MATH3322 Matrix Computation
Homework 4

Due date: 12 April, Monday

1. Let $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^n$ be two vectors satisfying $\|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2$.

   (a) Find a Householder matrix $\boldsymbol{H}$ such that $\boldsymbol{H}\boldsymbol{x} = \boldsymbol{y}$.

   (b) Prove $\boldsymbol{H}\boldsymbol{y} = \boldsymbol{x}$ as well.

2. As the first sub-step of the QR algorithm for $\boldsymbol{A}$, we used an orthogonal matrix $\boldsymbol{Q}_1$ to reduce the first column $\boldsymbol{a}_1$ to a multiple of $\boldsymbol{e}_1$. We have seen in class that two following approaches are used.

   (i) In the Householder QR algorithm, we choose $\boldsymbol{Q}_1$ to be the Householder matrix such that $\boldsymbol{Q}_1\boldsymbol{a}_1 = \|\boldsymbol{a}_1\|_2\boldsymbol{e}_1$. (Here without loss of generality, we assumed $\boldsymbol{a}_1$ is reduced to $\|\boldsymbol{a}_1\|_2\boldsymbol{e}_1$ rather than $-\|\boldsymbol{a}_1\|_2\boldsymbol{e}_1$.)

   (ii) In the Givens QR algorithm, we use the product of a series of Givens rotations $\boldsymbol{Q}_1 = \boldsymbol{G}_{n-1}^{(1)} \cdots \boldsymbol{G}_1^{(1)}$ such that $\boldsymbol{Q}_1\boldsymbol{a}_1 = \|\boldsymbol{a}_1\|_2\boldsymbol{e}_1$.

   Prove that the two $\boldsymbol{Q}_1$'s in (i) and (ii) are different if all entries of $\boldsymbol{a}_1$ are non-zero. (Actually, there are many other different ways to orthogonally reduce $\boldsymbol{a}_1$ to a multiple of $\boldsymbol{e}_1$.)

3. Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $m \geq n$ and have full column rank. Let $\begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{x}} \end{bmatrix}$ be the solution of

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{A} \\ \boldsymbol{A}^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$$

   Prove that $\hat{\boldsymbol{x}}$ is also the solution of the least squares problem $\min_{\boldsymbol{x}\in\mathbb{R}^n} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2$. This gives us a new method to solve the least squares problem.

4. Write a code to use QR least squares to solve the linear regression with polynomial functions. An incomplete sample code `QRLS_poly.m` is provided. Submit your code and the plotted graph. You will see *over-fitting* of the high-degree polynomial regression. (If you cannot see it, run the code again.)