1. $Ax = \begin{bmatrix} 6 \\ 15 \end{bmatrix}$

$A^T y = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 19 \\ 24 \end{bmatrix}$

$AB^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$,

2. $\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ \vdots & & & \\ A_{p1} & & \cdots & A_{pq} \end{bmatrix}^T = \begin{bmatrix} A_{11}^T & \cdots & A_{p1}^T \\ \vdots & & \\ A_{1q}^T & \cdots & A_{pq}^T \end{bmatrix} = C$

$B = \begin{array}{c|cccc} & n_1 & n_2 & \cdots & n_q \\ \hline m_1 & A_{11} & A_{12} & \cdots & A_{1q} \\ \vdots & \vdots & & & \\ m_p & A_{p1} & & \cdots & A_{pq} \end{array}$

Aim: $B_{ij} = C_{ji} \Rightarrow B^T = C$.

size of A_{ij} is $m_i \times n_j$.

(i,j)-th entry of B,

$i = \sum_{k=1}^{r_i} m_k + r. \qquad 1 \le r \le m_{r_i+1}.$

$j = \sum_{k=1}^{c_i} n_k + c \qquad 1 \le c \le n_{c_i+1}.$

$B_{ij} = (A_{m_{r_i+1}, n_{r_i+1}})_{r,c}$

$(A^T)_{ij} = A_{ji}$

(j,i)-th entry of C.

$C_{ji} = (A^T_{m_{r_i+1}, n_{r_i+1}})_{c,r} = (A_{m_{r_i+1}, n_{r_i+1}})_{r,c}$.

$\Rightarrow B_{ij} = C_{ji}$

$\Rightarrow \boxed{B^T = C.}$

4. (a) $C = AB$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$C_{ij} = A_{i1}\cdot B_{1j} + A_{i2}\cdot B_{2j}. \qquad \underline{i,j = 1,2.}$$

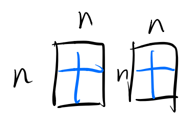
$2\times 4 = 8$ multiplications.

$1 \times 4 = 4$ additions.

(b). 7 multiplications and 18 additions.

(c) (a): $8\cdot 2\cdot(\frac{n}{2})^3 + 4\cdot(\frac{n}{2})^2 = 2n^3 + n^2$

(b) $7\cdot 2\cdot(\frac{n}{2})^3 + 18\cdot(\frac{n}{2})^2 = \frac{7}{4}n^3 + \frac{9}{2}n^2.$



requires $\rightarrow$ $F(n)$.

7 matrix multi of size $\frac{n}{2}\times\frac{n}{2}$

18 mat. add.

(d). Denote the computational complexity by $F(n)$. Then $F(1) = 1$.

multiplication addition

$$F(n) = \cancel{7\cdot F}(\tfrac{n}{2}) + 18(\tfrac{n}{2})^2$$

$$7\cdot \cancel{F}(\tfrac{n}{2}) = 7(\cancel{7\cdot F}(\tfrac{n}{4}) + 18(\tfrac{n}{4})^2)$$

$\vdots$

$$7^{k-1}\cancel{F(2)} = 7^{k-1}(7\cdot F(1) + 18\cdot 1^2)$$

$$= 7\cdot\left[7\cdot F(\tfrac{n}{4}) + 18\cdot(\tfrac{n}{4})^2\right] + 18(\tfrac{n}{2})^2$$

$$= \cdots$$

$$= \underline{7^{\log_2 n}}\ \underset{1}{\underline{F(1)}} + 18n^2\left(\frac{1}{4} + \frac{7}{4^2} + \cdots + \frac{7^{k-1}}{4^k}\right) \qquad k = \log_2 n.$$

$$= n^{\log_2 7} + 6\cdot n^{\log_2 7} - 6n^2$$

$$= \underline{O(n^{\log_2 7})}.$$

5. $f(x) = \sum_{i=1}^{n/2} x_{2i-1} \cdot x_{2i} \qquad \forall x \in \mathbb{R}^n.$

(a). Show $x^T y = \sum_{i=1}^{n/2} (x_{2i-1} + y_{2i})(x_{2i} + y_{2i-1}) - f(x) - f(y).$

$$RHS = \sum_{i=1}^{n/2} (x_{2i-1} \cdot x_{2i} + y_{2i-1} \cdot y_{2i} + x_{2i-1} y_{2i-1} + x_{2i} \cdot y_{2i}) - f(x) - f(y).$$

$$= \sum_{i=1}^{n/2} \left(x_{2i-1} \cdot y_{2i-1} + x_{2i} y_{2i}\right)$$

$$= \sum_{i=1}^{n} x_i \cdot y_i$$

$$= x^T y \quad = LHS.$$

(b). $C = AB$

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} \qquad B = [b_1, \cdots, b_n].$$

$$AB = \begin{bmatrix} a_1^T b_1 & \cdots & a_1^T b_n \\ \vdots & & \\ a_n^T b_1 & \cdots & a_n^T b_n \end{bmatrix}$$

$$a_i^T b_j = \sum_{l=1}^{n/2} \left((a_i)_{2l-1} + (b_j)_{2l}\right) \cdot \left((a_i)_{2l} + (b_j)_{2l-1}\right) - f(a_i) - f(b_j).$$

① Compute $f(a_i)$, $f(b_j)$, $\forall i, j = 1, \cdots, n$

This step requires $2n \times \frac{n}{2} = n^2$ multiplications

$$f(x) = \sum_{i=1}^{n/2} x_{2i-1} \cdot x_{2i}$$

② $a_i^T b_j \quad \forall i, j = 1, \cdots, n.$

$$n^2 \times \frac{n}{2} = \frac{n^3}{2}$$

$$\frac{n^3}{2} + n^2 = \frac{n^3}{2} + O(n^2)$$

b.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 4 & 5 & 3 & 2 \\ 2 & -2 & 3 & 7 \end{array}\right]$$

$$\begin{bmatrix} 2 & 1 & 1 & 2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 2 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_3 = 1 \\ x_2 = -1 \\ x_1 = 1 \end{cases}$$