Fundamental of Control Theory

Stability nonlinear system: linearization

Faculty of Engineering
University of Pavia

Stability – summary so far

Nonlinear System

$\dot{x}(t) = f\big(x(t), u(t)\big)$
$y(t) = g\,\big(x(t), u(t)\big)$

- Property of each equilibrium point
- Looking for general result

Linear System

$\dot{x}(t) = Ax(t) + Bu(t)$
$y(t) = Cx(t) + Du(t)$

- Property of the system
- General result (eigenvalues of A Matrix)

Nonlinear System: *Stability - I*

Given the system

$$\boxed{\begin{aligned}\dot{x}(t) &= f\big(x(t), u(t)\big)\\ y(t) &= g\,\big(x(t), u(t)\big)\end{aligned}}$$

And the equilibrium $(\bar{x}, \bar{u})$ point

$$\boxed{\begin{aligned}0 &= f(\bar{x}, \bar{u})\\ \bar{y} &= g\,(\bar{x}, \bar{u})\end{aligned}}$$

How evaluate the stability of $(\bar{x}, \bar{u})$?

Idea
Use a LTI approximation of the system valid "near"

Nonlinear System: *Linearization - I*

Apply Taylor decomposition around $(\bar{x}, \bar{u})$ of $f\big(x(t), u(t)\big)$

Introducing the distances $\boxed{\delta x(t) = x(t) - \bar{x}}$ $\boxed{\delta u(t) = u(t) - \bar{u}}$

$$\frac{d}{dt}(\bar{x} + \delta x(t)) = f(\bar{x}, \bar{u}) + \left.\frac{\partial f(x,u)}{\partial x}\right|_{\bar{x},\bar{u}} \delta x(t) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{\bar{x},\bar{u}} \delta u(t) + \ldots$$

$$\frac{d}{dt}\delta x(t) = 0 + A\delta x(t) + B\,\delta u(t)$$

Where $\boxed{A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\bar{x},\bar{u}}}$ $\boxed{B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{\bar{x},\bar{u}}}$

Nonlinear System: *Linearization - II*

Apply Taylor decomposition around $(\bar{x}, \bar{u})$ of $g\big(x(t), u(t)\big)$

Introducing the distance $\boxed{\delta y(t) = y(t) - \bar{y}}$

$$\bar{y} + \delta y(t) = g(\bar{x}, \bar{u}) + \left.\frac{\partial g(x,u)}{\partial x}\right|_{\bar{x},\bar{u}} \delta x(t) + \left.\frac{\partial g(x,u)}{\partial u}\right|_{\bar{x},\bar{u}} \delta u(t) + \ldots$$

$$\delta y(t) = -\bar{y} + \bar{y} + C\delta x(t) + D\,\delta u(t)$$

Where $\boxed{C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{\bar{x},\bar{u}}}$ $\boxed{D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{\bar{x},\bar{u}}}$

Nonlinear System: *Linearization - III*

Given $\boxed{\begin{array}{l} \dot{x}(t) = f\big(x(t), u(t)\big) \\ y(t) = g\,\big(x(t), u(t)\big) \end{array}}$ And the equilibrium $(\bar{x}, \bar{u})$ point

The linearized system is $\boxed{\begin{array}{l} \delta\dot{x}(t) = A\delta x(t) + B\delta u(t) \\ \delta y(t) = C\delta x(t) + C\delta u(t) \end{array}}$

$\boxed{\delta x(t) = x(t) - \bar{x}}$ $\boxed{A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\bar{x},\bar{u}}}$ $\boxed{B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{\bar{x},\bar{u}}}$

$\boxed{\delta u(t) = u(t) - \bar{u}}$

$\boxed{\delta y(t) = y(t) - \bar{y}}$ $\boxed{C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{\bar{x},\bar{u}}}$ $\boxed{D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{\bar{x},\bar{u}}}$

Nonlinear System: *Stability - II*

An equilibrium point $(\bar{x}, \bar{u})$ is asymptotically stable if the linearized system is asymptotically stable.

An equilibrium point $(\bar{x}, \bar{u})$ is unstable if at least one of the eigenvalues of the linearized system has a positive real part

Remark if the linearized system is only stable or has all the eigenvalues not positive, no conclusion can be given to the stability of the nonlinear system equilibrium point.

