# UNIMODULAR COVERS OF 3-DIMENSIONAL PARALLELEPIPEDS AND CAYLEY SUMS

ABSTRACT. We show that the following classes of lattice polytopes have unimodular covers, in dimension three: parallelepipeds, smooth centrally symmetric polytopes, and Cayley sums $\mathrm{Cay}(P, Q)$ where the normal fan of $Q$ refines that of $P$. This improves results of Beck et al. (2018) and Haase et al. (2008) where the last two classes were shown to be IDP.

## 1. INTRODUCTION

A lattice polytope $P \subset \mathbb{R}^d$ has the *integer decomposition property* if for every positive integer $n$, every lattice point $p \in nP \cap \mathbb{Z}^d$ can be written as a sum of $n$ lattice points in $P$. We abbreviate this by saying that "$P$ is IDP". Being IDP is interesting in the context of both enumerative combinatorics (Ehrhart theory) and algebraic geometry (projective normality of toric varieties). It falls into a hierarchy of several properties each stronger than the previous one; see, e.g., [3, Section 2.D], [10, Sect. 1.2.5], [13, p. 2097], [14, p. 2313]. Let us here only mention that

$P$ has a unimodular triangulation $\Rightarrow$ $P$ has a unimodular cover $\Rightarrow$ $P$ is IDP.

Remember that a *unimodular triangulation* is a triangulation of $P$ into unimodular simplices, and a *unimodular cover* is a collection of unimodular simplices whose union equals $P$.

Oda [15] posed several questions regarding smoothness and the IDP property for lattice polytopes. Following [9, 19], we say that a pair $(P, Q)$ of lattice polytopes has the integer decomposition property, or that *the pair* $(P, Q)$ *is IDP*, if

$$(P + Q) \cap \mathbb{Z}^d = P \cap \mathbb{Z}^d + Q \cap \mathbb{Z}^d,$$

where $A + B := \{a + b : a \in A, b \in B\}$ denotes the *Minkowski sum* of two sets $A, B \subset \mathbb{R}^d$.

A lattice polytope $Q$ is called *smooth* if it is simple and the primitive edge directions at every vertex form a linear basis for the lattice; equivalently, if the projective toric variety defined by the normal fan of $Q$ is smooth. The following versions of Oda's questions are now considered conjectures [11, 14], and they are open even in dimension three:

**Conjecture 1.1.**
1. (Related to problems 2 and 5 in [15]) Every smooth lattice polytope is IDP.
2. (Related to problems 1, 3, 4, 6 in [15]) Every pair $(P, Q)$ of lattice polytopes with $Q$ smooth and the normal fan of $Q$ refining that of $P$ is IDP.

When the normal fan of a polytope $Q$ refines that of another polytope $P$, as in the second conjecture, we say that $P$ *is a weak Minkowski summand of* $Q$, since this is easily seen to be equivalent to the existence of a polytope $P'$ such that $P + P' = kQ$ for some dilation constant $k > 0$. This property has the following algebraic implication for the projective toric variety $X_Q$: $P$ is a weak Minkowski summand of $Q$ if and only if the Cartier divisor defined by $P$ on $X_Q$ is *numerically effective*, or "nef" (see [5, Cor. 6.2.15, Thm. 6.3.12], but observe that what we here call "weak Minkowski summand" is called "$\mathbb{N}$-Minkowski summand" there).

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Motivated by these and other questions, several authors have studied the IDP property for different classes of lattice polytopes, with special attention to dimension 3 (in dimension 2 it is straightforward that every lattice polygon has unimodular triangulations). For example, very recently Beck et al. [1] proved that all smooth centrally symmetric 3-polytopes are IDP. More precisely, they show that any such polytope can be covered by lattice parallelepipeds (affine images of 3-cubes) and unimodular simplices, both of which are trivially IDP. In Section 2 we show:

**Theorem 1.2.** *Every 3-dimensional lattice parallelepiped has a unimodular cover.*

This, together with the mentioned result from [1], gives:

**Corollary 1.3.** *Every smooth centrally symmetric lattice 3-polytope has a unimodular cover.* □

These results leave open the following important questions regarding parallelotopes:

**Question 1.4.** Do 3-dimensional parallelepipeds have unimodular triangulations?

**Question 1.5.** Higher dimensional parallelotopes (affine images of cubes) are IDP. Do they have unimodular covers?

The two-dimensional case of Conjecture 1.1(2) is known to hold, with three different proofs by Fakhruddin [7], Ogata [16] and Haase et al. [11]. This last one actually shows that smoothness of $Q$ is not needed. In dimension three, however, the conjecture fails without the smoothness assumption. Indeed, if we let $P = Q$ be any non-unimodular *empty tetrahedron*, then $P$ is obviously a weak Minkowski summand of $Q$ but the pair $(P, Q)$ is not IDP. By an empty tetrahedron we mean a lattice tetrahedron containing no lattice points other than its vertices (see the proof of Lemma 2.2 for a classification of them).

An alternative approach to Conjecture 1.1(2) is via Cayley sums, which we discuss in Section 3. Recall that the *Cayley sum* of two lattice polytopes $P, Q \subset \mathbb{R}^d$ is the lattice polytope

$$\mathrm{Cay}(P, Q) := \mathrm{conv}\,(P \times \{0\} \cup Q \times \{1\}) \subset \mathbb{R}^{d+1}.$$

We normally require $\mathrm{Cay}(P, Q)$ to be full-dimensional (otherwise we can delete coordinates) but $P$ or $Q$ do not necessarily need to be full-dimensional. We only require the linear subspaces parallel to them to span $\mathbb{R}^d$.

