SOLVING THE NEUTRON STAR TOV EQUATIONS WITH POLYTROPIC EOS

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Abstract

In this paper a computational study was performed on the neutron star TOV equations. This study aimed to examine a polytropic EOS and it's usefulness, as well as present example solutions for such a model. Results shown in previous theoretical studies on the subject were confirmed, while many examples which show how the model can comply with more modern knowledge where shown. Plots for the Mass, Pressure and metric functions as a function of Radius and classic M-R plots were produced to showcase these findings. The computational study performed here was based on the paper shown in reference [18].

Περίληψη

Σε αυτήν την εργασία πραγματοποιήθηκε μια υπολογιστική μελέτη πάνω στις εξισώσεις TOV για αστέρες νετρονίων. Αυτή η μελέτη είχε σκοπό να εξετάσει μια πολυτροπική καταστατική εξίσωση και τη χρησιμότητά της, αλλά και να παρουσιάσει παραδείγματα λύσεων για ένα τέτοιο μοντέλο. Επιβεβαιώθηκαν αποτελέσματα προηγούμενων θεωρητικών ερευνών επί του θέματος, ενώ παρουσιάστηκαν και πολλά παραδείγματα που δείχνουν πως μπορεί να υπάρξει συμφωνία μεταξύ του εξεταζομένου μοντέλου και πιο σύγχρονων γνώσεων επί του θέματος. Γραφικές παραστάσεις της μάζας, πίεσης και των μετρικών συναρτήσεων λ και ν συναρτήσει της ακτίνας όπως και κλασσικές γραφικές παραστάσεις M-R παράχθηκαν ώστε να παρουσιαστούν καλύτερα αυτά τα ευρήματα. Η υπολογιστική αυτή μελέτη ήταν βασισμένη στην εργασία που δείχνεται στην αναφορά [18].

Part I

Introduction and Theory

CHAPTER 1

Introduction

The life cycle of a star has always been an object of interest for astrophysics. Of the many topics this subject holds however, one of the most interesting is by far the process of the star's eventual collapse and what comes of this process. Therefore it is obvious that neutron stars, being one of these objects that come from a star's collapse would be a subject of interest for astrophysicists. This is evident in the fact that their theoretical prediction in 1933 by Walter Baade and Fritz Zwicky came a total of 34 years before their actual detection by Ioshif Shklovsky. The study of neutron stars has since then, been in the forefront of research in astrophysics and has been shown to be linked to many other branches of physics like cosmology, fluid mechanics, nuclear physics etc.

In this paper, neutron stars are studied from a computational aspect. For starters their heretofore known properties are presented, as well as commonly used methods of approaching the problem of defining an *Equation Of State* (EOS). Afterwards, a computational study is performed, with a polyopic EOS, in an attempt to find the parameters of the EOS which best fit known Mass - Radius data gathered from observations.

In the first chapter, the theoretical aspect of the subject is presented, in all topics which hold interest for this research, ranging from the formation and structure of a neutron star, to the well known Tolman-Oppenheimer-Volkoff (TOV) equations which define them and the research which has been performed up to this point on the subject of the EOS.

In the second chapter the computational method followed for this paper is presented. Specifically the solution to the TOV equations is presented, the calculations performed on the metrics, as well as the method for producing the classic M-R diagrams which help evaluate the validity of a proposed polytropic EOS.

In the final chapter the results of the research are presented and discussed and finally an appendix is given which shows the Python codes written for this paper.

CHAPTER 2

Background

Neutron stars are super-dense, compact objects which are formed after the collapse of stars who's initial mass lies in the range of $8M_\odot$ up to $20\text{-}30M_\odot$.[18] They typically have a mass of around $1.4M_\odot$, radii of 10km and can reach supranuclear densities, up to the order of 10^{15}g cm^{-3}. With such extreme conditions, neutron star matter is considered by far the most dense form of matter which has been confirmed to exist thus far and as stellar objects they are second in density only to black holes.

2.1 Formation and Evolution

The formation of a neutron star, begins once it's progenitor reaches the end of its thermonuclear evolution. Assuming the original star's mass lies in the range mentioned above, then it can continue to burn heavier and heavier elements, until it reaches ^{56}Fe through

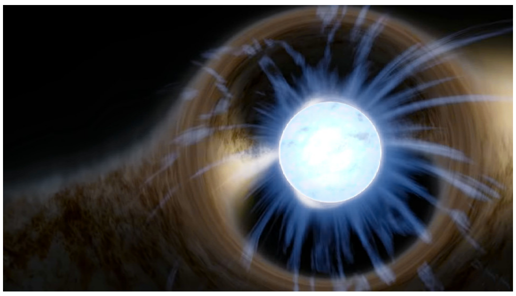


Figure 2.1: Simulation of a neutron star gravitationaly lensing the background. Adapted from https://en.wikipedia.org/wiki/Neutron_star

Silicon burning:[13]

$$
\begin{aligned}
&{}^{28}Si + {}^{28}Si \longrightarrow {}^{56}Ni + \gamma \\
&{}^{56}Ni \longrightarrow {}^{56}Co + e^{+} + \nu_e \qquad (2.1) \\
&{}^{56}Co \longrightarrow {}^{56}Fe + e^{+} + \nu_e
\end{aligned}
$$

As this process continues, it enhances both the inverse β-decay process, also known as electron catpure and the iron photodisintegration process, shown in the following equations.

$$
\begin{aligned}
&e^{-} + p^{+} \longrightarrow n + \nu_e \\
&\gamma^{+}\ {}^{56}Fe \longrightarrow 13\ {}^{4}He + 4\ n
\end{aligned}
\tag{2.2}
$$

What's important to note about the processes shown in (2.1), (2.2) is that all of these are endothermic reactions which subtract energy from the star. This in turn destabilizes the star, who's mass continues to increase. Eventually the core mass surpasses the Chandrasekhar Limit, which leads to the eventual collapse of the core, which in turn causes the phenomenon known as a supernova explosion. During the supernova, elements heavier than ^{56}Fe are created. The core on the other hand which is left behind, now has densities of the order $10^{14} gr cm^{-3}$, and consists mainly of neutrons, thus beggining the life cycle of a neutron star.[18]

After and during the supernova explosion there are many phenomena that can take place around the newly formed neutron star. An example of one of these processes is accretion. In 1989, as per reference [4], Roger Chevalier showed how, if a self similar flow is considered and using a power law density profile, *accretion*, i.e. the fall back of mass) that was scattered during the explosion unto the neutron star may occur. In fact after calculating the accretion rate, it was shown that most of the mass accretion occurs during the explosion, where at some point a reverse shockwave propagates mass towards the center of mass.

Accretion, as a phenomena can also contribute to many other properties of the neutron star, such as X-ray emission,[3] low mass black-hole[8] formation etc.

2.2 Structure

Current theories suggest, that a neutron star can be sufficiently studied, if we divide it into it's *atmosphere* as well as 4 main regions: the *outer* and *inner crust*, and the *outer* and *inner core*.[16] Although in this report, only the core's structure is examined when considering the EOS, at this point we shall make a brief mention of the properties of all these regions.

The *atmosphere*, is a thin plasma layer, where the thermal and electromagnetic radiation is formed. It's thickness varies from the scale of $10cm$ to the scale of $1mm$. The *atmosphere* plays an important role in observations, especially when pertaining to the masses and radii of neutron stars.

