Math 3322 HW 1 Lee Jae Yeol 20308109

1. $Ax = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 6 \\ 15 \end{bmatrix}}$

$A^T = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}^T = \underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}$

$A^T y = \underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 14 \\ 19 \\ 24 \end{bmatrix}}$

$AB^T = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}}$

2. Lets denote the shape of $(A_{pq}) \in R^{\alpha\times\beta}$

$$A_{ij} = \begin{cases} (A_{11})_{ij} & \text{if } i \le \alpha_1 \text{ and } j \le \beta_1 \\ (A_{12})_{i,j-\beta_1} & \text{if } i \le \alpha_1 \text{ and } j > \beta_1 \\ \vdots & \\ (A_{1q})_{i,j-\beta_1-\beta_2\cdots\beta_{q-1}} & \text{if } i \le \alpha_1 \text{ and } j > \beta_{q-1} \\ (A_{p1})_{i-\alpha_1-\alpha_2\cdots\alpha_{p-1},j} & \vdots \\ \vdots & \\ (A_{pq})_{i-\alpha_1-\alpha_2\cdots\alpha_{p-1},j-\beta_1-\beta_2\cdots-\beta_{q-1}} & \end{cases}$$

Since $A = [a_{ij}]_{ij} \iff A^T = [a_{ij}]_{j,i}$

$$A_{ij} = \begin{cases} (A_{11})_{j,i} & \text{if } i \le \alpha_1 \text{ and } j \le \beta_1 \\ (A_{12})_{j,i-\beta_1} & \text{if } i \le \alpha_1 \text{ and } j > \beta_1 \\ \vdots & \\ (A_{1q})_{j,i-\beta_1-\beta_2\cdots\beta_{q-1}} & \text{if } i \le \alpha_1 \text{ and } j > \beta_{q-1} \\ (A_{p1})_{j-\alpha_1-\alpha_2\cdots\alpha_{p-1},i} & \vdots \\ \vdots & \\ (A_{pq})_{j-\alpha_1-\alpha_2\cdots\alpha_{p-1},i-\beta_1-\beta_2\cdots-\beta_{q-1}} & \end{cases}$$

Therefore

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & & & \vdots \\ \vdots & & & \vdots \\ A_{p1} & - & \cdots & A_{pq} \end{bmatrix}^T = \begin{bmatrix} A_{11}^T & A_{21}^T & \cdots & A_{p1}^T \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ A_{1q}^T & & \cdots & A_{pq}^T \end{bmatrix}$$

% code attached

3. The missing code is `c(i) = c(i) + A(i,j) *b(j)`

and the result is :

```
>> LeeJaeYeol20308109Hw1
ij loop, time: 1.9984 seconds
ji loop, time: 0.60653 seconds
build-in function, time: 0.041393 seconds
>> LeeJaeYeol20308109Hw1
ij loop, time: 2.0824 seconds
ji loop, time: 0.5714 seconds
build-in function, time: 0.044018 seconds
```

4. $C = AB$ $\quad A, B \in \mathbb{R}^{n \times n}$ $\quad$ standard needs $2n^3$ flops.

a) Decompose matrix multiplication $C = AB$ into 2×2 block matrix multiplication.

$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j}$

For $C_{11} = A_{11}B_{11} + A_{12}B_{21} = 2\left(\frac{n}{2}\right)^3 + 1 \cdot \left(\frac{n}{2}\right)^2 + 2\left(\frac{n}{2}\right)^3 = \frac{n^3}{2} + \frac{n^2}{4}$

$\left(\frac{n}{2} \times \frac{n}{2}\right)$ $\quad \left(\frac{n^3}{2} + \frac{n^2}{4}\right) \cdot 4 = 2n^3 + n^2 = O(n^3)$

we need 2 sub-matrix multiplication and 1 addition.

Therefore, for all $C_{11}, C_{12}, C_{21}, C_{22}$,

we need 8 sub-matrix multiplication and 4 addition.

b)

	sub-matrix multiplication		sub-matrix addition	
P_1	1		2	
P_2	1		1	
P_3	1		1	
P_4	1		1	
P_5	1		1	
P_6	1		2	
P_7	1		2	
C_{11}	0		3	
C_{12}	0		1	
C_{21}	0		1	
C_{22}	0	/total = 7	3	/total = 18

c) For 4.a), $C_{11} = A_{11}B_{11} + A_{12}B_{21} = 2\left(\frac{n}{2}\right)^3 + 1 \cdot \left(\frac{n}{2}\right)^2 + 2\left(\frac{n}{2}\right)^3 = \frac{n^3}{2} + \frac{n^2}{4}$