As we note in Proposition 3.1, if the Cayley sum of $P$ and $Q$ is IDP then the pair $(P, Q)$ is IDP. Hence, the following statement proved in Section 3 is stronger than the afore-mentioned result of [7, 11, 16]:

**Theorem 1.6.** *Let $Q$ be lattice polygon, and $P$ a weak Minkowski summand of $Q$. Then the Cayley sum $\mathrm{Cay}(P, Q)$ has a unimodular cover.*

This has the following consequence, also proved in Section 3. Here a *prismatoid* is a polytope whose vertices all lie in two parallel facets.

**Corollary 1.7.** *Every smooth 3-dimensional lattice prismatoid has a unimodular cover.*

Let us mention that recent work of Gubeladze [8] shows another class of 3-polytopes admitting unimodular covers: the convex hulls of all lattice points inside an ellipsoid; these had previously been shown to be IDP by Bruns, Gubeladze and Michałek [4]. To date there are no known examples of IDP 3-polytopes without a unimodular cover, although such polytopes exist in higher dimension [2].

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## 2. Parallelepipeds

The main tool for the proof of Theorem 1.2 is what we call the parallelepiped circumscribed to a given tetrahedron, defined as follows:

**Definition 2.1.** Let $T$ be a tetrahedron with vertices $p_1$, $p_2$, $p_3$, and $p_4$. Consider the points $q_i = \frac{1}{2}(p_1 + p_2 + p_3 + p_4) - p_i$ for each $i \in \{1, 2, 3, 4\}$, and let

$$C(T) = \operatorname{conv}(p_i, q_i : i \in \{1, 2, 3, 4\}).$$

$C(T)$ is a parallelepiped with facets $\operatorname{conv}(p_i, p_j, q_k, q_l)$ for all choices of $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We call it the *parallelepiped circumscribed* to $T$.

For each $i \in \{1, 2, 3, 4\}$, let $T_i = \operatorname{conv}(q_i, p_j, p_k, p_l)$, with $\{i, j, k, l\} = \{1, 2, 3, 4\}$; we call the $T_i$ *corner tetrahedra* of $C(T)$. Together with $T$ they triangulate $C(T)$.

Modulo an affine transformation, the situation of $T$ and $C(T)$ is exactly that of the regular tetrahedron inscribed in a cube; see Figure 1.

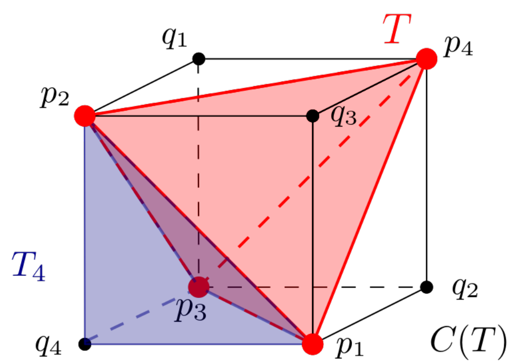


Figure 1. In red we have a tetrahedron $T$, in black its circumscribed parallelepiped $C(T)$, and in blue the corner simplex $T_4$.

Observe that the points $q_i$ need not be lattice points. However, the following lemma shows that we can find lattice points in each corner tetrahedron.

**Lemma 2.2.** *Let $T = \operatorname{conv}\{p_1, p_2, p_3, p_4\}$ be an empty lattice tetrahedron that is not unimodular. Let $C(T)$ be the parallelepiped circumscribed to $T$ and let $T_1, T_2, T_3$ and $T_4$ be the corresponding corner tetrahedra in $C(T)$. Then, every $T_i$ contains at least one lattice point different from $\{p_1, \ldots, p_4\}$.*

*Proof.* By White's classification of empty tetrahedra ([20], see also, e. g. [10, Sect. 4.1]), there is no loss of generality in assuming $T = \operatorname{conv}(p_1, p_2, p_3, p_4)$ with

$$p_1 = (0, 0, 0), \quad p_2 = (1, 0, 0), \quad p_3 = (0, 0, 1), \quad p_4 = (a, b, 1).$$

where $b \geq 2$ is the (normalized) volume of $T$, and $a \in \{1, \ldots, b-1\}$ satisfies $\gcd(a, b) = 1$. This gives

$$q_1 = \left(\frac{1+a}{2}, \frac{b}{2}, 1\right), \qquad q_2 = \left(\frac{a-1}{2}, \frac{b}{2}, 1\right),$$
$$q_3 = \left(\frac{1+a}{2}, \frac{b}{2}, 0\right), \qquad q_4 = \left(\frac{1-a}{2}, -\frac{b}{2}, 0\right).$$

Then, the inequalities $b \geq 1 + a \geq 2$ imply:

$$u := (1, 1, 0) \in \operatorname{conv}(p_1, p_2, q_3) \subset T_3, \quad v := (0, -1, 0) \in \operatorname{conv}(p_1, p_2, q_4) \subset T_4.$$

Observe that $u + v = p_1 + p_2 = q_3 + q_4$. Now, this implies that the quadrilateral $\operatorname{conv}(p_1, q_4, p_2, q_3)$ contains a fundamental domain for the lattice $\mathbb{Z}^2 \times \{0\}$. Hence, its translate $\operatorname{conv}(q_2, p_3, q_1, p_4)$ contains a fundamental domain for $\mathbb{Z}^2 \times \{1\}$ and, in particular, it contains at least one lattice point other than $p_3$ and $p_4$. By central symmetry around its center $\left(\frac{a}{2}, \frac{b}{2}, 1\right)$, $\operatorname{conv}(q_2, p_3, q_1, p_4)$ must contain lattice points in both triangles $\operatorname{conv}(q_2, p_3, p_4) \subset T_1$ and $\operatorname{conv}(q_1, p_3, p_4) \subset T_2$.