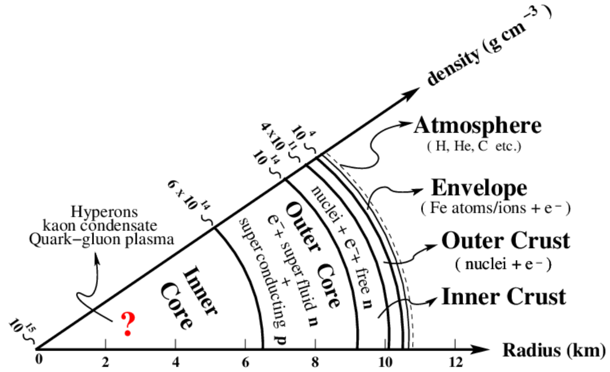


Figure 2.2: Structure regions of a neutron star. Adapted from Reference [25]

The *outer crust*, is the region of the star where the density takes values from $\sim 10^7 gcm^{-3}$ to $\rho_{ND} = 4 \times 10^{11} gcm^{-3}$, where ρ_{ND} is the neutron drip density. In this area the neutron star is composed mainly of an electron gas, which also provides most of the pressure, as well as a heavy nuclei lattice. It's thickness is a few hundred meters and it is also the last area of the star in which neutrons can live bound to nuclei. In desities higher than ρ_{ND}, all bound states in the nuclei available for neutrons are filled. Thus the neutrons, being fermions, start leaking out of the nuclei, hence the term "neutron drip". It has been extensively shown that in this density region, the matter is similar to white dwarf matter and that it's properties, such as nuclear energies, can be extrapolated from empirical formulas.[2]

The *inner crust*, consists of matter exceeding the neutron drip density, reaching values from ρ_{ND}, up to $0.56\rho_0$, where $\rho_0 \approx 2.67 \times 10^{14} gcm^{-3}$ is the nuclear density and its thickness is $\approx 1km$. Here the pressure is dominantly contributed by the neutron gas which forms, however the matter consists of 3 distinct phases: a neutron gas (NG), a phase of proton rich matter (PRM) and an electron gas, which ensures a neutral electric charge. In order to fully describe this region, it's necessary to know the densities of the PRM and the NG phases, which defines the way the 2 phases interact with each other, as well as the structures they form. These structures are generally formed due to sudden phase transitions in the matter and due to their geometric shapes, are often referred to as "nuclear pasta".[20]

The *outer* and *inner cores*, are where the largest fraction of the neutron star matter lies.[14] Each region's thickness can reach several km and their densities can range from $0.5\rho_0 < \rho < 2\rho_0$ for the inner core, and $2\rho_0 < \rho < \rho_c$ for the inner core, where ρ_c is the density in the center of the star.[16] Matter is generally assumed to be in a liquid or superfluid state, composed mainly by neutrons, electrons, protons and muons.[7] Because of the extremely high densities, although the exact nuclear composition of the core is unclear, it is highly likely that other compositions can appear. Depending on how the core is modeled, the appearance of hyperons, kaons and pions is possible, as well as a quark deconfined phase, where the quarks are not bound to nucleons or hadrons.[24]

On a final note regarding the structure, it should be noted that, according to recent studies, in order to properly model both the core and the crust of a neutron star, in a consistent manner, both sides of the core-crust interface must be modeled with the same many-body model and the same effective neutron-neutron interaction.[7] Therefore the crust region must also be assumed to be permeated by superfluid neutrons.[10]

2.3 Neutron Stars as Pulsars

At this point it is prudent before continuing, to briefly dicuss pulsars, since they are one of the main methods of observing neutron stars. It has been known since their early discovery that pulsars, are rotating stellar objects, mainly neutron stars, with a strong magnetic field of $B_0 \approx 10^{12}G$.[22] Due to this rotation a shortlived pulse of radio waves is emmited from the magnetic poles of the star.

Pulsars have been known to offer great estimates of neutron star masses, when these neutron stars are found as part of binary systems. One such interesting result is the fact that the $P_b - m_2$ relationship shows that most neutron star mass measurements only deviate slightly from $1.4M_\odot$,[6] specifically around 7%. This suggests in turn that any mass transfer greater than $0.1M_\odot$ in a binary pulsar system is unfeasible. Of course this result is only valid where the $P_B - m_2$ relation holds, a relation which is mainly a statistical tool rather than anything else. However the result seems to be confirmed by individual studies performed on binary systems.[11]

Other studies on the other hand, show a deviation from the canonical value of $1.4M_\odot$. In reference [19], it is suggested that pulsars exist with masses that reach up to $2.4M_\odot$ and that EOS models should have a maximum Mass of at the very least $2.1M_\odot$. Of course this estimate on the maximum Mass comes mostly from theoretical estimates, rather thatn observational data.

2.4 TOV Equations

The TOV equations are the set of differential equations, which describe the mass and pressure of static spherically symmetric objects of isotropic material in gravitational equilibrium. In this report the Newtonian form given in eq 2.3 shall be used, where geometrized units ($G = c = 1$) have been assumed.[15]

$$
\begin{aligned}
\frac{dm}{dr} &= 4\pi\epsilon r^2 \\
\frac{dp}{dr} &= -\frac{[\epsilon + p][m(r) + 4\pi\epsilon r^3]}{r^2(1 + 2\frac{m(r)}{r})}
\end{aligned}
\tag{2.3}
$$

In the equations above, the density $\rho(r)$, has been substituted for the energy density ϵ, since when the speed of light c is set to 1, the energy density and density are numerically equal, as per the relation $\epsilon = \rho c^2$.

The equations 2.3 are derived from the metric for a static spherically symmetric star which can be written as:

$$ds^2 = e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2) - e^{2\nu(r)}dt^2 \tag{2.4}$$

where, according to GR, the functions $\lambda(r)$ and $\nu(r)$ are referred to as *metric functions*. The derivation process for equations (2.3) is outlined thouroughly in reference [18], and will not be repeated here. What will be mentioned are the main properties of the TOV equations and the metric functions in the context of a neutron star.

As mentioned earlier the TOV equations define the mass and pressure as a function of the radius of the star. In theory, a complete analytical solution of these equations should provide a detailed description of these values in every point of the star, and by extension should give a large amount of information on it's structure. Of course, considering how unknown and complicated the structure of a neutron star is, this is something that has not been achieved in the almost 100 years since the equations where first derived.[1] As explained in the next section in greater detail, the existence or not of such an analytical solution is entirely dependent on the EOS of choice.

After solving the TOV equations, one is in a position to calculate the metric functions. For the $\lambda(r)$ and $\nu(r)$ metrics we find that, regardless of the EOS, for

$r \leq R$ the relations (2.5) hold

$$\begin{aligned} e^{-2\lambda} &= 1 - \frac{2m(r)}{r} \\ \nu'(r) &= \frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))} = \frac{-p'(r)}{\epsilon(r) + p(r)} \end{aligned} \tag{2.5}$$

where R is the radius where the pressure vanishes.[12] If $r \geq R$, then both of these are converted to

$$e^{-2\lambda} = e^{2\nu} = 1 - \frac{2M}{R} \tag{2.6}$$

where M is the total mass of the star.

It is often useful to split the total Mass $M(R)$, into contributions of 3 seperate energies:

1. E_{rest}, which represents the total rest mass-density of the baryons $m_N n$
2. E_{int}, which represents the internal energy of the star
3. E_B, which represents the binding energy, or the *gravitational potential energy* of the star.