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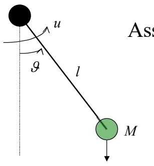
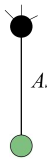
Pendulum - Equilibrium



Assuming $x_1(t) = \vartheta(t)$ $x_2(t) = \dot{\vartheta}(t)$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{g}{l}\sin(x_1(t)) - \frac{k}{Ml^2}x_2(t) + \frac{1}{Ml^2}u(t) \\ y(t) &= x_1(t) \end{aligned}$$

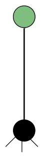


$Eq_1: ((0;0);0)$

Asymptotically Stable

$Eq_2: ((\pi;0);0)$

Unstable



Verify those result apply general result

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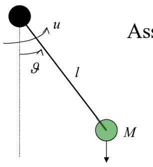
Pendulum - Linearization – I



Assuming $x_1(t) = \vartheta(t)$ $x_2(t) = \dot{\vartheta}(t)$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{g}{l}\sin(x_1(t)) - \frac{k}{Ml^2}x_2(t) + \frac{1}{Ml^2}u(t) \\ y(t) &= x_1(t) \end{aligned}$$

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\bar{x},\bar{u}} = \left.\begin{bmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \frac{\partial f_1(x,u)}{\partial x_2} \\ \frac{\partial f_2(x,u)}{\partial x_1} & \frac{\partial f_2(x,u)}{\partial x_2} \end{bmatrix}\right|_{\bar{x},\bar{u}} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & -\frac{k}{Ml^2} \end{bmatrix}\right|_{\bar{x},\bar{u}}$$

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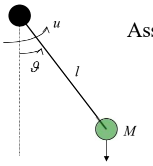
Pendulum - Linearization – II



Assuming $x_1(t) = \vartheta(t)$ $\quad x_2(t) = \dot{\vartheta}(t)$

$$\boxed{\begin{aligned} &\dot{x}_1(t) = x_2(t) \\ &\dot{x}_2(t) = -\tfrac{g}{l}\sin(x_1(t)) - \tfrac{k}{Ml^2}x_2(t) + \tfrac{1}{Ml^2}u(t) \\ &y(t) = x_1(t) \end{aligned}}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{\bar{x},\bar{u}} = \left.\begin{bmatrix} \dfrac{\partial f_1(x,u)}{\partial u} \\ \dfrac{\partial f_2(x,u)}{\partial u} \end{bmatrix}\right|_{\bar{x},\bar{u}} = \left.\begin{bmatrix} 0 \\ \dfrac{1}{Ml^2} \end{bmatrix}\right|_{\bar{x},\bar{u}} = \begin{bmatrix} 0 \\ \dfrac{1}{Ml^2} \end{bmatrix}$$

$$C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{\bar{x},\bar{u}} = \left.\begin{bmatrix} \dfrac{\partial g(x,u)}{\partial x_1} & \dfrac{\partial g(x,u)}{\partial x_2} \end{bmatrix}\right|_{\bar{x},\bar{u}} = \left.\begin{bmatrix} 1 & 0 \end{bmatrix}\right|_{\bar{x},\bar{u}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

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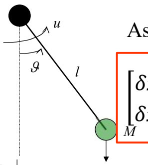
Pendulum - Stability – I



Assuming $x_1(t) = \vartheta(t)$ $\quad x_2(t) = \dot{\vartheta}(t)$

$$\boxed{\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \left.\begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l}\cos(x_2) & -\dfrac{k}{Ml^2} \end{bmatrix}\right|_{\bar{x},\bar{u}} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{Ml^2} \end{bmatrix}} u(t)$$

Eq_1: $((0;0);0)$

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l} & -\dfrac{k}{Ml^2} \end{bmatrix}$$

$$\lambda^2 + \frac{k}{Ml^2}\lambda + \frac{g}{l} = 0$$

$$\lambda_{1/2} = \frac{-\dfrac{k}{Ml^2} \pm \sqrt{\left(\dfrac{k}{Ml^2}\right)^2 - 4\dfrac{g}{l}}}{2}$$

$$det(\lambda I - A) = \lambda\left(\lambda + \frac{k}{Ml^2}\right) - \frac{g}{l}(-1) = 0$$

