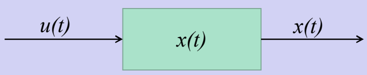
Fundamentals of Control Theory

System definition & Taxonomy

Faculty of Engineering
University of Pavia

What is a system in C.T.?

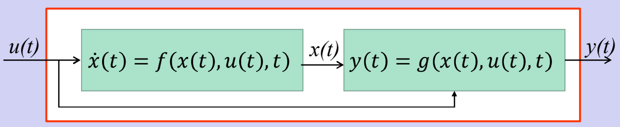


A system is an entity having some internal quantities *x(t)* supposedly observable from an external observer.
The *x(t)* evolution is regulated by external quantities *u(t)* by means of a differential equation such as:

$$\frac{dx(t)}{dt} = f(x(t), u(t), t)$$

Dynamic Equation
State Equation

Model



Generally, the state is not the measured output of the system, which is usually described by a function

$$y(t) = g(x(t), u(t), t)$$ *Output Equation*

The overall model of the system is then:

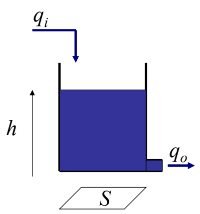
$$\dot{x}(t) = f(x(t), u(t), t)$$
$$y(t) = g(x(t), u(t), t)$$

Variables

$$u(t) \longrightarrow \boxed{\begin{array}{c}\dot{x}(t) = f(x(t), u(t), t)\\ y(t) = g(x(t), u(t), t)\end{array}} \longrightarrow y(t)$$

- *Input* ($u(t) \in \mathbb{R}^m$): m independent actions taken on the system
- *Output* ($y(t) \in \mathbb{R}^p$): p measurement of interest
- *State* ($x(t) \in \mathbb{R}^n$): several interpretation
  - Memory of the past input
  - What is needed, given time and input to evaluate the output
- n: order of the system

Water tank with forced flow



h = *water level* $= x(t)$
S = *tank section (costant value)*
q_i = *volumetric flow inrate (forced)* $= u_1(t)$
q_o = *volumetric flow outrate (forced)* $= u_2(t)$

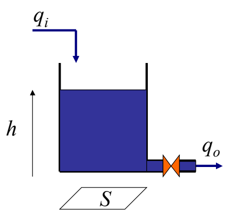
Mass balance

$$\boxed{\frac{dh(t)}{dt} = \frac{1}{S}(q_i(t) - q_o(t))}$$

$$\frac{dh(t)}{dt} Sm = mq_i(t) - mq_o(t)$$

$$\boxed{\begin{array}{c}\dot{x}(t) = \dfrac{1}{S}(u_1(t) - u_2(t))\\ y(t) = x(t)\end{array}}$$

Water tank with forced inflow



h = *water level* $= x(t)$
S = *tank section (constant value)*
q_i = *volumetric flow rate (forced)* $= u(t)$
q_o = *volumetric flow rate (free)*

From Bernoulli law $q_o(t) = \sqrt{2gh(t)}$

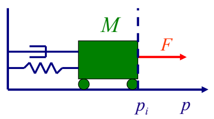
Mass balance

$$\boxed{\frac{dh(t)}{dt} = \frac{1}{S}\left(q_i(t) - \sqrt{2gh(t)}\right)}$$

$$\frac{dh(t)}{dt} = \frac{1}{S}\left(q_i(t) - q_o(t)\right)$$

$$\boxed{\begin{array}{c}\dot{x}(t) = \dfrac{1}{S}\left(u(t) - \sqrt{2gx(t)}\right)\\ y(t) = x(t)\end{array}}$$

Spring-mass-dumper - I



p = position
M = mass
k = spring constant
h = friction constant
p_i = quiet position (no force applied)
F = traction force (imposed)