There relations for these are given in reference [18]

$$\begin{aligned} E_{rest} &= 4\pi \int_0^R \frac{m_N n(r) r^2}{e^\lambda} \, dr \\ E_{int} &= 4\pi \int_0^R \frac{[\epsilon - m_N n(r)] r^2}{e^\lambda} \, dr \\ E_B &= 4\pi \int_0^R r^2 \epsilon [1 - \frac{1}{e^\lambda}] \, dr \end{aligned} \tag{2.7}$$

where the baryon mass-density is given $m_N n(r)$ is given by the relation $m_N n(r) = (\epsilon(r) + p(r)) e^{\frac{\nu(r) - \nu(R)}{2}} - n_0 e_0$, where e is the internal energy per nucleon.[12]

2.5 Equations of State

In order to make a complete theoretical model of a neutron star, one must first consider the EOS that is used to describe the star. There have been many proposed forms of the neutron star EOS up to this point, and here these proposed EOS's shall be described. Along with the EOS, it's effectiveness shall be described as well as the model's shortcomings.

2.5.1 Constant Density EOS

First is the solution for homogeneous stars stated by Schwarzschild. In this EOS the density, or in geometrized units the energy density, is considered to be constant, or $\epsilon = const$. This solution is one of the few EOS models thus far to produce an analytical solution for the TOV equations. Although it fails to describe the physics of the problem accurately, it is still a good approximation for the core of very dense stars to consider the density as a constant. As shown in reference [18], under these conditions the TOV equations (2.3) reduce to:

$$\frac{dp}{dr} = -\frac{4}{3}\pi r \frac{(\epsilon + p)(\epsilon + 3p)}{1 - \frac{8\pi}{3}r^2\epsilon} \tag{2.8}$$

since for a constant energy density

$$\frac{dm}{dr} = 4\pi r^2 \epsilon \Rightarrow m(r) = \frac{4\pi}{3} r^3 \epsilon \tag{2.9}$$

Now that the mass is known, the TOV equations can be solved to find the pressure which yields the result:

$$p = \epsilon \frac{y - y_1}{3y - y_1} \tag{2.10}$$

where $y = \sqrt{1 - \frac{2m(r)}{r}}$ and y_1 is the value of y at the radius of the star R, where $p = 0$.

It is also noteworthy that, given this EOS the condition

$$\sqrt{1 - \frac{2m(R)}{R}} > 0 \Rightarrow R > \frac{9}{4}M \tag{2.11}$$

must hold. This result is known as Buchdal's Theorem and holds true for any EOS.[5]

2.5.2 Barotropic EOS

A generalization on the previous EOS, barotropic EOSs are independent of the of the Temperature (T). Mostly useful for perfect fluid models for neutron stars, since the thermodynamical properties of perfect fluids can be described fully through 2 variables and a given EOS. Generally barotropic EOSs, when concerning stellar objects, come most commonly in the polytropic form $P = K\epsilon^{\gamma} + D$, where K,γ and D are constants.[9]

Polytropes are easy to handle and solve which is partially the reason why they are so famous among astrophysicists, even though they are not the best approximation for fluid behaviour. The exact method of solving the TOV equations with a polytropic EOS is also one of the topics of this paper, so it will not be discussed at length at this point. It will be mentioned however, that the form of solutions they produce is expected similar to images 2.3,2.4. This EOS can produce many different stars based on the initial conditions and the exact form of the EOS. When the total mass of the stars described are plotted against the radius R where the pressure vanishes, a plot similar to 2.5 is produced.[12] The area of interest in this plot is right branch of the plot, after the value of M_{max}, since this area is normally the area where the stars are considered stable, as explained in 3.4. Although the criteria shown there, namely $\frac{dM}{dR} < 0$ does not hold for the entire area, this is due to the fact that the barotropic EOS does not describe areas that aren't close to the core properly.

2.5.3 Modern EOS Models

More recent attempts to describe the EOS of a neutron star, lie less in an attempt to theoretically describe the star, and more in attempting to construct computational models for each region of the star. These models attempt to compute

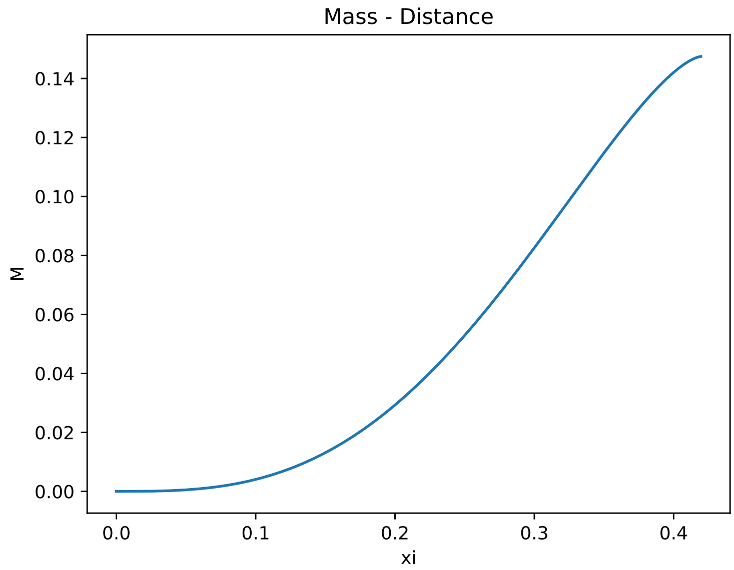


Figure 2.3: Example plot of Mass as a function of the radius. Both values are expressed in dimensionless units M for Mass and Xi for distance.

each region individually, while fitting the results so that the solution remains coherent. Examples of these are presented in image 2.6 Such models and the way they work are explained in much larger detail in reference [16]

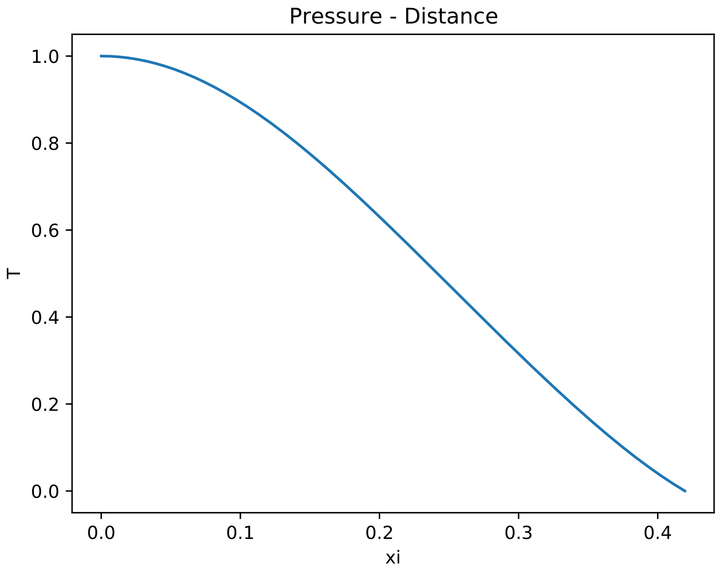


Figure 2.4: Example plot of Pressure as a function of the radius. Both values are expressed in dimensionless units T for Pressure and Xi for distance.

