of an entire function of one or more variables. These systems are called *infinite dimensional* o *distribuited element systems*, where the state is hold by a differential equation with partial derivatives e not total derivatives anymore.

  A simple relevant example is a *delay of time*: a SISO system in which the output replicates the input delaying it of a time $\tau > 0$. This is described by

$$y(t) = u(t - \tau)$$

  that connects the output directly with the input but it does not explain the behaviour for $0 \leq t < \tau$, if you do not know the input for $-\tau \leq t < 0$.



Linear Systems - Superposition principle

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), x(t_0) = x_{t_0}, \quad t \geq t_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Superposition Principle

$$\begin{aligned} \dot{x}'(t) &\equiv Ax'(t) + Bu'(t) \quad , \quad x'(t_0) \equiv x'_{t_0} \\ y'(t) &\equiv Cx'(t) + Du'(t) \\ \dot{x}''(t) &\equiv Ax''(t) + Bu''(t) \quad x''(t_0) \equiv x''_{t_0} \\ y''(t) &\equiv Cx''(t) + Du''(t) \end{aligned}$$

$$\begin{aligned} u'''(t) &= \alpha u'(t) + \beta u''(t) \\ x'''_{t_0} &= \alpha x'_{t_0} + \beta x''_{t_0} \end{aligned}$$

$$\Downarrow$$

$$x'''(t) = \alpha x'(t) + \beta x''(t)$$
$$y'''(t) = \alpha y'(t) + \beta y''(t)$$

Dynamic in linear systems

Consider a continuous linear time-invariant system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

The state dynamic corrisponding to the input $u(t)$ defined for $t \geq t_0$ and at the initial state $x(t_0) = x_{t_0}$ is described by the following equation, commonly known as *Lagrange's formulae:*

$$x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

The corrisponding output dynamic is

$$y(t) = Ce^{A(t-t_0)}x_{t_0} + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

Linear Systems - Free and forced dynamics

Free dynamic: $u(t) = 0$ (dependent only on the initial state)
Forced dynamic: $x_{t_0} = 0$ (dependent only on the input)

Exercize

A water tank is filled by a river canal through the opening of a rolling shutter that allows 1 m^3 of water per hour into the tank. The water tank has a 2 m^2 section and 5 m height, and a volume water leak proportional to the level inside it with a coefficient of $K_1 = 0.2m^2$/hour

- Find (starting from an empty water tank) if and when the tank will overflow.
- Find if there is an x level of water that, with the alternative opening and closing of the rolling shutter every hour, corresponds in a periodic condition.

Solution
The equation that describes the water tank system is

$$A\frac{dx}{dt} = -K_1 x + u$$

$$\dot{x}\frac{-0.2}{2}x + \frac{u}{2}$$

$$g_x(t) = e^{At}B = e^{-0.1(t-\tau)}0.5$$

Exercize

- Finding the integral with null initial condition and input $u = sca(t)$ the result is:

$$\begin{aligned}x(t) &= \int_0^t g_x(t-\tau)\,u(\tau)\,d\tau\\ &= \int_0^t e^{-0.1(t-\tau)}0.5sca(\tau)d\tau = 5e^{-0.1(t-\tau)}\Big|_0^t = 5(1-e^{-0.1t})sca(t)\end{aligned}$$

  The maximum value is 5 so the water tank will never overflow.
- An equilibrium condition exists if the following exists

$$[\bar{x}_m e^{-0.1} + 5(1-e^{-0.1})]e^{-0.1} = \bar{x}_m$$

  Where $\bar{x}$ is the minimum value reached by the equilibrium solution.

$$\bar{x}_m = \frac{5(1-e^{-0.1})}{e^{0.1}-e^{-0.1}} = 2.37$$

  The maximum value is

$$\bar{x}_M = \bar{x}_m e^{-0.1} + 5(1-e^{-0.1}) = 2.62$$

Dynamic and equilibrium

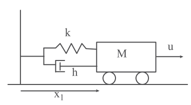
$$\begin{aligned}\dot{x}(t) &= f(x(t),u(t)), \quad x(t_0) = x_{t_0},\\ y(t) &= g(x(t),u(t))\end{aligned}$$

Given $x(t_0)$ and $u(t)$, $t \geq t_0$ is possible to know the *state dynamic* $x(t)$ and the *output dynamic* $y(t)$, $t \geq t_0$
Given a constant input $u(t) = \bar{u}$, the corresponding equilibrium state $\bar{x}$ is the value of the state for which $\dot{x}(t) = 0$. Thus,

$$\begin{aligned}0 &= f(\bar{x},\bar{u})\\ \bar{y} &= g(\bar{x},\bar{u})\end{aligned}$$

where $\bar{y}(t)$ is the output equilibrium.

Example: Cart



The system depicted in the picture, where coefficient $k = k_0 e^{-x_1}$ is a non-linear function of x, is described by the following model

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -\frac{k_0}{M}e^{-x_1}x_1 - \frac{h}{M}x_2 + \frac{u}{M}\\ y = x_1\end{cases}$$

where $M = 1$, $h = 1.1$, $k_0 = 0.33$.
Find the $\bar{u}$ value, where the equilibrium is found when $\bar{x}_1 = 0.5$

$$\begin{cases}0 = \bar{x}_2\\ \bar{u} = k_0 e^{-\bar{x}_1}\bar{x}_1\end{cases}$$

Stability

Given a dynamical system

$$\dot{x} = f(x(t)),\ x(t_0) = x_0$$

Definition

An equilibrium $\bar{x}$ is defined *stable* if, given $\varepsilon > 0$, there exists a $\delta > 0$ so that for every initial state x_0 that fulfills the relation

$$\|x_0 - \bar{x}\| \leq \delta$$

it results in

$$\|x(t) - \bar{x}\| \leq \varepsilon$$

for every $t \geq 0$. ■

Definition

An equilibrium $\bar{x}$ is defined *unstable* if it is not stable.

Stability

Definition

An equilibrium $\bar{x}$ is called *asymptotically stable* if it is stable and

$$\lim_{t\to\infty} \|x(t) - \bar{x}\| = 0$$

Lyapunov's theorem

Definition

A function $V(\cdot)$ is called positive (negative) if a (circular) neighbourhood of the origin exists in which $V(x) > (<) 0$ for $x \neq 0$ and $V(0) = 0$.

Definition

A square and symmetric matrix P is called positive (negative) if $V(x) = x'Px$ is a function called positive (negative).

Theorem

A square and symmetric matrix of order n is called positive if and only if all the n principle minors $D_1, \ldots, D_n$ extracted from the matrix are positive

$$D_1 = \det(p_{11})$$

$$D_2 = \det\left(\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\right), D_j = \det\left(\begin{bmatrix} p_{11} & \cdots & p_{1j} \\ \vdots & \ddots & \vdots \\ p_{1j} & \cdots & p_{jj} \end{bmatrix}\right), j \leq n$$

Lyapunov theorem

Theorem

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x(t))$. Let V: $D \subset R^n \to R$ be a function continuously differentiable so that

$$\begin{aligned} V(0) &= 0 \text{ e } V(x) > 0 \text{ in } D - \{0\} \\ \dot{V}(x) &\leq 0 \text{ in } D \end{aligned}$$

then, $x = 0$ is a stable equilibrium point. Moreover, if

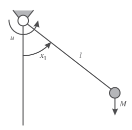
$$\dot{V}(x) < 0 \text{ in } D \text{ -} \{0\}$$

then $x = 0$ is asymptotically stable. Lastly if

$$\dot{V}(x) > 0 \text{ in } D$$

then, $x = 0$ is an unstable equilibrium point.

Pendolum



M = mass, l = rod lenght x_1 = angular position with regard to the vertical, x_2 angular velocity, friction torque proportional to the velocity with a friction coefficient $k > 0$

$$Ml^2\ddot{\theta} = -k\dot{\theta} - M\,\lg\sin(\theta)$$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{g}{l}\sin(x_1(t)) - \frac{k}{Ml^2}x_2(t) \end{aligned}$$

where g is the gravitational acceleration.

Pendolum

$$V(x) = \frac{g}{l}(1 - \cos x_1) + \frac{1}{2}x_2^2$$

$$\begin{aligned} \dot{V}(x) &= \frac{g}{l}\dot{x}_1\sin(x_1) + x_2\dot{x}_2 \\ &= \frac{g}{l}x_2\sin(x_1) - x_2\frac{g}{l}\sin(x_1) - \frac{k}{Ml^2}x_2^2(t) = -\frac{k}{Ml^2}x_2^2(t) \end{aligned}$$

If $k = 0$ than $\dot{V}(x) = 0$ and accordingly the origin is stable and not asymptotically stable.
When $k > 0$ $\dot{V}(x)$ is semi-definite positive because it is independent from x_1.
Than with this Lyapunov function we can only conclude that the origin is stable and not asymptotically stable.

Pendolum

Let's try with a different positive-definite function

$$\begin{aligned} V(x) &= \frac{g}{l}(1-\cos x_1) + \frac{1}{2}x'Px \\ &= \frac{g}{l}(1-\cos x_1) + \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

where P is a positive-definite matrix (i.e. $p_{11} > 0; p_{11}p_{22} - p_{12}^2 > 0$)

$$\begin{aligned} \dot{V}(x) &= x^{'}P\dot{x} + \frac{g}{l}\sin x_1 \dot{x}_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \frac{g}{l}\sin x_1 \dot{x}_1 \\ &= \left[p_{11}x_1 + p_{12}x_2 + \frac{g}{l}\sin(x_1)\right]\dot{x}_1 + (p_{12}x_1 + p_{22}x_2)\dot{x}_2 \\ &= \left[p_{11}x_1 + p_{12}x_2 + \frac{g}{l}\sin(x_1)\right]x_2 \\ &\quad +(p_{12}x_1 + p_{22}x_2)\left(-\frac{g}{l}\sin(x_1) - \frac{k}{Ml^2}x_2\right) \end{aligned}$$

Pendolum

$$\begin{aligned} \dot{V}(x) &= -p_{12}\frac{g}{l}x_1\sin(x_1) + (p_{12} - p_{22}\frac{k}{Ml^2})x_2^2 \\ &\quad +(p_{11} - p_{12}\frac{k}{Ml^2})x_2 + (\frac{g}{l}\sin(x_1) - p_{22}\frac{g}{l}\sin(x_1))x_2 \end{aligned}$$

With $p_{22} = 1$, (removing $sin(x_1)x_2$) $p_{11} = p_{12}\frac{k}{Ml^2} > 0$ (removing x_2 and complying with positive-definite 1), $p_{12} - p_{22}\frac{k}{Ml^2} < 0$ (negative x_2^2), $0 < p_{12} < \frac{k}{Ml^2}$ (negative x_1^2+previous condition), $p_{12}\frac{k}{Ml^2} - p_{12}^2 > 0$ (positive-definite condition 2 given p_{11} e p_{22}), $p_{12} = 0.5\frac{k}{Ml^2}$

$$\dot{V}(x) = -0.5\frac{k}{Ml^2}\frac{g}{l}x_1\sin(x_1) - 0.5\frac{k}{Ml^2}x_2^2$$

As $x_1 sinx_1 > 0$ per $0 < |x_1| < \pi$, $V(x)$ the Lyapunov's theorem is satisfied for $D = \left\{x \in R^2 | |x_1| < \pi\right\}$ and accordingly the origin is an asymptotically stable equilibrium point.

System stability of linear systems

Suppose you want to study the stability of the equilibrium $\bar{x}$ of a linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $u(t) = \bar{u}$, $t \geq 0$, and $x(0) = \bar{x}$. Using the superposition principle with $u'(t) = u''(t) = \bar{u}$, $x_0' = \bar{x}$, $x_0'' = \bar{x} + \delta x_0$, $\alpha = -1$ e $\beta = 1$, you can find that the difference $\delta x(t) = x(t) - \bar{x}$ between the perturbed dynamic and nominal dynamic is hold by the following equation

$$\begin{aligned} \dot{x}'(t) &\equiv Ax'(t) + B\bar{u} \quad , \quad x'(t_0) \equiv \bar{x} \\ \dot{x}''(t) &\equiv Ax''(t) + B\bar{u} \quad x''(t_0) \equiv \bar{x} + \delta x_0 \\ \delta\dot{x}(t) &= A\delta x(t) \quad , \quad \delta x(0) = \delta x_0 \end{aligned}$$

Theorem

An equilibrium of a linear time-invariant system is stable, asymptotically stable or unstable, if and only if all the equilibria are respectively, stable, asymptotically stable or unstable. ■

Stability, asymptotic stability or instability of the system.

Lyapunov theorem for linear systems

Theorem

A linear time-invariant system is asymptotically stable if and only if given a symmetrical positive-definite matrix Q exists a symmetrical positive-definite matrix P that satisfies the Lyapunov equation.

$$PA + A'P = -Q$$

Moreover, if the system is asymptotically stable, than P is the only solution.

Please note that $V(x) = x'Px$ is a Lyapunov function for the system $\dot{x} = Ax$. As a matter of fact

$$\dot{V}(x) = x'PAx + x'A'P.x = x'(PA + A'P)x = -x'Qx$$

Example

$$\begin{cases} \dot{x}_1 = -2x_1 - 3x_2 \\ \dot{x}_2 = 4x_1 - 2x_2 \end{cases}$$

Studying the stability with the Lyapunov equation

$$A = \begin{bmatrix} -2 & -3 \\ 4 & -2 \end{bmatrix}, Q = I$$

$$\begin{aligned} \begin{bmatrix} -2 & 4 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 4 & -2 \end{bmatrix} &= -I \\ \begin{bmatrix} -2a + 4b - 2a + 4b & -2b + 4c - 3a - 2b \\ -3a - 2b - 2b + 4c & -3b - 2c - 3b - 2c \end{bmatrix} &= -I \end{aligned}$$

$$\begin{cases} -4a + 8b = -1 \\ -4b + 4c - 3a = 0 \\ -6b - 4c = -1 \end{cases} \begin{cases} a = \frac{1+8b}{4} \\ c = \frac{1-6b}{4} \\ -4b + 1 - 6b - \frac{3}{4}(1 + 8b) = 0 \end{cases}$$

Example

$$\left\{-16b + \frac{1}{4} = 0 \right. \begin{cases} b = \frac{1}{64} \\ a = \frac{9}{32} \\ c = \frac{58}{64*4} = \frac{29}{128} \end{cases}$$

$$P = \frac{1}{128}\begin{bmatrix} 36 & 2 \\ 2 & 29 \end{bmatrix} > 0 \Rightarrow Asymptotically\ Stable$$

$$P = Lyap(A', Q)$$

Eigenvalues and diagonalizability

Given a complex variable λ and a square matrix A, $n \times n$, it is possible to associate the *characteristic polynomial*, of grade n,

$$\varphi(\lambda) = \det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \ldots + \alpha_{n-1}\lambda + \alpha_n$$

and the *characteristic equation*

$$\varphi(\lambda) = 0$$

The n solutions λ_i of the characteristic equation are called eigenvalues of A; if the latest is formed of real numbers, also the coefficients α_i are reals and the eigenvalues are reals or complex conjugate pairs. For each eigenvalue λ_i it is possible to associate a column vector v_i, called eigenvector, satisfying $(\lambda I - A)v_i = 0$.
Assuming that the eigenvalues are all different:

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

Stability

Theorem

*A **linear time-invariant system is asymptotically stable** if and only if all the eigenvalues have negative real part.* ■

Theorem

*A **linear time-invariant system** is unstable if at least one eigenvalue has a positive real part.* ■

Theorem

*A **linear time-invariant system** with eigenvalues that have negative and null real part, is unstable if and only if among the eigenvalues with null real part, there is at least one that belongs to a mini Jordan block of dimension major then 1.* ■

Linearization in a neighbourhood of an equilibrium

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \quad x(t_0) = x_{t_0}, \\ y(t) &= g(x(t), u(t)) \end{aligned}$$

Given an equilibrium point $(\bar{x}, \bar{u})$ and $\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$, $\delta y = y - \bar{y}$, $\delta x_{t_0} = x_{t_0} - \bar{x}_{t_0}$ variations on state, input and output variables w.r.t. to the equilibrium, supposing that f and g are sufficiently regular, they can be developed in a Taylor series around $(\bar{x}, \bar{u})$ to obtain a linearized system

$$\begin{aligned} \delta\dot{x}(t) &= A\delta x(t) + B\delta u(t), \quad \delta x(t_0) = \delta x_{t_0}, \\ \delta y(t) &= C\delta x(t) + D\delta u(t) \end{aligned}$$

$$\begin{aligned} A &= \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=\bar{x}, u=\bar{u}}, \quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=\bar{x}, u=\bar{u}} \\ C &= \left.\frac{\partial g(x,u)}{\partial x}\right|_{x=\bar{x}, u=\bar{u}}, \quad D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{x=\bar{x}, u=\bar{u}} \end{aligned}$$

Stability of the nonlinear system

Theorem

The state equilibrium $\bar{x}$, associated with the input $\bar{u}$, of a nonlinear system is asymptotically stable if the linearized system is asymptotically stable.