$\left(\frac{n}{2} \times \frac{n}{2}\right)$ $\quad$ multiplication $\quad$ addition

Since we have $C_{11}, C_{12}, C_{21}, C_{22}$, $\quad 4 \cdot \left(\frac{n^3}{2} + \frac{n^2}{4}\right) = 2n^3 + n^2 = O(n^3)$

flops needed

For 4.b) Since all of them have size $\left(\frac{n}{2} \times \frac{n}{2}\right)$,

$2\left(\frac{n}{2}\right)^3 \times 7 + \left(\frac{n}{2}\right)^2 \times 18 = \frac{7n^3}{4} + \frac{9n^2}{2}$

flops needed

5. $f(x) = \sum_{i=1}^{n/2} x_{2i-1} \cdot x_{2i} \quad \forall x \in \mathbb{R}^n$

a) $\sum_{i=1}^{n/2} (x_{2i-1} + y_{2i})(x_{2i} + y_{2i-1}) - f(x) - f(y)$

$= \sum_{i=1}^{n/2} x_{2i-1} \cdot x_{2i} + x_{2i-1} \cdot y_{2i-1} + y_{2i} \cdot x_{2i} + y_{2i} \cdot y_{2i-1} - \sum_{i=1}^{n/2} x_{2i-1} \cdot x_{2i} - \sum_{i=1}^{n/2} y_{2i-1} \cdot y_{2i}$

$= \sum_{i=1}^{n/2} (\underbrace{x_{2i-1} \cdot y_{2i-1}}_{\text{odd num}} + \underbrace{y_{2i} \cdot x_{2i}}_{\text{even number}})$

$= \sum_{i=1}^{n} (x_i \cdot y_i)$

$= x^T y$

b) $x^T y$ is a vector vector multiplication with complexity $O(2n)$.

$x^T y = \sum_{i=1}^{n/2} (x_{2i-1} + y_{2i})(x_{2i} + y_{2i-1}) - f(x) - f(y)$

scalar addition : $1+1+1$

scalar multiplication : 1 ✓

$= \frac{n}{2}$

Since $x^T y$ happens $n \times n$ times, ($\because A, B \in \mathbb{R}^{n \times n}$)

$\therefore \frac{n}{2} \times n^2 = \frac{n^3}{2}$

$f(x) = \sum_{i=1}^{n/2} x_{2i-1} \, x_{2i} \quad \forall x \in \mathbb{R}^n$

$f(y) = \frac{n}{2}$

in a simmilar manner

scalar multiplication : 1 ✓

addition : 1

$= \frac{n}{2}$

Since $C = AB$, $A, B \in \mathbb{R}^{n \times n}$, we need n $f(x)$ and $f(y)$.

$n \times \frac{n}{2} + n \times \frac{n}{2} = n^2$

There fore, it uses only $\frac{n^3}{2} + O(n^2)$ scalar multiplications.

Algorithm

for $i = 1 : n$

  for $j = 1 : n$

    $c_{ij} = 0$

    for $k = 1 : \frac{n}{2}$

      $c_{ij} = c_{ij} + (A_{i,2k-1} + B_{2k,j})(A_{i,2k} + B_{2k-1,j})$

    end

    $c_{ij} = c_{ij} - f(A_i) - f(B_j)$

  end

end

6. Gaussian Elimination

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 4 & 5 & 3 & 2 \\ 2 & -2 & 3 & 7 \end{array}\right]$$

$A \qquad x \qquad b$

$$\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 4 & 5 & 3 & 2 \\ 2 & -2 & 3 & 7 \end{array}\right]$$

$$\begin{array}{cccc} -4 & -2 & -2 & -4 \\ 4 & 5 & 3 & 2 \\ \hline 0 & 3 & 1 & -2 \end{array}$$

$$\begin{array}{cccc} -2 & -1 & -1 & -2 \\ 2 & -2 & 3 & 7 \\ \hline 0 & -3 & 2 & 5 \end{array}$$

$$\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & 1 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & 2 & 5 \end{array}\right]$$

$$\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & 1 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$\begin{cases} 2x_1 + x_2 + x_3 = 2 \\ \quad 3x_2 + x_3 = -2 \\ \qquad 3x_3 = 3 \end{cases}$$

backward substitution

$x_1 = 1$
$x_2 = -1$
$x_3 = 1$

$$\therefore\ x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$