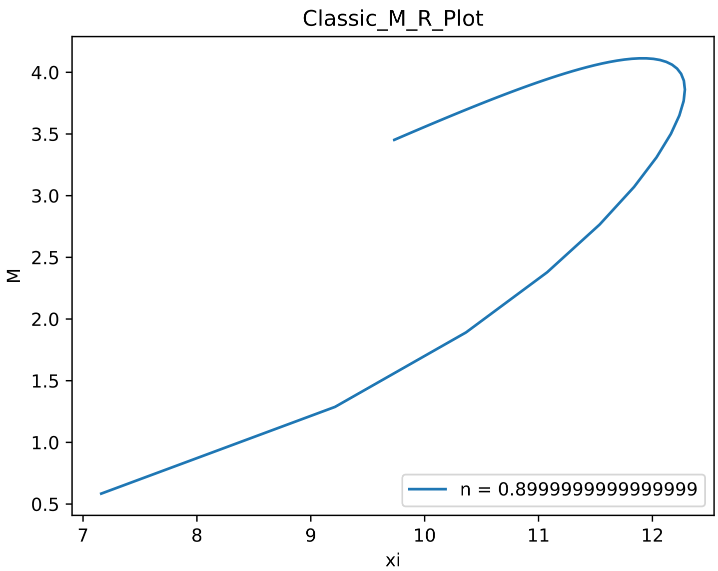


Figure 2.5: M-R plot example in dimensionless units

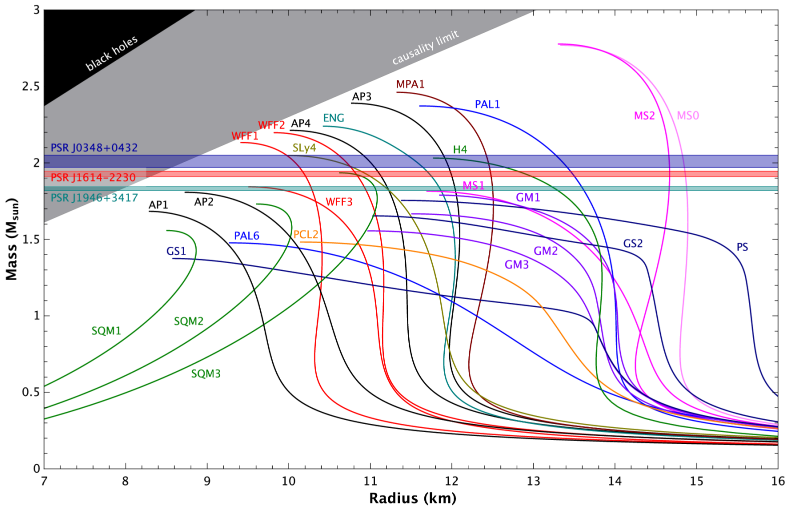


Figure 2.6: M-R plots for different EOSs

Part II

Computations and Results

CHAPTER 3

Method

3.1 Solving the TOV Equations

When solving the TOV equations (2.3) the primary point to consider is the EOS of choice. In this paper a polytropic EOS is considered, mainly for it's simplicity and flexibility as a model. The EOS of choice is taken to be

$$p = K\epsilon^{\gamma} \tag{3.1}$$

with ϵ,γ and p expressible as

$$\begin{aligned} &\gamma = 1 + \frac{1}{n} \\ &\epsilon = \epsilon_0 \Theta^n(r) \\ &p = K\rho^{\gamma}\Theta^{n+1}(r) = p_0\Theta^{n+1}(r) \end{aligned} \tag{3.2}$$

as shown in reference [18]. Given the current circumstances, the solution for the neutron star is controllable through 3 parameters:

1. ϵ_0, which represents the initial condition for the integration
2. K and n, the coefficient and order of the polytrope respectively, which represent for set of values the EOS of choice.

After integration functions for Mass and Pressure are obtained, which can easily be adjusted through the 3 control parameters in order to best fit currently obtained data for neutron star Passes and Pressures. In this thesis the main focus shall be the mass.

As for the actual solution of the equations first they shall be adjusted slightly to best fit a more generalized solution. Specifically, the following values are set and substituted into eq (2.3):

$$\begin{aligned} a_0 &= \frac{\epsilon_0}{p_0} \\ M &= m\sqrt{\epsilon_0} \\ \xi &= r\sqrt{\epsilon_0} \end{aligned} \tag{3.3}$$

The explanation for these substitutions is rather simple: in geometrized units the value ϵ_0 has dimension $[l^{-2}]$, whereas the values m and r are both of dimension $[l]$. Thus the new values M and ξ are both dimensionless, which both simplifies and generalizes the results obtained through this research. Of course in order to obtain presentable results, one must later restore the units, the process of which is explained in the next section. Unlike the values of M and ξ, the value a_0 simply serves to simplify the form the TOV equations take after these substitutions.

Plugging eq (3.2),(3.3) into (2.3), the TOV equations now take on the form

$$\begin{aligned} \frac{dM}{d\xi} &= 4\pi\xi^2\Theta^n(r) \\ \frac{d\Theta}{d\xi} &= -\frac{a_0+\Theta}{n+1}\frac{M(\xi)+4\pi\xi^3\frac{\Theta^{n+1}}{a_0}}{\xi[\xi-2M(\xi)]} \end{aligned} \tag{3.4}$$

The solution is nearly complete. All that is left is to find the initial conditions for integration for the functions M, the dimensionless mass, and Θ, the dimensionless pressure. Generally it is known, based on what is covered in chapter 2 and the relation for ϵ and Θ, that $M(0) = 0$, and $\Theta(0) = 1$. However as initial conditions these do not suffice, since using these initial conditions will give an undefined value for $\frac{d\Theta}{\xi}$ on the first step.

A Taylor expansion for the equations at $\xi = 0$ gives

$$\begin{aligned} M(\xi) &\approx m_3\xi^3 + m_5\xi^5 + O(\xi^7) \\ \Theta(\xi) &\approx 1 + \Theta_2\xi^2 + \Theta_4\xi^4 + O(\xi^6) \end{aligned} \tag{3.5}$$

After inserting eq (3.5) into (3.4), the following results are obtained for the values m_3,m_5,Θ_2,Θ_4:

$$\begin{aligned} m_3 &= \frac{4\pi}{3} \\ \Theta_2 &= -2\pi\frac{(1+a_0)(3+a_0)}{3a_0(n+1)} \\ m_5 &= \frac{4\pi n\Theta_2}{5} \\ \Theta_4 &= -\frac{\Theta_2}{2(n+1)}(m_3 + \frac{4\pi}{a_0}) \end{aligned} \tag{3.6}$$

Now one can computationaly solve the TOV equations.

3.2 Calculating the Metrics

As mentioned in in section 2.4, after solving the equations it is possible to also calculate the metric functions λ and ν and define thus the metric ds of the spacetime around the neutron star.