Theorem

The state equilibrium $\bar{x}$, associated with the input $\bar{u}$, of a nonlinear system is asymptotically stable if all the eigenvalues of the corresponding linearized system have a real negative part.

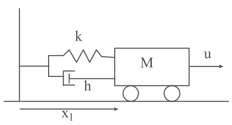
Theorem

The state equilibrium $\bar{x}$, associated with the input $\bar{u}$, of a nonlinear system is unstable if at least one of the eigenvalues of the corresponding linearized system has a positive real part.

The condition is only sufficient: $\dot{x} = x^3, \dot{x} = -x^3$

Stability/Equilibrium - Stability linearized

Example (without solutions)



The system depicted in the picture is described by the following model

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k_0}{M} e^{-x_1} x_1 - \frac{h}{M} x_2 + \frac{u}{M} \\ y = x_1 \end{cases}$$

where $M = 1$, $h = 1.1$, $k_0 = 0.33$.

- Find the value $\bar{u}$ whereby in the equilibrium $\bar{x}_1 = 0.5$.
- Study the stability of this point of equilibrium.
- Find the value $\bar{u}$ whereby in the equilibrium $\bar{x}_1 = 2$.
- Study the stability of this point of equilibrium.

Equivalent representations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Consider a non singular constant matrix $T \in R^{n \times n}$, a new state vector $\hat{x}$ can be defined as

$$\hat{x}(t) = Tx(t)$$
$$x(t) = T^{-1}\hat{x}(t)$$

$$\begin{aligned} \dot{\hat{x}}(t) &= T\dot{x}(t) = T(Ax(t) + Bu(t)) = T(AT^{-1}\hat{x}(t) + Bu(t)) \\ &= TAT^{-1}\hat{x}(t) + TBu(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ y(t) &= Cx(t) + Du(t) = CT^{-1}\hat{x}(t) + Du(t) = \hat{C}\hat{x}(t) + \hat{D}u(t) \end{aligned}$$

$$\hat{A} = TAT^{-1} \quad , \quad \hat{B} = TB \quad , \quad \hat{C} = CT^{-1} \quad , \quad \hat{D} = D$$

Equivalent representations

The system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is *equivalent* to the system (A, B, C, D), meaning that for an input $u(t)$, $t \geq 0$, and two initial states x_0 and $\hat{x}_0$ linked by the condition $\hat{x}_0 = Tx_0$, state dynamics of the systems are linked by the relation $\hat{x}(t) = Tx(t)$, $t > 0$, and the output dynamics are identical.
Please note that the A and $\hat{A}$ matrices are similar and so they have the same eigenvalues.

Eigenvalues and diagonalization

Suppose that all the eigenvalues of A are *distinct*.

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

It can be proved that the *eigenvectors matrix*, $n \times n$, is not singular and so it is possible to set

$$T_D^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

so that we define the diagonal form of A

$$A_D = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = T_D A T_D^{-1}$$

Modes

Free state dynamic

$$\begin{aligned}
\hat{x}_l(t) &= e^{\hat{A}_D t}\hat{x}_0 = \sum_{k=0}^{\infty} \frac{\left(\hat{A}_D t\right)^k}{k!}\hat{x}_0 = \\
&= diag\left\{\sum_{k=0}^{\infty}\frac{(s_1 t)^k}{k!}, \sum_{k=0}^{\infty}\frac{(s_2 t)^k}{k!}, \ldots, \sum_{k=0}^{\infty}\frac{(s_n t)^k}{k!}\right\}\hat{x}_0 = \\
&= diag\left\{e^{s_1 t}, e^{s_2 t}, \ldots, e^{s_n t}\right\}\hat{x}_0
\end{aligned}$$

Subsequently, free dynamics of the state and of the output are

$$\begin{aligned}
x_l(t) &= T_D^{-1}\hat{x}_l(t) = T_D^{-1}\,\mathrm{diag}\left\{e^{s_1 t}, e^{s_2 t}, \ldots, e^{s_n t}\right\} T_D x_0 \\
y_l(t) &= C T_D^{-1} diag\left\{e^{s_1 t}, e^{s_2 t}, \ldots, e^{s_n t}\right\} T_D x_0
\end{aligned}$$

They are linear combinations with coefficients dependent from x_0, T_D and C, of exponential terms $e^{s_i t}$, $i = 1, 2, \ldots, n$, called *modes*. It is important to observe that pairs $s_i = \sigma_i + j\omega_i$, $\bar{s}_i = \sigma_i - j\omega_i$ of complex conjugate eigenvalues of A produce terms that added together create for x_l and y_l a single real term of $e^{\sigma_i t}\sin(\omega_i t + \varphi_i)$ kind.

Example

Consider a system with

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Compute the free dynamic

$$\begin{aligned}
[A_D, T_D^{-1}] &= eig(A) \\
A_D &= \begin{bmatrix} 1+j & 0 \\ 0 & 1-j \end{bmatrix} \\
T_D^{-1} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \\
T_D &= \frac{1}{\sqrt{2}}\begin{bmatrix} +1 & -j \\ +1 & j \end{bmatrix}
\end{aligned}$$

$$x_l(t) = T_D^{-1}\hat{x}_l(t) = T_D^{-1} diag\left\{e^{s_1 t}, e^{s_2 t}, \ldots, e^{s_n t}\right\} T_D x_0$$

Example

$$\begin{aligned}
x_l(t) &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}\begin{bmatrix} e^{(1+j)t} & 0 \\ 0 & e^{(1-j)t} \end{bmatrix}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} x_0 = \\
&= \frac{1}{2}e^t\begin{bmatrix} e^{jt} & e^{-jt} \\ je^{jt} & -je^{-jt} \end{bmatrix}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} x_0 \\
&= \frac{1}{2}e^t\begin{bmatrix} e^{jt}+e^{-jt} & -je^{jt}+je^{-jt} \\ je^{jt}-je^{-jt} & e^{jt}+e^{-jt} \end{bmatrix} x_0 \\
&= e^t\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} x_0
\end{aligned}$$

$$\begin{aligned}
e^{j\theta} &= \cos(\theta) + j\sin(\theta) \\
e^{-j\theta} &= \cos(-\theta) + j\sin(-\theta) = \cos(\theta) - j\sin(\theta) \\
sen(\theta) &= \frac{e^{j\theta} - e^{-j\theta}}{2j}, \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}
\end{aligned}$$

Example

$$\begin{aligned}
y_l(t) &= e^t\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} x_0 \\
&= e^t\begin{bmatrix} \cos(t) - \sin(t) & \sin(t) + \cos(t) \end{bmatrix} x_0 \\
&= \sqrt{2}e^t\begin{bmatrix} \cos(t + \pi/4) & \cos(t - \pi/4) \end{bmatrix} x_0
\end{aligned}$$

keep in mind that $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp sin(\alpha)\sin(\beta)$.

Calculus of the exponential matrix

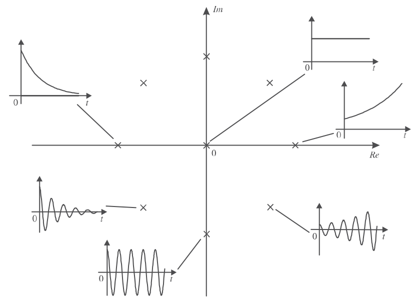
Calculus of the free dynamic for systems with distinct eigenvalues

$$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t}\mathcal{F}(\lambda_i), \quad \mathcal{F}(\lambda_i) = \prod_{j=0, j\neq i}^{n} \frac{A-\lambda_j I}{\lambda_i - \lambda_j}$$

In the example

$$\begin{aligned}
e^{At} &= \frac{e^{(1+j)t}}{(1+j)-(1-j)}\begin{bmatrix} 1-(1-j) & 1 \\ -1 & 1-(1-j)\end{bmatrix} \\
&+ \frac{e^{(1-j)t}}{(1-j)-(1+j)}\begin{bmatrix} 1-(1+j) & 1 \\ -1 & 1-(1+j)\end{bmatrix} \\
&= \frac{1}{2j}\begin{bmatrix} je^{(1+j)t}+je^{(1-j)t} & e^{(1+j)t}-e^{(1-j)t} \\ -e^{(1+j)t}+e^{(1-j)t} & je^{(1+j)t}+je^{(1-j)t}\end{bmatrix} \\
&= -\frac{j}{2}e^{t}\begin{bmatrix} je^{jt}+je^{-jt} & e^{jt}-e^{-jt} \\ -e^{jt}+e^{-jt} & je^{jt}+je^{-jt}\end{bmatrix} \\
&= \frac{1}{2}e^{t}\begin{bmatrix} e^{jt}+e^{-jt} & -je^{jt}+je^{-jt} \\ je^{jt}-je^{-jt} & e^{jt}+e^{-jt}\end{bmatrix}
\end{aligned}$$

Modes of systems with distinct eigenvalues

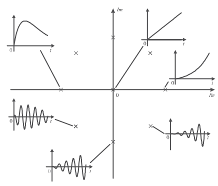


Matrix of the non-diagonalizable dynamic

When the matrix A has multiple eigenvalues, that is when not all the eigenvalues s_i, $i = 1, 2, ..., n$, are different from one another, it can happen that it is impossible to make it *diagonal*; anyway it exists a transformation matrix $T = T_J$ able to transform A in the so-called *Jordan normal form* $\hat{A}_J$. The Modes that form the free dynamics of the state and of the output are in the following form

$$t^{\eta-1}e^{s_i t} \text{ o } t^{\eta-1}e^{\sigma_i t}\sin(\omega_i t + \varphi_i)$$

if s_i is real and where η is any integer number included between 1 and the maximum dimension $(\bar{\eta})$ of Jordan's miniblocks with s_i e φ_i in a proper fase.



Example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} x_2(t) &= x_2(0) \\ x_1(t) &= x_1(0) + x_2(0)t \end{aligned}$$

From the previous formulae with $\bar{\eta} = 2$ and $\lambda_i = 0$ you get that the modes are 1 and t.

Impulse

An impulse is a signal such that

$$\begin{aligned} imp(t) &= 0 \quad , \quad t \neq 0 \\ \int_{-\infty}^{+\infty} imp(t)\,dt &= 1 \end{aligned}$$

where the integral can be limited to any range enclosing the point $t = 0$. Given any function $\varphi(t)$ continuous for $t = \tau$, you have

$$\varphi(t)\,imp(t-\tau) = \varphi(\tau)\,imp(t-\tau)$$

and so

$$\int_{-\infty}^{+\infty} \varphi(t)\,imp(t-\tau)\,dt = \varphi(\tau)\int_{-\infty}^{+\infty} imp(t-\tau)\,dt = \varphi(\tau)$$

Impulse response and forced dynamic

Assume that $m = 1$ e $u(t) = imp(t)$.

$$\begin{aligned} g_x(t) &= e^{At}B \\ g_y(t) &= Ce^{At}B + D imp(t) \end{aligned}$$

These dynamics are called *impulse response of the state* and *of the output*. Please note that, for $t > 0$, they overlap with the free dynamics produced from the initial state $x(0) = B$ and as such they are made of linear combinations of the system modes (applied in $t_0 = 0$).

$$\begin{aligned} g_x(t) * u(t) &= \int_{-\infty}^{+\infty} g_x(t-\tau)\,u(\tau)\,d\tau = \int_0^t g_x(t-\tau)\,u(\tau)\,d\tau = \\ &= \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau = x_f(t) \end{aligned}$$

Impulse response and forced dynamic

$$
\begin{aligned}
g_y(t) * u(t) &= \int_{-\infty}^{+\infty} g_y(t-\tau)\, u(\tau)\, d\tau = \int_0^t g_y(t-\tau)\, u(\tau)\, d\tau = \\
&= \int_0^t \left(Ce^{A(t-\tau)} B + D imp(t-\tau) \right) u(\tau)\, d\tau = \\
&= C \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau + Du(t) = y_f(t)
\end{aligned}
$$

Routh - Hurwitz stability criterion

$$\varphi(s) = \varphi_0 s^n + \varphi_1 s^{n-1} + \varphi_2 s^{n-2} + \ldots + \varphi_{n-1} s + \varphi_n \quad , \quad \varphi_0 \neq 0$$

Routh's table, made from the characteristic polynomial. It has $n+1$ rows and a triangular structure as every two rows, excluding the first if n is even, the number of elements decreases of one:

$$
\begin{array}{ccccccc}
\varphi_0 & \varphi_2 & \varphi_4 & \cdot & \cdot & \cdot & \cdot \\
\varphi_1 & \varphi_3 & \varphi_5 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
h_1 & h_2 & h_3 & \cdot & \cdot & & \\
k_1 & k_2 & k_3 & \cdot & \cdot & & \\
l_1 & l_2 & l_3 & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & & &
\end{array}
$$

$$l_i = -\frac{1}{k_1} \det\left(\begin{bmatrix} h_1 & h_{i+1} \\ k_1 & k_{i+1} \end{bmatrix} \right) = h_{i+1} - \frac{h_1 k_{i+1}}{k_1}$$

Moreover, if $k_1 = 0$, the Routh's table is conventionally said to be not well-defined.

Routh - Hurwitz stability criterion

Theorem

The system is asymptotically stable if and only if the Routh's table related to the characteristic polynomial is well-defined and all the elements of its first column have the same sign. ■

Example

Consider a system with characteristic polynomial

$$\varphi(s) = s^3 + (2+\beta) s^2 + (1+2\beta) s + \alpha + \beta$$

The corrisponding Routh's table is

$$
\begin{array}{cc}
1 & 1+2\beta \\
2+\beta & \alpha+\beta \\
\dfrac{2(\beta+1)^2 - \alpha}{2+\beta} & \\
\alpha+\beta &
\end{array}
$$

Therefore the system is asymptotically stable for the pairs of value α, β that satisfy $\beta > -2$, $2(\beta+1)^2 > \alpha$, $\beta > -\alpha$.

Eigenvalues criterion

$$\varphi(s) = \varphi_0 s^n + \varphi_1 s^{n-1} + \varphi_2 s^{n-2} + \ldots + \varphi_{n-1} s + \varphi_n \quad , \quad \varphi_0 \neq 0$$

Theorem

If the system is asymptotically stable, than the coefficients φ_i, $i = 0, 1, \ldots, n$, of the characteristic polynomial have all the same sign. ■

It is simple to verify that this condition is also sufficient for systems of order $n = 1$ and $n = 2$.

$$\varphi_1/\varphi_0 = -tr(A) = -\sum_{i=1}^{n} s_i \quad > 0, \quad \varphi_n/\varphi_0 = \det(-A) = (-1)^n \prod_{i=1}^{n} s_i > 0$$

Example

Applying the theorem we can conclude the lack of asymptotic stability for systems with characteristic polynomials such as

$$\varphi(s) = s^3 + 3s^2 - s - 3$$

Eigenvalues criterion

Example

or

$$\varphi(s) = s^4 + 5s^2 + 4$$

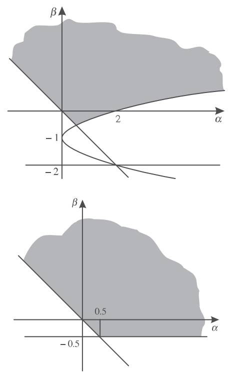
to which we can find the corresponding eigenvalues -3, -1, $+1$ e $\pm j$, $\pm 2j$. The condition is not also sufficient for $n > 2$: for example the roots of the following equation

$$\varphi(s) = s^3 + s^2 + s + 1 = 0 \tag{1}$$

are -1 e $\pm j$.

Eigenvalues criterion

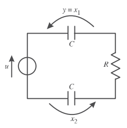
In order to make a comparison, please note that the necessary condition is verified when the following inequalities are verified $\beta > -2$, $\beta > -\frac{1}{2}, \beta > -\alpha$.



Properties of the asymptotically stable systems

- $\lim_{t\to\infty} x(t)$ is independent from the initial state
- the impulse response, both of the state and of the output, asymptotically tends to zero
- In the same way every response to any input of limited time duration tends to zero.
- Given $\det(A) \neq 0$, because an asymptotically stable system cannot have eigenvalues equal to zero, the equilibrium associated with the constant input $u(t) = \bar{u}$ is $\bar{x} = -A^{-1}B\bar{u}$. Dynamics produced by $u(t) = \bar{u}sca(t)$ tend to it at any initial state.
- *External stability* or *BIBO (Bounded Input Bounded Output) stability*: produces bounded output in relation to every limited input.