$Re(\lambda_{1,2})<0$ iff h, k>0 ⇔ *asymptotically stable*

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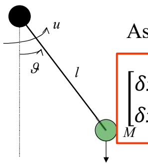
Pendulum - Stability – II



Assuming $x_1(t) = \vartheta(t)$ $\quad x_2(t) = \dot{\vartheta}(t)$

$$\boxed{\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \left.\begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l}\cos(x_2) & -\dfrac{k}{Ml^2} \end{bmatrix}\right|_{\bar{x},\bar{u}} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{Ml^2} \end{bmatrix}} u(t)$$

Eq_2: $((\pi;0);0)$

$$A = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{l} & -\dfrac{k}{Ml^2} \end{bmatrix}$$

$$\lambda^2 + \frac{k}{Ml^2}\lambda - \frac{g}{l} = 0$$

$$\lambda_{1/2} = \frac{-\dfrac{k}{Ml^2} \pm \sqrt{\left(-\dfrac{k}{Ml^2}\right)^2 + 4\dfrac{g}{l}}}{2}$$

$$det(\lambda I - A) = \lambda\left(\lambda + \frac{k}{Ml^2}\right) - (-1)\left(-\frac{g}{l}\right) = 0$$

$Re(\lambda_1)<0$, $Re(\lambda_2)>0$ ⇔ *unstable*

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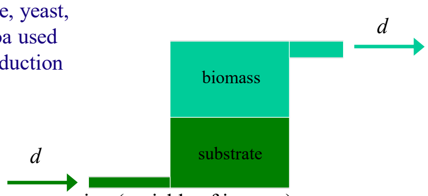
Microbial Growth

Colonies of algae, yeast, bacteria, protozoa used for enzymes production



$b =$ biomass concentration (variable of interest)
$S =$ substrate concentration
$d =$ volumetric flow rate of nutrients (constant)
$S_i =$ concentration of nutrients in substrate (forced)

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Microbial Growth - *equation*

$$\frac{db(t)}{dt} = \frac{\mu_0 S(t)}{K_s + S(t)} b(t) - d\, b(t)$$
$$\frac{dS(t)}{dt} = -k_1 \frac{\mu_0 S(t)}{K_s + S(t)} b(t) + d\,(S_i(t) - S(t))$$

$b =$ biomass concentration
$S =$ substrate concentration: nutrients (S_i) and microbial (S)
$d =$ volumetric flow rate of nutrients

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Microbial Growth - *exercise*

Given

$$K_s = 0.4 \quad k_1 = 0.35 \quad S_i = 0.2$$
$$d = 0.1 \quad \mu_0 = 0.4$$

1. Identify state, input, output and function *f* and *g*
2. Classify the system
3. Find equilibrium points
4. Evaluate stability of the equilibrium points

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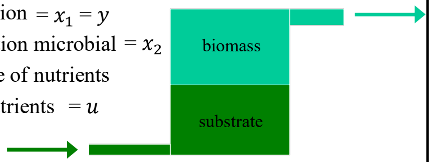
Microbial Growth - system

$b =$ biomass concentration $= x_1 = y$
$S =$ substrate concentration microbial $= x_2$
$d =$ volumetric flow rate of nutrients
$S_i =$ concentration of nutrients $= u$



$$\boxed{\begin{aligned} \frac{dx_1(t)}{dt} &= \frac{\mu_0 x_2(t)}{K_s + x_2(t)} x_1(t) - d x_1(t) \\ \frac{dx_2(t)}{dt} &= -k_1 \frac{\mu_0 x_2(t)}{K_s + x_2(t)} x_1(t) + d\,(S_i(t) - x_2(t)) \\ y(t) &= x_1(t) \end{aligned}}$$

Microbial Growth - Classification

$$\boxed{\begin{aligned} \frac{dx_1(t)}{dt} &= \frac{\mu_0 x_2(t)}{K_s + x_2(t)} x_1(t) - d x_1(t) \\ \frac{dx_2(t)}{dt} &= -k_1 \frac{\mu_0 x_2(t)}{K_s + x_2(t)} x_1(t) + d\,(S_i(t) - x_2(t)) \\ y(t) &= x_1(t) \end{aligned}}$$