After finding the mass and pressure dimensionless functions M and Θ, the easiest to calculate is the metric function λ, which is given by the first equation of (2.5) as

$$e^{-2\lambda} = 1 - \frac{2M(\xi)}{\xi} \tag{3.7}$$

On the other hand the remaining metric $\nu(r)$, can be calculated according to (2.5)

$$\nu(r) = \int_0^{\xi} -\frac{p'(\xi)}{\epsilon + p}\, d\xi + \nu_0 = \int_0^{\xi} -\frac{(n+1)\frac{d\Theta}{d\xi}}{a_0 + \Theta}\, d\xi + \nu_0 = \ln\,[\frac{a_0+1}{a_0+\Theta(\xi)}]^{(n+1)} + \nu_0 \tag{3.8}$$

Because the metric at the radius R, must reduce the value shown in equation (2.6), the relation

$$e^{2\nu_0}(\frac{a_0+1}{a_0+\Theta})^{2(n+1)} = 1 - \frac{2M(\xi_1)}{\xi_1} \tag{3.9}$$

is deduced, where ξ_1 is the dimensionless radius at which the pressure vanishes, or $\xi_1 = R\sqrt{\epsilon_0}$. Therefore

$$e^{2\nu_0} = (1 - \frac{2M(\xi_1)}{\xi_1})(\frac{a_0+\Theta}{a_0+1})^{2(n+1)} \tag{3.10}$$

The value $M(\xi_1)$ can be calculated from the value of $\Theta'(\xi_1)$ as shown below

$$\begin{aligned} \Theta'(\xi_1) &= (\frac{a_0}{n+1})(\frac{M(\xi_1)}{\xi_1(\xi_1 - 2M(\xi_1))}) \\ \Rightarrow M(\xi_1) &= \frac{(n+1)\xi_1^2|\Theta'(\xi_1)|}{a_0 + 2\xi_1(n+1)|\Theta'(\xi_1)|} \end{aligned} \tag{3.11}$$

and therefore, the metric $\nu(\xi)$ is now calculated as

$$\begin{aligned} e^{2\nu_0} &= (\frac{a_0}{a_0 + 2\xi_1(n+1)|\Theta'(\xi_1)|})(\frac{a_0}{a_0+1})^{2(n+1)} \\ \Rightarrow e^{2\nu(\xi)} &= (\frac{a_0}{a_0+1})^{2(n+1)}(\frac{a_0}{a_0 + 2\xi_1(n+1)|\Theta'(\xi_1)|})(\frac{a_0+1}{a_0+\Theta(\xi)})^{2(n+1)} \end{aligned} \tag{3.12}$$

The expected plots of the metrics λ and ν are shown in figures 3.1,3.2

3.3 Reintroducing proper units

After solving the equations with the method described in section 3.1, it is necessary to present the results in meaningful units. For this report, the Solar Mass $M_\odot$ was chosen as the unit of mass, and kilometers km where chosen as the unit of distance. Then the process for moving from geometrized units $G = c = 1$ to the chosen unit system is as follows.

Let a star be considered, with central density $\rho_c = 5 \times 10^{14} gcm^{-3}$, polytropic index $\gamma = 2.75$ and $K = 1.98183 \times 10^{-6}[cgs]$.[21] First, all known parameters are

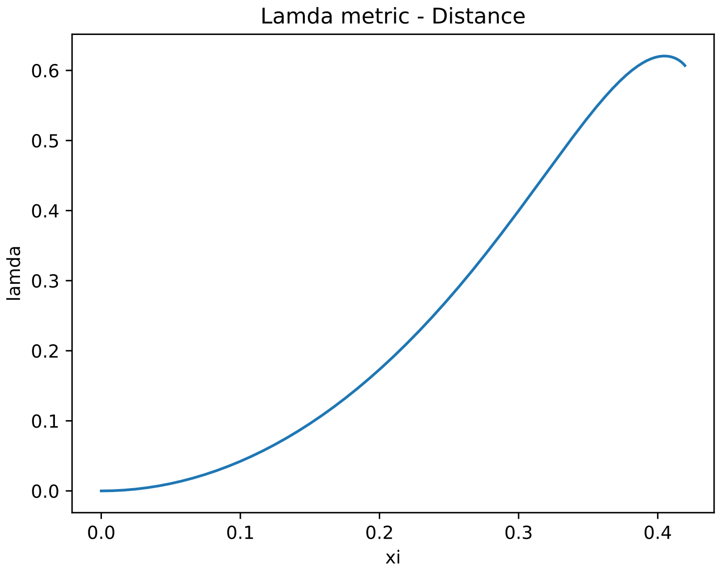


Figure 3.1: λ metric as a function of the dimensionless radius ξ

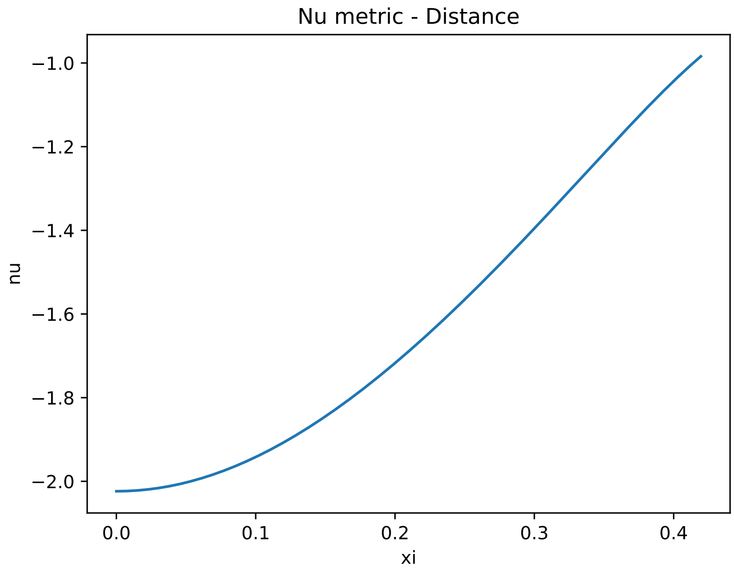


Figure 3.2: ν metric as a function of the dimensionless radius ξ

converted to units $G = c = 1$. To perform this conversion table 3.1 was used. After the conversions the given values become:

$$\begin{aligned} \rho_c &= 3.6231 \times 10^{-14} cm^{-2} \\ K_{G=c=1} &= 2.3759 \times 10^{14} \end{aligned} \tag{3.13}$$

To obtain the value for K in geometrized units, the conversion for pressure was used, since $p = K\epsilon_0^\gamma$, which must hold for any system of units. Hence the conversion for K becomes

$$\begin{aligned} p_{cgs} &= f_p p_{geom} \\ \Rightarrow K_{cgs}\epsilon_{0_{cgs}}^\gamma &= f_p K_{geom}\epsilon_{0_{geom}}^\gamma \\ \Rightarrow K_{geom} &= \frac{K_{cgs}\epsilon_{0_{cgs}}^\gamma}{f_p \epsilon_{0_{geom}}} \end{aligned} \tag{3.14}$$

where f_p is the conversion factor for pressure, as well as the energy density, since pressure and energy density share common units. Now converting the mass and radius into dimensionless units is simple, since all that is necessary is to divide with $\sqrt{\epsilon_0}$. For the computational research in this thesis, $a_0 = \frac{\epsilon_0}{p_0}$, K and γ in geometrized units were considered the known parameters. So to restore the units, the reverse process was followed.

In the previous section the method for calculating the metric functions λ and ν in dimensionless units was shown. Doing the same in geometrized units, is simply a matter of first restoring the units for the values ξ, M, Θ. These functions however are taken to be arbitrary. Hence all results for the metric functions shall be shown in dimensionless units.

3.4 Stability Testing

After solving the TOV equations and restoring the units, one comes up with a plot, which represents the set of stars produced for a given solution. This plot, known as a classic M-R plot, shows which stars are capable of being produced with the given EOS and initial conditions. One such plot is shown in figure 2.5. Each point in these plots represents a star in a state of equilibrium. However not all points represent a *stable* equilibrium. It can be shown that in order for a star to have *stable* equilibrium, it is necessary for have $\frac{dM}{d\epsilon_0} > 0$.[18] Since

$$\frac{dM}{dR} = \frac{dM}{d\epsilon_0}\frac{d\epsilon_0}{dR} \tag{3.15}$$

and because of the fact that a star's radius decreases as the central density increases, it must be true that $\frac{dM}{dR} < 0$ for the star to achieve *stable* equilibrium. This condition, although necessary is not a sufficient criterion for choosing which stars are stable from a given M-R plot. However it is sufficient for excluding the stars that do not fir the criteria. Thus the final step in this research was to exclude from the results, all points on the M-R plots, in the areas where $\frac{dM}{dR} > 0$, i.e. on the left side of the M_{max} point.