Electrical circuit - non-reachable



$$i_C = C\frac{dv_C}{dt}$$

$$\begin{aligned}\dot{x}_1(t) &= -\frac{1}{RC}(x_1(t) + x_2(t) - u(t))\\ \dot{x}_2(t) &= -\frac{1}{RC}(x_1(t) + x_2(t) - u(t))\\ y(t) &= x_1(t)\end{aligned}$$

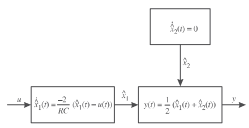
$$\begin{aligned}\hat{x}_1(t) &= x_1(t) + x_2(t)\\ \hat{x}_2(t) &= x_1(t) - x_2(t)\end{aligned}$$

Eletrical circuit - 2

Please note that $\hat{x}_1$ and $\hat{x}_2$ can be considered state variable as the equations establish a bijective relation between them and the variables previously used. The system is equivalent to the following system

$$\begin{aligned}\dot{\hat{x}}_1(t) &= -\frac{2}{RC}(\hat{x}_1(t) - u(t))\\ \dot{\hat{x}}_2(t) &= 0\\ y(t) &= \frac{1}{2}(\hat{x}_1(t) + \hat{x}_2(t))\end{aligned}$$

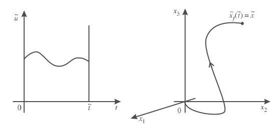
The relations between the variables u, $\hat{x}_1$, $\hat{x}_2$ and y are represented in



Electrical Circuit - 3

In this particular case you can easily note that the difference $\hat{x}_2$ between the voltages on the capacitors does not depend in any way from the input u, but you can easily conclude that you can choose an input u so that, starting from the null-state, the sum of the tensions $\hat{x}_1$ at time $\tilde{t} > 0$ is equal to $\hat{x}_{1fin}$.

Reachability



Definition

A state $\tilde{x}$ is called *reachable* if there exist a finite time $\tilde{t} > 0$ and an input $\tilde{u}$, defined between 0 and $\tilde{t}$, so that $\tilde{x}_f\left(\tilde{t}\right) = \tilde{x}$, where $\tilde{x}_f(t)$, $0 \leq t \leq \tilde{t}$, is the so called forced dynamic of the state produced by $\tilde{u}$.
A system where all the state are reachable is called *completely reachable*. ■

The reachability is assigned to the pair (A, B).
Reachability Matrix is defined as

$$M_r = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in R^{n\times mn}$$

Reachability

Theorem

The system is completely reachable if and only if the rank of the reachability matrix is equal to n, so that

$$\rho(M_r) = n$$

If a system is non completely reachable it is possible to decompose the reachable part from the unreachable one.

Electrical network

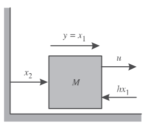
With $R = C = 1$

$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\rho(M_r) = \rho\left(\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\right) = 1$$

Exercize: compute the impulse response of the output.

Mass on rail - not observable



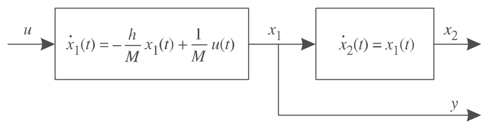
The picture represents an object of M mass moving on a straight rail characterized by a viscous friction coefficient h. Using as input variable u the external force applied to the object, as state variables x_1 and x_2 the speed and the object's position and as an output variable the speed, it is possible to write

$$u = Ma + hv$$

$$\begin{aligned} \dot{x}_1(t) &= -\frac{h}{M}x_1(t) + \frac{1}{M}u(t) \\ \dot{x}_2(t) &= x_1(t) \\ y(t) &= x_1(t) \end{aligned}$$

Mass on rail - not observable

The dependencies between the variables u, x_1, x_2 and y are here represented in the following scheme



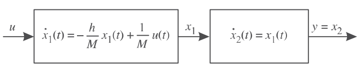
In particular, please note that the exam of the output does not allow to get informations of initial position as the speed y does not depend on this last variable. Vice versa, knowing the output y allows to easily define the initial value of the state variable x_1. It can be observed that if the position is chosen as an output variable so that

$$y(t) = x_2(t)$$

Mass on rail - not observable

it is possible to estimate the initial state starting from the output dynamic knowledge

$$\begin{aligned} x_1(0) &= \dot{x}_2(0) = \dot{y}(0) \\ x_2(0) &= y(0) \end{aligned}$$



Observability

Definition

A state $\tilde{x} \neq 0$ of a system is called *unobservable* if, for any finite $\tilde{t} > 0$, called $\tilde{y}_l(t)$, $t \geq 0$, the free output dynamic produced by $\tilde{x}$, is $\tilde{y}_l(t) = 0$, $0 \leq t \leq \tilde{t}$.
A system with no unobservable states is called *completely observable.* ■

The observability property is assigned to the (A, C) pair.
Observability Matrix is defined as

$$M_o = \begin{bmatrix} C' & A'C' & A'^2C' & \cdots & A'^{n-1}C' \end{bmatrix} \in R^{n \times pn}$$

where the vertex stands for the operation of transposition.

Theorem

The system (A,C) is completely observable if and only if the observability matrix rank is equal to n, so that

$$\rho(M_o) = n \qquad (2)$$

■

Mass on rail

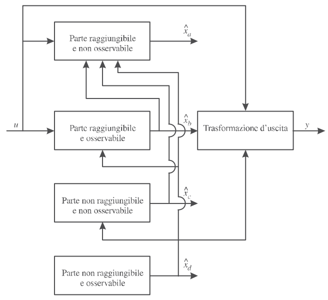
Con $M = h = 1$

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

$$\rho(M_o) = \rho\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = 1$$

Exercise: compute the output impulse response.

Canonical decomposition



Theorem

The output forced dynamic of the system, of all the four part of the system decomposition, depends only from the reachable and observable part.

Canonical decomposition

A reachable and observable system is considered to be in its *minimal form* as it is not possible to use a number of variables lower than its order to describe the relation between input and output that it establishes.

Theorem

Assuming that a system is in its minimal form, it is also BIBO stable if and only if it is asymptotically stable.

Laplace transform

Given a complex function f of the real variable t; Consider $s = \sigma + j\omega \in C$ a complex variable with a real part σ and coefficient of the imaginary unit j equal to ω. If the function

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st} dt$$

exists for at least a value of s, it is called *Laplace transform* of $f(t)$. Usually the transform is indicated with a capital of the same letter, which stands for the transforming time function and it is written as $F(s) = \mathcal{L}[f(t)]$ and $f(t) = \mathcal{L}^{-1}[F(s)]$.

Impulse transform

Example

Given $f(t) = imp(t)$; then for every s it is possible to get

$$\mathcal{L}[imp(t)] = \int_{0^-}^{+\infty} imp(t)\, e^{-st} dt = e^{-s0} = 1$$

Laplace tranform's properties

Linearity
Consider two function f and g. In this case $\forall\alpha \in C$, $\forall\beta \in C$

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha F(s) + \beta G(s)$$

it comes out that the Laplace transform is a linear operator.
Time shifting property
For any $\tau > 0$ consider a function $\hat{f}(t) = f(t - \tau)$ obtained shifting forward the function $f(t)$, assumed null for negative times, of a time equal to τ. You get the following

$$\mathcal{L}\left[\hat{f}(t)\right] = \mathcal{L}[f(t - \tau)] = e^{-\tau s} F(s)$$

Time domain derivation
Suppose that the function $f(t)$ is derivable, meaning generalized functions, for all the $t \geq 0$ or at least with left and right derivatives (for $t > 0$). It than follows

$$\mathcal{L}\left[\dot{f}(t)\right] = sF(s) - f(0)$$

Laplace tranform's properties

Time domain integration
Suppose that the function $f(t)$ is integrable between 0 and $+\infty$. Than

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{1}{s}F(s)$$

so that dividing using s in the complex variable domain is like integrate in the time domain and $1/s$ can be read as *integration operation*.

Laplace tranform's properties

Initial value theorem
If a real function f has a rational transform F where the denominator grade is higher than the numerator grade, thus

$$f(0) = \lim_{s\to\infty} sF(s)$$

If the function f is a first kind discontinuos for $t=0$, $f(0)$ has to be considered as $f\left(0^+\right)$.
This is valid under broad assumptions even if F is not rational only if $f(0)$, or at least $f\left(0^+\right)$, exists.
Final value theorem
If a real function f has a rational transform F where the denominator's order is higher of the numerator's order and the poles have null or negative real part, than

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

This is valid under broad assumptions even if F is not rational when $\lim_{t\to\infty} f(t)$ exists.

Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Laplace tranform

$f(t)$	$F(s)$
$imp(t)$	1
$sca(t)$	$\frac{1}{s}$
$ram(t)$	$\frac{1}{s^2}$
$par(t)$	$\frac{1}{s^3}$
$e^{\alpha t} sca(t)$	$\frac{1}{s-\alpha}$
$te^{\alpha t} sca(t)$	$\frac{1}{(s-\alpha)^2}$
$\sin(\omega t)\, sca(t)$	$\frac{\omega}{s^2+\omega^2}$

$f(t)$	$F(s)$
$\cos(\omega t)\, sca(t)$	$\frac{s}{s^2+\omega^2}$
$t\sin(\omega t)\, sca(t)$	$\frac{2\omega s}{\left(s^2+\omega^2\right)^2}$
$t\cos(\omega t)\, sca(t)$	$\frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$
$e^{\sigma t}\sin(\omega t)\, sca(t)$	$\frac{\omega}{(s-\sigma)^2+\omega^2}$
$e^{\sigma t}\cos(\omega t)\, sca(t)$	$\frac{s-\sigma}{(s-\sigma)^2+\omega^2}$
$te^{\sigma t}\sin(\omega t)\, sca(t)$	$\frac{2\omega(s-\sigma)}{\left[(s-\sigma)^2+\omega^2\right]^2}$
$te^{\sigma t}\cos(\omega t)\, sca(t)$	$\frac{(s-\sigma)^2-\omega^2}{\left[(s-\sigma)^2+\omega^2\right]^2}$

Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Transfer function

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

where $u \in R^m$, $x \in R^n$, $y \in R^p$ and call $U(s)$, $X(s)$ and $Y(s)$, functions of the complex variable s, Laplace transforms of $u(t)$, $x(t)$ and $y(t)$. Remembering the linearity property of the Laplace transform and the formula for the calculus of the transform of a derivative function, you get

$$\begin{aligned} sX(s) - x(0) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

As $\det(sI-A)$ is a polynomial for s not identicaly null, $(sI-A)^{-1}$ exists and

$$\begin{aligned} X(s) &= (sI-A)^{-1}BU(s) + (sI-A)^{-1}x(0) \\ Y(s) &= (C(sI-A)^{-1}B + D)U(s) + C(sI-A)^{-1}x(0) \end{aligned}$$

Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Transfer function

The matrix $p \times m$

$$G(s) = C(sI-A)^{-1}B + D$$

is called *transfer function*. By multiplying $G(s)$ by the Laplace tranform of the input $U(s)$, we get Laplace transform of the output $Y(s)$ corresponding to the initial null state (or of the forced output). For null initial condition the system can be represented with its *input-output representation*

$$Y(s) = G(s)U(s)$$

Hence the transfer function is an *external representation* of the system, opposed to the internal representation expressed by the state representation.

Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Tranfer function and impulse response

Consider a SISO system with input $u(t) = imp(t)$. Thus, $U(s) = 1$ and it results $Y(s) = G(s)$, so that the transfer function can be interpreted as a Laplace transform of the output impulse respond g_y.

$$y(t) = \int_0^t g_y(t-\tau)u(\tau)d\tau$$

You get $\mathcal{L}[y(t)] = Y(s) = G(s)U(s)$.

Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Electrical network

When $R = C = 1$

$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

$$\varphi(s) = \det(sI - A) = \det\left(\begin{bmatrix} s+1 & 1 \\ 1 & s+1 \end{bmatrix}\right) = s^2 + 2s$$

$\Downarrow$

simply stable

$$\rho(M_r) = \rho\left(\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\right) = 1, \quad \rho(M_o) = \rho\left(\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\right) = 2$$

$\Downarrow$

not reachable, observable

Electrical network

$$\begin{aligned} G(s) &= \frac{1}{s^2+2s}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s+1 & -1 \\ -1 & s+1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \\ &= \frac{s}{s^2+2s} = \frac{1}{s+2} \end{aligned}$$

$\Downarrow$

externally stable

Representation and parameters

The elements of the matrix $(sI - A)^{-1}$ are rational functions w.r.t. s, all with a common denominator of order n, the characteristic polynomial of A.
Referring to SISO systems you can write

$$G(s) = \frac{N_G(s)}{D_G(s)} = \frac{\beta_\nu s^\nu + \beta_{\nu-1}s^{\nu-1} + \ldots + \beta_1 s + \beta_0}{\alpha_n s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0}$$

where $\nu \leq n$, and for strictly proper systems $\beta_n = 0$. Moreover, without loss of generality, it is possible to assume $\alpha_n = 1$. In the SISO systems, the difference between the denominator and the numerator orders is called *relative degree* of $G(s)$.
Dynamical systems for which $n < \nu$ are called *improper*.

Poles and zeros

$\hat{s}$ is a *zero* of $G(s)$ (and of the system), if it is a root of its numerator, that is if $N_G(\hat{s}) = 0$.
$\hat{s}$ is a *pole* of $G(s)$ (and of the system), if it is a root of its denominator, that is if $D_G(\hat{s}) = 0$.
Poles of $G(s)$ are also the roots of the equation $det(sI - A) = 0$ and thus eigenvalues of A.
Nervertheless, there could be eigenvalues that are not poles of $G(s)$ if there have been cancellations between numerator and denominator of $G(s)$.
Moreover, if there has not been any cancellation, the multiplicity of a pole coincides with the one of the corresponding eigenvalue.

Cancellation and stability

$G(s) = \mathcal{L}\left[g_y(t)\right]$ depends only on the reachable and observable part of the system. In the SISO case poles of $G(s)$ are also eigenvalues of the reachable and observable part of the system.
It can be proved that the order of the polynomial $D_G(s)$ (the number of poles), coincides with the order of the reachable and observable part of the system.
Moreover

Theorem

Poles of a SISO system coincide with the eigenvalues of the reachable and observable part of the system itself including the multiplicity.

Hence system eigenvalues that are non poles of the transfer function $G(s)$, are necessarily eigenvalues of the unreachable and unobservable part of the system.
They are not present among poles of $G(s)$ after concellations.

Theorem

Considering a system in its minimal form, it is asymptotically stable if and only if the poles of $G(s)$ have all negative real part.

