1. $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$

$$\Big\downarrow P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 8 & 14 \\ 1 & 2 & 4 \\ 2 & 6 & 13 \end{bmatrix} \xrightarrow{L_1 = \begin{bmatrix} 1 & & \\ -1/3 & 1 & \\ -2/3 & & 1 \end{bmatrix}} \begin{bmatrix} 3 & 8 & 14 \\ 0 & -2/3 & -2/3 \\ 0 & 2/3 & 11/3 \end{bmatrix}$$

$$\Big\downarrow L_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 8 & 14 \\ 0 & -2/3 & -2/3 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} 1 & & \\ 2/3 & 1 & \\ 1/3 & -1 & 1 \end{bmatrix}$$

$PA = LU$

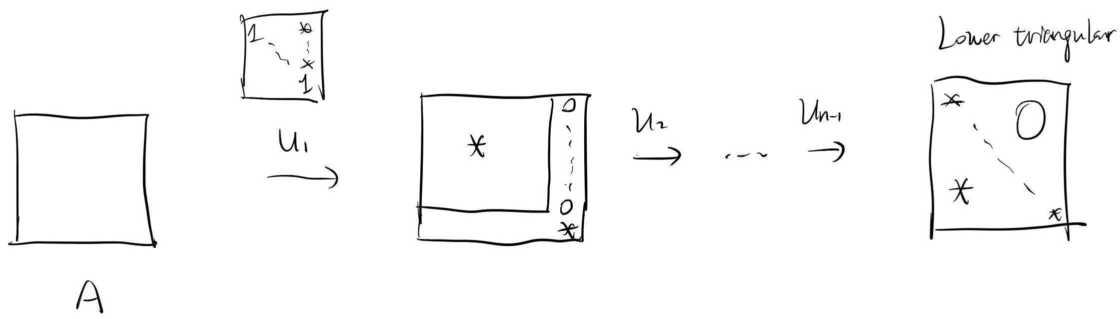
$$U = \begin{bmatrix} 3 & 8 & 14 \\ 0 & -2/3 & -2/3 \\ 0 & 0 & 3 \end{bmatrix}$$

2. (UL decomposition.)

```
for i = n : -1 : 1
    A(1:i-1, i) = A(1:i-1, i) · A(i,i)^-1;
    A(1:i-1, 1:i-1) = A(1:i-1, 1:i-1) - A(1:i-1, i) · A(i, 1:i-1)^T;
end
```

$$U(i,j) = \begin{cases} 0 & i > j \\ 1 & i = j \\ A(i,j) & i < j \end{cases} \qquad L(i,j) = \begin{cases} A(i,j) & i \geq j \\ 0 & i < j. \end{cases}$$

idea:



$$U_{n-1} \cdots U_1 A = L.$$

$$A = \underbrace{U_1^{-1} \cdots U_{n-1}^{-1}}_{\text{Upper triangular}} L$$

3. (a) $A = \begin{bmatrix} a_{11} & & \text{symmetric} \\ a_{21} & a_{22} & \\ & \ddots & \ddots \\ & & a_{n,n-1} & a_{nn} \end{bmatrix}$ Suppose $L = \begin{bmatrix} l_{11} & & \\ l_{21} & \ddots & \\ \vdots & & \\ l_{n1} & \cdots & l_{nn} \end{bmatrix}$.

$\parallel$

$$LL^T = \begin{bmatrix} l_{11} & & \\ l_{21} & \ddots & \\ \vdots & & \\ l_{n1} & \cdots & l_{nn} \end{bmatrix} \cdot \begin{bmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ & \ddots & & \vdots \\ & & & l_{nn} \end{bmatrix} = \begin{bmatrix} l_{11}^2 & & & \\ l_{21}\cdot l_{11} & l_{21}^2+l_{22}^2 & & \\ l_{31}\cdot l_{11} & & \ddots & \\ \vdots & & & \\ l_{n1}\cdot l_{11} & & & l_{n1}^2+\cdots+l_{nn}^2 \end{bmatrix}$$

To show $l_{ij}=0$ for all $i-j>1$, we use induction on j.
i.e. for each fixed j, we show $l_{ij}=0$ for all $i>j+1$.

Base case, when $j=1$.

$(LL^T)_{i1} = l_{i1}\cdot l_{11} = a_{i1}$.

$a_{11}>0$ since A is SPD. so $l_{11}=\sqrt{a_{11}}\neq 0$.

and so for $i>2$, $l_{i1}\cdot l_{11} = a_{i1} = 0$. $\Rightarrow l_{i1}=0$.

suppose for all $j\le j_0-1$, $l_{ij}=0$ for $i>j+1$.

consider $j=j_0$. and $i\ge j_0$.

$$(LL^T)_{ij_0} = l_{i1}l_{j_01} + \cdots + l_{ij_0}\cdot l_{j_0j_0}. = a_{ij_0}.$$

when $i=j_0$, $a_{j_0j_0} = l_{j_0,1}^2+\cdots+l_{j_0,j_0}^2$
since $l_{j_0,1},\cdots,l_{j_0,j_0-1}$ is known, we have $l_{j_0j_0} = \sqrt{\underbrace{a_{j_0j_0}-\sum_{k=1}^{j_0-1} l_{j_0,k}^2}_{\text{positive since } A \text{ is SPD (need to be proved).}}} > 0$

$i=j_0+1$, we can solve l_{j_0+1,j_0}.

$i>j_0+1$. $(LL^T)_{ij_0} = l_{i1}l_{j_01}+\cdots+l_{ij_0}\cdot l_{j_0j_0} = 0$.

$$= \underbrace{l_{ij_0-1}}_{0}\cdot l_{j_0j_0-1} + l_{ij_0}l_{j_0j_0}$$

$$= l_{ij_0}\cdot l_{j_0j_0}$$

$\Rightarrow l_{ij_0}=0$ for all $i>j_0+1$. $\boxtimes$

(b). Now we know from (a) that L has the form

$$L = \begin{bmatrix} l_{11} & & & \\ l_{21} & \ddots & & \\ & \ddots & \ddots & \\ & & l_{n,n-1} & l_{nn} \end{bmatrix}.$$

for $i = 1:n$

$\quad l_{ii} = \sqrt{a_{ii} - l_{i,i-1}^2}$ 3

$\quad l_{i+1,i} = a_{i+1,i} / l_{ii}$ 1

end.

$4n + O(1)$.

(c). $LL^T x = b$.

for $i = 1:n$

$\quad b(i) = b(i) - L(i, i-1) \cdot b(i-1)$ 3

$\quad b(i) = b(i) / L(i,i)$

end.

for $i = n-1:1$.

$\quad b(i) = b(i) - L(i+1, i) \cdot b(i+1)$ 3

$\quad b(i) = b(i) / L(i,i)$

end.

$4n + 6n = 10n$.

4. (a). Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.

Consider the quadratic form $\quad x^T A x = 2 \sum_{i=1}^{n} x_i^2 - 2 \sum_{i=1}^{n-1} x_i x_j$

$$= \sum_{i=1}^{n-1} (x_i - x_j)^2 + x_1^2 + x_n^2$$

$x^T A x = 0$ iff. $x_1 = \cdots = x_n = 0$.

$x^T A x > 0$ iff. $x \neq 0$.