Values	cgs	$G = c = 1$	Conversion factor
Time	$1sec$	$2.9979 \times 10^{10} cm$	c
Length	$1cm$	$1cm$	1
Mass	$1g$	$7.4263 \times 10^{-29} cm$	Gc^{-2}
Pressure	$1gcm^{-1}sec^{-2}$	$8.623 \times 10^{-50} cm^{-2}$	Gc^{-4}
Density	$1gcm^{-3}$	$7.4263 \times 10^{-29} cm^{-2}$	Gc^{-2}

Table 3.1: Conversions from cgs to $G = c = 1$. In this table, the values of the constants G and c were taken as, $G = 6.6743 \times 10^{-8} g^{-1} cm^3 sec^{-2}$ and $c = 2.9979 \times 10^{10} cm$.

CHAPTER 4

Results

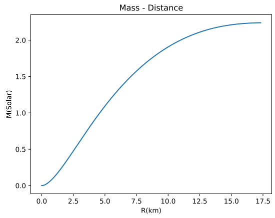
As explained in the following sections separately, although plots for many different values of K and n can be produced, the plots which more clearly represent the capabilities of the chosen EOS type were chosen to be included. Thus the values $n = 1.5$ and $K = 220000000, 280000000, 340000000$ in $G = c = 1$ geometrized units were chosen. This is because:

1. The value $n = 1.5 \Rightarrow \gamma = 3$ is shown to have good results in for a polytropic EOS. One other such value could be chosen to be $n = \frac{4}{7} \Rightarrow \gamma = 2.75$, $\gamma = 2.93 \Rightarrow n = \frac{1}{1.93}$ etc[17].[21]

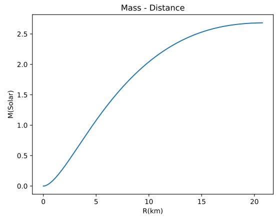
2. The values of K that were chosen show the highest and lowest bounds for the maximum mass M_{max} and EOS should produce, as explained in section 4.2, as well as a representative middle value.

4.1 TOV results and Metrics

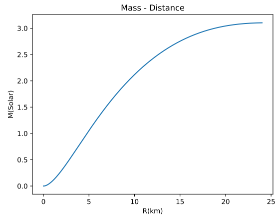
Using the code A.2, the solution for the TOV equations was produced using the method proposed in section 3.1. The plots, produced by the code A.3 are shown in 4.1.



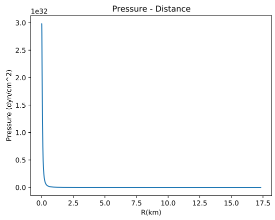
(a)



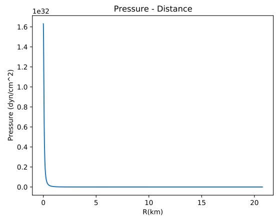
(b)



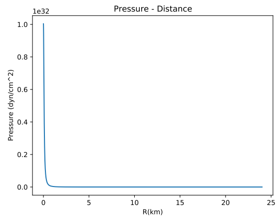
(c)



(d)



(e)



(f)

Figure 4.1: Figures which show Mass and Pressure of a neutron star as a function of the Radius. Mass is measured in $M_{\odot}$, Pressure in $dyncm^{-2}$ and R in km. The values $K = 220000000, 280000000, 340000000$ are represented by the figure (a) and (d), (b) and (e), (c) and (f) respectively. For all plots $n = 1.5$.

From the plots presented in 4.1, the relationship which a polytropic EOS gives for the Mass and Pressure as a function of the Radius is evident. The Mass begins from the value $m(r = 0) = 0$, which is the condition that was originally set for the center mass. Then it scales smoothly to a max value at which it settles. This value M_{max}, represents the total mass of the star, or the solution to the equation

$$M(R) = 4\pi \int_0^R r^2 \epsilon dr \tag{4.1}$$

where R is the radius at which $\Theta(r) \approx 0$. The pressure on the other hand begins from a value p_0 and slowly decreases until it reaches 0 [1].

The metrics λ and ν were calculated by the code A.4, through the method described in section 3.2. Plots for the metrics are presented in figure 4.2

4.2 Classic M - R Diagrams

Using the code A.5, the plots shown in figure 4.3 were produced, which as stated in section 3.4, represent all the possible neutron stars that can be predicted by a given EOS, based on the initial conditions. It should be noted that in these plots, the K and a_0 values are given in $G = c = 1$ units. Of the many produced plots in this paper, the ones presented here are the most characteristic, considering that according to modern theories any EOS should predict a maximum mass of $2.1M_\odot \approx M_1 \leq M_{max} \leq M_2 \approx 3.2M_\odot$.

[1] In reality the pressure does not reach 0, but an extremely small value. This is because the pressure tends to 0 asymptotically

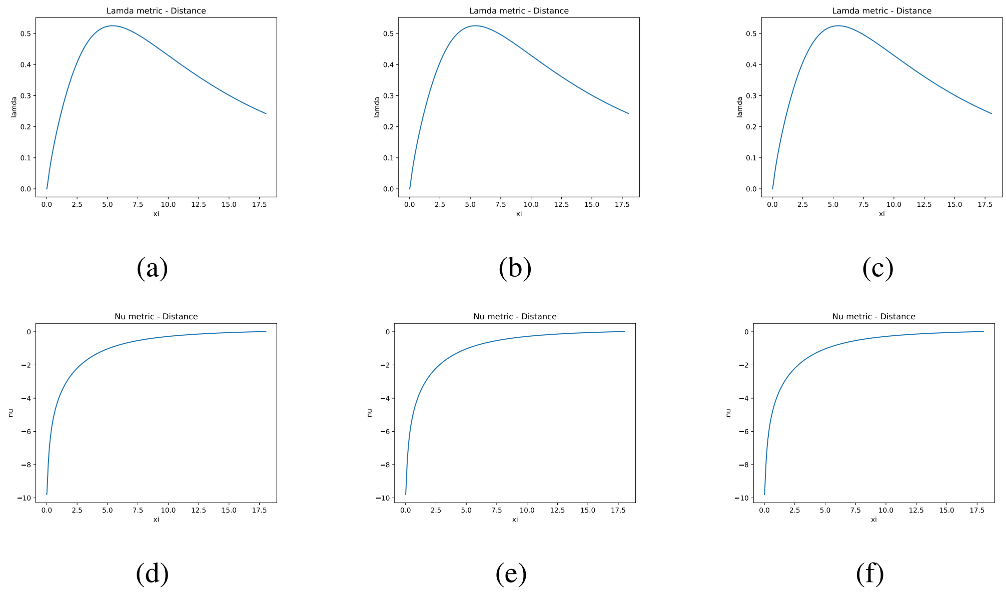


Figure 4.2: Figures which show the λ and ν metric functions as a function of the dimensionless radius ξ for $n = 1.5$. The values $K = 220000000, 280000000, 340000000$ are represented by the plots (a) and (d), (b) and (e), (c) and (f) respectively.

