Fundamentals of Control Theory

Equilibrium

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University of Pavia

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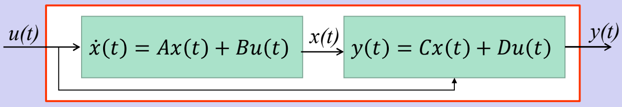
Superposition principle- LTI



- Simulation 1
  - $t_0 = 0,$
  - $x(t_0) = x'_0,$
  - $u(t) = u'(t)$
- Simulation 2
  - $t_0 = 0,$
  - $x(t_0) = x''_0,$
  - $u(t) = u''(t)$
- Simulation 3
  - $t_0 = 0,$
  - $x(t_0) = \alpha x'_0 + \beta x''_0,$
  - $u(t) = \alpha u'(t) + \beta u''(t)$

- Solution 1
  $x(t) = \phi'(t)$
- Solution 2
  $x(t) = \phi''(t)$
- Solution 3
  $x(t) = \alpha\phi'(t) + \beta\phi''(t)$

The result holds for time varying Linear System

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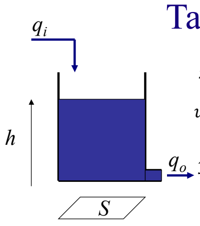
Tank with forced flow



$x(t) = h(t)$

$u(t) = \begin{bmatrix} q_i(t) \\ q_o(t) \end{bmatrix}$

$y(t) = h(t)$

$S = 0.5$

$$\boxed{\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}}$$

$$\boxed{\begin{aligned} &A = 0 \quad B = [1/S \quad -1/S] \\ &C = 1 \qquad\quad D = 0 \end{aligned}}$$

Evaluate state movement

$t_0 = 0,$
$x(t_0) = 0,$
$u(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
$x(t) = 4t - 2t = 2t$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$\begin{cases} t_0 = 0, \\ x(t_0) = 0, \\ u'(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{cases} \qquad x'(t) = \int_0^t \frac{1}{0.5} 2\,d\tau = 4t$$

$$\begin{cases} t_0 = 0, \\ x(t_0) = 0, \\ u''(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \qquad x''(t) = \int_0^t -\frac{1}{0.5} 1\,d\tau = -2t$$

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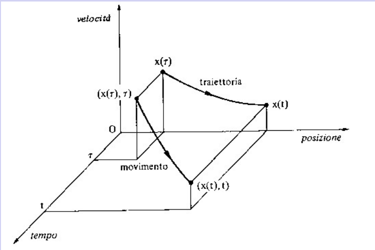
Trajectory

The state movement projection on the state plane is called *State Trajectory*



Likewise, the *Output Trajectory* is the projection of the output movement on the output plane

Equilibrium

- An notable behaviour for the system is the Constant movement $x(t) = x_{eq}$
- This correspond to single a point as trajectory (x_{eq})
- A common problem is to find a constant input $u(t) = u_{eq}$ that generate this equilibrium

$\dot{x}(t) = f(x(t), u(t), t)$ ➡ $\boxed{0 = f(x_{eq}, u_{eq}, t)}$

Equilibrium point $\boxed{(x_{eq}, u_{eq}) \textit{ such that } f(x_{eq}, u_{eq}, t) = 0,\ t > t_0}$

$\boxed{(x_{eq}, u_{eq}) \textit{ such that } f(x_{eq}, u_{eq}) = 0}$ *TI*

LTI $\boxed{(x_{eq}, u_{eq}) \textit{ such that } Ax_{eq} + Bu_{eq} = 0 \rightarrow u_{eq} = -B^{-1}Ax_{eq}}$

An economic model - Equilibrium

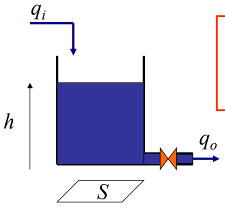
Y = Gross Domestic Product = y(t)
I = Investment = u(t)
C = Family Consumption = x(t)

$$\boxed{\begin{aligned} \dot{x}(t) &= (b-a)x(t) + bu(t) \\ y(t) &= x(t) + u(t) \end{aligned}}$$

Evaluate the constant investment ammount $\bar{u}$
to keep a constant consmption level of $\bar{x}$

$$(b-a)\bar{x} + b\bar{u} = 0 \qquad \bar{u} = -\frac{b-a}{b}\bar{x}$$

Tank with forced inflow - Equilibrium



$$\boxed{\begin{aligned}\dot{x}(t) &= \frac{1}{S}\left(u(t) - \sqrt{2gx(t)}\right)\\ y(t) &= x(t)\end{aligned}}$$