Invariance of the transfer function

$\hat{A} = TAT^{-1}$, $\hat{B} = TB$, $\hat{C} = CT^{-1}$, $\hat{D} = D$.

$$\begin{aligned} \hat{G}(s) &= \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} = CT^{-1}(sI - TAT^{-1})^{-1}TB + D = \\ &= C(sT^{-1}IT - T^{-1}TAT^{-1}T)^{-1}B + D = C(sI - A)^{-1}B = G(s) \end{aligned}$$

This result is obvious as the transfer function represents the link between Laplace transforms of the input and of the output and it has to be independent from the characteristic internal rapresentation used.

$$(AB)^{-1} = B^{-1}A^{-1}$$

Time delay

It is possible to define a transfer function for many classes of Infinite-Dimensional Dynamical System as long as they have the time-linearity and time-invariat properties. For example the time delay described by the input-output relation $(y(t) = u(t-\tau))$ is linear and stable. Thus, transforming with Laplace both members and remembering the rules of translation in the domain of time for the transforms themself, you get $Y(s) = e^{-\tau s}U(s)$, so that

$$G(s) = e^{-\tau s}$$

Example

If

$$Y(s) = \frac{s-10}{(s+2)(s+5)}$$

it is possible to write

$$\frac{s-10}{(s+2)(s+5)} \equiv \frac{P_1}{s+2} + \frac{P_2}{s+5}$$

where

$$\begin{aligned} P_1(s+5) + P_2(s+2) &= s-10 \\ (P_1+P_2)s + 5P_1 + 2P_2 &= s-10 \end{aligned}$$

$$\begin{cases} P_1 + P_2 = 1 \\ 5P_1 + 2P_2 = -10 \end{cases} \quad \begin{cases} P_1 = 1 - P_2 \\ 5 - 5P_2 + 2P_2 = -10 \end{cases} \quad \begin{cases} P_1 = 1 - P_2 \\ 5 - 5P_2 + 2P_2 = -10 \end{cases}$$

$$\begin{cases} P_1 = -4 \\ P_2 = 5 \end{cases}$$

Example

ultimately

$$y(t) = \mathcal{L}^{-1}\left[\frac{s-10}{(s+2)(s+5)}\right] = \left(-4e^{-2t} + 5e^{-5t}\right) sca(t)$$

Second method

$$\begin{aligned} P_2(-5+2) &= -5-10 \\ P_1(-2+5) &= -2-10 \\ P_1 &= -4 \quad , \quad P_2 = 5 \end{aligned}$$

Tranfer function representations

$$G(s) = \frac{\rho \prod_i (s+z_i) \prod_i (s^2 + 2\zeta_i \alpha_{ni} s + \alpha_{ni}^2)}{s^g \prod_i (s+p_i) \prod_i (s^2 + 2\xi_i \omega_{ni} s + \omega_{ni}^2)}$$

$$G(s) = \frac{\mu \prod_i (1+\tau_i s) \prod_i (1 + 2\zeta_i s/\alpha_{ni} + s^2/\alpha_{ni}^2)}{s^g \prod_i (1+T_i s) \prod_i (1 + 2\xi_i s/\omega_{ni} + s^2/\omega_{ni}^2)}$$

ρ is called *transfer constant*, g is called *type*
$z_i \neq 0$ e $p_i \neq 0$ are the real *zeros* and *poles* different from null, with the changed sign.
$\alpha_{ni} > 0$ and $\omega_{ni} > 0$ are the *natural modes* of the complex pairs of zeros and poles and the scalars ζ_i and ξ_i, in modulus lower than 1, the correspondents *damping* factors
μ is called *gain*, while the scalars $\tau_i \neq 0$ and $T_i \neq 0$ are the *time constants*

$$\mu = \frac{\rho \prod_i z_i \prod_i \alpha_{ni}^2}{\prod_i p_i \prod_i \omega_{ni}^2} \quad , \quad \rho = \frac{\mu \prod_i \tau_i \prod_i \omega_{ni}^2}{\prod_i T_i \prod_i \alpha_{ni}^2}$$

$$\tau_i = \frac{1}{z_i} \quad , \quad T_i = \frac{1}{p_i}$$

Gain

Given an asymptotically stable system with $g = 0$ subjected to a constant input $\bar{u}$ (Laplace transform $U(s) = \bar{u}/s$), the output tends to a steady-state value $\bar{y}$ that can be found using the final value theorem

$$\begin{aligned} \bar{y} &= \lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)\frac{\bar{u}}{s} = \lim_{s\to 0} s\left(C(sI-A)^{-1}B + D\right)\frac{\bar{u}}{s} = \\ &= G(0)\bar{u} = \left(-CA^{-1}B + D\right)\bar{u} \end{aligned}$$

or using

$$\bar{y} = \lim_{s\to 0} s\frac{\mu \prod_i (1+\tau_i s) \prod_i (1 + 2\zeta_i s/\alpha_{ni} + s^2/\alpha_{ni}^2)}{\prod_i (1+T_i s) \prod_i (1 + 2\xi_i s/\omega_{ni} + s^2/\omega_{ni}^2)} \frac{\bar{u}}{s} = \mu\bar{u}$$

This results in

$$\mu = \bar{y}/\bar{u} = G(0) = -CA^{-1}B + D$$

When $g \neq 0$ the coefficient μ is still called gain, or *generalized gain*, and its value is given by

$$\mu = \lim_{s\to 0} s^g G(s)$$

Example

Write the transfer function of a system with a gain equal to 3, a zero in 2, a pole in -2 and a pole in -5.

$$G(s) = \frac{3(1-\frac{s}{2})}{(1+\frac{s}{2})(1+\frac{s}{5})} = \frac{-15(s-2)}{(s+2)(s+5)}$$

Ideal derivative and integration

Ideal derivative
Consider a system described by a transfer function of $g<0$ kind. Using the final value theorem you get that the steady-state value of the output, given a constant input, is null, that is $\bar{y}=0$. This means that the system works on the input as a simple derivative action if $g=-1$, or multiple if $g<-1$.
Integration
In order to value the behaviour of a system with poles in the origin ($g>0$), consider the simple case with $G(s)=1/s$. The final result is than $C(sI-A)^{-1}B=1/s$, so that $A=0$, and $CB=1$. For example $B=1$, $C=1$. The system in state variable can be represented by its minimal form

$$\begin{aligned}\dot{x}(t) &= u(t)\\ y(t) &= x(t)\end{aligned}$$

The output is the integral of the input, so a system with $G(s)=1/s$ is commonly called *integrator*.

Time constants

$$\tau_i=\frac{1}{z_i} \quad , \quad T_i=\frac{1}{p_i}$$

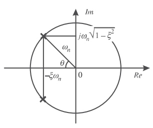
The impulse response is a linear combination of the $t^j e^{-t/T_i}$, $j=0,1,\ldots,\nu_i-1$ modes, where ν_i is the multiplicity of the studied pole.
Denominator time constants are linked to the speed of the system transients: with an higher value of a time constant the contribution to the output dynamic from the corresponding pole will slower run out.

Normal Mode and Damping factor

Given $a\pm jb$, as the roots of the characteristic equation $s^2+2\xi\omega_n s+\omega_n^2=0$. You can easily find that

$$a=-\xi\omega_n \quad , \quad b=\omega_n\sqrt{1-\xi^2}$$

The poles magnitude coincides with the normal mode ω_n, while the damping factor ξ is the cosine of the θ angle formed by the poles with the origin and the real negative semi-axis. Therefore, considering a fixed value of ω_n, when ξ changes from -1 to $+1$ the poles move on a circumference of ω_n radius centered on the origin. In particular poles are pure imaginary for $\xi=0$ and, on the extremes $\xi=\pm1$, they are real and they coincide in the $-\omega_n$ point for $\xi=1$ and they are real and coincide in ω_n point for $\xi=-1$. Also the poles have a real negative part when $\xi>0$, a null real part when $\xi=0$, a positive real part when $\xi<0$.



Normal Mode and Damping factor

When the system has just a pair of complex conjugate poles and a unit gain

$$G(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

the impulse response is

$$\begin{aligned}Y(s) &= G(s)U(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\\ &= \frac{\omega_n}{\sqrt{1-\xi^2}}\frac{\omega_n\sqrt{1-\xi^2}}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}\end{aligned}$$

$$y(t)=\frac{\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\left(\omega_n t\sqrt{1-\xi^2}\right)$$

It is made by a sinusoidal term with a mode given by the imaginary part of the poles, multiplied by an exponential that depends on the poles real part.

Realization

$$G(s)=\frac{\beta_{n-1}s^{n-1}+\beta_{n-2}s^{n-2}+\ldots+\beta_1 s+\beta_0}{s^n+\alpha_{n-1}s^{n-1}+\ldots+\alpha_1 s+\alpha_0}+\beta_n$$

Canonical forms, characterized by having all the elements independent from α_i and β_i, excluding $2n+1$ elements.
The *reachability canonical form*, or *control canonical form*, results from

$$A=\begin{bmatrix}0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ldots & 0\\ 0 & 0 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 1\\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \ldots & -\alpha_{n-1}\end{bmatrix} \quad , \quad B=\begin{bmatrix}0\\0\\0\\ \vdots\\0\\1\end{bmatrix}$$

$$C=\begin{bmatrix}\beta_0 & \beta_1 & \beta_2 & \ldots & \beta_{n-1}\end{bmatrix} \quad , \quad D=\beta_n$$

Example

$$\begin{aligned}G(s) &= \frac{2s^2+s+3}{s^2+2s+2}=\frac{\alpha s+\beta}{s^2+2s+2}+\gamma\\ &= \frac{\alpha s+\beta+\gamma s^2+2\gamma s+2\gamma}{s^2+2s+2}\\ \gamma &= 2,\ \alpha+2\gamma=1,\ \beta+2\gamma=3\\ \gamma &= 2, \alpha=-3, \beta=-1\end{aligned}$$

$$G(s)=-\frac{3s+1}{s^2+2s+2}+2$$

$$A=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}, B=\begin{bmatrix}0\\1\end{bmatrix}, C=\begin{bmatrix}-1 & -3\end{bmatrix}, D=2$$

Example

$$G(s) = \begin{bmatrix} -1 & -3 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2 =$$
$$\frac{\begin{bmatrix} -1 & -3 \end{bmatrix}}{s^2+2s+2} \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2$$
$$= \frac{\begin{bmatrix} -1 & -3 \end{bmatrix}}{s^2+2s+2} \begin{bmatrix} 1 \\ s \end{bmatrix} + 2 = \frac{-1-3s}{s^2+2s+2} + 2$$
$$= \frac{-1-3s+2s^2+4s+4}{s^2+2s+2} = \frac{2s^2+s+3}{s^2+2s+2}$$

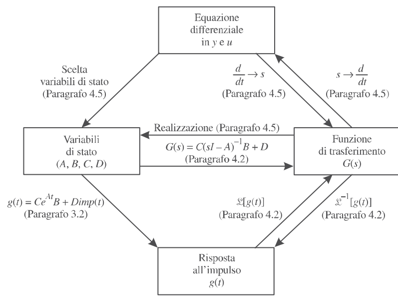
Differential equation for y and u

$$\frac{d^n y(t)}{dt^n} + \alpha_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + \alpha_1 \frac{dy(t)}{dt} + \alpha_0 y(t) =$$
$$= \beta_n \frac{d^n u(t)}{dt^n} + \beta_{n-1}\frac{d^{n-1}u(t)}{dt^{n-1}} + \ldots + \beta_1 \frac{du(t)}{dt} + \beta_0 u(t)$$

With the right hypothesis on the initial value and applying the Laplace transformation operator you get

$$s^n Y(s) + \alpha_{n-1} s^{n-1} Y(s) + \ldots + \alpha_1 s Y(s) + \alpha_0 Y(s)$$
$$= \beta_n s^n U(s) + \beta_{n-1} s^{n-1} U(s) + \ldots + \beta_1 s U(s) + \beta_0 U(s)$$

System representation



Sinusoid response

Theorem

Suppose that the system (A,B,C,D) with transfer function $G(s)$ does not have eigenvalues for $\pm j\omega_0$ and apply the input

$$\tilde{u}(t) = U \sin(\omega_0 t + \varphi) \quad , \quad t \geq 0$$

Therefore it exists an initial state where the output is a sinusoid and it is worth

$$\tilde{y}(t) = Y \sin(\omega_0 t + \psi) \quad , \quad t \geq 0$$

with

$$Y = |G(j\omega_0)|\, U \quad , \quad \psi = \varphi + \arg G(j\omega_0)$$

Moreover, if the system is asymptotically stable, for any initial state it results

$$\lim_{t\to\infty} (y(t) - \tilde{y}(t)) = 0$$

It is important to observe that the output of a SISO linear system with a sinusoidal input with pulse $\omega_0 > 0$ can be null only if the transfer function of the system has a pair of complex zeros in $\pm j\omega_0$ ($G(j\omega_0) = 0$).

Frequency response

The complex function

$$G(j\omega) = C(j\omega I - A)^{-1}B + D$$

defined for non negative values of the variable ω and so that $j\omega$ is not a pole of $G(s)$ is called *frequency response* associated with the (A, B, C, D) system. It coincides with the restriction of the transfer function $G(s)$ in the points of the imaginary positive semi-axis, excluding at most a finite number of point corresponding to the eventual poles of $G(s)$ on the imaginary axis. On the other hand the theory of analytic function (that includes as subset the rational functions of a complex variable) says that to a frequency response $G(j\omega)$, $\omega \geq 0$, corresponds a single transfer function $G(s)$.

Pure delay

The notion of frequency response can be extended to multivariable systems and infinitive dimension systems when these are linear and stable. In general, given a system with transfer function $G(s)$, the function $G(j\omega)$ defined for all the non negative values of ω is called frequency response.
For example, the frequency response associated to a time delay is

$$G(j\omega) = e^{-j\omega\tau} \quad , \quad \omega \geq 0$$

applying the input

$$\tilde{u}(t) = U \sin(\omega_0 t + \varphi) \quad , \quad t \geq 0$$

to the time delay, it clearly results

$$y(t) = U \sin(\omega_0 t - \omega_0 \tau + \varphi) \quad , \quad t \geq \tau$$

We get $|G(j\omega_0)| = 1$ and $\arg G(j\omega_0) = -\omega_0 \tau$.

Identification of the frequency response

- $u(t) = U\sin(\omega_0 t + \varphi)$
- Input signals with “more” harmonics. You need to part the harmonics of the output signals (using a numerical calculus of the spectrum or using appropriate filters) and relate them with the ones in the input.
- An extreme case, not achievable in practice, is where $u(t) = imp(t)$, to which it corresponds $\mathcal{F}[u(t)] = 1$ and so $\mathcal{F}[y(t)] = G(j\omega)$. A single experiment would be enough to find the whole frequency response.

Bode plot

$$G(s) = \frac{\mu \prod_i (1+\tau_i s) \prod_i (1 + 2\zeta_i s/\alpha_{ni} + s^2/\alpha_{ni}^2)}{s^g \prod_i (1+T_i s) \prod_i (1 + 2\xi_i s/\omega_{ni} + s^2/\omega_{ni}^2)}$$

$$G(j\omega) = \frac{\mu \prod_i (1+j\omega\tau_i) \prod_i (1 + 2j\zeta_i\omega/\alpha_{ni} - \omega^2/\alpha_{ni}^2)}{(j\omega)^g \prod_i (1+j\omega T_i) \prod_i (1 + 2j\xi_i\omega/\omega_{ni} - \omega^2/\omega_{ni}^2)}$$

$$|G(j\omega)|_{dB} = 20\log|G(j\omega)|$$

$$\begin{aligned} |G(j\omega)|_{dB} &= 20\log|\mu| - 20g\log|j\omega| + \sum_i 20\log|1+j\omega\tau_i| + \\ &+ \sum_i 20\log\left|1 + 2j\zeta_i\omega/\alpha_{ni} - \omega^2/\alpha_{ni}^2\right| - \sum_i 20\log|1+j\omega T_i| + \\ &- \sum_i 20\log\left|1 + 2j\xi_i\omega/\omega_{ni} - \omega^2/\omega_{ni}^2\right| \end{aligned}$$

$$\begin{aligned} \arg G(j\omega) &= \arg\mu - g\arg(j\omega) + \sum_i \arg(1+j\omega\tau_i) + \\ &+ \sum_i \arg(1 + 2j\zeta_i\omega/\alpha_{ni} - \omega^2/\alpha_{ni}^2) - \sum_i \arg(1+j\omega T_i) + \\ &- \sum_i \arg(1 + 2j\xi_i\omega/\omega_{ni} - \omega^2/\omega_{ni}^2) \end{aligned}$$

Bode plot

$$\begin{aligned} G_a(s) &= \mu \\ G_b(s) &= \frac{1}{s} \\ G_c(s) &= \frac{1}{1+Ts} \\ G_d(s) &= \frac{1}{1 + 2\xi s/\omega_n + s^2/\omega_n^2} \end{aligned}$$

$$|G_a(j\omega)|_{dB} = 20\log|\mu|$$

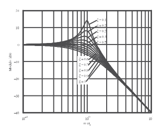
$$|G_b(j\omega)|_{dB} = 20\log\left|\frac{1}{j\omega}\right| = -20\log\omega$$

$$|G_c(j\omega)|_{dB} = 20\log\left|\frac{1}{1+j\omega T}\right| = -20\log\sqrt{1+\omega^2T^2}$$

$$|G_c(j\omega)|_{dB} \simeq \begin{cases} -20\log 1 = 0 & , \quad \omega \ll 1/|T| \\ -20\log\omega|T| & , \quad \omega \gg 1/|T| \end{cases}$$

Bode plots

The dynamic of all these plots are independent from the sign of the time constant T.

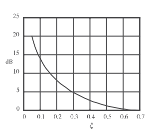


$$\begin{aligned} |G_d(j\omega)|_{dB} &= 20\log\left|\frac{1}{1 + 2j\xi\omega/\omega_n - \omega^2/\omega_n^2}\right| = \\ &= \simeq \begin{cases} -20\log 1 = 0 & , \quad \omega \ll \omega_n \\ -40\log(\omega/\omega_n) & , \quad \omega \gg \omega_n \end{cases} \end{aligned}$$

Bode plots

This function graph depends on the modulus but not on the sign of the damping ξ. The maximum is $\omega \neq 0$ when $|\xi| < 1/\sqrt{2} \simeq 0.707$ and it called *resonance peak*, corresponding to the *resonance pulse*

$$\omega_r = \omega_n\sqrt{1-2\xi^2}, |G_d(j\omega_r)| = \frac{1}{2|\xi|\sqrt{1-\xi^2}}$$

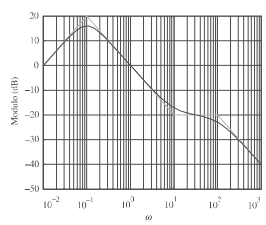


Asymptotic Bode plotting of the modulus

With lower modes of all the elements $1/|\tau_i|$, $1/|T_i|$, α_{ni} e ω_{ni} the only elements of $G(j\omega)$ that define it are μ and $1/(j\omega)^g$.

