1. $A = \begin{bmatrix} 1 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$

$r_{11} = \sqrt{3}$, $q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

$r_{12} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = -\sqrt{3}$

$q_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} - (-\sqrt{3}) \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$.

$r_{22} = 2\sqrt{2}$.

$q_2 = \frac{1}{2\sqrt{2}} \cdot \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & 0 \end{bmatrix}}_{= Q} \underbrace{\begin{bmatrix} \sqrt{3} & -\sqrt{3} \\ 0 & 2\sqrt{2} \end{bmatrix}}_{= R}.$$

2. $v = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. $a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $a_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Wrong answer.

Cannot take

$\langle v, a_1 \rangle a_1 + \langle v, a_2 \rangle a_2$

as the projection!

$$A = [a_1 \ a_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$r_{11} = \sqrt{2}. \quad q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$r_{12} = \frac{1}{\sqrt{2}} [1 \ 0 \ 1] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}, \quad q_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

$$r_{22} = \frac{\sqrt{6}}{2}. \quad q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}}_{\parallel \atop Q} \cdot \underbrace{\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} \end{bmatrix}}_{\parallel \atop R}.$$

$$Q =: [q_1 \ q_2]$$

$$\langle v, q_1 \rangle \cdot q_1 + \langle v, q_2 \rangle q_2 = \left([2, 1, -1] \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \left([2, 1, -1] \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}.$$

$$= \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \left(\frac{5}{\sqrt{6}} \right) \cdot \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{5}{6} \\ \frac{5}{3} \\ -\frac{5}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix}$$

If you want to project a vector v onto the column space of some matrix A with full column ranks, then the projection is given by

$$A(A^TA)^{-1}A^T \cdot v.$$

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$A \cdot (A^TA)^{-1} A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

$$A \cdot (A^TA)^{-1} A^T \cdot v = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix}.$$

$$A = QR$$

$$A^TA = R^T \underline{\underline{Q^TQ}} R = R^TR$$

$$A \cdot (A^TA)^{-1} \cdot A^T = QR \cdot (R^TR)^{-1} R^TQ^T$$

$$= QQ^T$$

4.

$$\underset{n\times n}{A} = \underset{n\times n}{Q}R \iff [a_1, \cdots, a_n] = Q[r_1, \cdots, r_n]$$

$$= [Qr_1, \cdots, Qr_n].$$

$A = [a_1, \cdots, a_n]$

$R =: [r_1, \cdots, r_n]$

$$\Updownarrow$$

$$a_i = Q \cdot r_i \quad \forall i \in \{1, \cdots, n\}, \quad (*) \qquad R = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ 0 & & r_{nn} \end{bmatrix}$$

$$|\det(A)| = |\det(QR)| = \underbrace{|\det(Q)|}_{1.} \cdot |\det(R)|$$

$$= |\det(R)|$$

$$= |r_{11} \cdot r_{22} \cdots r_{nn}| \leq \|r_1\|_2 \cdots \|r_n\|_2 = \|a_1\|_2 \cdots \|a_n\|_2.$$

From $(*)$, we have $\|a_i\|_2^2 = \|Qr_i\|_2^2 = |r_i^T \underline{Q^T Q} r_i|$

$$= |r_i^T r_i|$$

$$= \|r_i\|_2^2$$

$$\Rightarrow \quad \|a_i\|_2 = \|r_i\|_2 \geq |r_{ii}|$$

$$\uparrow \quad r_i = \begin{bmatrix} r_{1i} \\ \vdots \\ r_{ii} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$