□

**Lemma 2.3.** *Let $P$ be a lattice parallelepiped and let $T \subset P$ be a tetrahedron. Then, at least one of the four corner tetrahedra $T_i$ of the circumscribed parallelepiped $C(T)$ is fully contained in $P$.*

*Proof.* Let us denote the vertices of $T$ by $p_1, p_2, p_3, p_4$ and the vertices of $C(T)$ not in $T$ by $q_1, q_2, q_3, q_4$, with the conventions of Definition 2.1.

We call *band* any region of the form $f^{-1}([\alpha, \beta])$ for some linear functional $f \in (\mathbb{R}^3)^* \setminus \{0\}$ and closed interval $[\alpha, \beta] \subset \mathbb{R}$. We claim that any band containing $T$ must contain at least three of the $q_i$s. This claim implies that the parallelepiped $P$, which is the intersection of three bands, contains at least one of the $q_i$s and hence it fully contains the corresponding $T_i$.

To prove the claim, suppose that $q_1 \notin B := f^{-1}([\alpha, \beta])$ for a certain band $B \supset T$. Without loss of generality, say $f(q_1) < \alpha$. Then the equalities $q_1 + q_i = p_j + p_k$ and $q_1 + p_1 = q_i + p_i$, where $\{i, j, k\} = \{2, 3, 4\}$, respectively give:

$$f(q_i) = f(p_j + p_k - q_1) = f(p_j) + f(p_k) - f(q_1) > 2\alpha - \alpha = \alpha, \tag{1}$$

$$f(q_i) = f(q_1 + p_1 - p_2) = f(q_1) + f(p_1) - f(p_i) < \alpha + \beta - \alpha = \beta, \tag{2}$$

so that $q_i \in B$ for $i \in \{2, 3, 4\}$. This concludes the proof of the claim, and of the lemma.

□

**Corollary 2.4.** *Let $T$ be an empty lattice tetrahedron contained in a lattice parallelepiped $P$. Then, $T$ can be covered by unimodular tetrahedra contained in $P$.*

*Proof.* We proceed by induction on the (normalized) volume of $T$, which is a positive integer. If this volume equals 1 then $T$ is unimodular and there is nothing to prove, so we assume $T$ is not unimodular. Let $p_1, p_2, p_3, p_4$ denote the vertices of $T$.

Lemma 2.3 guarantees that one of the corner tetrahedra $T_i$ of the parallelepiped $C(T)$ is contained in $P$. Without loss of generality, suppose $T_4 = \operatorname{conv}(p_1, p_2, p_3, q_4)$ is in $P$. By Lemma 2.2, we know that $T_4$ contains a lattice point other than the $p_i$s, which we denote by $u$. Then $S = \operatorname{conv}(T \cup \{u\})$ can be triangulated in two different ways: $S = T \cup T_4'$, where $T_4' = \operatorname{conv}(p_1, p_2, p_3, u) \subseteq T_4$ and $S = S_1 \cup S_2 \cup S_3$, with

$$S_1 = \operatorname{conv}(p_2, p_3, p_4, u), S_2 = \operatorname{conv}(p_1, p_3, p_4, u), S_3 = \operatorname{conv}(p_1, p_2, p_4, u).$$

Each of the tetrahedra $S_i$ has lattice volume strictly smaller than $T$ because, for each $i$, $p_i$ is the unique point of $C(T)$ maximizing the distance to the opposite facet $\operatorname{conv}(p_j, p_k, p_l)$ of $T$. Thus, $S_1$, $S_2$ and $S_3$ cover $T$ and have volume strictly smaller than $T$. The $S_i$ may not be empty, but we can triangulate them into empty tetrahedra, which by inductive hypothesis can be covered unimodularly. □

*Proof of Theorem 1.2.* Arbitrarily triangulate the parallelepiped into empty lattice tetrahedra and apply Corollary 2.4 to these tetrahedra. □

Let us say that a lattice 3-polytope $P$ *has the circumscribed parallelepiped property* if it satisfies the conclusion of Lemma 2.3: "for every empty tetrahedron $T$ contained in $P$ at least one of the four corner tetrahedra in $C(T)$ is contained in $P$". If this holds then $P$ has a unimodular cover, since then the proofs of Corollary 2.4 and Theorem 1.2 work for $P$. In turn, our proof that parallelepipeds have the property (Lemma 2.3) is based on the fact that they have only three (pairs of) normal vectors. In the following two examples we show a smooth 3-polytope and two 3-polytopes with four normal vectors that do not have the property. The latter are not IDP:

**Example 2.5** (A smooth 3-polytope without the circumscribed parallelepiped property)**.** Let $P$ be the Cayley embedding of a long horizontal rectangle and a long vertical rectangle. That is, $P = \operatorname{conv}([0,a]\times[0,1]\times\{0\}\cup[0,1]\times[0,b]\times\{1\})$, for big $a$ and $b$. This is smooth and contains a big empty tetrahedron $T$ with vertices $(0,0,0)$, $(a,1,0)$, $(0,0,1)$, $(1,b,1)$ which occupies most of its volume. In particular, none of the corner tetrahedra of $T$ is contained in $P$.