Force balance

$M\ddot{p}(t) = -k(p(t) - p_i) - h\dot{p}(t) + F(t)$

Assuming $\quad p_i = 0 \qquad M\ddot{p}(t) = -kp(t) - h\dot{p}(t) + F(t)$

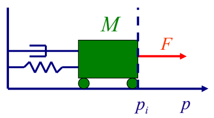
Introducing $\qquad \dot{p}(t) = q(t)$

$q(t) = \dot{p}(t) \qquad M\dot{q}(t) = -kp(t) - hq(t) + F(t)$

Fundamentals of Control Theory — *Università degli Studi di Pavia*

7

Spring-mass-dumper - II



p = position
M = mass
k = spring constant
h = friction constant
p_i = quiet position (no force applied) =0
F = traction force (imposed)

$\dot{p}(t) = q(t)$

$\dot{q}(t) = -\frac{k}{M}p(t) - \frac{h}{M}q(t) + \frac{1}{M}F(t)$

Posing

$x_1(t) = p(t)$
$x_2(t) = q(t)$
$u(t) = F(t)$

$$\boxed{\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{k}{M}x_1(t) - \frac{h}{M}x_2(t) + \frac{1}{M}u(t) \\ y(t) &= x_1(t) \end{aligned}}$$

Fundamentals of Control Theory — *Università degli Studi di Pavia*

8

System Taxonomy - I

$[u_1(t) \ \dots \ u_m(t)]' \longrightarrow \boxed{[x_1(t) \ \dots \ x_n(t)]'} \longrightarrow [y_1(t) \ \dots \ y_p(t)]'$

Number of signals

SISO: single input, single output — SIMO: single input, multi-output
MIMO: multi-input, multi-output — MISO: multi-input, single output
Autonomous : no-input

Proper & Strictly proper system

In a *proper system* the output is directly related to the input

$$\boxed{\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), u(t), t) \end{aligned}}$$

In a *strictly proper system* the output is NOT directly related to the input

$$\boxed{\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), t) \end{aligned}}$$

Fundamentals of Control Theory — *Università degli Studi di Pavia*

9

System Taxonomy - II

$[u_1(t) \quad \dots \quad u_m(t)]' \rightarrow [x_1(t) \quad \dots \quad x_n(t)]' \rightarrow [y_1(t) \quad \dots \quad y_p(t)]'$

Dynamic & Static Systems

Static: no state is present
$y(t) = g(u(t), t)$

Dynamic: state is present
$\dot{x}(t) = f(x(t), u(t), t)$
$y(t) = g(x(t), u(t), t)$

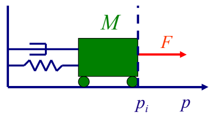
Time-variant & Time invariant

Explicit time dependence
$\dot{x}(t) = f(x(t), u(t), t)$
$y(t) = g(x(t), u(t), t)$

Implicit time dependence
$\dot{x}(t) = f\big(x(t), u(t)\big)$
$y(t) = g\big(x(t), u(t)\big)$

10

Spring-mass-dumper - III



p = *position*
M = *mass*
k = *spring constant*
h = *friction constant*
p_i = *quiet position (no force applied)*
F = *traction force (imposed)*

Time varing system

Assuming that the spring constantly decrease over time

$k(t) = k_0 e^{-\alpha t}$

$M\ddot{p}(t) = -k(t)p(t) - h\dot{p}(t) + F(t)$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{k(t)}{M}x_1(t) - \frac{h}{M}x_2(t) + \frac{1}{M}u(t)$$
$$y(t) = x_1(t)$$

11

System Taxonomy - III

$[u_1(t) \quad \dots \quad u_m(t)]' \rightarrow [x_1(t) \quad \dots \quad x_n(t)]' \rightarrow [y_1(t) \quad \dots \quad y_p(t)]'$

Linear Systems

Both system functions (f and g) are linear

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$
$y(t) = C(t)x(t) + D(t)u(t)$

$A(t) \in \mathbb{R}^{n\times n}$ $B(t) \in \mathbb{R}^{n\times m}$
$C(t) \in \mathbb{R}^{p\times n}$ $D(t) \in \mathbb{R}^{p\times n}$