$x(t) = h(t)$
$u(t) = q_i(t)$
$y(t) = h(t)$

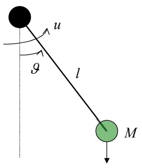
$S = 2m^2 \qquad g = 10\,\frac{m}{s^2}$

Evaluate the constant inflow $\bar{u}$ to keep the level at 80 m

$(x_{eq} = 80, u_{eq} = \bar{u}): f(80, \bar{u}) = 0 \qquad 0 = \frac{1}{S}\left(\bar{u} - \sqrt{2 \cdot g \cdot 80}\right)$

$0 = \bar{u} - \sqrt{160g} \qquad \bar{u} = 4\sqrt{10g} \quad = 4\sqrt{10 \cdot 10} = 40$

Pendulum



A rod pivoted at a fulcrum with a mass M.
We are interested in the rod angle ϑ.
Subject to a friction proportional to speed $\dot{\vartheta}$.
A force couple u may be applied to the rod.

$$Ml^2\ddot{\vartheta}(t) = -glM\sin\left(\vartheta(t)\right) - k\dot{\vartheta}(t) + u(t)$$

Find equilibrium point/s when the couple is not in place (u=0)

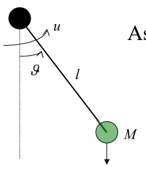
Assuming

$x_1(t) = \vartheta(t)$

$x_2(t) = \dot{\vartheta}(t)$

$$\boxed{\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -\frac{g}{l}\sin(x_1(t)) - kx_2(t) + \frac{1}{Ml^2}u(t)\\ y(t) &= x_1(t)\end{aligned}}$$

Pendulum - Classification



Assuming $\quad x_1(t) = \vartheta(t) \qquad x_2(t) = \dot{\vartheta}(t)$

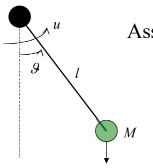
$$\boxed{\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -\frac{g}{l}\sin(x_1(t)) - kx_2(t) + \frac{1}{Ml^2}u(t)\\ y(t) &= x_1(t)\end{aligned}}$$

Dynamic system (n=2) *Order 2*

SISO (m=1, q=1) *Not linear*

Time invariant *Strictly Proper*

Pendulum - Equilibrium



Assuming $x_1(t) = \vartheta(t)$ $\quad x_2(t) = \dot{\vartheta}(t)$

$$\boxed{\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{g}{l}\sin(x_1(t)) - \frac{k}{Ml^2}x_2(t) + \frac{1}{Ml^2}u(t) \\ y(t) &= x_1(t) \end{aligned}}$$

Find equilibrium
Points for $\bar{u} = 0$

$Eq: (\bar{x}; \bar{u})\ s.t.\ f(\bar{x}; \bar{u}) = 0$

$$\begin{cases} 0 = \overline{x_2} \\ 0 = -\frac{g}{l}\sin(\overline{x_1}) - \frac{k}{Ml^2}\overline{x_2} \end{cases}$$

$0 = -\frac{g}{l}\sin(x_1(t))$

$\boxed{Eq_1: ((0; 0); 0) \quad Eq_2: ((\pi; 0); 0)}$

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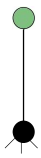
Pendulum - Consideration

$Eq_1: ((0; 0); 0)$ $\quad Eq_2: ((\pi; 0); 0)$



Are the two Equilibrium points equivalent?

Consider to apply the *same input* (no force applied)
but a *small variation* to equilibrium *state*:

Eq_1 : nothing change
Eq_2 : the system move away from

Stability exploits this difference

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Fundamentals of Automatic Control

Stability definition & theorem for LTI system

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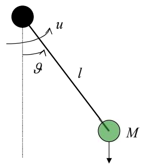
Pendulum



A rod pivoted at a fulcrum with a mass M.
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Subject to a friction proportional to speed $\dot{\vartheta}$.
A force couple u may be applied to the rod.

$$Ml^2\ddot{\vartheta}(t) = -glMsin\big(\vartheta(t)\big) - k\dot{\vartheta}(t) + u(t)$$

Assuming

$$x_1(t) = \vartheta(t)$$
$$x_2(t) = \dot{\vartheta}(t)$$

$$\boxed{\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\tfrac{g}{l}\sin(x_1(t)) - kx_2(t) + \tfrac{1}{Ml^2}u(t) \\ y(t) &= x_1(t) \end{aligned}}$$

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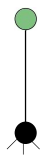
Pendulum - Consideration

$Eq_1: \big((0;0);0\big)$ $Eq_2: \big((\pi;0);0\big)$



Are the two Equilibrium points equivalent?