Example

$$G(s) = \frac{100s(1-0.1s)}{(1+16s+100s^2)(1+0.01s)}$$



Phase plots

The agreement applied in order to compute the input of every single contribution is to use the minimum value among all the possible values, that will mean considering the value when modulus not higher than $180°$.

$$\arg G_a(j\omega) = \arg\mu = \begin{cases} 0° & , \quad \mu > 0 \\ -180° & , \quad \mu < 0 \end{cases}$$

Please note that the choice to force the phase displacement at $-180°$, and not at $+180°$, when $\mu < 0$ is just a convention.

$$\arg G_b(j\omega) = \arg\left(\frac{1}{j\omega}\right) = -90°$$

$$\arg G_c(j\omega) = -\arg(1 + j\omega T) = -\arctan(\omega T)$$

$$\arg G_c(j\omega) \simeq \begin{cases} -\arg(1) = 0° \quad , \quad \omega \ll 1/|T| \\ -\arg(j\omega T) = \begin{Bmatrix} -90° & , & T > 0 \\ +90° & , & T < 0 \end{Bmatrix} \quad , \quad \omega \gg 1/|T| \end{cases}$$

Phase plots

$$\arg G_d(j\omega) = -\arg\left(1 + 2j\xi\omega/\omega_n - \omega^2/\omega_n^2\right) = -\arctan\frac{2\xi\omega/\omega_n}{1 - \omega^2/\omega_n^2}$$

Asymptotic plot, composed for $\omega \leq \omega_n$ by the horizontal half-line of zero ordinate and for $\omega > \omega_n$ by the horizontal half-line of ordinate $-180°$ when $\xi \geq 0$ and at $+180°$ when $\xi < 0$.

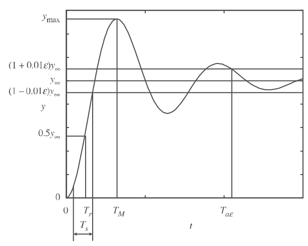
Asymptotic plot of the phases

With modes lower than all the elements $1/|\tau_i|$, $1/|T_i|$, α_{ni} and ω_{ni} the only factors of $G(j\omega)$ that define it are μ and $1/(j\omega)^g$. Therefore the initial line of the plot have an ordinate at $\arg\mu - g90°$. When modes increase, the ordinate changes in the following way: when modes are equals to $1/|\tau_i|$ and $1/|T_i|$ the ordinate increases ($\tau_i > 0$ and $T_i < 0$) or decreases ($\tau_i < 0$ and $T_i > 0$) of $90°$. With normal modes α_{ni} or ω_{ni} the ordinate increases ($\zeta_i \geq 0$ e $\xi_i < 0$) or decreases ($\zeta_i < 0$ and $\xi_i \geq 0$) of $180°$. The exact plot can be obtained adding up the exact plots of $G(s)$ single factors.

Pure delay

$G(s) = e^{-\tau s}$, $\tau > 0$. In particular it results $|G(j\omega)| = 1$, $\forall\omega$, and than $|G(j\omega)|_{dB} = 0$, $\forall\omega$, while the phase expresses in grades is $\arg G(j\omega) = -\omega\tau 180/\pi$, so that, for example, $\arg G(j/\tau) = -180/\pi \simeq -57°$.

The step response



a) *equilibrium value* y_∞: output value after the transient has died out; considering what have said it is equal to μ or to zero ($g < 0$);

b) *maximum value* $y_{\max}$: maximum output value;

The step response

c) *maximum overshoot percentage* $S\%$: amplitude, percentage, of the maximum overshoot relating with the equilibrium value, that is

$$S\% = 100\frac{y_{\max} - y_\infty}{y_\infty}$$

d) *maximum overshoot time* T_M: first instant where $y = y_{\max}$;

e) *upgrade time* T_s: time required so that the output switches for the first time from 10% to 90% of its equilibrium value;

f) *delay time* T_r: time required so that the output reaches the value $0.5y_\infty$ for the first time;

g) *settling time* $T_{a\varepsilon}$: time required so that the difference between the output and the equilibrium value y_∞ will keep under $\varepsilon\%$, that is the output stays in the range $[(1-0.01\varepsilon)\,y_\infty, (1+0.01\varepsilon)\,y_\infty]$; for example T_{a1} will point the time required to the output to get in the amplitude range of $\pm 0.01 y_\infty$ in the space near the equilibrium value y_∞. Moreover we can say that $T_{a\varepsilon}$ represents the settling time at $(100 - \varepsilon)\,\%$.

First Order Systems

$$G(s) = \frac{\mu}{1+Ts}$$

$$\begin{aligned} Y(s) &= G(s)\frac{1}{s} = \frac{\mu}{(1+Ts)\,s} = \frac{\mu/T}{(1/T+s)\,s} \\ &= \frac{A}{1/T+s} + \frac{B}{s} = \frac{As + B\,(1/T+s)}{(1/T+s)\,s} \\ -\frac{A}{T} &= \mu/T,\ A = -\mu,\ \frac{B}{T} = \mu/T,\ B = \mu \end{aligned}$$

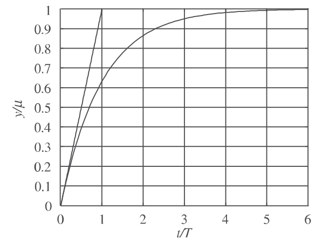
$$y(t) = \mu\left(1 - e^{-t/T}\right) \quad , \quad t \geq 0$$

Verify the initial and final value and the initial derivative value.

$$\begin{aligned} y(T_{a_\varepsilon}) &= \mu\left(1 - e^{-T_{a_\varepsilon}/T}\right) = (1 - \varepsilon * 0.01)\mu \\ e^{-T_{a_\varepsilon}/T} &= \varepsilon * 0.01 \\ T_{a_\varepsilon} &= -T\,\mathrm{lg}(\varepsilon * 0.01) \end{aligned}$$

First Order Systems

y_∞	T_s	T_r	T_{a5}	T_{a1}
μ	$\simeq 2.2T$	$\simeq 0.7T$	$\simeq 3T$	$\simeq 4.6T$



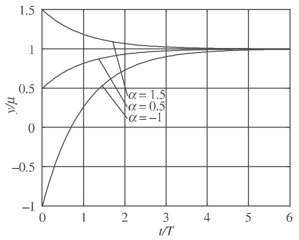
Proper first order systems

$$G(s) = \frac{\mu(1+\tau s)}{1+Ts}$$

Set $\tau = \alpha T$, $\alpha \neq 0$, $\alpha \neq 1$. The output trend is given by (exercize not developed)

$$y(t) = \mu\left(1 + (\alpha - 1)\,e^{-t/T}\right) \quad , \quad t \geq 0$$

Verify the initial and final value.



Proper first order systems

$$T_{a\varepsilon} = T \ln \frac{|1-\alpha|}{0.01\varepsilon}$$

It is not really important to define the upgrade time T_s, the delay time T_r and the maximum overshoot percentage $S\%$.

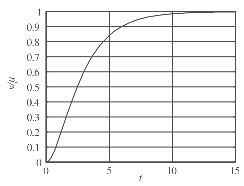
Second order systems - distinct poles

$$G(s) = \frac{\mu}{(1+T_1 s)(1+T_2 s)} \quad , \quad T_1 > T_2$$

$$y(t) = \mu\left(1 - \frac{T_1}{T_1 - T_2}e^{-t/T_1} + \frac{T_2}{T_1 - T_2}e^{-t/T_2}\right) \quad , \quad t \geq 0$$

Verify initial and final value and initial derivative value.
The initial value and the first derivative initial value are equal to zero, while the secon derivative initial value is $d^2y(0)/dt^2 = \mu/T_1T_2$.
If the system is asymptotically stable ($T_1 > T_2 > 0$), the system is faster with smaller time constant T_1 e T_2, while $T_{a\varepsilon}$, T_r e T_s are not simple function of T_1 and T_2. Moreover it is possible to show that the step response does not present overshoot.

Second order systems - distinct poles



The slower exponential term, that is the one with time constant T_1, has a coefficient with magnitude larger than the one that multiplies the faster exponential term, that is the one with time constant T_2. If T_1 is much larger than T_2 then $T_1/(T_1 - T_2) \simeq 1$ and $T_2/(T_1 - T_2) \simeq 0$, while e^{-t/T_2} goes to zero much faster than e^{-t/T_1}. For t not too small ($t \simeq 4 \div 5T_2$),

$$y(t) \simeq \mu\left(1 - e^{-t/T_1}\right) \quad , \quad t \geq 0$$

that is the system can be well approximate with a first order system with time constant T_1.

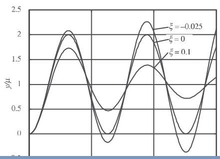
Second order systems - complex poles

$$G(s) = \frac{\mu\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

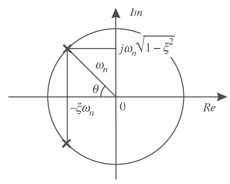
Analytic step response

$$y(t) = \mu\left(1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\omega_n t\sqrt{1-\xi^2} + \arccos(\xi)\right)\right) \quad , \quad t \geq 0$$

that is formed by a sinusoidal term multiplied by an exponential term. In particular for $\xi > 0$ the exponential is decreasing and the answer asymptotically tends to the steady state value μ, for $\xi < 0$ the system is unstable and the output diverges, while for $\xi = 0$ the system is stable but not asymptotically stable.



Maximum overshoot percentage



For asymptotically stable system $(0 < \xi < 1)$.
The time instant with derivative equal to zero are

$$-\frac{\mu}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}(\cos()\omega_n\sqrt{1-\xi^2} - \xi\omega_n \sin()) = 0$$

$$tg(\omega_n t\sqrt{1-\xi^2} + \arccos(\xi)) = \frac{\sqrt{1-\xi^2}}{\xi}$$

and from the picture

$$tg(\arccos(\xi)) = \frac{\sqrt{1-\xi^2}}{\xi}$$

Maximum overshoot percentage

is a solution, then the other are periodic with period π

$$\bar{t}_k = \frac{k\pi}{\omega_n\sqrt{1-\xi^2}} \quad , \quad k \text{ intero positivo.}$$

They are maximum for odd k and minimum for even k. The corresponding output values are

$$y(\bar{t}_k) = \mu\left(1 - (-1)^k e^{-\xi k\pi/\sqrt{1-\xi^2}}\right)$$

so that

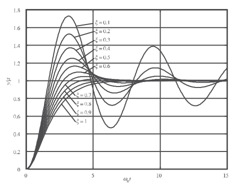
$$y_{\max} = \mu\left(1 + e^{-\xi\pi/\sqrt{1-\xi^2}}\right)$$

$$T_M = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$S\% = 100e^{-\xi\pi/\sqrt{1-\xi^2}}$$

Maximum overshoot percentage

The maximum overshoot percentage depends only from the damping factor and is a decreasing function of ξ.



Settling time

A settling time $(T_{a\varepsilon})$ approximation can be achieved by noting that the minimum and maximum values of y are on the functions

$$y_M(t) = 1 + e^{-\xi\omega_n t}$$

and

$$y_m(t) = 1 - e^{-\xi\omega_n t}$$

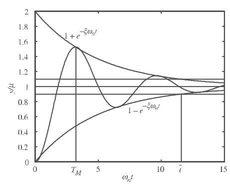
Hence, if at a time instant $\bar{t}$ the functions $y_M(\bar{t})$ e $y_m(\bar{t})$ enter the range $[(1-0.01\varepsilon)y_\infty, \ (1+0.01\varepsilon)y_\infty]$, the output response $y(t)$ also enters the same range for $t \geq \bar{t}$. To compute this settling time estimation is then sufficient to impose

$$e^{-\xi\omega_n \bar{t}} = 0.01\varepsilon$$

whence

$$\bar{t} = -\frac{1}{\xi\omega_n}\ln 0.01\varepsilon$$

Settling time



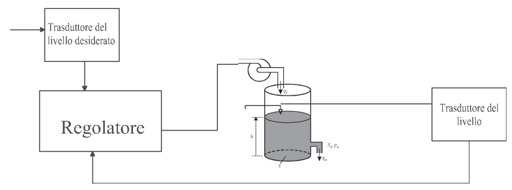
y_∞	$S\%$	T_M	$T_{a\varepsilon}$ estimation
μ	$100e^{-\xi\pi/\sqrt{1-\xi^2}}$	$\frac{\pi}{\omega_n\sqrt{1-\xi^2}}$	$-\frac{1}{\xi\omega_n}\ln 0.01\varepsilon$

Exercise

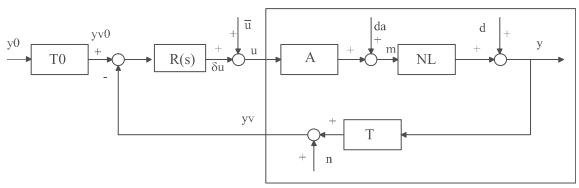
$$G_1(s) = \frac{s}{s^2+2s+1}, \quad G_2(s) = \frac{-5}{s-5}, \quad G_3(s) = \frac{10}{s^2+0.3s+10},$$
$$G_4(s) = \frac{-3s+2}{s+2}, \quad G_5(s) = \frac{1}{s^2+2s+1}, \quad G_6(s) = \frac{20}{s+20},$$
$$G_7(s) = \frac{-1}{s^2+2s+1}, \quad G_8(s) = \frac{80}{s^2+1.2s+80}$$

AGGIUNGERE GRAFICI

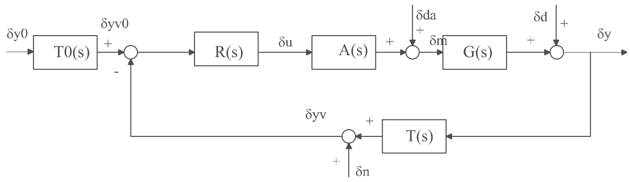
Example of a control scheme



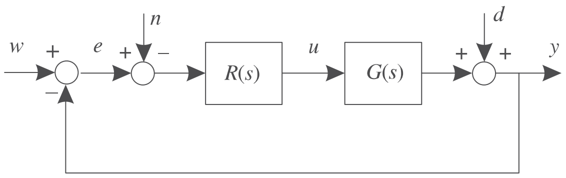
Example of a control scheme



Example of a control scheme

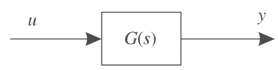


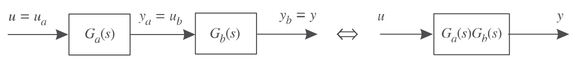
Control system requirements



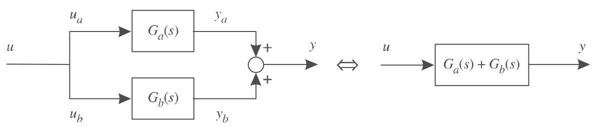
- Asymptotic stability (otherwise Initial-State Dependence)
- Strong stability (w.r.t. model uncertainties)
- Static performances in nominal conditions (signals of interest are signals whose Laplace transform has poles with non negative real part, specifically with a zero real part, A/s,A/s^2, sinusoidal signals)
- Nominal dynamic performances (step response), disturbances attenuation

Block diagrams



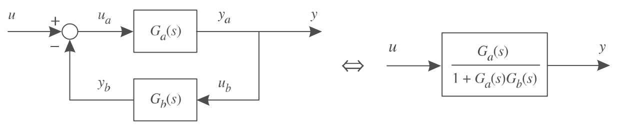


$$Y(s) = G_b(s)Y_a(s) = G_b(s)G_a(s)U(s)$$



$$Y(s) = Y_a(s) + Y_b(s) = G_a(s)U_a(s) + G_b(s)U_b(s) = (G_a(s) + G_b(s))U(s)$$

System in a negative feedback



$$Y(s) = G_a(s)(U(s) - Y_b(s)) = G_a(s)(U(s) - G_b(s)Y(s))$$

The overall transfer function results to be

$$G(s) = \frac{Y(s)}{U(s)} = \frac{G_a(s)}{1 + G_a(s)G_b(s)}$$

that is equal to the relation between the transfer function of the subsystem, that appears on the ingoing line between u and y and the sum between 1 and the so called *transfer function of the loop*

$$L(s) = G_a(s)G_b(s)$$

Stability of interconnected systems

$$G(s) = \frac{N(s)}{D(s)} = G_a(s)G_b(s) = \frac{N_a(s)N_b(s)}{D_a(s)D_b(s)}$$
$$G(s) = \frac{N(s)}{D(s)} = G_a(s) + G_b(s) = \frac{N_a(s)D_b(s) + N_b(s)D_a(s)}{D_a(s)D_b(s)}$$

Theorem

The eigenvalues of a system made of any number of subsystems serial connected and to the parallel there are the eigenvalues,considered with their multiplicities, of all the subsystems components, and so the overall system is asymptotic stable if and only all the subsystems that it is made of are also asymptotic stables. ■

Stability of a feedback system

$$G(s) = \frac{N(s)}{D(s)} = \frac{G_a(s)}{1 + G_a(s)G_b(s)} = \frac{N_a(s)D_b(s)}{D_a(s)D_b(s) + N_a(s)N_b(s)}$$

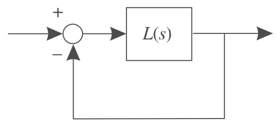
It can be observed that the poles of a feedback system (that is D roots) do not have an explict link with the subsystem poles (D_a and D_b roots).
It is possible to cancel the common roots between numerator and denominator if and only if N_a and D_b polynomials have common factors, that is subsystem zeros exist in the ingoing system coincidents with the poles of the feedback system. Suppose there are no cancellations. The eventual common roots between D_a and N_b of the ingoing system coincidents with the feedback system zeros, are also D roots. They are equivalent to the eigenvalus on which the feedback has no effect. The other D roots are the one of the equation

$$1 + L(s) = 0$$

The feedback system is asymptotic stable if and only if all the roots have negative real part, using the hypothesis that eventual common roots between the N_a and D_b polynomes or between D_a and N_b polynomes are pertinent to singularity with a real negative part.