$n = 2,$
$m = 1,$
$q = 1$

- Dynamic System
- Second order
- SISO
- Time-invariant
- Non linear
- Proper

Microbial Growth – Equilibrium - I

$$\boxed{\begin{aligned} 0 &= \frac{\mu_0 \bar{x}_2}{K_s + \bar{x}_2} \bar{x}_1 - d \bar{x}_1 \\ 0 &= -k_1 \frac{\mu_0 \bar{x}_2}{K_s + \bar{x}_2} \bar{x}_1 + d\,(\bar{u} - \bar{x}_2) \end{aligned}}$$

Eq_1: $((0; 0.2); 0.2)$

$$0 = \left(\frac{\mu_0 \bar{x}_2}{K_s + \bar{x}_2} - d\right) \bar{x}_1 \;\Rightarrow\; \bar{x}_1 = 0$$

$$0 = -k_1 \frac{\mu_0 \bar{x}_2}{K_s + \bar{x}_2} \bar{x}_1 - d\,(\bar{u} - \bar{x}_2) \;\Rightarrow\; 0 = -d\,(\bar{u} - \bar{x}_2)$$

$$0 = (\bar{u} - \bar{x}_2) \;\Rightarrow\; \bar{x}_2 = \bar{u}$$

Microbial Growth – Equilibrium - II

$$0 = \frac{\mu_0 \bar{x}_2}{K_s + \bar{x}_2}\bar{x}_1 - d\bar{x}_1$$
$$0 = -k_1 \frac{\mu_0 \bar{x}_2}{K_s + \bar{x}_2}\bar{x}_1 + d\,(\bar{u} - \bar{x}_2)$$

Eq_1: $((0; 0.2); 0.2)$

Eq_2: $((0.19; 0.13); 0.2)$

$$0 = \left(\frac{\mu_0 \bar{x}_2}{K_s + \bar{x}_2} - d\right)\bar{x}_1 \rightarrow \frac{\mu_0 \bar{x}_2}{K_s + \bar{x}_2} = d \rightarrow \mu_0 \bar{x}_2 = dK_s + d\bar{x}_2$$

$$\mu_0 \bar{x}_2 - d\bar{x}_2 = dK_s \rightarrow \bar{x}_2 = \frac{dK_s}{\mu_0 - d} = \frac{0.1 * 0.4}{0.4 - 0.1} = \frac{4}{30}$$

$$0 = -\frac{k_1 \mu_0 \bar{x}_2}{K_s + \bar{x}_2}\bar{x}_1 + d\,(\bar{u} - \bar{x}_2) \rightarrow k_1\, d\, \bar{x}_1 = d\,(\bar{u} - \bar{x}_2)$$

$$k_1\, \bar{x}_1 = (\bar{u} - \bar{x}_2) \rightarrow \bar{x}_1 = \frac{\bar{u} - \bar{x}_2}{k_1} = \frac{0.2 - 4/30}{0.35} = \frac{4}{21}$$

Microbial Growth – Linearization I

$$\frac{dx_1(t)}{dt} = \frac{\mu_0 x_2(t)}{K_s + x_2(t)}x_1(t) - dx_1(t)$$
$$\frac{dx_2(t)}{dt} = -k_1 \frac{\mu_0 x_2(t)}{K_s + x_2(t)}x_1(t) + d\,(S_i(t) - x_2(t))$$
$$y(t) = x_1(t)$$

$$A = \left.\begin{bmatrix} \frac{\mu_0 x_2}{K_s + x_2} - d & \frac{\mu_0 x_1 (K_s + x_2) - \mu_0 x_1 x_2 (1)}{(K_s + x_2)^2} \\ -k_1 \frac{\mu_0 x_2}{K_s + x_2} & -k_1 \frac{\mu_0 x_1 (K_s + x_2) - \mu_0 x_1 x_2 (1)}{(K_s + x_2)^2} - d \end{bmatrix}\right|_{\bar{x},\bar{u}}$$

$$A = \left.\begin{bmatrix} \frac{\mu_0 x_2}{K_s + x_2} - d & \frac{\mu_0 x_1 K_s}{(K_s + x_2)^2} \\ -k_1 \frac{\mu_0 x_2}{K_s + x_2} & -k_1 \frac{\mu_0 x_1 K_s}{(K_s + x_2)^2} - d \end{bmatrix}\right|_{\bar{x},\bar{u}}$$