Consider to apply the *same input* (no force applied)
but a *small variation* to equilibrium *state*:
if I am in Eq_1 nothing change
if I am in Eq_2 the system move away from

Stability exploits this difference

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Dynamic System
equilibrium stability - I

Concept
$\bar{x}$ is a *stable equilibrium state* if the state signals generated by every "small" perturbance of $\bar{x}$ is always close to $\bar{x}$

- "small" perturbance of $\bar{x}$ $\|x_0 - \bar{x}\| < \varepsilon$
- always close to $\bar{x}$ $\|x(t) - \bar{x}\| < \delta$

Definition
An equilibrium $\bar{x}$ is defined *stable* if, given an ε>0, $\exists\delta$>0 such that for each initial state x_0 fulfilling $\|x_0 - \bar{x}\| < \varepsilon$ results in $\|x(t) - \bar{x}\| < \delta$ for every $t > 0$.

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Dynamic System
equilibrium stability - II

Concept

$\bar{x}$ is an *asymptotically stable equilibrium state* if the state signals generated by applying the equilibrium input to every “small” perturbance of $\bar{x}$ tend to return to $\bar{x}$

Definition

An equilibrium $\bar{x}$ is defined *asymptotically stable* if is stable and

$$\lim_{t\to\infty} \|x(t) - \bar{x}\| = 0 .$$

Definition

An equilibrium $\bar{x}$ is *unstable* if is not stable

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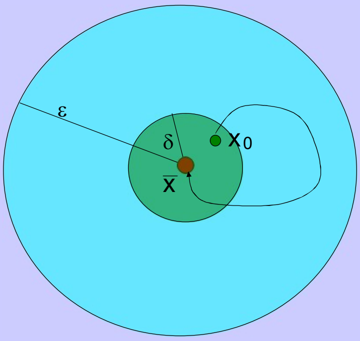
Asymptotic stability



δ: *region of attraction*

δ can be really small

if δ is the whole state space

$\bar{x}$: global asymptotical stable equilibrium state

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Temperatura della terra
stabilità dell’equilibrio

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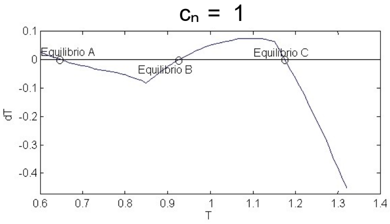
Stabilità dell'equilibrio



Punto di equilibrio B

Comunque perturbi il sistema mi allontano dall'equilibrio

Equilibrio Instabile

Punti di equilibrio A e C

$x_0 < \bar{x}$ $\quad \frac{dx}{dt} > 0$) $\quad x(t)$ crescente

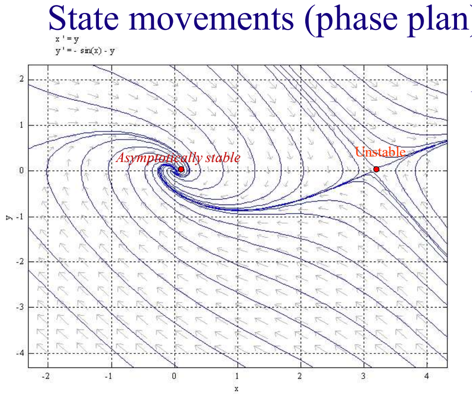
$x_0 > \bar{x}$ $\quad \frac{dx}{dt} < 0$) $\quad x(t)$ decrescente

Equilibrio Stabile

Pendulum

Equlibrium Stability

State movements (phase plan)



M=g=l=1

u(t)=0

LTI system - *stability*

State dynamic $\quad x(t) = e^{At}x_o + \int_0^t e^{A(t-\tau)}B\bar{u}\,d\tau$

Equilibrium dynamic $\quad \bar{x} = e^{At}\bar{x} + \int_0^t e^{A(t-\tau)}B\bar{u}\,d\tau$

$$\boxed{x(t) - \bar{x} = e^{At}(x_o - \bar{x})}$$

Stability is related ONLY to the matrix A → *For LTI Stability is a Property of the System instead of the equilibrium point*

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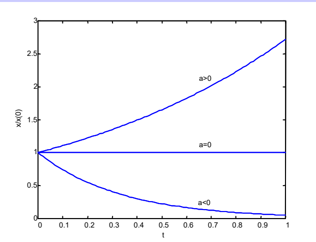
Ist Order LTI - Stability

$$\frac{dx(t)}{dt} = ax(t)$$

La matrice $A = a$ coincide con il suo autovalore



$a > 0$ Unstable System

$a = 0$ Stable System

$a < 0$ *asymptotically stable system*

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An economic model - Equilibrium