More explicitly, the remaining vertices $q_i$ of the circumscribed parallelepiped are $(\frac{a+1}{2},\frac{b+1}{2},1),(\frac{a+1}{2},\frac{b+1}{2},0),(\frac{1-a}{2},\frac{b-1}{2},1),(\frac{a-1}{2},\frac{1-b}{2},1)$. None of these points are contained in $P$, and therefore none of the corner tetrahedra are either.

**Example 2.6** (Non-IDP polytopes with four facet directions)**.** The following triangular prism $P$ and centrally symmetric octahedron $Q$ are not IDP:

$$P = \operatorname{conv}((0,1,1),(1,0,1),(1,1,0),(-1,0,0),(0,-1,0),(0,0,-1)), \tag{3}$$

$$Q = \operatorname{conv}((0,1,1),(1,0,1),(1,1,0),(0,-1,-1),(-1,0,-1),(-1,-1,0)). \tag{4}$$

Indeed, in both cases the point $(1,1,1)$ lies in the second dilation but is not the sum of two lattice points in the polytope.

An affirmative answer to the following question (weaker than the circumscribed parallelepiped property) would still imply that smooth 3-polytopes can be unimodularly covered and, hence, Conjecture 1.1(1) in dimension three:

**Question 2.7.** If $T$ is an empty tetrahedron contained in a smooth 3-polytope $P$, can one guarantee that there is a lattice point of $P$ in the circumscribed parallelepiped of $T$ (apart of the vertices of $T$)?

## 3. Cayley sums

Let $P$ and $Q$ be two lattice polytopes in $\mathbb{R}^d$. We do not require them to be full-dimensional, but we assume their Minkowski sum is. Remember that the *Minkowski sum* $P+Q$ and the *Cayley sum* of $P$ and $Q$ are defined as:

$$P+Q := \{p+q\in\mathbb{R}^d : p\in P, q\in Q\}\subset\mathbb{R}^d,$$
$$\operatorname{Cay}(P,Q) = \operatorname{conv}(P\times\{0\}\cup Q\times\{1\})\subset\mathbb{R}^{d+1}.$$

The so-called *Cayley Trick* is the isomorphism

$$2\operatorname{Cay}(P,Q)\cap(\mathbb{R}^d\times\{1\}) = (P+Q)\times\{1\}\cong P+Q,$$

which easily implies:

**Proposition 3.1** (see, e.g. [19, Thm. 0.4])**.** *If* $\operatorname{Cay}(P,Q)$ *is IDP then the pair* $(P,Q)$ *is IDP.*

The Cayley Trick also provides the following canonical bijections:

| | | |
|---|---|---|
| polyhedral subdivisions of $\operatorname{Cay}(P,Q)$ | $\leftrightarrow$ | mixed subdivisions of $P+Q$ |
| triangulations of $\operatorname{Cay}(P,Q)$ | $\leftrightarrow$ | fine mixed subdivisions of $P+Q$ |
| unimodular simplices in $\operatorname{Cay}(P,Q)$ | $\leftrightarrow$ | unimodular prod-simplices in $P+Q$. |

See [6] for more details on the Cayley Trick and on triangulations and polyhedral subdivisions of polytopes. In fact these bijections can be taken as definitions of the objects in the right-hand sides. In particular, we call *prod-simplices* in $P+Q$ the Minkowski sums $T_1+T_2$ where $T_1\subset P$ and $T_2\subset Q$ are simplices with complementary affine spans. A prod-simplex is *unimodular* if the union of edge-vectors from any vertex of $T_1$ and any vertex of $T_2$ form a unimodular basis.

We now turn our attention to $d=2$, in order to prove Theorem 1.6. A triangulation of $\operatorname{Cay}(P,Q)\subset\mathbb{R}^3$ consists of tetrahedra of types $(1,3)$, $(2,2)$ and $(3,1)$, where the type denotes how many vertices they have in $P$ and in $Q$. Empty tetrahedra of types $(1,3)$ or $(3,1)$, which are Cayley sums of an empty (hence unimodular) triangle in $P$ and a point in $Q$, or viceversa, are automatically unimodular. The case that we need to study are therefore tetrahedra of type $(2,2)$, which are Cayley sums of a segment $p \subset P$ and a segment $q \subset Q$. These correspond to prod-simplices of two segments in $P+Q$, which are *parallelograms*. The following lemma, whose proof we postpone to Section 4, is crucial to understand how to unimodularly cover these tetrahedra.

We use the following conventions: if $a, b$ are points, we denote by $[a,b]$ and $(a,b)$ respectively the closed and open line segments with endpoints $a, b$. Given a segment $s = [a,b]$, we denote $\overrightarrow{s}$ the vector $b-a$ and denote $\langle \overrightarrow{s} \rangle$ the line spanned by $\overrightarrow{s}$. A lattice parallelogram $p+q$ is called *unimodular* if it is a fundamental domain for the lattice. Equivalently, if $\mathrm{Cay}(p,q)$ is a unimodular tetrahedron.