Stability of a feedback system

The stability of a linear system does not depend on the inputs. The analys that follows will refer to the following system.



A first necessary condition for the asymptotic stability of the overall feedback system is that the system associated with $L(s)$ does not have positive or nil real eigenvalues relating to the not observable and/or reachable part. NUsing the hypothesis that every subsystem does not have unreachable parts and/or not observable parts or that the eigenvalues associated with them have a negative real part, that implies that there cannot be cancellation between poles with a part higher or equal to zero between the transfer function that contribuites to form $L(s)$.

Stability of a feedback system

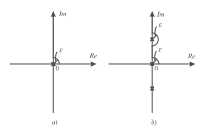
The asymptotic stability of a feedback system is analized through verifying that all the roots of the feedback system characteristic equation, that is

$$1 + L(s) = 0$$

have real part lower than zero.

Polar patterns

A polar pattern is an image through $G(s)$ of the points of the imaginary positive semi axis (excluding $G(s)$ poles), of the points $\varepsilon e^{j\theta}$, $\theta \in [0, \pi/2)$ and ε infinitesimal, if $s = 0$ is a pole of $G(s)$, and of the points $j\bar{\omega} + \varepsilon e^{j\theta}$, $\theta \in (-\pi/2, \pi/2)$ and ε infinitesimal, if $j\bar{\omega}$ is a pole of $G(s)$.



Given a certain function $G(s)$, the polar pattern of the associeted frequency can be obtain using or a proper graph construction or directly from the Bode diagrams of $G(j\omega)$.

Graphical determination

$$G(j\omega) = \frac{\rho \prod_{i=1}^{q} (j\omega + z_i) \prod_{i=1}^{r} (-\omega^2 + 2j\zeta_i\alpha_{ni}\omega + \alpha_{ni}^2)}{(j\omega)^g \prod_{i=1}^{k} (j\omega + p_i) \prod_{i=1}^{h} (-\omega^2 + 2j\xi_i\omega_{ni}\omega + \omega_{ni}^2)}$$

It is easy to verify that

$$G(j\omega) = Me^{jf}$$

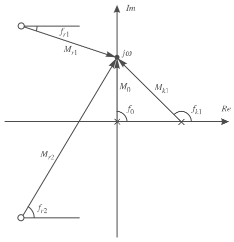
where

$$M = \frac{|\rho| \prod_{i=1}^{q} M_{qi} \prod_{i=1}^{2r} M_{ri}}{M_o^g \prod_{i=1}^{k} M_{ki} \prod_{i=1}^{2h} M_{hi}}$$

Graphical determination

$$f = f_\rho + \sum_{i=1}^{q} f_{qi} + \sum_{i=1}^{2r} f_{ri} - gf_o - \sum_{i=1}^{k} f_{ki} - \sum_{i=1}^{2h} f_{hi}$$

f_ρ is the argument of the transfer constant ρ.



Example

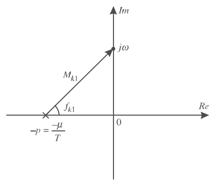
Consider the transfer funtion

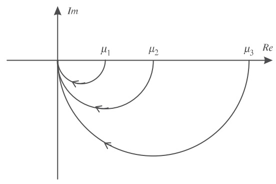
$$G(s) = \frac{\mu}{1 + Ts} = \frac{\rho}{s + p}$$

where $\mu > 0$ and $T > 0$, $\rho = \mu/T$ and $p = 1/T$. It is necessary to value the M_{k1} module and the f_{k1} angle of the vector designed in the Figure when the point on the corrent point on the imaginary non negative semi axis varies. In particulat it results

$$|G(j\omega)| = \frac{\rho}{M_{k1}} \quad , \quad \arg G(j\omega) = -f_{k1}$$

Example

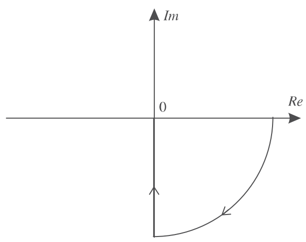




Example of an integrator

The polar diagram of the frequency response of the integrator

$$G(s) = \frac{1}{s}$$



coincides with the imaginary negative semi axis. In fact, as it is possible to deduce from the corresponding Bode diagrams, the module decreases when ω, while the phase keeps constant and equal to $-90°$.

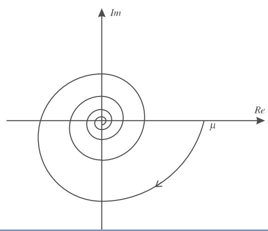
Example of delay

The polar diagram associated to the time delay

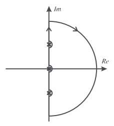
$$G(s) = e^{-\tau s}$$

is formed of a unit circle that has been gone through clockwise an infinite number of time starting from the positive real semi axis. In fact the form is constant and equal to one for every ω, while the phase is decreasing with ω. The funcion polar diagram

$$G(s) = \frac{\mu}{1 + Ts} e^{-\tau s} \quad , \quad \mu > 0 \ , \ T > 0$$



Nyquist plot



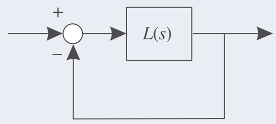
Nyquist plot, associated with the standard transfer funcion $G(s)$, the image through $G(s)$ of the *Nyquist path*
Please remember that in order to track the Nyquisit plot of $G(s)$, $G(-j\omega) = \bar{G}(j\omega)$ and, if $G(s)$ is strictly proper, the image through $G(s)$ of the semi circumference of infinite ray corresponds to the point $G(j\infty) = 0$. Nyquist plot can be determined directly from the polar diagram associated with $G(j\omega)$; in fact it is formed by the polar diagram itself and by its specular image of the real axis. The diagram is a construction of a closed line in the complex plan whose orientation is conventionally fixed following increasing values of ω.

Nyquist criterion

P = number of $L(s)$ poles with real part higher than zero
N = number of laps made by the Nyquist plot of the loop function $L(s)$ around the -1 point, positive if done counterclockwise and negative if clockwise. If the diagram goes through the point -1 the N value is not well defined.
Nyquist criterion.

Theorem

Necessary and sufficient condition so that the feedback system is asymptotic stable is that



N is well defined and results $N = P$.

Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Example

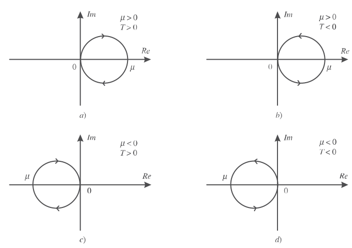
$$L(s) = \frac{\mu}{1+Ts}$$

Caso 1: $\mu > 0$, $T > 0$. As $T > 0$, we have $P = 0$, $N = 0$. The conditions of the Nyquist criterion are so verified and the system in closed loop is asymptotic stable.
Caso 2: $\mu > 0$, $T < 0$. In this case $P = 1$ and $N = 0$. The system is a feedback and so it is unstable, and so $L(s)$.
Caso 3: $\mu < 0$, $T > 0$. It results $P = 0$ and it is necessary to ditinguish between three subcases. If $\mu > -1$ we have $N = 0$ and the feedback system is asymptotic stable. If $\mu = -1$ N and the feedback system is asymptotic stable; in particular the pole in closed loop is $\hat{s} = 0$. If $\mu < -1$ we have $N = -1$ and the feedback system is unstable.
Caso 4: $\mu < 0$, $T < 0$. In this case $P = 1$ and again, is necessary to ditinguish between three subcases. If $\mu > -1$ we have $N = 0$ and the feedback system is unstable. If $\mu = -1$ the N value is not well defined and the feedback system is not asymptotic stable; in particular the closed loop pole is $\hat{s} = 0$. If $\mu < -1$ we have $N = 1$ and the feedback system is asymptotic stable even when $L(s)$ is unstable.

Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Example

$$L(s) = \frac{\mu}{1+Ts}$$



Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Study on the stability when ρ varies

$$\begin{aligned}
L(s) &= \rho\frac{(s+1)(s+3)}{(s+2)(s^2+1)(s+0.5)(s+5)},\\
L(s) &= \rho\frac{(s-2)(s+4)}{(s+2.5)(s+1)(s^2+1)(s+0.1)}\\
L(s) &= \rho\frac{s(s+8)}{(s^2+1)(s+2)^2(s^2+10s+26)}\\
L(s) &= \rho\frac{(s+2)(s^2+16)}{(s-1)^2(s+8)(s+12)}
\end{aligned}$$

Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Asymptotic stable Open loop systems

Corollary

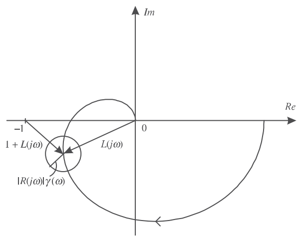
Given a negative feedback system with a asymptotic stable transfer function of $L(s)$ loop, sufficient condition for the asymptotic stability of the system in open loop is that $|L(j\omega)| < 1$, $\forall\omega$.

Corollary

Given a negative feedback system with a asymptotic stable transfer function of $L(s)$ loop, sufficient condition for the asymptotic stability of the system in closed loop is that it is verified the condition $|\arg L(j\omega)| < 180^\circ$, $\forall\omega$.

Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

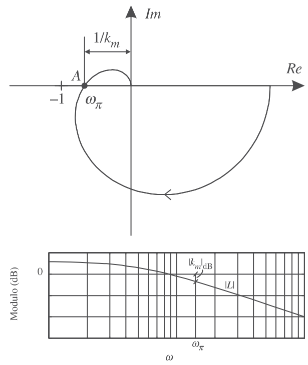
Strong stability



Gain Margin

$P = 0$, and it goes through the real negative axis only one time.

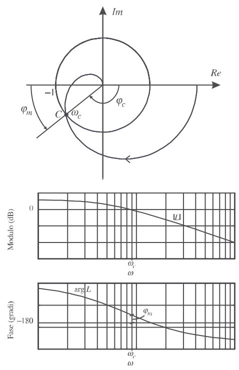
$$k_m = \frac{1}{|L(j\omega_\pi)|} \quad , \quad \arg L(j\omega_\pi) = -180^\circ$$



Phase Margin

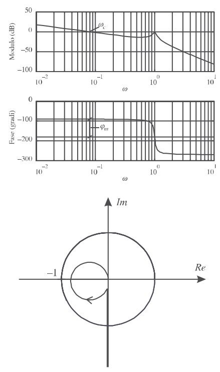
$P = 0$, and it goes through the unit's circle only one time.

$$\varphi_m = 180^\circ - |\varphi_c| \quad , \quad \varphi_c = \arg L(j\omega_c) \quad , \quad |L(j\omega_c)| = 1$$

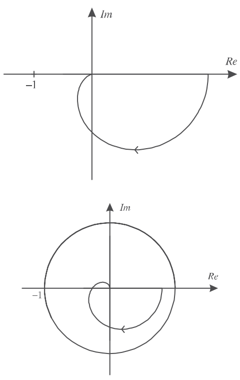


Phase margin validity

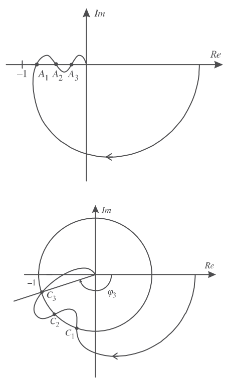
$$L(s) = \frac{0.08}{s(1 + 0.1s + s^2)}$$



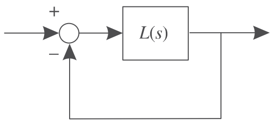
Phase and Gain Margin



Phase and Gain Margin



Bode criterion



Theorem

Consider the feedback system in the picture and suppose that:

a) *$L(s)$ does not have poles with positive real part ($P = 0$);*
b) *the bode diagram of the module of $L(j\omega)$ crosses the 0 dB axis just once.*

Than, indicating with μ the $L(s)$ gain and with φ_m the phase margin ($\varphi_m = 180 - |\varphi_c|$, $\varphi_c = \arg L(j\omega_c)$, $|L(j\omega_c)| = 1$), necessary and sufficient condition for the asymptotically stability of the system is that $\mu > 0$ and $\varphi_m > 0°$. ■

Studying the stability

$$L(s) = \frac{10}{(1+s)(1+10s)}$$
$$L(s) = \frac{10}{s(1+s)(1+10s)}$$
$$L(s) = \frac{10e^{-0.1s}}{(1+s)(1+10s)}$$
$$L(s) = \frac{10(1+100s)}{(1+s)(1+10s)}$$
$$L(s) = \frac{10(1-100s)}{(1+s)(1+10s)}$$
$$L(s) = \frac{1}{1-s}$$
$$L(s) = \frac{-5}{1-s}$$

$$L(s) = e^{-\tau s} \Longrightarrow |L(j\omega)| = 1, \arg G(j\omega) = -\omega\tau 180/\pi$$

Sensitivity functions

- *Sensitivity function*
$$S(s) = \frac{1}{1+R(s)G(s)}$$
- *Complementary sensitivity function*
$$F(s) = \frac{R(s)G(s)}{1+R(s)G(s)}$$
- *Sensitivity control function*
$$Q(s) = \frac{R(s)}{1+R(s)G(s)} = F(s)G(s)^{-1} = R(s)S(s)$$
-
$$\begin{bmatrix} Y(s) \\ U(s) \\ E(s) \end{bmatrix} = \begin{bmatrix} F(s) & S(s) & -F(s) \\ Q(s) & -Q(s) & -Q(s) \\ S(s) & -S(s) & F(s) \end{bmatrix} \begin{bmatrix} W(s) \\ D(s) \\ N(s) \end{bmatrix}$$

Limits of Performance

- Nominal performances on the output y and error e result of the disturbance d
$$F(s) = 1$$
$$S(s) = 0$$
- It would imply missing an n attenuation on the output and
$$Q(s) = G(s)^{-1}$$
but being $G(s)$ strictly proper the $Q(s)$ module would increase when ω are high
$$\Downarrow$$
amplification of exogenus signals on u
-
$$F(s) + S(s) = 1$$

Limits of Performance

- If $G(s)$ is strictly proper and $R(s)$ is somehow proper than
$$s \to \infty \quad results \quad F(s) \to 0 \quad while \quad S(s) \to 1$$
- The specifics can be expressed on
$$L(s) = R(s)G(s)$$

Complementary sensitivity function analysis

$$F(s) = \frac{L(s)}{1+L(s)}$$

Represents:

a) transfer function between the reference w and the output y;
b) transfer function, with a changed sign, between the disturbance n and the output y;
c) transfer function between the disturbance n and the error e.