Y = Gross Domestic Product = y(t)
I = Investment = u(t)
C = Family Consumption = x(t)

$$\boxed{\begin{aligned}\dot{x}(t) &= (b-a)x(t) + bu(t)\\ y(t) &= x(t) + u(t)\end{aligned}}$$

$Eq(\bar{x}, \bar{u})$ $\qquad \bar{u} = -\frac{b-a}{b}\bar{x}$

Dynamic matrix
A = b-a

b-a > 0 Unstable System

b-a = 0 Stable System

b-a < 0 *asymptotically stable system*

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LTI system
stability – I

We need to evaluate the stability of the perturbation

$$x(t) - \bar{x} = e^{At}(x_0 - \bar{x})$$

Hypothesis: *eigenvalues of A are distinct*

Reminder *the eigenvalues of a n x n matrix A are the n roots of the characteristic equation*

$$det(\lambda I - A) = 0$$

LTI system
stability - II

For each distinct eigenvalue λ_i there is an eigenvector v_i such that

$$Av_i = \lambda_i v_i \qquad i = 1, \dots, n$$

given $T = [v_i \;\; \cdots \;\; v_n]$ we have: $AT^{-1} = T^{-1}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$

Introducing $x = T^{-1}\tilde{x}$, the state equation $\dot{x}(t) = Ax(t)$, become

$$T^{-1}\frac{d\tilde{x}(t)}{dt} = AT^{-1}\tilde{x}(t) \Rightarrow \frac{d\tilde{x}(t)}{dt} = TAT^{-1}\tilde{x}(t) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}\tilde{x}(t)$$

LTI system
stability - III

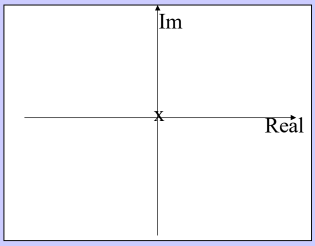
$$\frac{d\tilde{x}_i(t)}{dt} = \lambda_i \tilde{x}(t) \qquad \Rightarrow \qquad \tilde{x}_i(t) = e^{\lambda_i t}\tilde{x}_i(0)$$

$$x(t) = T^{-1}\tilde{x}(t) = T^{-1}\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{bmatrix}\tilde{x}(0)$$

$$x(t) = T^{-1}\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{bmatrix}Tx(0)$$

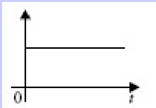
Free dynamic is a linear combination of exponential called modes

LTI systems
modes



$\lambda_i = 0$

$\tilde{x}_i(t) = e^{0t}\tilde{x}_i(0)$



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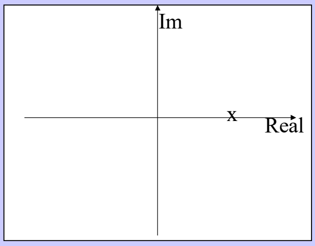
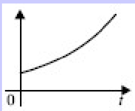
LTI systems
modes



$\lambda_i = a \qquad a > 0$

$\tilde{x}_i(t) = e^{at}\tilde{x}_i(0)$



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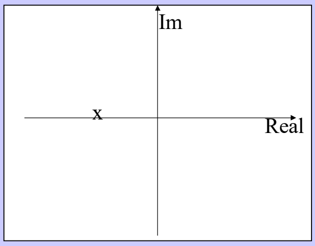
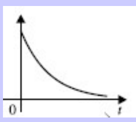
LTI systems
modes



$\lambda_i = a \qquad a < 0$

$\tilde{x}_i(t) = e^{at}\tilde{x}_i(0)$



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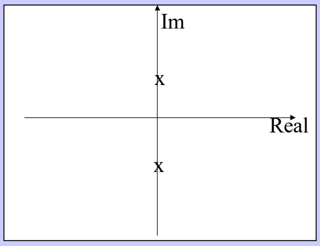
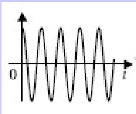
LTI systems
modes



$$\lambda_1 = bi \qquad \lambda_2 = -bi$$

$$\begin{bmatrix} \widetilde{x_1}(t) \\ \widetilde{x_2}(t) \end{bmatrix} = \begin{bmatrix} e^{bit} & 0 \\ 0 & e^{-bit} \end{bmatrix} \begin{bmatrix} \widetilde{x_1}(0) \\ \widetilde{x_2}(0) \end{bmatrix}$$