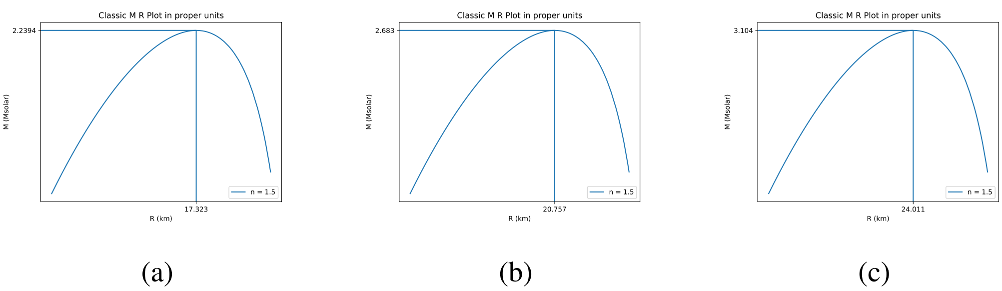


Figure 4.3: Classic M-R plots for $n = 1.5$ and $K = 220000000, 280000000, 340000000$ shown in plots (a), (b), (c) respectively.

CHAPTER 5

Summary

In this thesis a computational study was performed, based on the studies of neutron star TOV equations shown in reference [18]. In this study the method for solving the equations was outlined, examples of solving them with the Runge-Kutta method in Python for a number of polytropic EOS models. A number of examples show how tweaking values can change the maximum mass as well the exact relation of mass and pressure as a function of the radius.

However as outlined in this thesis the model is shown to have its weaknesses. Mainly how the polytropic EOS fails to accurately describe the star outside of areas near the core, as shown in figure 2.5. Specifically the mass values on the right of the maximum mass should in reality decrease monotonically with the increase of the radius. The fact that this does not happen illustrates one of the weaknesses of this model.

Future Research

As part of future research on the subject, mainly the objective should be to make improvements on the EOS models chosen to describe the neutron star.

Many such models have been suggested up to this point, for which a computational study at length has not been performed, such as considering an ultra-dense quark-degenerate matter composition for the core, strange degenerate matter as well as many other models which remain to be explored.

Acknowledgements

Lao Tzu once said:

The journey of a thousand miles, begins with one step.

This journey was but one of the many small steps, taken in a the thousand mile journey of understanding the universe around us. And as all journeys, it is never taken by a single person alone. Whether they're around for it's entirety, or just a small part of it and whether they're main or just recurring characters, the many people who take place in it are what make any journey interesting. Here I mean to thank all those who took place in this one and made it possible.

First and foremost I would like to thank professor Moustakidis Charalampos, who oversaw the writing of this thesis. Without his guidance I would have been lost since day one. He put up with the many times the process was halted and paused, and stood by nonetheless. And for that he deserves my utmost respect.

I would like to also thank all the rest of my professors and mentors who gave me the knowledge and skills I needed to write this thesis. They're teachings, whether scientific or different kind of teachings, where integral to my progress thus far. So for they're indirect and subtle contributions they have my eternal gratitude.

Lastly I would like to thank the people I hold closest to my heart. Not the scientists or teachers who gave knowledge, but the family and friends who gave me wisdom and love. They weren't the people who showed me the way for this thesis, but who helped me get this far and pushed me when I needed it the most. For the team effort that took I am tempted to list them, but the list would grow so long it would take forever. I hope this brief mention will be enough of a thanks, for a service that cannot be compensated by any material mean. For they're constant support, they have my undying love.

Appendices

APPENDIX A

Code Listings

```
import numpy as np

# Constants initialization
K = 3.0*10**(8)
n = 1.51
a_0 = 0.02
m_3 = 4*np.pi/3
theta_2 = -2*np.pi*((1+a_0)*(3+a_0))/(3*a_0*(n+1))
m_5 = 4*np.pi*n*theta_2/5
theta_4 = -theta_2*(m_3+4*np.pi/a_0)/(2*(n+1))

# Taylor Expansions
def Theta(xi):
    """Gives Taylor expansion for dimensionless pressure.

    :xi: Dimensionless distance.
    :returns: T where p = p_0*T(xi)^(n+1), p0 = K*rho_0^gamma, where gamma = n + 1/n

    """
```

```
    return 1 + theta_2*(xi**2) + theta_4*(xi**4)
def Mu(xi):
    """Gives Taylor expansion for dimensionless mass.

    :xi: Dimensionless distance.
    :returns: Mu where Mu = sqrt(epsilon_0)*m, m: mass of the
   star.

    """
    return m_3*(xi**3) + m_5*(xi**5)

# Differential Equations
def dM(xi, t):
    """Differential equation for dimensionless mass

    :xi: Dimensionless distance
    :t: Dimensionless pressure
    :returns: dM/dxi = 4*Pi*xi^2*(T(xi)^n)

    """
    return 4*np.pi*xi**2*t**n
def dT(xi, t, mu):
    """Differential equation for dimensionless pressure

    :xi: Dimensionless distance
    :t: Dimensionless pressure
    :mu: Dimensionless mass
    :returns: dT/dxi = -[(a_0+T(xi))/(n+1)]*[(M(xi)+4Pi*xi^3(T(
   xi)^(n+1))/a_0)/(xi(xi-2M(xi)))]

    """
    return -1.0*(a_0+t)/(n+1)*(mu+4*np.pi*xi**3*t**(n+1)/a_0)/(
   xi*(xi-2*mu))
```

```
# Unit Conversion from G = c = 1 to SI, with Length [km] and M [
    M_solar]
def MUnits(M):
    """Conversion of Mass units from dimensionless in G = c = 1
    to Solar Masses

    :M: Dimensionless mass
    :returns: m: Mass in Solar Masses

    """
    return M/np.sqrt((a_0*K)**(-1.0*n))
    *1.3466*10**28*5*10**(-34)
def RUnits(xi):
    """Conversion of Distance from dimensionless in G = c = 1 to
     km

    :xi: Dimensionless Distance
    :returns: R: Distance in km

    """
    return xi/np.sqrt((a_0*K)**(-1.0*n))*10**(-5)
def PrUnits(t):
    """Converion of Pressure from dimensionless in G = c = 1 to
    cgs

    :t: Dimensionless Pressure
    :returns: P: Pressure in cgs

    """
    return (a_0*K)**(-1.0*(n+1))*t**(n+1)/(8.263*10**(-50))
```

Listing A.1: Definitions

```python
import numpy as np
from definitions import *

# File for Data
data = open("TOV.dat","w+")
data.write("xi: Dimensionless distance\ntheta: Dimensionless
    pressure T\nM: Dimensionless mass M\n\n\nxi\ttheta\tM\n")

# numerical integration
h = 0.00001
xi = 0.00001
theta = Theta(xi)
m = Mu(xi)
XI = []
THETA = []
MU = []
while theta>=0.0001:
    XI.append(xi)
    MU.append(m)
    THETA.append(theta)
    data.write(str(xi)+"\t"+str(theta)+"\t"+str(m)+"\n")
    m_k1 = dM(xi, theta)
    theta_k1 = dT(xi, theta, m)
    m_k2 = dM(xi+h, theta+theta_k1*h)
    theta_k2 = dT(xi+h, theta+theta_k1*h, m+m_k1*h)
    m = m + (m_k1+m_k2)*h/2
    theta = theta + (theta_k1+theta_k2)*h/2
    xi = xi + h
data.close()

R = []
M = []
P = []
```

```python
for i in range(0, len(XI)-1, 1):
    R.append(RUnits(XI[i]))
    M.append(MUnits(MU[i]))
    P.append(PrUnits(THETA[i]))
```