**Lemma 3.2.** *Let $Q \subset \mathbb{R}^2$ be a two-dimensional lattice polytope and $P \subset \mathbb{R}^2$ a weak Minkowski summand of it. Let $p = [p_1, p_2] \subset P$ and $q = [q_1, q_2] \subset Q$ be two primitive and non-parallel lattice segments, and let $\langle \overrightarrow{p} \rangle$ and $\langle \overrightarrow{q} \rangle$ be the lines spanned by them. If the parallelogram $p+q$ is not unimodular, then at least one of the regions*

$$((p_1, p_2) + \langle \overrightarrow{q} \rangle) \cap P, \qquad \text{and} \qquad ((q_1, q_2) + \langle \overrightarrow{p} \rangle) \cap Q$$

*contains a lattice point.*

See Figure 2 for an illustration of the two regions in the statement, which we call *strips*. In this figure and the forthcoming ones in Section 4 we draw $p$ as a vertical segment and $q$ as a horizontal one for convenience. This is always possible via a linear transformation (which of course changes the lattice; in the proof we do not assume the lattice to be $\mathbb{Z}^2$).

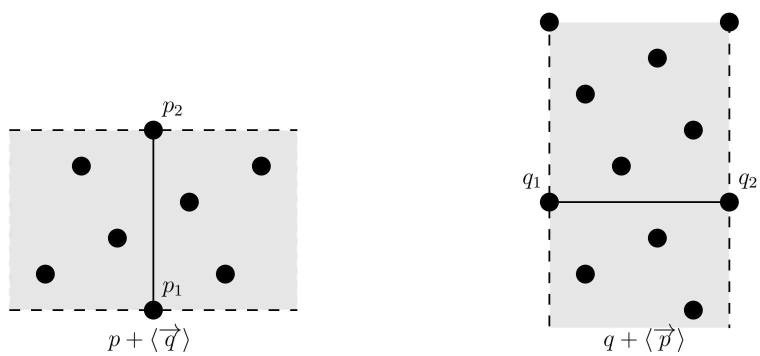


FIGURE 2. The strips of Lemma 3.2

**Corollary 3.3.** *Let $T$ be an empty lattice tetrahedron contained in the Cayley sum* $\mathrm{Cay}(P,Q)$*, where $Q$ is a lattice polygon and $P$ is a weak Minkowski summand of $Q$. Then, $T$ can be covered by unimodular tetrahedra contained in* $\mathrm{Cay}(P,Q)$*.*

*Proof.* The proof is by induction on the normalized volume of $T$, which we assume to be at least 2. This implies that $T$ is of type $(2,2)$, since empty tetrahedra of types $(1,3)$ and $(3,1)$ are unimodular. Thus, $T$ is the Cayley sum of primitive segments $p = [p_1, p_2] \subset P$ and $q = [q_1, q_2] \subset Q$. Let $u$ be the lattice point whose existence is guaranteed by Lemma 3.2. Assume (the other case is similar) that

$$u \in ((p_1, p_2) + \langle \overrightarrow{q} \rangle) \cap P,$$

and call $t$ the triangle $t = \mathrm{conv}\,(u, p_1, p_2) \subset P$.

Let us denote $\tilde{u}$, $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{q}_1$, $\tilde{q}_2$ the points corresponding to $u, p_1, p_2, q_1, q_2$ in $\mathrm{Cay}(P,Q)$. That is, $\tilde{p}_i = p_i \times \{1\}$, $\tilde{q}_i = q_i \times \{0\}$, and $\tilde{u} = u \times \{1\}$. Observe that the assumption $u \in ((p_1, p_2) + \langle \overrightarrow{q} \rangle$ implies that one of the segments $[\tilde{u}, \tilde{q}_i]$ crosses the interior of one of the triangles $\operatorname{conv}(\tilde{p}_1, \tilde{p}_2, \tilde{q}_j)$, where $\{i, j\} = \{1, 2\}$. Without loss of generality assume that $[\tilde{u}, \tilde{q}_2]$ crosses $\operatorname{conv}(\tilde{p}_1, \tilde{p}_2, \tilde{q}_1)$, as in Figure 3.

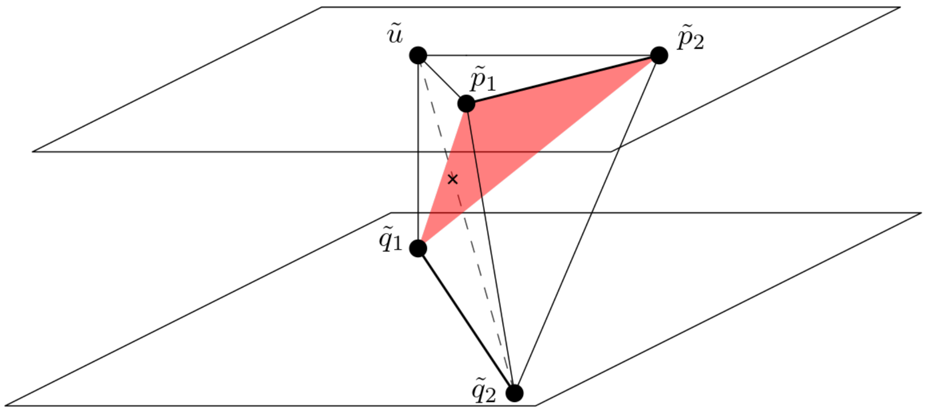


FIGURE 3. $[\tilde{u}, \tilde{q}_2]$ intersects $\operatorname{conv}(\tilde{p}_1, \tilde{p}_2, \tilde{q}_1)$

In turn, this means that the polytope $\operatorname{conv}(\tilde{u}, \tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2) = \operatorname{Cay}(t, q)$ has the following two triangulations:

$$\mathcal{T}^+ := \{\operatorname{Cay}(p, q), \operatorname{Cay}(t, \{q_1\})\},$$
$$\mathcal{T}^- := \{\operatorname{Cay}([p_1, u], q), \operatorname{Cay}([p_2, u], q), \operatorname{Cay}(t, \{q_2\})\}.$$