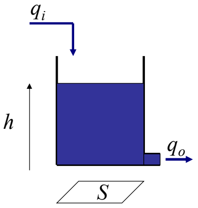
Linear, Time Invariant system

$\dot{x}(t) = Ax(t) + Bu(t)$
$y(t) = Cx(t) + Du(t)$

Classes will be focused on LTI systems

12

Water tank with forced flow - Classify



$$\boxed{\begin{aligned}\dot{x}(t) &= \frac{1}{S}(u_1(t) - u_2(t))\\ y(t) &= x(t)\end{aligned}}$$

$x(t) = h(t)$

$u(t) = \begin{bmatrix} q_i(t) \\ q_o(t) \end{bmatrix}$

$y(t) = h(t)$

n=1, m=2, p =1

1st order system *MISO system*

LTI System

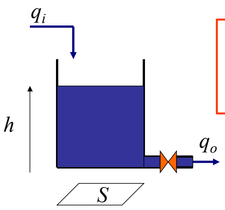
$$\boxed{\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}}$$

$$\boxed{\begin{aligned}A &= 0 \quad B = [1/S \quad -1/S]\\ C &= 1 \qquad\quad D = 0\end{aligned}}$$

Fundamentals of Control Theory *Università degli Studi di Pavia*

13

Water tank with forced inflow - Classify



$$\boxed{\begin{aligned}\dot{x}(t) &= \frac{1}{S}\left(u(t) - \sqrt{2gx(t)}\right)\\ y(t) &= x(t)\end{aligned}}$$

$x(t) = h(t)$

$u(t) = q_i(t)$

$y(t) = h(t)$

n=1, m=1, p =1

1st order system *SISO system*

Time Invariant System

Non linear System

Fundamentals of Control Theory *Università degli Studi di Pavia*

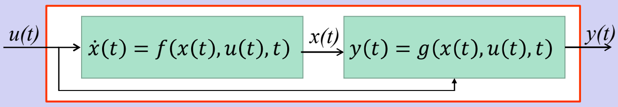
14

Fundamentals of Control Theory

Simulation

Faculty of Engineering
University of Pavia

Simulation - I

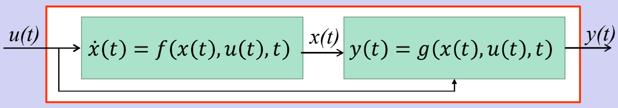


Simulation concept: given *u(t)* get *y(t)*

Fact: f(x(t),u(t),t) shows the state derivate. → state value, in at least one time, has to be known $x(t_0)$

Fact: the input is known only from a given time instant → the input is assumed known since the initial time t_0

Simulation - II



Simulation problem

given the initial state $x(t_0) = x_0$ and the input signal $u(t), t \geq t_0$, find state $x(t), t > t_0$ and output $y(t), t \geq t_0$ signals.

State Movement: $x(t) = \phi\big(t, t_0, x_0, u(.)\big), \ \ t > t_0$

Ouptut Movement: $y(t) = g\big(\phi\big(t, t_0, x_0, u(.)\big), u(t), t\big), \ \ t > t_0$

Solving Simulation problem

$u(t)$ → $\dot{x}(t) = f(x(t), u(t), t)$ → $x(t)$ → $y(t) = g(x(t), u(t), t)$ → $y(t)$

In its generical form, simulation is a Cauchy problem which has no close form

Remark: LTI systems

$u(t)$ → $\dot{x}(t) = Ax(t) + Bu(t)$ → $x(t)$ → $y(t) = Cx(t) + Du(t)$ →

have general theoretical results

Lagrange formula - LTI

$u(t)$ → $\dot{x}(t) = Ax(t) + Bu(t)$ → $x(t)$ → $y(t) = Cx(t) + Du(t)$ → $y(t)$

For LTI system the state movement is given by the *Lagrange formula*:

$$\boxed{x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau}$$

Two main components:

- $e^{A(t-t_0)}x(t_0)$ called *free movement* connected to initial state
- $\int_{t_0}^{t} e^{A(t-\tau)}u(\tau)\,d\tau$ called *forced movement* connected to input