Microbial Growth – Linearization II

$$A = \left.\begin{bmatrix} \frac{\mu_0 x_2}{K_s + x_2} - d & \frac{\mu_0 x_1 K_s}{(K_s + x_2)^2} \\ -k_1 \frac{\mu_0 x_2}{K_s + x_2} & -k_1 \frac{\mu_0 x_1 K_s}{(K_s + x_2)^2} - d \end{bmatrix}\right|_{\bar{x},\bar{u}}$$

$K_s = 0.4 \quad k_1 = 0.35$

$d = 0.1 \quad \mu_0 = 0.4$

$$A_1 = \begin{bmatrix} \frac{0.4 \cdot 0.2}{0.4 + 0.2} - 0.1 & 0 \\ -0.35 \frac{0.4 \cdot 0.2}{0.4 + 0.2} & 0 - 0.1 \end{bmatrix} = \begin{bmatrix} 0.133 - 0.1 & 0 \\ -0.35 \cdot 0.133 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.033 & 0 \\ -0.467 & -0.1 \end{bmatrix}$$

$$\det(sI - A_1) = \det \begin{bmatrix} s - 0.033 & 0 \\ 0.467 & s - 0.1 \end{bmatrix} = (s - 0.033)(s + 0.1) - 0 = 0$$

$$s_1 = -0.1, s_2 = 0.033$$

Microbial Growth – Linearization III

$$A=\begin{bmatrix}\frac{\mu_0 x_2}{K_s+x_2}-d & \frac{\mu_0 x_1 K_s}{(K_s+x_2)^2}\\ -k_1\frac{\mu_0 x_2}{K_s+x_2} & -k_1\frac{\mu_0 x_1 K_s}{(K_s+x_2)^2}-d\end{bmatrix}\Bigg|_{\bar{x},\bar{u}}$$

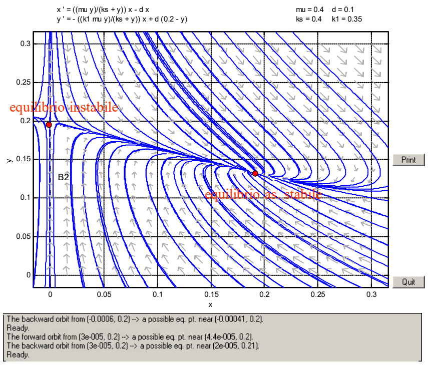
$K_s = 0.4 \quad k_1 = 0.35$
$d = 0.1 \quad \mu_0 = 0.4$

$$A_2=\begin{bmatrix}\frac{0.4\cdot 0.133}{0.4+0.133}-0.1 & \frac{0.4\cdot 0.19\cdot 0.4}{(0.4+0.133)^2}\\ -0.35\frac{0.4\cdot 0.133}{0.4+0.133} & -0.35\frac{0.4\cdot 0.19\cdot 0.4}{(0.4+0.133)^2}+0.1\end{bmatrix}=\begin{bmatrix}0 & 0.170\\ -0.035 & -0.135\end{bmatrix}$$

$$\det(sI-A_2)=\det\begin{bmatrix}s & -0.17\\ 0.035 & s-0.135\end{bmatrix}=s(s-0.135)-0.035*(-0.17)=0$$

$s^2-0.135s+0.037=0 \qquad s_1=-0.1, s_2=-0.0375$

piano di fase

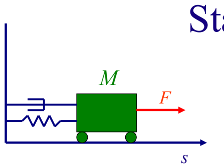


Dynamical System for Industrial Automation

Laplace Transform & Transfer function

Faculty of Engineering
University of Pavia

State definition



Physical equation

$$\ddot{s}(t) = \frac{F(t) - ks(t) - h\dot{s}(t)}{M}$$

State choice

$$x_1(t) = s(t)$$
$$x_2(t) = \dot{s}(t)$$

$$\widetilde{x_2}(t) = 2s(t)$$
$$\widetilde{x_2}(t) = s(t) + \dot{s}(t)$$

State equation

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{h}{M} \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{2k+h}{M} & -\frac{h}{M} \end{bmatrix}$$