The tetrahedra $\operatorname{Cay}(t, \{q_1\})$ and $\operatorname{Cay}(t, \{q_2\})$ are unimodular, which implies that $T = \operatorname{Cay}(p, q)$ has volume equal to the sum of the volumes of $\operatorname{Cay}([p_1, u], q)$ and $\operatorname{Cay}([p_2, u], q)$. In particular, we have covered $T$ by the three tetrahedra in $\mathcal{T}^-$, which are of smaller volume and hence have unimodular covers by induction hypothesis. □

*Proof of Theorem 1.6.* Arbitrarily triangulate $\operatorname{Cay}(P, Q)$ into empty lattice tetrahedra and apply Corollary 3.3 to these tetrahedra. □

Let us now show how to derive Corollary 1.7 from this theorem. *Prismatoids* were defined in [17] as polytopes whose vertices all lie in two parallel facets. In particular, a *lattice prismatoid* is any $d$-polytope $SL(\mathbb{Z}, d)$-equivalent to one of the form

$$\operatorname{conv}(Q_1 \times \{0\} \cup Q_2 \times \{k\}),$$

where $Q_1, Q_2$ are lattice $(d-1)$-polytopes and $k \in \mathbb{Z}_{>0}$. This is almost a generalization of Cayley sums, which would be the case $k = 1$, except the definition of prismatoid requires $Q_1$ and $Q_2$ to be full-dimensional, while the Cayley sum only requires this for $Q_1 + Q_2$.

**Proposition 3.4.** *Let $Q_1$, $Q_2$ be two lattice polygons and consider the prismatoid*

$$P := \operatorname{conv}(Q_1 \times \{0\} \cup Q_2 \times \{k\}),$$

*with $k \geq 2$. If $P \cap (\mathbb{R}^2 \times \{1\})$ is a lattice polygon then $P$ has a unimodular cover.*

*Proof.* The condition that $P \cap (\mathbb{R}^2 \times \{1\})$ is a lattice polygon implies the same for $P \cap (\mathbb{R}^2 \times \{i\})$, for every $i$. Indeed, the condition implies that every edge of $\operatorname{Cay}(P, Q)$ of the form $[u \times \{0\}, v \times \{k\}]$ has a lattice point in $\mathbb{R}^2 \times \{i\}$, and hence it has a lattice point in $P \cap (\mathbb{R}^2 \times \{i\})$, for every $i$.

Observe that for every $i \in \{1, \ldots, k-1\}$ the intersection $P \cap (\mathbb{R}^2 \times \{i\})$ has the same normal fan as $Q_1 + Q_2$. Thus, each slice

$$P \cap (\mathbb{R}^2 \times [i-1, i])$$

is a Cayley polytope. For $i \in \{2, \dots, k-1\}$, both bases have the same normal fan (and therefore each is a weak Minkowski summand of the other); for $i \in \{1, k\}$ one base is a weak Minkowski summand of the other. We can therefore apply Theorem 1.6 to each slice and combine the covers thus obtained to get a unimodular cover of $P$. □

*Proof of Corollary 1.7.* The polytope under study satisfies the hypotheses of Proposition 3.4: the smoothness of the prismatoid implies that every edge of the form $[u \times \{0\}, v \times \{k\}]$ has lattice points in all slices. Hence,

$$kP \cap (\mathbb{R}^2 \times \{1\}) = (k-1)Q_1 + Q_2.$$ □

## 4. Proof of Lemma 3.2

Let $f_q$ be the primitive lattice functional constant on $q$ and $f_p$ the one constant on $p$. We assume that $f_q(p_1) < f_q(p_2)$ and $f_p(q_1) < f_p(q_2)$.

Since we can perform without loss of generality respective lattice translations to $P$ and to $Q$, we assume that $q_1$ is the origin and that $f_q(p_1) = -1$. Observe that then $f_q(p_2)$ must be strictly positive, since $f_q(p_2) = 0$ would imply that $p + q$ is a unimodular parallelogram. Moreover, since $p$ is primitive $p_1$ and $p_2$ cannot be in the boundary of $q + \langle \overrightarrow{p} \rangle$, which implies that $p_1$ is the unique lattice point with $f_q(x) = -1$ in $q + \langle \overrightarrow{p} \rangle$. Similarly, the unique lattice point in the strip with $f_q(x) = 1$ is $q_1 + q_2 - p_1$.

We let $H_1 = \{f_q(x) \leq 0\}$ and $H_2 = \{f_q(x) \geq 0\}$; similarly let $V_1 = \{f_p(x) \leq f_p(p)\}$ and $V_2 = \{f_p(x) \geq f_p(p)\}$.[1]

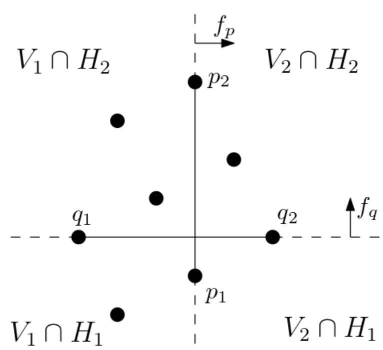


Figure 4. Setup for the proof of Lemma 3.2

Let $w = \text{area}(p+q) \geq 2$, where area denotes the area normalized to a fundamental domain. In what follows, the width of a functional $f$ on a set $S$, denoted $\text{width}_f(S)$ is defined as the difference $\sup_{x \in S} f(x) - \inf_{x \in S} f(x)$. Then:

$$w = \text{width}_{f_q}(p + \langle \overrightarrow{q} \rangle) = \text{width}_{f_q}(p) = \text{width}_{f_p}(q) = \text{width}_{f_p}(q + \langle \overrightarrow{p} \rangle).$$