An economic model

Y = Gross Domestic Product
I = Investment
C = Family Consumption
The state:

- *may act on I (public investment)*
- *want to raise Y*

Assuming closed economy (no import-export)

$$\mathrm{Y}(t) = C(t) + I(t)$$

Keynes model

$$\boxed{\frac{dC(t)}{dt} = -aC(t) + bY(t)}$$

Exercise

- Classify the system
- Simulate the system posing initial state and a constant input
- Simulate the system if the current investment doubles

An economic model - Classification

Y = Gross Domestic Product = y(t)
I = Investment = u(t)
C = Family Consumption = x(t)

$$Y(t) = C(t) + I(t)$$
$$\frac{dC(t)}{dt} = -aC(t) + bY(t)$$

$$\frac{dC(t)}{dt} = -aC(t) + b\big(C(t) + I(t)\big)$$

$$\boxed{\begin{aligned}\dot{x}(t) &= (b-a)x(t) + bu(t)\\ y(t) &= x(t) + u(t)\end{aligned}}$$

n=1, m=1, p=1

1st order system *SISO system*
Time Invariant System
Linear System
Proper System

An economic model – Simulation I

Y = Gross Domestic Product = y(t)
I = Investment = u(t)
C = Family Consumption = x(t)
a=3 *b=2* x_0*=10* *u(t)=2*

$$\boxed{\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\\ A = b - a = -1 &\quad C = 1\\ B = b = 2 &\quad D = 1\end{aligned}}$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau)\, d\tau \quad = e^{-t}10 + \int_0^t e^{-(t-\tau)} \cdot 2 \cdot 2\, d\tau$$
$$= 10e^{-t} + 4\int_0^t 1 \cdot e^{(\tau-t)}\, d\tau \quad = 10e^{-t} + 4[e^{(\tau-t)}]_0^t$$
$$= 10e^{-t} + 4(e^{(t-t)} - e^{(0-t)}) \quad = 10e^{-t} + 4(1 - e^{-t}) = 4 + 6e^{-t}$$

An economic model – Simulation I

Y = Gross Domestic Product = y(t)
I = Investment = u(t)
C = Family Consumption = x(t)
a=3 *b=2* x_0*=10* *u(t)=2*

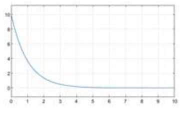
$$\boxed{\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\\ A = b - a = -1 &\quad C = 1\\ B = b = 2 &\quad D = 1\end{aligned}}$$

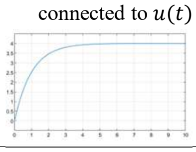
$x(t) = 10e^{-t}$ + $4(1 - e^{-t})$ = $x(t) = 4 + 6e^{-t}$

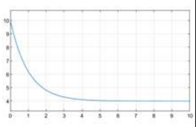
free movement connected to x_0

forced movement connected to $u(t)$

state movement







An economic model – Simulation II

Y = Gross Domestic Product = y(t)
I = Investment = u(t)
C = Family Consumption = x(t)

$$\dot{x}(t) = (b-a)x(t) + bu(t)$$
$$y(t) = x(t) + u(t)$$

$$A = b - a \qquad C = 1$$
$$B = b \qquad D = 1$$

$$s' \begin{cases} t_0 = 0, \\ x(t_0) = \bar{x}, \\ u'(t) = \bar{u} \end{cases}$$

$$x'(t) = e^{(b-a)t}\bar{x} + \int_0^t e^{(b-a)(t-\tau)} b\bar{u}\, d\tau$$
$$y'(t) = e^{(b-a)t}\bar{x} + \int_0^t e^{(b-a)(t-\tau)} b\bar{u}\, d\tau + \bar{u}$$

$$s'' \begin{cases} t_0 = 0, \\ x(t_0) = \bar{x}, \\ u''(t) = 2\bar{u} \end{cases}$$

$$y''(t) = e^{(b-a)t}\bar{x} + 2\int_0^t e^{(b-a)(t-\tau)} b\bar{u}\, d\tau + \overline{2u}$$