Static analysis

$$L(s) = \frac{\mu \prod_i (1+\tau_i s) \prod_i (1 + 2\zeta_i s/\alpha_{ni} + s^2/\alpha_{ni}^2)}{s^g \prod_i (1+T_i s) \prod_i (1 + 2\xi_i s/\omega_{ni} + s^2/\omega_{ni}^2)}$$

Results

$$\lim_{s\to 0} F(s) = \lim_{s \to 0} \frac{\mu}{s^g + \mu}$$
$$\lim_{s\to 0} F(s) = \begin{cases} \frac{\mu}{1+\mu} & , \quad g = 0 \\ 1 & , \quad g > 0 \\ 0 & , \quad g < 0 \end{cases}$$

Complementary sensitivity function analysis

$w(t) = Asca(t)$ is the result of the final value theorem and so from

$$y_\infty = \lim_{s\to 0} sF(s)\frac{A}{s} = A\lim_{s\to 0} F(s)$$

- Poles and zeros

$$F(s) = \frac{N_L(s)}{D_L(s)+N_L(s)}$$

- Frequency response

$$|F(j\omega)| = \frac{|L(j\omega)|}{|1+L(j\omega)|} \simeq \begin{cases} 1 & , \quad \omega \le \omega_c \\ |L(j\omega)| & , \quad \omega > \omega_c \end{cases}$$

Complementary sensitivity function analysis

- Damping and phase margin

$$\begin{aligned} |F(j\omega_c)| &= \frac{|L(j\omega_c)|}{|1+L(j\omega_c)|} = \frac{1}{|1+e^{j\varphi_c}|} = \frac{1}{|1+\cos\varphi_c + j\sin\varphi_c|} = \\ &= \frac{1}{\sqrt{(1+\cos\varphi_c)^2+\sin^2\varphi_c}} = \frac{1}{\sqrt{2(1+\cos\varphi_c)}} \\ &= \frac{1}{\sqrt{2(1-\cos\varphi_m)}} = \frac{1}{2\sin(\varphi_m/2)} \quad \left\{ sin^2\theta = \frac{1}{2}[1-\cos 2\theta] \right\} \end{aligned}$$

- Approximation to dominant poles assuming that $F(s)$

$$|F(j\omega_c)| = \frac{1}{2\xi}$$

$$\xi = \sin(\varphi_m/2)$$

$$\xi = \frac{\varphi_m}{2}\frac{\pi}{180} \simeq \frac{\varphi_m}{100}$$

only up to φ_m values around 75°. When $\varphi_m > 75^\circ$ $F(s)$ has a dominant real pole and in this ω_c case it forms an estimate of the associated time constant reverse.

Sensitivity function analysis

$$S(s) = \frac{1}{1+L(s)}$$

a) transfer function between the disturbance d and the output y;

b) transfer function between the reference w and the error e;

c) transfer function, with a changed sign, between the disturbance d and the error e.

$S(s) = 0$
A more realistic requirement is to ask that the frequency response $S(j\omega)$ has a sufficiently small module at the pulses where the signals spectrum of w and d shows significative components.

$$\lim_{s\to 0} S(s) = \lim_{s\to 0}\frac{s^g}{s^g+\mu} = \begin{cases} \frac{1}{1+\mu} & , \quad g=0 \\ 0 & , \quad g>0 \\ 1 & , \quad g<0 \end{cases}$$

Sensitivity function analysis

Step response

$$e_\infty = \lim_{s\to 0} sS(s)\frac{A}{s} = A\lim_{s\to 0} S(s) = \begin{cases} \frac{A}{1+\mu} & , \quad g=0 \\ 0 & , \quad g>0 \\ A & , \quad g<0 \end{cases}$$

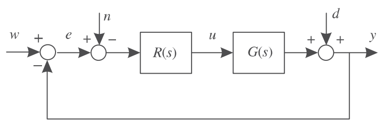
- Poles and zeros

$$S(s) = \frac{D_L(s)}{D_L(s)+N_L(s)}$$

- Frequency response

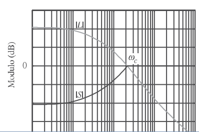
$$|S(j\omega)| = \frac{1}{|1+L(j\omega)|} \simeq \begin{cases} \frac{1}{|L(j\omega)|} & , \quad \omega \le \omega_c \\ 1 & , \quad \omega > \omega_c \end{cases}$$

Attenuation of the additive disturbance



$$Y(s) = \frac{1}{1+L(s)}D(s) = S(s)D(s)$$

$$|S(j\omega)| = \frac{1}{|1+L(j\omega)|} \simeq \begin{cases} \frac{1}{|L(j\omega)|} & , \quad \omega \le \omega_c \\ 1 & , \quad \omega > \omega_c \end{cases}$$



Attenuation of the additive disturbance

- $|S(j\omega)|$ it has to be small where the disturbance spectrum has significative components
- in the situation of canonical disturbances (steps, ramps,..) the loop function has to have an appropriate type or gain

Control sensitivity function analysis

$$Q(s) = \frac{R(s)}{1 + R(s)G(s)} = R(s)S(s) = F(s)G(s)^{-1}$$

a) transfer function between the disturbance w, d and n and the control u;

It would be better that, ceteris paribus, the $Q(j\omega)$ module results the smallest possible, at least in the pulse range where the inputs have significative components, in order to reduce the stresses to which the control variable u is submitted.

$$\lim_{s\to 0} Q(s) = \lim_{s\to 0} \mu_R S(s) = \frac{\mu_R}{1+\mu} = \lim_{s\to 0} \frac{F(s)}{\mu_G} = \frac{\mu}{\mu_G(1+\mu)}$$

- Poles and zeros

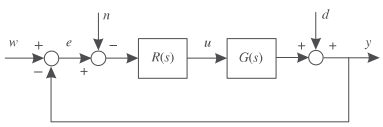
$$Q(s) = \frac{N_R(s)D_G(s)}{D_R(s)D_G(s) + N_R(s)N_G(s)} = \frac{N_R(s)D_G(s)}{D_L(s) + N_L(s)}$$

Control sensitivity function analysis

- Frequency response

$$|Q(j\omega)| = \frac{|R(j\omega)|}{|1 + L(j\omega)|} \simeq \begin{cases} \dfrac{1}{|G(j\omega)|} & , \quad \omega \le \omega_c \\ |R(j\omega)| & , \quad \omega > \omega_c \end{cases}$$

Main requirements



Please note we are talking about openloop stable systems
Nominal conditions stability
Bode Criterion: asymptotic stability is ensured if $\mu > 0$ $\varphi_m > 0$. In the product $R(s)G(s)$ **no** cancellations between singularities with real part higher than zero.
Stability in disrupted conditions
High level of strenght: high φ_m and k_m, ω_c not too high (time delay phase displacement τ is given by $-\omega_c\tau 180/\pi$).
Static precision
In order to reduce the amplitude of the error with an exhausted-transitional with e_∞ in presence of canonical w and d inputs (steps, ramps, etc.) it is necessary to increase the type g and the μ of the loop function.

Main requirements

Dynamical precision
Enlarge the band going through $F(s)$ and ω_c sufficiently high.
Second order aproximation: $S\% = 100e^{-\xi\pi/\sqrt{1-\xi^2}}$ (grafico) $\xi \simeq \frac{\varphi_m}{100}$
$T_{a\varepsilon} = -\frac{1}{\xi\omega_c}\ln 0.01\varepsilon$
First Order aproximation ($\varphi_m > 75°$): $T_{a1} = 5/\omega_c$
Attenuation of the disturbance effect d
From the sensitivity function $S(s)$, $|L(j\omega)| \gg 1$ in the pulses range on which the disturbance spectrum d stresses. Tipically that means a link to the minimum value of the critical pulse ω_c.
Attenuation of the disturbance effect n
If the disturbance spectrum n is confined to high frequency, that implies a link to the maximum value of ω_c
Moderation on the control variable
It is possible to notice studying the sensitivity function of the control $Q(s)$, the moderation property is linked to the $|R(j\omega)|$ value when pulses are higher than ω_c. Avoid that in the same group $|L(j\omega)| \gg |G(j\omega)|$.
Regulator feasibility
In order to respect that the condition that $R(s)$ is the transfer function of a proper dynamic system.

Synthesis

Following this reasoning, it is easier to think about the transfer function $R(s)$ of the regulator as $R(s) = R_1(s)R_2(s)$, where

$$R_1(s) = \frac{\mu_R}{s^r}$$

is called *static part* of the regulator and

$$R_2(s) = \frac{\prod_i (1 + \tau_i s)}{\prod_i (1 + T_i s)}$$

the *dynamical part* or better *stabilizer net*

Synthesis

Example

In case you have to design a regulator for the system described by

$$G(s) = \frac{10}{(1+10s)(1+5s)(1+s)}$$

considering the following characteristics:

a) $|e_\infty| \le 0.1$ when $w(t) = Asca(t)$ and $d(t) = Bsca(t)$ with $|A| \le 1$ and $|B| \le 5$;
b) $0.2 \le \omega_c \le 2$;
c) $\varphi_m \ge 50°$.

Choosing $R_1(s) = \frac{\mu_R}{s^r}$ than

$$L'(s) = R_1(s)G(s) = \frac{\mu_R 10}{s^r(1+10s)(1+5s)(1+s)}$$

$$e_\infty = \lim_{s\to 0} s\left(\frac{1}{1 + \frac{\mu_R 10}{s^r}}\frac{A}{s} + \frac{1}{1 + \frac{\mu_R 10}{s^r}}\frac{B}{s}\right)$$

Synthesis

Example

$r=0$ is enough to have a finite e_∞, $r=1$ to have $e_\infty=0$. Scelgo $r=0$ e quindi

$$|e_\infty| \leq \frac{6}{1+\mu_R 10} \leq 0.1 \Longrightarrow \mu_R \geq 5.9$$

I choose $\mu_R = 10$.
It is necessary to cut with $\omega_c = 1$ of -1 inclination: it is necessary to put a pole in 0.01, I choose three zeros to cancel G(s) poles, I have to add two poles to make the regulator achievable. I try not to use a regulator with too high values on high frequencies: I put two poles in 3
$\varphi_m = 180° - |-90° - 19° - 19°| = 52°$

$$R_2(s) = \frac{(1+10s)(1+5s)(1+s)}{(1+100s)(1+0.33s)^2}$$

$$R(s) = \frac{10(1+10s)(1+5s)(1+s)}{(1+100s)(1+0.33s)^2}$$

Made also with only two poles, with a supplement $+2$ poles, supplement $+1$ pole.

Synthesis - Right half plane zero

Example

Design a regulator for the system described by

$$G(s) = \frac{0.1(1-2s)}{s(1+10s)(1+0.1s)}$$

with the following requirements:

a) as ω_c large as possible;
b) $\varphi_m \geq 45°$.

Calculate the gain margin

I do not have specific static requirements, so I choose $R_1(s) = 1$ (I can still choose μ_R)
I cannot cancel the integrator and the zero on the right because in doing so I would loose the stability. Then I cannot cut over the zero frequency (I would have $\varphi_m < 180° - |-90° - 45° - 45°| < 0$)
I choose to cut at 0.316 (-10dB) with $\mu_R = 3.16$ nd cancelling the pole at 0.1, to make achievable the regulator and I place a pole at 3.

Synthesis - Right half plane zero

$$R(s) = \frac{3.16(1+10s)}{(1+0.33s)}$$

$\varphi_m = 180° - |-90° - 32° - 6° - 2°| = 50°$
$k_m = 4dB = 10^{(4/20)} = 1.58$

PID Regulator

$$u(t) = K_P e(t) + K_I \int_{t_0}^{t} e(\tau)d\tau + K_D \frac{de(t)}{dt}$$

K_P is called *coefficient of the proportional action*
K_I *integral action coefficient*
K_D *derivative action coefficient*

$$R_{PID}(s) = K_P + \frac{K_I}{s} + K_D s = \frac{K_D s^2 + K_P s + K_I}{s}$$

Alternative representation

$$R_{PID}(s) = K_P(1 + \frac{1}{T_I s} + T_D s) = K_P \frac{T_I T_D s^2 + T_I s + 1}{T_I s}$$

$T_I = K_P/K_I$ is the *integral time*
$T_D = K_D/K_P$ is the *derivative time*

PID Regulators

Achievable version

$$R_D^a(s) = \frac{K_P T_D s}{1 + \frac{T_D}{N}s} = \frac{K_D s}{1 + \frac{K_D}{K_P N}s}$$

where the positive costant N is chosen so the pole $s = -N/T_D$, added for the reachability, is on the outside of the frequency band of interest in the control. Typical values of N are $5 \div 20$. The PID in *real form* has the following transfer function

$$R_{PID}^a(s) = K_P(1 + \frac{1}{T_I s} + \frac{T_D}{1 + \frac{T_D}{N}s}s) = K_P + \frac{K_I}{s} + \frac{K_D s}{1 + \frac{K_D}{K_P N}s}$$

PID

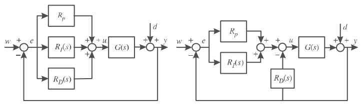
The PIDs have a pole in the origin of the complex plan and in their ideal form, two zeros in position

$$s = \frac{-T_I \pm \sqrt{T_I(T_I - 4T_D)}}{2T_I T_D} \quad (3)$$

If you consider the real form, zeros change and there is a pole with an higher pulse. Than the PID can be interpreted as a lead-lag network and his use allows the control of various classes of systems (check Paragrafo 12.5.3). From 3 please also note that zeros are real when $T_I \geq 4T_D$ and when $T_I = 4T_D$ they coincide at $s = -1/2T_D$; this choice is often used to simplify the automatic tuning.

Creating PID Regulators

Limits for the integral action



Error derivation

$$Y(s) = \frac{R_{PID}(s)G(s)}{1+R_{PID}(s)G(s)}W(s) + \frac{1}{1+R_{PID}(s)G(s)}D(s)$$

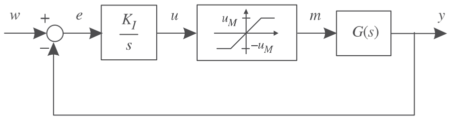
$$U(s) = \frac{R_{PID}(s)}{1+R_{PID}(s)G(s)}W(s) - \frac{R_{PID}(s)}{1+R_{PID}(s)G(s)}D(s)$$

Output derivation

$$Y(s) = \frac{R_{PI}(s)G(s)}{1+R_{PID}(s)G(s)}W(s) + \frac{1}{1+R_{PID}(s)G(s)}D(s)$$

$$U(s) = \frac{R_{PI}(s)}{1+R_{PID}(s)G(s)}W(s) - \frac{R_{PID}(s)}{1+R_{PID}(s)G(s)}D(s)$$

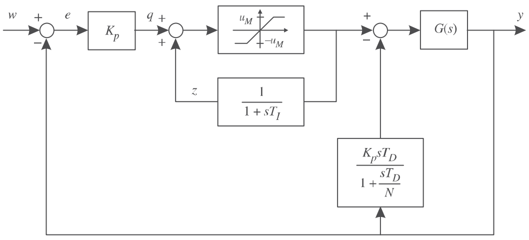
Integral action desaturation



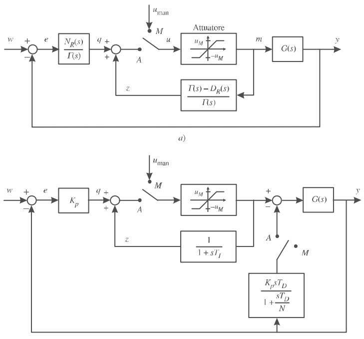
$$m(t) = \begin{cases} -u_M & , \quad u(t) < -u_M \\ u(t) & , \quad |u(t)| \le u_M \\ u_M & , \quad u(t) > u_M \end{cases}$$

Desaturation schematics

Please note Desaturation Schematics for PI+D



Industrial Production of PID



Synthesis: delay

Example

Project a regulator for the system described by

$$G(s) = \frac{e^{-4s}}{(1+s)^2}$$

with the following requirements:

a) $e_\infty = 0$ where there is the largest possible step variation of the reference ω_c;

b) $\omega_c \geq 0.1$

c) $\varphi_m \geq 30°$

$$R_1(s) = \frac{1}{s}$$

Is still possible to choose μ_r. I choose to cut at 0.2 and so I choose $\mu_r = 0.2$.

Synthesis: delay

I estimate the phase margin using

$$R(s) = \frac{0.2}{s}$$

and the result is $\varphi_m = 180° - |-90° - 2*11° - 4*0.2*180/\pi| = 22°$, is not sufficient and so I can cancel also the pole in one, the regulator is anyway reachable.

$$R(s) = \frac{0.2(1+s)}{s}$$

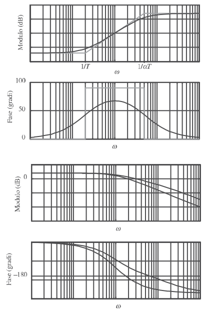
and the result is $\varphi_m = 180° - |-90° - 11° - 4*0.2*180/\pi| = 33°$.