*Proof of Lemma 3.2.* Suppose by contradiction that there is no lattice point as described in the lemma. In particular, no lattice point on the boundary of $Q$ can be in the interior of the strip $q + \langle \overrightarrow{p} \rangle$. Thus the boundary of $Q$ contains two primitive segments which each have one vertex on each side of the strip $q + \langle \overrightarrow{p} \rangle$; we will call these $b = [b_1, b_2], t = [t_1, t_2]$, with $b$ and $t$ crossing the strip in $H_1$ and

[1] Since in our figures $p$ and $q$ are vertical and horizontal, we use the letters $V$ and $H$ for the half-planes they defined. Similarly, later in the proof we use the letters $b$, $t$, $r$, and $l$ for certain points and segments meaning "bottom", "top", "right" and "left".

$H_2$ respectively and the convention that $f_p(b_2) > f_p(b_1)$ and $f_p(t_2) > f_p(t_1)$. This readily implies

$$\begin{aligned} f_p(t_1) &\leq f_p(q_1), \quad f_p(t_2) \geq f_p(q_2), \\ f_p(b_1) &\leq f_p(q_1), \quad f_p(b_2) \geq f_p(q_2). \end{aligned} \tag{5}$$

The same holds for $P$ and the strip $p + \langle \overrightarrow{q} \rangle$, and we call the segments $l = [l_1, l_2]$ and $r = [r_1, r_2]$, with $l$ and $r$ crossing the strip $p + \langle \overrightarrow{q} \rangle$ in $V_1$ and $V_2$ respectively. The only difference is that in the case that $P$ is one dimensional we have $l = r = p$. Again we have

$$\begin{aligned} f_q(l_1) &\leq f_q(p_1), \quad f_q(l_2) \geq f_q(p_2), \\ f_q(r_1) &\leq f_q(p_1), \quad f_q(r_2) \geq f_q(p_2). \end{aligned} \tag{6}$$

Observe that a priori one of $l$ and $r$ can coincide with $p$, if $p$ is on the boundary of $P$, and similarly one of $t, b$ might be $q$, if $q$ is on the boundary of $Q$.

**Claim 4.1.** *The following inequalities hold,*

$$\mathrm{width}_{f_q}(l), \mathrm{width}_{f_q}(r), \mathrm{width}_{f_p}(t), \mathrm{width}_{f_p}(b) \geq w.$$

*Each inequality is strict, unless the segment in question coincides with $p$ or $q$.*

*Proof.* The inequality $\geq w$ follows in each case from (6) and (5).

If one of the inequalities, say the one for $l$, is not strict, then $l$ has one endpoint on each of the boundary lines of $(p + \langle \overrightarrow{q} \rangle)$. Unless $l = p$, one of the endpoints of $l$ is not an endpoint of $p$, say $l_1 \neq p_1$. Thus the triangle $T = \mathrm{conv}(p_2, p_1, l_1)$ is contained in $P$ and its edge $[p_1, l_1]$ is an integer dilation of $q$.

Since $T$ contains $p$ and a copy of $q$, its area (normalized to a fundamental domain) is $w/2 \geq 1$, and by Pick's theorem it must contain a lattice point other than its vertices. Since $p$ and $q$ are primitive, this lattice point must lie in the interior of the strip. ☐

**Claim 4.2.** *$f_q(b_2 - b_1)$ and $f_q(t_2 - t_1)$ are non-zero and have the same sign. That is, $f_q$ achieves its maximum over $b$ and over $t$ on the same halfplane $V_1$ or $V_2$.*

*Proof.* Both $t$ and $b$ must cross the interior of $p$, or otherwise $p_1$ or $p_2$ are the lattice points we are looking for in $Q$. To seek a contradiction assume, as in Figure 5, that

$$f_q(t_2 - t_1) \leq 0 \leq f_q(b_2 - b_1).$$

That is, $f_q$ decreases (perhaps weakly) along $t$ and increases along $b$, as $f_p$ increases on both. This implies that $Q \cap V_2$ is contained in the open strip $\{f_q(p_1) < f_q(x) < f_q(p_2)\}$, of width $w$. This strip cannot contain a translated copy of $r$, since $\mathrm{width}_{f_q}(r) \geq w$, which gives a contradiction: since $P$ is a weak Minkowski summand of $Q$, $Q$ must have edges parallel to $r$ both in $V_1$ and $V_2$. ☐

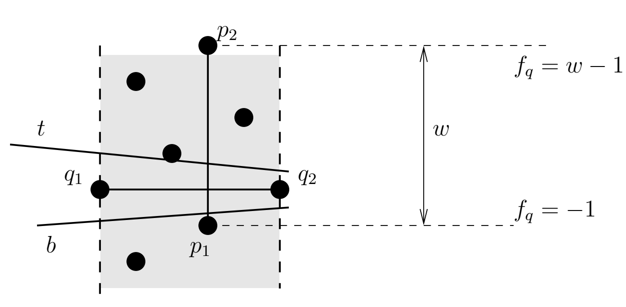


FIGURE 5. Illustration of the proof of Claim 4.2

We assume without loss of generality that the maximum on $t$ (and hence on $b$) is achieved in $V_2$, that is to say, $f_p$ and $f_q$ increase in the same direction along $t$ (and hence along $b$). Otherwise the following considerations can be applied to $V_1$.