The state definition determines the A matrix

Problem definition

Goal: to find a representation state-independent

Consider only Input => Output relation

For LTI this representation exists

- Called *Transfer function*
- Based on *Laplace Transform*

Laplace Transform

Mathematical fundamental

Laplace transform

Given a function $f: \mathbb{R} \mapsto \mathbb{R}$, the *Laplace transform* $F: \mathbb{C} \mapsto \mathbb{C}$ is

$$\boxed{F(s) = \int_0^\infty f(t)e^{-st}dt}$$

Remark: *F(s)* is defined if $\exists s \in \mathbb{C}$ s. t. the integral is finite

Notation:

- lower case => time domain ($t \in \mathbb{R}$)
- upper case => Laplace domain ($s = \sigma + \omega i \in \mathbb{C}$)

$$f(t) = \mathcal{L}^{-1}[F(s)] \qquad F(s) = \mathcal{L}[f(t)]$$

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Laplace Transform – nomenclature - I

Interested in rational Laplace transform

$$F(s) = \frac{N(s)}{D(s)} = \frac{\alpha_z s^v + \alpha_{q-1}s^{v-1} + \cdots + \alpha_1 s + \alpha_0}{\beta_n s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}$$

where $n >= v$,

We call

- Poles p_i the n roots of *D(s)*
- Zeros z_i the v roots of *N(s)*

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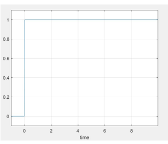
Step – Laplace transform

Step: sca(t)



$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty 1e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_{t=0}^{\lim t\to\infty}$$

choose a *s>0*

$$F(s) = \lim_{t\to\infty} -\frac{e^{-st}}{s} - \left(-\frac{e^{-s0}}{s}\right) = \frac{1}{s}$$

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Exponential – Laplace transform

Step: $e^{at}sca(t)$

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} e^{at}e^{-s}\,dt = \left[\frac{e^{(a-s)t}}{a-s}\right]_{t=0}^{\lim t\to\infty}$$

choose a $s>a$

$$F(s) = \lim_{t\to\infty}\frac{e^{(a-s)t}}{a-s} - \left(\frac{e^{(a-s)0}}{a-s}\right) = 0 - \frac{1}{a-s} = \frac{1}{s-a}$$

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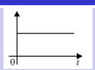
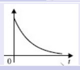
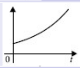
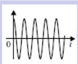
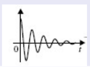
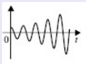
Main Laplace transform

f(t)		F(s)
step(t)		$\frac{1}{s}$
$e^{at}sca(t)$	 	$\frac{1}{s-a}$
$\sin(\omega t)sca(t)$		$\frac{\omega}{s^2+\omega^2}$
$e^{at}\sin(\omega t)sca(t)$	 	$\frac{\omega}{(s-a)^2+\omega^2}$

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Laplace Transform – properties I

Linearity

Given two functions *f(t)*, $g(t)$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\boxed{\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha F(s) + \beta G(s)}$$

Time shift

Given a functions *f(t)* supposed *null for negative time,* and a time shift $\tau > 0$ we have

$$\boxed{\mathcal{L}[f(t-\tau)] = e^{-\tau s}\mathcal{L}[f(t)] = e^{-\tau s}F(s)}$$

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Laplace Transform – properties II

Time domain derivation

Given functions *f(t)*, derivable $\forall t \geq 0$ we have

$$\boxed{\mathcal{L}\left[\frac{df(t)}{dt}\right] = \mathcal{L}[\dot{f}(t)] = sF(s) - f(0)}$$