Listing A.2: TOV solution

```python
from matplotlib import pyplot as plt
from TOV import R, M, P
from definitions import n,K

# Plots
plt.plot(R,M)
plt.xlabel("R(km)")
plt.ylabel("M(Solar)")
plt.title("Mass - Distance")
name = "Plots/M_n_{}_K_{}.pdf".format(n,K)
plt.savefig(name)
plt.clf()
plt.plot(R,P)
plt.xlabel("R(km)")
plt.ylabel("Pressure (dyn/cm^2)")
plt.title("Pressure - Distance")
name = "Plots/Pressure_n_{}_K_{}.pdf".format(n,K)
plt.savefig(name)
```

Listing A.3: Plot creation

```python
from definitions import *
from matplotlib import pyplot as plt
import numpy as np
from TOV import XI, THETA, MU

# metrics
data = open("metric.dat","w+")
data.write("\n\nxi\tlamda(xi)\tnu(xi)\n")
lamda = []
nu = []
count = 0
for i in XI:
    lamda.append(np.log(1/(1-2*MU[count]/i))/2)
    nu.append(np.log((a_0/(a_0+1))**(2*(n+1))*a_0/(a_0+2*XI[-1]*(n-1)*dT(XI[-1], THETA[-1], MU[-1]))*((a_0+1)/(a_0+THETA[count])**(2*(n+1))))/2)
    data.write(str(i)+"\t"+str(lamda[count])+"\t"+str(nu[count])+"\n")
    count += 1

# Plots
plt.plot(XI, lamda)
plt.xlabel("xi")
plt.ylabel("lamda")
plt.title("Lamda metric - Distance")
name = "Plots/Lambda_n_{}_K_{}.pdf".format(n,K)
plt.savefig(name)
plt.clf()
plt.plot(XI, nu)
plt.xlabel("xi")
plt.ylabel("nu")
plt.title("Nu metric - Distance")
name = "Plots/Nu_n_{}_K_{}.pdf".format(n,K)
```

```
plt.savefig(name)
data.close()
```

Listing A.4: Metric Calculation

```
from matplotlib import pyplot as plt
from matplotlib import ticker as ticker
from matplotlib.collections import LineCollection
import numpy as np
import csv

# Constants Initialiation
h = 0.0001
m_3 = 4*np.pi/3

# Taylor Expansions
def Theta(xi):
    """Gives Taylor expansion for dimensionless pressure.

    :xi: Dimensionless distance.
    :returns: T where p = p_0*T(xi)^(n+1), p0 = K*rho_0^gamma,
    where gamma = n + 1/n

    """
    return 1 + theta_2*(xi**2) + theta_4*(xi**4)
def Mu(xi):
    """Gives Taylor expansion for dimensionless mass.

    :xi: Dimensionless distance.
    :returns: Mu where Mu = sqrt(epsilon_0)*m, m: mass of the
    star.

    """
    return m_3*(xi**3) + m_5*(xi**5)

# Differential Equations
def dM(xi, t):
    """Differential equation for dimensionless mass
```

```

    :xi: Dimensionless distance
    :t: Dimensionless pressure
    :returns: dM/dxi = 4*Pi*xi^2*(T(xi)^n)

    """
    return 4*np.pi*xi**2*t**n
def dT(xi, t, mu):
    """Differential equation for dimensionless pressure

    :xi: Dimensionless distance
    :t: Dimensionless pressure
    :mu: Dimensionless mass
    :returns: dT/dxi = -[(a_0+T(xi))/(n+1)]*[(M(xi)+4Pi*xi^3(T(xi)^(n+1))/a_0)/(xi(xi-2M(xi)))]

    """
    return -1.0*(a_0+t)/(n+1)*(mu+4*np.pi*xi**3*t**(n+1)/a_0)/(xi*(xi-2*mu))

# Unit Conversions
def M_to_solar(M):
    """Conversion of Mass units from dimensionless in G = c = 1 to Solar Masses

    :M: Dimensionless mass
    :returns: m: Mass in Solar Masses

    """
    return M/np.sqrt(epsilon_0)*1.3466*10**28*5*10**(-34)
def R_to_km(xi):
    """Conversion of Distance from dimensionless in G = c = 1 to km
```

```

    :xi: Dimensionless Distance
    :returns: R: Distance in km

    """
    return xi/np.sqrt(epsilon_0)*10**(-5)

# Double loop for Plots
for K in np.arange(3*10**8, 4*10**8, 2*10**7):
    print("\n\nK={}".format(K))
    for n in np.arange(1.40, 1.52, 0.01):
        XI = []
        MU = []
        print("\n\nn={}".format(n))
        for a_0 in np.arange(0.01, 0.04, 0.0001):
            xi = 0.00001
            theta_2 = -2*np.pi*((1+a_0)*(3+a_0))/(3*a_0*(n+1))
            m_5 = 4*np.pi*n*theta_2/5
            theta_4 = -theta_2*(m_3+4*np.pi/a_0)/(2*(n+1))
            theta = Theta(xi)
            M = Mu(xi)
            epsilon_0 = (a_0*K)**(-1.0*n)

        # TOV Solutions
            while theta>=0.0001:
                M_k1 = dM(xi, theta)
                theta_k1 = dT(xi, theta, M)
                M_k2 = dM(xi+h, theta+theta_k1*h)
                theta_k2 = dT(xi+h, theta+theta_k1*h, M+M_k1*h)
                M = M + (M_k1+M_k2)*h/2
                theta = theta + (theta_k1+theta_k2)*h/2
                xi = xi + h
            print("R={}\tM={}\ta_0={}".format(R_to_km(xi),
```

```
M_to_solar(M), a_0))

    # Keep total Mass and Radius in XI and MU respectively
        XI.append(R_to_km(xi))
        MU.append(M_to_solar(M))

    # Finding the Max Value of MU
    i = 0
    while True:
        if MU[i] > MU[i+1]:
            break
        i = i + 1

    # Finding the value of R for which M[R] = 1.4
    # if MU[i] > 1.4:
        # j = 0
        # while True:
            # if MU[j] > 1.4:
                # break
            # j = j + 1
        # if 1.4-MU[j-1] < MU[j] - 1.4:
            # j = j-1

    # Plots
    leg = "n = {}".format(n)
    plt.plot(XI,MU,label=leg)
    ax = plt.axes()
    title = "Classic M R Plot in proper units"
    plt.title(title)
    plt.legend(loc="lower right")
    plt.xlabel("R (km)")
    plt.ylabel("M (Msolar)")
    ax.xaxis.set_major_locator(ticker.FixedLocator([XI[i]]))
```

```
        ax.yaxis.set_major_locator(ticker.FixedLocator([1.4,MU[i]]))
        l1 = [(XI[i],0),(XI[i],MU[i])]
        l2 = [(0,MU[i]),(XI[i],MU[i])]
        lc1 = LineCollection([l1,l2])
        plt.gca().add_collection(lc1)
        # if MU[i] > 1.4:
            # ax.xaxis.set_major_locator(ticker.FixedLocator([XI[i],XI[j]]))
            # l3 = [(XI[j],0),(XI[j],1.4)]
            # l4 = [(0,1.4),(XI[j],1.4)]
            # lc2 = LineCollection([l3,l4])
            # plt.gca().add_collection(lc2)
        # else:
            # ax.xaxis.set_major_locator(ticker.FixedLocator([XI[i]]))
        name = "Plots/n_{}_K_{}_proper_units.pdf".format(n, K)
        plt.savefig(name)
        plt.clf()
        print("\n\n")
```

Listing A.5: Classic M-R plots production for specific K and for a cetain range of n

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