**Claim 4.3.** *Assume without loss of generality that $b$ and $t$ either are parallel or their affine spans cross in $V_2$ (if they cross in $V_1$, the same claim can be reworded for $V_1$ and $l$). Then,*

(1) *The intersection of $Q$ with any line parallel to $p$ in $V_2$ has width with respect to $f_q$ strictly smaller than $w$.*
(2) *$f_p(r_2) > f_p(r_1)$, that is, $f_p$ achieves its maximum over $r$ in $H_2$.*

*Proof.* Both $t$ and $b$ must intersect $p$, as said in the proof of Claim 4.2. Their intersections with $p$ are thus endpoints of a segment of width with respect to $f_q$ less than $w$, the width of $p$. Since $t$ and $b$ cross in $V_2$, the same is true for any segment parallel to $p$ contained in $Q \cap V_2$.

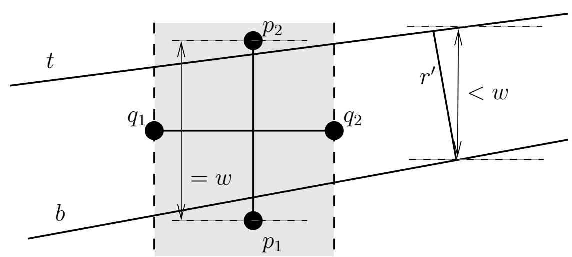


FIGURE 6. Illustration of the proof of Claim 4.3

For part (2), recall that by Claim 4.1, $\text{width}_{f_q}(r) \geq w$. If $f_p(r_2) \leq f_p(r_1)$, it would be impossible to fit a translated copy $r'$ of $r$ in the correct side of $Q$: since $f_q$ increases along $t$, $r'$ has width in direction $f_q$ smaller than the segment parallel to $p$ with endpoint $r'_1$. This segment by part (1) has width less than $w$ in direction $q$, a contradiction.

□

The last two claims can be summarized as saying that in the pictures $b$, $t$ and $r$ have positive slope. Observe that this implies that $q$ is not in the boundary of $Q$ and $p \neq r$, so both $P$ and $Q$ are full dimensional.

Let $g$ be the primitive lattice functional constant on $[p_1, q_2]$ (and therefore constant also on $[q_1, q_1 + q_2 - p_1]$). By the assumption on $p_1$, the values of $g$ on these segments differ by 1. We choose the sign of $g$ so that

$$g([p_1, q_2]) = g([q_1, q_1 + q_2 - p_1]) - 1.$$

**Claim 4.4.** *$g(t_1) > g(t_2)$, $g(b_1) > g(b_2)$, and $g(r_1) < g(r_2)$.*

*Proof.* Since $b$ and $t$ must respectively separate $p_1$ and $q_1 + q_2 - p_1$ from the other two vertices of the parallelogram $\text{conv}(q_1, p_1, q_2, q_1 + q_2 - p_1)$, they must respectively intersect its (parallel) edges $[p_1, q_2]$ and $[q_1, q_1 + q_2 - p_1]$, which implies the stated inequalities for $b$ and $t$. The same argument applied to the parallelogram $\text{conv}(p_1, q_2, p_2, p_1 + p_2 - q_2)$, yields the inequalities for $r$. □

We are now ready to show a contradiction. Since the normal fan of $Q$ refines that of $P$, $Q$ must have an edge $r'$ which is a translated copy of $r$. Let $r'_1$ and $r'_2$ be its endpoints.

Consider the segment $s$ contained in $r'_1 + \langle \overrightarrow{p} \rangle$ with endpoints $s_1 = r'_1$ on $d$ and $s_2$ on the line spanned by $t$. The width of $s$ with respect to $g$ is $g(s_2) - g(s_1) < 1$, because $s$ is shorter than $Q \cap p$, which is strictly contained between the consecutive

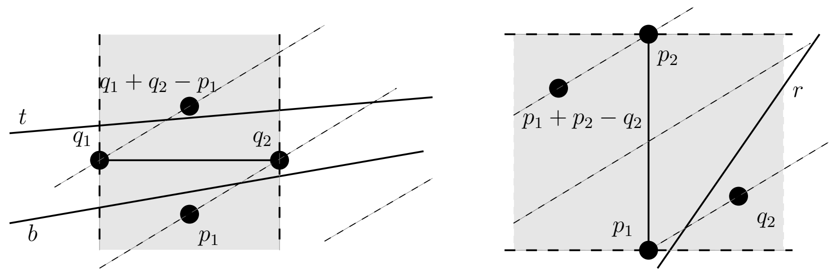


FIGURE 7. Illustration of the proof of Claim 4.4

parallel lattice lines through $p_1$ and $q_2$ and through $q_1$ and $q_1 + q_2 - p_1$ where $g$ is constant.

We now observe that $g(r_2) < g(s_2)$, because $r_2$ lies to the right of $s_2$ ($r$ has positive slope) and below the line spanned by $t$, and $g$ decreases moving to the right along $t$ (Claim 4.4). Thus $g(r_2) - g(r_1) = g(r'_2) - g(r'_1) < g(s_2) - g(s_1) < 1$, that is, $g(r_2) = g(r_1)$. This contradicts Claim 4.4. □

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