MATH3322 Matrix Computation
Homework 5

Due date: 26 April, Monday

1. Compute an eigenvalue decomposition of $\boldsymbol{A}$ as follows

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

2. Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a non-symmetric matrix with eigenvalues λ_i, $i = 1, \ldots, n$ satisfying

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|.$$

   Prove that λ_1 is real. (*Hint: Eigenvalues are roots of a polynomial of real coefficients, so complex eigenvalues are in conjugate pairs.*)

3. Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a non-symmetric matrix with eigenvalues λ_i, $i = 1, \ldots, n$ satisfying

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|.$$

   Prove that λ_1 is real.

4. In this problem, we convert the eigenvalue decomposition of a complex Hermitian matrix to the eigenvalue decomposition of a real matrix. Let $\boldsymbol{C}$ be an $n \times n$ complex matrix. Define $\boldsymbol{C}^* = \overline{\boldsymbol{C}^T}$ be the conjugate transpose of $\boldsymbol{C}$. Assume that $\boldsymbol{C}^* = \boldsymbol{C}$ (i.e., $\boldsymbol{C}$ is Hermitian).

   (a) Prove that all eigenvalues of $\boldsymbol{C}$ are real.

   (b) Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n \times n}$ be the real and imaginary parts of $\boldsymbol{C}$ respectively, i.e., $\boldsymbol{C} = \boldsymbol{A} + i\boldsymbol{B}$. Prove that $\boldsymbol{C}$ is Hermitian if and only if the following matrix $\boldsymbol{M}$ is symmetric

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix}$$

   (c) Show that if $\lambda \in \mathbb{R}$ is an eigenvalue of $\boldsymbol{C}$, then λ is also an eigenvalue of $\boldsymbol{M}$.

   (d) Let $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y} \in \mathbb{C}^n$ with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ be an eigenvector of $\boldsymbol{C}$ associated to the eigenvalue λ. Construct two orthogonal eigenvectors of $\boldsymbol{M}$ of the same eigenvalue λ.

5. Write a program of power iteration. Use the criteria $\|A\boldsymbol{y}^{(k)} - \mu^{(k)}\boldsymbol{y}^{(k)}\|_2 \leq \epsilon$ to stop the iteration. Check that a random initialization will converge to the same solution $(\lambda_1, \pm\boldsymbol{x}_1)$ almost surely. A sample code is `PowerIter.m` provided.