MATH3322 Matrix Computation
Homework 2

Due date: 8 March, Monday

1. Let
$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$
Find the LU decompsotion with row pivoting of $\boldsymbol{A}$.

2. Instead of the LU decomposition, we can also use a *UL decomposition* to solve the system of linear equations. In particular, given $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, we decompose $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{L}$, where $\boldsymbol{U} \in \mathbb{R}^{n\times n}$ is *unit upper triangular* and $\boldsymbol{U} \in \mathbb{R}^{n\times n}$ is *lower triangular*. Propose an algorithm for computing the UL decomposition of $\boldsymbol{A}$.

3. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a tridiagonal matrix, i.e., $a_{ij} = 0$ if $|i-j| > 1$. We also assume that $\boldsymbol{A}$ is symmetric positive definite (SPD).
   (a) Prove that the Cholesky decompostion $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T$ satisfies $l_{ij} = 0$ for all $i-j > 1$. In other words, $\boldsymbol{L}$ is bi-diagonal.
   (b) Propose an $O(n)$ algorithm for computing the Cholesky decomposition of $\boldsymbol{A}$. What is the number of operations needed of your algorithm? Your answer should be in the form of $Cn + O(1)$ with explicit constant C.
   (c) Based on the Cholesky decomposition, construct an $O(n)$ algorithm to solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Express the number of operations needed in the form of $Cn + O(1)$ with explicit C.

4. We consider a discrete 1-D Laplacian equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where
$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n\times n} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$$
   (a) Prove that $\boldsymbol{A}$ is SPD. (*Hint: Write the quadratic form into a sum of squares.*)
   (b) Since $\boldsymbol{A}$ is also tridiagonal, the algorithms in Question 3 can be applied. Write a Matlab code to implement your algorithm in Question 3(b)(c) for solving $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where A template file `spdtridiagsolver.m` is provided. Plot the solution you obtained with $n = 500$.