Math 3322 HW2 Lee Jae Yeol 20308109

1. $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$

pivot entry of the lower triangle (L) always has higher absolute values than its subcomponents.

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} 3 & 8 & 14 \\ 1 & 2 & 4 \\ 2 & 6 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 8 & 14 \\ \frac{1}{3}\ \cancel{1} & -\frac{2}{3}\ \cancel{2} & 4 \\ \frac{2}{3}\ \cancel{2} & \frac{2}{3}\ \cancel{6} & 13 \end{bmatrix}$$

$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 3 & 8 & 14 \\ \frac{1}{3} & -\frac{2}{3} & \cancel{4}\ -\frac{2}{3} \\ \frac{2}{3} & -1\ \cancel{\frac{2}{3}} & 13 \end{bmatrix}$$

$4 - \frac{14}{3} = -\frac{2}{3}$

$\frac{2}{3} \div (-\frac{2}{3}) = -1$

no row pivoting necessary

$$\rightarrow \begin{bmatrix} 3 & 8 & 14 \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -1 & \cancel{13}\ 3 \end{bmatrix}$$

$13 - \begin{bmatrix} \frac{2}{3} & -1 \end{bmatrix} \begin{bmatrix} 14 \\ -\frac{2}{3} \end{bmatrix} = 3$

$\therefore\ P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & & \\ \frac{1}{3} & 1 & \\ \frac{2}{3} & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 8 & 14 \\ & -\frac{2}{3} & -\frac{2}{3} \\ & & 3 \end{bmatrix}$

and we can check $LU = PA$.

2. Given $A \in R^{n \times n}$, decompose $A = UL$ where $U \in R^{n \times n}$ unit upper triangular
$L \in R^{n \times n}$ lower triangular.

We partition the matrix A

$$A = \begin{bmatrix} A(1:k-1, 1:k-1) & A(1:k-1, k) & A(1:k-1, k+1:n) \\ A(k, 1:k-1) & A(k,k) & A(k, k+1:n) \\ A(k+1, 1:k-1) & A(k+1, k) & A(k+1, k+1:n) \end{bmatrix}$$

$$U = \begin{bmatrix} U(1:k-1, 1:k-1) & U(1:k-1, k) & U(1:k-1, k+1:n) \\ 0 & 1 & U(k, k+1:n) \\ 0 & 0 & U(k+1:n, k+1:n) \end{bmatrix}$$

$$L = \begin{bmatrix} L(1:k-1, 1:k-1) & 0 & 0 \\ L(k, 1:k-1) & L(k,k) & 0 \\ L(k+1:n, 1:k-1) & L(k+1:n, k) & L(k+1:n, k+1:n) \end{bmatrix}$$

$$A(k, 1:k-1) = L(k, 1:k-1) + U(k, k+1:n) \cdot L(k+1:n, 1:k-1)$$
$$A(k,k) = L(k,k) + U(k, k+1:n) \cdot L(k+1:n, k)$$

$$\Rightarrow A(k, 1:k) = L(k, 1:k) + U(k, k+1:n) \cdot L(k+1:n, 1:k)$$

Since $a_{ij}$ can be overwritten by $\begin{cases} U_{ij} & \text{if } i < j \\ L_{ij} & \text{if } i \geq j \end{cases}$

$$\Rightarrow L(k, 1:k) = A(k, 1:k) - U(k, k+1:n) \cdot L(k+1:n, 1:k)$$

$$\Rightarrow \underline{A(k, 1:k) = A(k, 1:k) - A(k, k+1:n) * A(k+1:n, 1:k)}$$

For

$$A(1{:}k-1,k) = U(1{:}k-1,k)\cdot L(k,k) + U(1{:}k-1,k+1{:}n)\cdot L(k+1{:}n,k)$$

$$\Rightarrow U(1{:}k-1,k) = \Big(A(1{:}k-1,k) - U(1{:}k-1,k+1{:}n)\,L(k+1{:}n,k)\Big)\,/\,L(k,k)$$

$$\Rightarrow A(1{:}k-1,k) = \Big(A(1{:}k-1,k) - A(1{:}k-1,k+1{:}n)\cdot A(k+1{:}n,k)\Big)\,/\,A(k,k)$$

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Algorithm for $A = UL$ decomposition

```
for k = n:-1:1
    A(k, 1:k) = A(k, 1:k) - A(k, k+1:n) * A(k+1:n, 1:k);
    A(1:k-1, k) = [A(1:k-1, k) - A(1:k-1, k+1:n) · A(k+1:n, k)] / A(k,k);
end
```

Example : $A = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & \overset{2}{\cancel{3}} & 1 \\ 2\,\cancel{3} & 2\,\cancel{3} & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & \overset{2\div 2=1}{1\,\cancel{2}} & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad 5 - \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5 - 3 = 2$$

$$\rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \therefore\ U = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix},\quad L = \begin{bmatrix} 2 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{bmatrix}$$

and the algorithm coded gives the same result.