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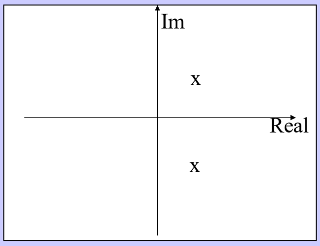
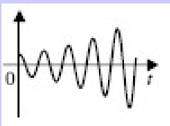
LTI systems
modes



$$\lambda_1 = a + bi \qquad a > 0$$
$$\lambda_2 = a - bi$$

$$\begin{bmatrix} \widetilde{x_1}(t) \\ \widetilde{x_2}(t) \end{bmatrix} = e^{at} \begin{bmatrix} e^{bit} & 0 \\ 0 & e^{-bit} \end{bmatrix} \begin{bmatrix} \widetilde{x_1}(0) \\ \widetilde{x_2}(0) \end{bmatrix}$$



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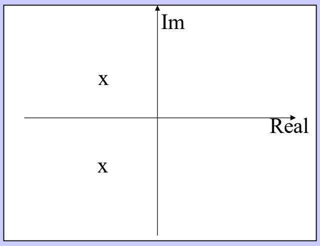
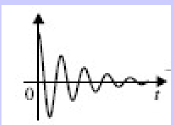
LTI systems
modes



$$\lambda_1 = a + bi \qquad a < 0$$
$$\lambda_2 = a - bi$$

$$\begin{bmatrix} \widetilde{x_1}(t) \\ \widetilde{x_2}(t) \end{bmatrix} = e^{at} \begin{bmatrix} e^{bit} & 0 \\ 0 & e^{-bit} \end{bmatrix} \begin{bmatrix} \widetilde{x_1}(0) \\ \widetilde{x_2}(0) \end{bmatrix}$$



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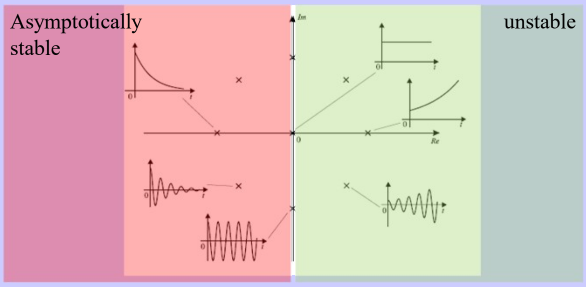
LTI systems
modes

Hypothesis: *eigenvalues of A are distinct*



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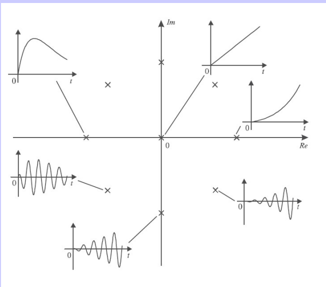
LTI systems
modes

Hypothesis: *eigenvalues of A have multiplicity = 2*



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LTI system: *Stability - II*

A LTI system is asymptotically stable if every eigenvalue has real part lower than zero.

A LTI system is unstable if at least one of the eigenvalues has a STRICTLY positive real part

If the eigenvalues of a LTI system are not positive, the system can be stable or unstable but it can not be asymptotically stable.

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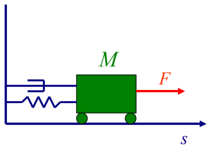
Mass-spring



$$M\ddot{s}(t) = -ks(t) - h\dot{s}(t) + F(t)$$

Set $x_1(t) = s(t)$, $x_2(t) = \dot{s}(t)$, $u(t) = F(t)$ and $y(t) = s(t)$

$$\boxed{\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{1}{M}\{-kx_1(t) - hx_2(t) + u(t)\} \end{aligned}}$$

$$y(t) = x_1(t)$$

Mass-spring *stability*

$$\boxed{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{h}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t)}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{h}{M} \end{bmatrix}$$

$$det(\lambda I - A) = 0$$

$$det\left(\begin{bmatrix} \lambda & -1 \\ \frac{k}{M} & \lambda + \frac{h}{M} \end{bmatrix}\right) = 0$$

$$\lambda(\lambda + \frac{h}{M}) - (-1)\frac{k}{M} = 0$$

$$\lambda^2 + \frac{h}{M}\lambda + \frac{k}{M} = 0$$

$$\lambda_{1/2} = \frac{-\frac{h}{M} \pm \sqrt{\left(\frac{h}{M}\right)^2 - 4\frac{k}{M}}}{2}$$

Re($\lambda_{1,2}$)<0 iff h, k>0 ⇔
asymptotically stable