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1. $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

The characteristic Polynomial is

$$P_A(z) = \det(zI - A)$$
$$= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right)$$
$$= \det\left(\begin{bmatrix} z-1 & -1 \\ 0 & z-2 \end{bmatrix}\right)$$
$$= (z-1)(z-2)$$

Therefore, $P_A(\lambda) = (\lambda-1)(\lambda-2) = 0$ $\therefore \lambda_1 = 2, \lambda_1 = 1$

For eigenvector of $\lambda_1 = 2$,

$(\lambda_1 I - A)x = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} x = 0 \Rightarrow x = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \forall \alpha \in \mathbb{C}$

For eigenvector of $\lambda_2 = 1$,

$(\lambda_2 I - A)x = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x = 0 \Rightarrow x = \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$X \Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = A$$

$\therefore$ Eigenvalue $(\Lambda) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, eigenvector $(X) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

or eigenpair 1 $\left(2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, eigenpair 2 $\left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

2. $A \in \mathbb{R}^{n \times n}$ non-symmetric matrix with eigenvalues $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \cdots \geq |\lambda_n|$

Let λ_1 be a dominant complex eigenvalue of matrix A.

Since eigenvalues are roots of a polynomial of real coefficients,

Complex eigenvalues are in conjugate pairs.

Then, there exists another complex eigenvalue λ_n which equals to $\overline{\lambda_1}$.

This contradicts to the assumption that λ_1 is a dominant eigenvalue

(i.e. $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \cdots\cdots$) since $\lambda_n = \overline{\lambda_1}$ will have the same magnitude

as λ_1. Therefore, λ_1 is real.

3. Same answer to Q.2

4. Hermitian $C = C^* = \overline{C^T}$ conjugate transpose of $C \in \mathbb{C}^{n \times n}$

a) Eigenvalue decomposition $Cx = \lambda x$

※ Hermitian (self-adjoint) matrix is a complete square matrix that is equal to its conjugate transpose. $a_{ij} = \overline{a_{ji}}$

$A = \overline{A^T}$ $\begin{matrix} \cdot & 4-2i \\ 4+2i & \cdot \end{matrix} \rightarrow \begin{matrix} \cdot & 4+2i \\ 4-2i & \cdot \end{matrix}$

Conjugate transpose : transpose $\rightarrow$ complex conjugate

$$x^* Cx = \lambda x^* x$$
$$= \lambda \cdot \sum_{i=1}^{n} \overline{x_i}^T x_i$$
$$= \lambda \cdot \|x\| \qquad (\because \overline{x_i}^T = x_i)$$

$x^* C x = \lambda \cdot \|x\|$ taking conjugate transpose on both side,

$$\overline{(x^* C x)}^T = \overline{(\lambda \|x\|)}^T \Rightarrow \overline{x^T}\, \overline{C^T} x = \overline{\lambda} \cdot \|x\|$$
$$= x^* C x \qquad (\because C \text{ is Hermitian } C = \overline{C^T})$$
$$= \lambda \cdot \|x\|$$

$$\Rightarrow \begin{cases} \overline{(x^* C x)}^T = \overline{\lambda} \cdot \|x\| \\ \overline{(x^* C x)}^T = x^* C x = \lambda \cdot \|x\| \end{cases} \Rightarrow \overline{\lambda} \cdot \|x\| = \lambda \cdot \|x\|$$

Since $\|x\| \in \mathbb{R}$, $\overline{\lambda}$ must equal to λ.

$\therefore$ $\lambda \in \mathbb{R}$

All eigenvalues of C are real.

b) $C = A + iB$

$\Rightarrow$: The definition of a Hermitian matrix C is that C is equal to its conjugate transpose $C = C^* = \overline{C^T}$

$$C = A + iB = C^* = (\overline{A+iB})^T$$
$$= A^T - iB^T$$

$$\Rightarrow A = A^T,\ B = -B^T$$

$$M = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix} = M^T$$

$\therefore$ M is symmetric

$\Leftarrow$ If M is a symmetric matrix,

$$M = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = M^T = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix}$$

$$\Rightarrow B = -B^T,\ A = A^T.$$

Then, $C = A + iB = A^T - iB^T$

$$= (\overline{A+iB})^T$$
$$= C^*$$

$\therefore$ C is Hermitian.

$\therefore$ $C = A + iB$ is Hermitian $\Longleftrightarrow$ $M = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric

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c) $\lambda \in \mathbb{R}$ is an eigenvalue of C $\Rightarrow$ $Cz = \lambda z$ where $\lambda = x + iy$

$$(A + iB)(x + iy) = \lambda(x + iy)$$

$$\Rightarrow Ax + iBx + iAy - By = \lambda x + i\lambda y$$

$$= Ax - By + i(Ay + Bx) = \lambda x + i\lambda y$$

$$\therefore \begin{cases} Ax - By = \lambda x \\ Ay + Bx = \lambda y \end{cases} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = M\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore$ λ is also an eigenvalue of M.

d) $\lambda, \begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenpair of M,

find another eigenvector orthogonal to $\begin{bmatrix} x \\ y \end{bmatrix}$ with eigenvalue λ.

Since $\begin{bmatrix} y \\ -x \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$ are orthogonal, $\left(\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix} = 0 \right)$

$$M \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}$$

$$= \begin{matrix} Ay + Bx \\ By - Ax \end{matrix}$$

$$= \begin{bmatrix} \lambda y \\ -\lambda x \end{bmatrix} \qquad \left(\because \begin{cases} Ax - By = \lambda x \\ Ay + Bx = \lambda y \end{cases} \right)$$

$$= \lambda \begin{bmatrix} y \\ -x \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ -x \end{bmatrix}$ are two orthogonal eigenvectors of M

with same eigenvalue.