3. $A \in \mathbb{R}^{n \times n}$ a tridiagonal matrix (i.e. $a_{ij} = 0$ if $|i-j| > 1$) and SPD.

Since $A$ is a SPD tridiagonal matrix, for all $A_{ij}$ such that

$i \geq j+2$, their values should equal to 0.

Also, due to SPD, by Cholesky decomposition, $A$ can be decomposed into $A = LL^T$ where $L$ is a lower triangular matrix with positive diagonals.

Assume $L_{ij}$ for $i \geq j+2$ is nonzero.

$A_{ij} = L_{i\text{th row}} \cdot L^T_{j\text{th column}}$ ($\because A = LL^T$)

Then, since $L_{ij} \neq 0$ and $L^T_{jj}$ is always nonzero,

therefore $A_{ij \, (i \geq j+2)} \neq 0$ which contradicts that $A$ is a tridiagonal matrix.

$\therefore$ $\underline{L \text{ is bi-diagonal.}}$

b) Since in a) we proved that $L$ is a bi-diagonal matrix for the Cholesky decomposition of tridiagonal, SPD matrix $A$,

the substep calculations which were originally vector vector multiplication has reduced to scalar scalar multiplication.

for $k = 1 : n$

$A(k,k) = \left( A(k,k) - A(k,k-1)^2 \right)^{\frac{1}{2}}$

$A(k+1,k) = A(k+1,k) / A(k,k)$

end

$\Big\}$ Pseudo code Concept

for k = 1:n

Actual Matlab Code

```
if k == 1
    A(k,k) = (A(k,k))^(1/2)
    A(k+1,k) = A(k+1,k) / A(k,k)
```

Complexity

$1+1=2$

```
elseif k == n
    A(k,k) = (A(k,k) - A(k,k-1)^2)^(1/2)
```

$1+1+1=3$

```
else
    A(k,k) = (A(k,k) - A(k,k-1)^2)^(1/2)
    A(k+1,k) = A(k+1,k) / A(k,k)
end
```

$4(n-2)$

$= 4n-3$

end

$\therefore \boxed{4n + O(1)}$

c) $Ax = b$

Continuing from 3. b),

```
% forward substitution
for i = 1:n
    if i == 1
        b(i) = b(i) / A(i,i)
```

$1$

```
    else
        b(i) = (b(i) - A(i,i-1) . b(i-1)) / A(i,i)
    end
end
```

$3 \cdot (n-1)$

$= 3n-2$

```
% back ward substitution
for i = n:-1:1
    if i == n
        b(i) = b(i) / A(i,i)
```

$1$

```
    else
        b(i) = (b(i) - b(i+1) . A(i,i+1)) / A(i,i)
    end
end
```

$3(n-1)$

$= 3n-2$

$\therefore$ Total : $4n-3 + 3n-2 + 3n-2 = 10n-7 \approx 10n + O(1)$

4. Discrete 1-D Laplacian Equation $Ax = b$

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \in R^{n\times n}, \quad b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in R^n$$

a) It can directly be shown that $A$ is a symmetric matrix (i.e. $A = A^T$)

$$x^T A x = \underset{1\times n}{\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}} \underset{n\times n}{\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

$$= \underset{1\times n}{\begin{bmatrix} 2x_1 - x_2, & -x_1 + 2x_2 - x_3, & -x_2 + 2x_3 - x_4 & \cdots & -x_{n-1} + 2x_n \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}$$

$$= \underbrace{2x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2}_{x_1^2 + x_1^2 - 2x_1x_2 + x_2^2 + x_2^2} - x_2x_3 - x_2x_3 + 2x_3^2 \cdots - x_{n-1}x_n + 2x_n^2$$

$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots (x_{n-1} - x_n)^2 + x_n^2$$

Since all of them are square terms, $x^T A x > 0$ for any $x \in R^n$ and $x \neq 0$.

$\therefore$ Since $A = A^T$ (symmetric) and $x^T A x > 0$ (positive definite),

$A$ is a symmetric positive definite (SPD) matrix.

b)