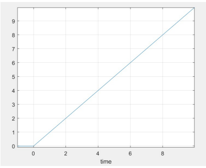
Time domain integration

Given functions *f(t)*, integrable between 0 and ∞, we have

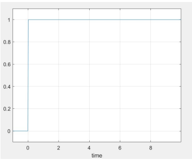
$$\boxed{\mathcal{L}\left[\int_0^\infty f(t)dt\right] = \frac{1}{s}F(s)}$$

ramp – Laplace transform

$$f(t) = ramp(t) = \int_0^\infty sca(t)dt$$



$ramp(t)$



$sca(t)$

$$F(S) = \frac{1}{s}\mathcal{L}[sca(t)] = \frac{1}{s^2}$$

cosine – Laplace transform

$$f(t) = \cos(\omega t)$$

$$\frac{d\sin(\omega t)}{dt} = \omega\cos(\omega t) \qquad \cos(\omega t) = \frac{1}{\omega}\frac{d\sin(\omega t)}{dt}$$

$$F(s) = \mathcal{L}\left[\frac{1}{\omega}\frac{d\sin(\omega t)}{dt}\right] = \frac{1}{\omega}\mathcal{L}\left[\frac{d\sin(\omega t)}{dt}\right]$$

$$= \frac{1}{\omega}(s\mathcal{L}[\sin(\omega t)] - \sin(\omega 0)) = \frac{1}{\omega}\left(s\frac{\omega}{s^2+\omega^2} - 0\right)$$

$$F(s) = \frac{s}{s^2+\omega^2}$$

Laplace Transform – properties III

Given a function *f(t)* that have a rational transform $F(s)$ where the denominator order is greater than numerator we have

Initial Value Theorem

$$\boxed{f(0) = \lim_{s\to\infty} sF(s)}$$

Moreover, the real parts of all the poles are negative or null, we have

Final Value Theorem

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

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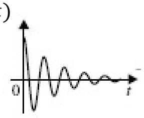
Initial and final value - examples

Consider the signal $f(t) = e^{-3t}\sin(10t)sca(t)$

What is the "final value" of the signal?



$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\frac{10}{(s+3)^2+10^2} = \lim_{s\to 0} \frac{10}{s^2+6s+109} = \frac{0}{109}$$

Check hypothesis

$(s+3)^2 + 10^2 = 0 \Rightarrow s^2 + 6s + 109 = 0 \qquad s_{1,2} = \frac{-6\pm\sqrt{36-4*109}}{2}$

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Transfer Function

LTI System

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Input – Output representation

Consider a system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$

$$u(t) \rightarrow \boxed{x(t)} \rightarrow y(t)$$

Apply Laplace transform

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$U(s) \rightarrow \boxed{C(sI-A)^{-1}B + D} \rightarrow Y(s)$$

suppose $x(0) = 0$

$$(sI - A)X(s) = BU(s) \quad \Longrightarrow \quad X(s) = (sI - A)^{-1}BU(s)$$

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s) = [C(sI - A)^{-1}B + D]U(s)$$

Transfer Function

Given a *LTI* system described by matrixes *A, B, C, D* the $m \times q$ matrix

$$\boxed{G(s) = C(sI - A)^{-1}B + D}$$

is called transfer function.

For null initial state $x(0) = 0$ the output of the system fed by the input vector $u(t)$

$$\boxed{Y(s) = G(s)U(s)}$$

where $U(s) = \mathcal{L}[u(t)]$ and $Y(s) = \mathcal{L}[y(t)]$

Transfer Function - Invariance

Is it state independent? YES, IT IS

What if I apply a different state representation $\tilde{x}(t) = T^{-1}x(t)$

$$\tilde{A} = TAT^{-1}, \quad \tilde{B} = TB, \quad \tilde{C} = CT^{-1}, \quad \tilde{D} = D$$

$$\tilde{G}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = CT^{-1}(sI - TAT^{-1})^{-1}TB + D$$

$$= C(sT^{-1}I - T^{-1}TAT^{-1})^{-1}TB + D = C(sT^{-1} - AT^{-1})^{-1}TB + D$$

$$= C(sT^{-1}T - AT^{-1}T)^{-1}B + D = C(sI - A)^{-1}B + D = G(s)$$

Transfer function – SISO Example I

$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, D= 0

$n = 2$ $m = 1$ $q = 1$, Strictly proper sys.