Lead Networks

A lead network is described by the transfer function

$$R(s) = \mu_R \frac{1 + Ts}{1 + \alpha Ts}$$

where $\mu_R > 0$, $T > 0$, $0 < \alpha < 1$.

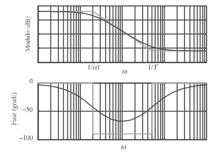


Lag network

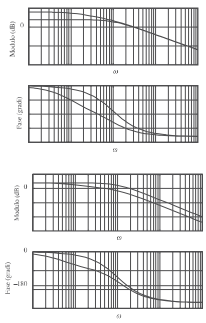
A lag network is described by the tranfer function

$$R(s) = \mu_R \frac{1 + Ts}{1 + \alpha Ts}$$

where $\mu_R > 0$, $T > 0$, $\alpha > 1$.



Lag network

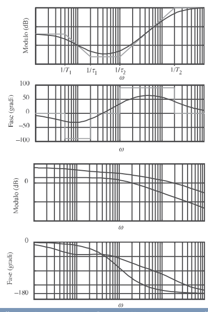


Lead-lag network

The combination of a lead network and of a lag network creates the so-called lead-lag network, described the transfer funcion

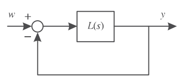
$$R(s) = \mu_R \frac{(1 + \tau_1 s)(1 + \tau_2 s)}{(1 + T_1 s)(1 + T_2 s)}$$

when $\mu_R > 0$, $T_1 > \tau_1 \geq \tau_2 > T_2 > 0$



Root locus

For the system



with the following loop function transfer

$$L(s) = \rho \frac{\prod_{i=1}^{m} (s + z_i)}{\prod_{i=1}^{n} (s + p_i)} = \rho \frac{N^*(s)}{D(s)}$$

the location described on the complex plan by the roots of the characteristics location is called "Root locus"

$$1 + L(s) = 0$$

when the real paramenter ρ varies from $-\infty$ and $+\infty$, and it is $\rho \neq 0$. Specifically, the part of the location corresponding to $\rho > 0$ pets the name of direct location *direct location (LD)*, otherwise it is called *reverse location (LI)* the one corresponding to $\rho < 0$.

Root locus

Complex relation

$$\frac{N^*(s)}{D(s)} = -\frac{1}{\rho}$$

is equivalent to the following two equations, expressed with in terms of modulus and phases:

$$\frac{|N^*(s)|}{|D(s)|} = \frac{1}{|\rho|}$$

$$\begin{aligned} \arg N^*(s) - \arg D(s) &= 180^\circ - \arg \rho = \\ &= \begin{cases} (2k+1)180^\circ & , \ \rho > 0 \ , \ k \ integer \quad (LD) \\ 2k180^\circ & , \ \rho < 0 \ , \ k \ integer \quad (LI) \end{cases} \end{aligned}$$

It will be demostrated that the relations between the arguments is sufficient to characterize completely the geometric aspect of the location, while the relation on the "moduli" is needed to determine the punctuation comparing to ρ.

Tracking rules

Rule 1
The location of roots is made of $2n$ *branches*: n of those are part of the direct location and the remaning n are part of the reverse location.

Rule 2
The location of roots is symmetrical comparing with the real axis.

Infact the l'Eq.

$$D(s) = \rho N^*(s) \tag{4}$$

has real coefficient, his roots are reals or complex conjugates.

Rule 3
The branches "start" from the $L(s)$ poles.

Infact for $|\rho| \to 0$ the roots of 4 converge towards the poles of the loop function.

Tracking rules

Rule 4
Both in the direct location than in the reverse location, for $|\rho| \to \infty$, m branches "reach" the $L(s)$ zeros and the remaining branches ν rami tend to infinity.

LThe reasoning of this results is based on the observation that, when $|\rho| \to \infty$, l'Eq. 4 becomes an equation of order m

$$N^*(s) = 0$$

that only has m roots, equals to the zeros of $L(s)$. When $|\rho| \to \infty$, these are the only roots to the finite of Eq. 4.

Rule 5
Branches that tend to infinity use the asymptotes that meet the real axis on the point of the abscissa

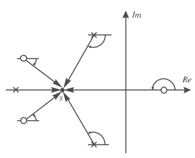
$$x_a = \frac{1}{\nu}\left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i\right)$$

and they form with the real axis angles equals to

Tracking rules

$$\psi_{ak} = \begin{cases} \dfrac{(2k+1)180^\circ}{\nu} & , \ k = 0, 1, \ldots, \nu - 1 \quad , \ \rho > 0 \quad (LD) \\ \dfrac{2k180^\circ}{\nu} & , \ k = 0, 1, \ldots, \nu - 1 \quad , \ \rho < 0 \quad (LI) \end{cases}$$

Rule 6



All the real axis points belong to the location of roots. Specifically all the points on left of an odd number of the $L(s)$ singularity are part of the direct location; they are part of the reverse location all the points on the right of a even number of the $L(s)$ singularity.

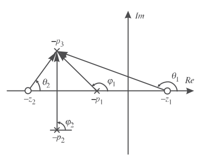
Tracking rules

Rule 7
When $\nu \geq 2$, the center of gravity does not depend on ρ and coincide with the real point of the abscissa

$$x_b = -\frac{1}{n}\sum_{i=1}^{n} p_i$$

Tracking rules

Rule 8



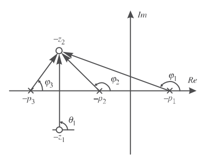
Think about pole $-p_j$ of $L(s)$ nd suppose that it has simple multiplicity, the trigonometric function of the output branch offers an angle

$$\alpha_j = \begin{cases} 180^\circ + \sum_{i=1}^{m} \theta_i - \sum_{i \neq j} \varphi_i & , \ \rho > 0 \quad (LD) \\ \sum_{i=1}^{m} \theta_i - \sum_{i \neq j} \varphi_i & , \ \rho < 0 \quad (LI) \end{cases}$$

where the angles φ_i and θ_i are the one computed following the standard mode considering the vectors that join the point $s = -p_j$ to the other poles and zeros. If the pole $-p_j$ has multiplicity h_j (check the Course Book).

Tracking rules

Rule 9



Think about the zero $-z_j$ of $L(s)$ and suppose that it has simple multiplicity, the trigonometric function of the output branch offers an angle

$$\beta_j = \begin{cases} 180^\circ - \sum_{i \neq j} \theta_i + \sum_{i=1}^{n} \varphi_i & , \ \rho > 0 \quad (LD) \\ -\sum_{i \neq j} \theta_i + \sum_{i=1}^{n} \varphi_i & , \ \rho < 0 \quad (LI) \end{cases}$$

where the angles θ_i and φ_i are the one computed following the standard mode considering the vectors that join the point $s = -z_j$ to the other poles and zeros. If $-z_j$ has multiplicity h_j (check the Course Book).

Tracking rules

Rule 10
In every point of the location, the value of $|\rho|$ is given by

$$|\rho| = \frac{\prod_{i=1}^{n} \eta_i}{\prod_{i=1}^{m} \lambda_i}$$

where λ_i and η_i represent the distances from the point to the zero and poles of $L(s)$.

Studying the stability with the variation of ρ

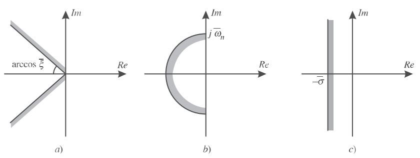
$$L(s) = \rho \frac{(s+1)^2(s+2)}{s(s+20)(s+16)^2(s^2+4)}$$
$$L(s) = \rho \frac{(s+2)(s^2+9)}{s(s+10)(s+6)(s^2+4s+8)}$$
$$L(s) = \rho \frac{(s^2+1)}{s(s-1)(s+1)^2(s+5)^2}$$
$$L(s) = \rho \frac{(s+1)}{(s+4)^3(s-1)}$$

Use of the root locus in the synthesis

- Dominant poles hypothesis: complementary sensitivity function

$$F(s) = \frac{\mu\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad , \quad \omega_n > 0 \ , \ |\xi| < 1$$

- Requirement for the step response overshoot $\longrightarrow \xi \geq \bar{\xi}$
- Requirements for the assesment time $\longrightarrow \xi\omega_n \geq -\frac{1}{T_{a\varepsilon}} \ln 0.01\varepsilon$
- Requirements for the "tempo di salita" $\longrightarrow \omega_n \geq \bar{\omega}_n$



Example

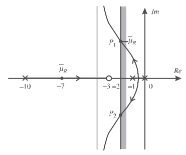
Given

$$G(s) = \frac{s+3}{s(s+1)(s+10)} \quad , \quad R(s) = \mu_R$$

determine μ_R so that the adjustment time is equal or less than $T_{a1} = 2.5$

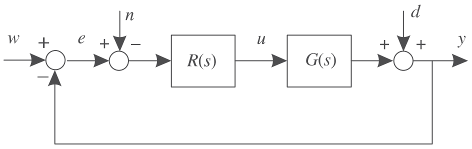
$$T_{a1} \simeq \frac{5}{\xi\omega_n}$$

poles on the left of $-\xi\omega_n = -2$



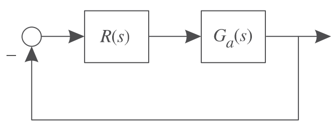
$$\mu_R > 31.5$$

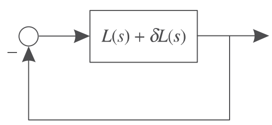
Stability in nominal conditions



- Stability $\longrightarrow$ absence of cancellations between zeros and poles of $R(s)$ and $G(s)$ with real part ≥ 0 and, when $L(s)$, $N = P$

Stability with perturbated conditions





$$G_a(s) = G(s) + \delta G(s) \quad , \quad |\delta G(j\omega)| \leq \gamma(\omega) \quad , \quad \forall\omega$$

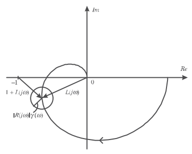
Hypothesis

- $G(s)$ and $G_a(s)$ have the same number of poles with real positive part
- absence of cancellations between zeros and poles of $R(s)$ and $G_a(s)$ with real part ≥ 0
- stability is present if $N_a = N$, where N_a is refered to $L_a(s) = L(s) + \delta L(s)$, where $\delta L(s) = R(s)\delta G(s)$

$$\Downarrow$$

the sufficient and necessary condition is that

$$|R(j\omega)|\gamma(\omega) < |1 + L(j\omega)| \quad , \quad \forall\omega$$



Hypothesis

Alternative writing:

$$\frac{\gamma(\omega)}{|G(j\omega)|} < \frac{1}{|F(j\omega)|} \quad , \quad \forall\omega, \text{ where } F(s) = \frac{L(s)}{1 + L(s)}$$

Example

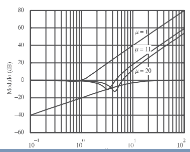
$$G(s) = \frac{1}{s(s+1)} \quad , \quad R(s) = \mu \quad , \quad F(s) = \frac{\mu}{s^2 + s + \mu}$$

Asympotic stability for every $\mu > 0$

$$G_a(s) = \frac{1}{s(1+s)(1+0.1s)}$$

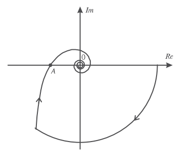
Stability of the perturbed system for $\mu < 11$

$$\frac{\delta G(s)}{G(s)} = \frac{-0.1s}{1 + 0.1s}$$



Time delay effect

$$L(s) = e^{-\tau s}\frac{\mu}{s} \quad , \quad \mu > 0$$



- asympotic stability if and only if $x_A > -1$ (x_A is the abscissa of A)
- group of values ω_k, $k = 0, 1, 2, \ldots$ so that $\arg L(j\omega_k) = -90^\circ - \omega_k \tau 180/\pi = -180^\circ - k360^\circ$ è

$$\omega_k = (4k + 1)\pi/2\tau$$

$$\Downarrow$$

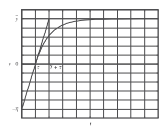
$$|L(j\omega_k)| = 2\mu\tau/(4k + 1)\pi \quad , \quad x_A = -2\mu\tau/\pi$$

Time delay effect

- With the increase of τ the Nyquist plot moves clockwise for an increasing number of loops around -1
- asympotic stability only for $\tau < \pi/2\mu$

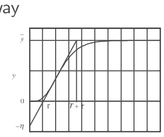
Approximate models: Tangent's method

Step response of amplitude $\bar{u}$ of the system with transfer function $G(s) = \frac{\mu}{1 + Ts}e^{-\tau s}$



$$\mu = \bar{y}/\bar{u} \quad , \quad \eta = -\tau\bar{y}/T$$

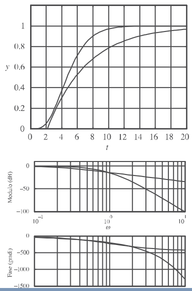
Given the response (non - oscillating) to the step of an unknown system of amplitude $\bar{u}$, a model approximate with the previous transfer function is obtained graphically in the following way



Approximate models: Tangent's method

Transfer function of the true system ($G(s)$) and the approximate model ($G_a(s)$)

$$G(s) = \frac{1}{(s+1)^5} \quad , \quad G_a(s) = \frac{e^{-2.1s}}{1+5.12s}$$

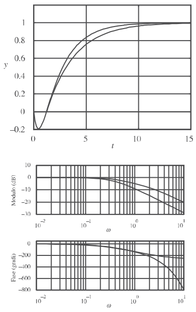


Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Approximate models: Tangent's method (Newton's method)

Transfer function of the true system ($G(s)$) and the approximate model ($G_a(s)$)

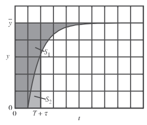
$$G(s) = G(s) = \frac{1-s}{(1+2s)(1+0.5s)} \quad , \quad G_a(s) = G_a(s) = \frac{e^{-1.22s}}{1+2.67s}$$



Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

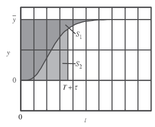
Approximate models: areas method

Response to the step of amplitude $\bar{u}$ of the system with transfer function $G(s) = \frac{\mu}{1+Ts}e^{-\tau s}$



$$\mu = \bar{y}/\bar{u} \quad , \quad S_1 = \bar{y}(\tau + T) \quad , \quad S_2 = \bar{y}T/e$$

Given the response (non - oscillating) to the step of an unknown system of amplitude $\bar{u}$, an approximate model with the previous transfer function is obtained in the following way

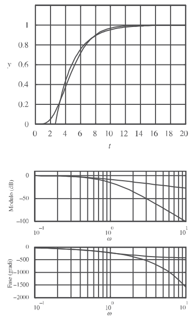


Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Approximate models: areas method

Transfer function of the true system ($G(s)$) and the approximant model ($G_a(s)$)

$$G(s) = \frac{1}{(s+1)^5} \quad , \quad G_a(s) = \frac{e^{-2.62s}}{1+2.385s}$$

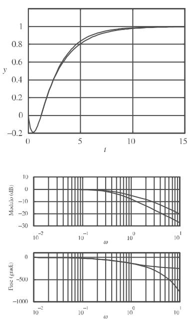


Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

Approximate models: areas method

Transfer function of the true system ($G(s)$) and the approximant model ($G_a(s)$)

$$G(s) = \frac{1-s}{(1+2s)(1+0.5s)} \quad , \quad G_a(s) = \frac{e^{-1.2s}}{1+2.3s}$$



Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill

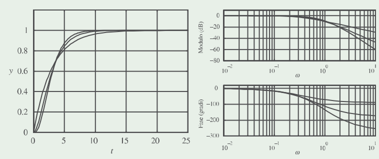
Approximation models of systems with real poles

$$G(s) = \frac{\mu}{(1+T_1 s)(1+T_2 s)\dots(1+T_n s)}$$

$$G_{a1}(s) = \frac{\mu}{1+(T_1+\dots+T_n)s} \, , \, G_{a2}(s) = \frac{\mu}{(1+0.5(T_1+\dots+T_n)s)^2}$$

Example

$$G(s) = \frac{1}{(1+s)^3} \quad , \quad G_{a1}(s) = \frac{1}{1+3s} \quad , \quad G_{a2}(s) = \frac{1}{(1+1.5s)^2}$$



Adapted from Bolzern, Scattolini, Schiavoni, "Fondamenti di Controlli Automatici", McGraw Hill