Math 3322 HW4

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1. $x, y \in \mathbb{R}^n$ s.t $\|x\|_2 = \|y\|_2$

a) Since Household matrix is of form $H = I - 2vv^T$

$= I - 2\frac{uu^T}{u^Tu}$ $\left(u = \frac{v}{\|v\|}\right)$

Lets assume u is equal to $x-y$

Then, $Hx = x - \dfrac{2(x-y)(x-y)^T x}{(x-y)^T(x-y)}$

$$= \frac{x(x-y)^T(x-y) - 2(x-y)(x-y)^T x}{(x-y)^T(x-y)} \quad \cdots (1)$$

$x(x-y)^T(x-y) = (xx^T - xy^T)(x-y) = xx^Tx - xy^Tx - xx^Ty + xy^Ty$

$(x-y)(x-y)^T x = xx^Tx - xy^Tx - yx^Tx + yy^Tx$

$$(1) : \frac{2yx^Tx - 2yy^Tx}{(x-y)^T(x-y)} = \frac{y(x-y)^T(x-y)}{(x-y)^T(x-y)} = y$$

In other words, when $\|x\|_2 = \|y\|_2$, $x-y$ passes the origin.

$$\therefore H = I - \frac{2(x-y)(x-y)^T}{(x-y)^T(x-y)}$$

b) $Hy = y - \dfrac{2(x-y)(x-y)^T y}{(x-y)^T(x-y)}$

$$= \frac{y(x-y)^T(x-y) - 2(x-y)(x-y)^T y}{(x-y)^T(x-y)} \quad \cdots (2)$$

$(yx^T - yy^T)(x-y) = yx^Tx - yy^Tx - yx^Ty + yy^Ty$

$-2(x-y)(x-y)^T y = -2(xx^Ty - xy^Ty - yx^Ty + yy^Ty)$

$$(2) : \frac{2xy^Ty - 2xx^Ty}{(x-y)^T(x-y)} = \frac{x(x-y)^T(x-y)}{(x-y)^T(x-y)} = x \qquad \therefore Hy = x$$

2. The entries of a_1 are all non-zero.

Lets denote H as $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

and G as $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ which G is just a Givens Rotation.

To check whether matrix H is a householder matrix, we need to verify that it can be rewritten into $H = I - 2vv^T$.

By choosing $v = \left[\sqrt{\frac{1-\cos\theta}{2}}, \sqrt{\frac{1+\cos\theta}{2}} \right]^T$,

$H = I - 2vv^T = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. Therefore, H is indeed a householder matrix.

Now, $H \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $G \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

With nonzero entries $a_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $H(a_1) \neq G(a_1)$ is shown although the output $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is same.

3.

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

minimize $\|Ax-b\|^2 = \|\sum_{j=1}^{n} a_j x_j - b\|^2$

b

$r = A\hat{x} - b$ is orthogonal to range(A)

$A\hat{x}$

Since $A \in \mathbb{R}^{m \times n}$ with $m \geq n$, rank$(A) = n$, there exists a decomposition $A = QR$.

we got $\begin{cases} r + Ax = b \\ A^T r = 0 \end{cases}$ from the above matrix.

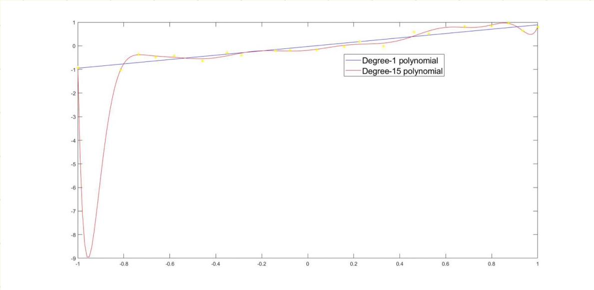
Multiply A^T to the first equation : $\underbrace{A^T r}_{=0} + A^T A x = A^T b$

$$\Rightarrow A^T A x = A^T b$$

$$\Rightarrow x = (A^T A)^{-1} A^T b$$

which is exactly the closed form solution of $\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2$, meaning the same solution $\hat{x}$ for the given matrix and $\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2$.

4.



Overfitting observed when degree = 15.