$$G(s) = C(sI - A)^{-1}B + D \qquad \boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$\det(sI - A) = \det \begin{bmatrix} s+1 & 1 \\ 1 & s+1 \end{bmatrix} = (s+1)(s+1) - 1 = \boxed{s(s+2)}$$

2 eigenvalues (0, -2)

$$(sI - A)^{-1} = \frac{1}{s(s+2)} \begin{bmatrix} s+1 & -1 \\ -1 & s+1 \end{bmatrix}$$

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Transfer function – SISO Example II

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s(s+2)} \begin{bmatrix} s+1 & -1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 =$$

$$G(s) = \frac{1}{s(s+2)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$G(s) = \frac{1}{s(s+2)} \begin{bmatrix} s+1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{s(s+2)}(s+1-1)$$

$$G(s) = \frac{s}{s(s+2)} = \frac{1}{(s+2)}$$

Pole/zero cancellation

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Transfer function – MIMO Example I

$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, D= 0

$n = 2$ $m = 2$ $q = 2$, Strictly proper system

$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{s(s+2)} \begin{bmatrix} s+1 & -1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0$$

$$G(s) = \frac{1}{s(s+2)} I \begin{bmatrix} -1 & s+1 \\ s+1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{s(s+2)} & \frac{s+1}{s(s+2)} \\ \frac{s+1}{s(s+2)} & -\frac{1}{s(s+2)} \end{bmatrix}$$

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Transfer Function - MIMO

Given a *LTI MIMO* system described by matrixes *A, B, C, D,*

$$\boxed{G(s) = C(sI - A)^{-1}B + D}$$

The transfer function is a $q \times m$ matrix

It can be demonstrated that matrix $(sI - A)^{-1}$

- has $n \times n$ elements
- each element is rational function
- all elements share a common denominator coincident with the characteristic polynomial of A.

Transfer Function – SISO I

Given a *LTI SISO* system we have

$$G(s) = \frac{N_G\,(s)}{D_G(s)} = \frac{\alpha_z s^{\upsilon} + \alpha_{q-1}s^{\upsilon-1} + \cdots + \alpha_1 s + \alpha_0}{\beta_n s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}$$

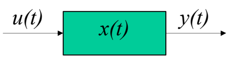
where

- for strictly proper system (D = 0) $\upsilon < n$
- poles (D_G roots) are eigenvalues
- Remark: there can be eigenvalues that are not poles (pole/zeros cancellation)

Response - Example I

Consider a system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$



where $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1 \quad 0]$, D= 0

Determine the "final value" from a

- step response
- ramp response

Response - Example I

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \quad 0], \quad D = 0$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det(sI - A) = \det \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix} = s(s+2) + 1 = (s+1)(s+1)$$

$$(sI - A)^{-1} = \frac{1}{(s+1)^2} \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}$$

Response - Example II

$$G(s) = [1 \quad 0] \frac{1}{(s+1)^2} \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0$$

$$G(s) = \frac{1}{(s+1)^2} [1 \quad 0] \begin{bmatrix} s+2 & 1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$G(s) = \frac{1}{(s+1)^2} [s+2 \quad 1] \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$G(s) = \frac{s+2}{(s+1)^2} = \frac{s+2}{(s+1)^2}$$

Response - Example III

step response

$$Y(s) = G(s)U(s) = \frac{s+2}{(s+1)^2} \frac{1}{s}$$

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s \frac{s+2}{(s+1)^2} \frac{1}{s} = \lim_{s \to 0} \frac{s+2}{(s+1)^2} = \frac{2}{(1)^2} = 2$$

ramp response

$$Y'(s) = G(s)U'(s) = \frac{s+2}{(s+1)^2} \frac{1}{s^2}$$

$$\lim_{t \to \infty} y'(t) = \lim_{s \to 0} sY'(s) = \lim_{s \to 0} s \frac{s+2}{(s+1)^2} \frac{1}{s^2} = \lim_{s \to 0} \frac{s+2}{(s+1)^2} \frac{1}{s} = \lim_{s \to 0} \frac{2}{1} \frac{1}{s} = \infty$$