Report on "Unimodular covers of 3-dimensional parallelepipeds and Cayley sums"

1 Summary and Recommendation

This work studies unimodular covers of 3-dimensional parallelepipeds and Cayley sums. A lattice polytope $P \subset \mathbb{R}^d$ is said to have a *unimodular cover* if there exists a collection of unimodular simplices whose union coincides with P. The existence of unimodular covers is a driving question in combinatorics with important implications in toric geometry, commutative algebra, and optimization theory. Even for relatively "simple" classes of lattice polytopes (i.e., with "manageable" combinatorial complexity) it is not known if unimodular covers exists. The work under review closes this gap for 3-dimensional parallelepipeds and Cayley sums.

With work by Beck et al. [BHH+19], the proven result implies that smooth centrally symmetric 3-dimensional lattice polytopes admit unimodular covers. This statement (for the 3-dimensional centrally symmetric case) is a strengthening of Oda's famous question whether smooth lattice polytopes have the integer decomposition property [Oda08]. A lattice polytope $P \subset \mathbb{R}^d$ is said to have the *integer decomposition property* (or *IDP* for short) if for every $k \in \mathbb{Z}_{\geq 1}$ and every lattice point $x \in kP \cap \mathbb{Z}^d$ there exist lattice points $x_1, \ldots, x_k \in P \cap \mathbb{Z}^d$ such that $x = x_1 + \cdots + x_k$. It is easy to see that the existence of unimodular covers of P implies that P has the IDP. Indeed, a stronger conjecture by Oda is verified, namely whether "certain" pairs (P, Q) of lattice polytopes have the IDP. An affirmative answer is given when P is a weak Minkowski summand of a lattice polygon Q.

The proof of these beautiful results relies on the investigation of the fundamental parallelepiped circumscribed to a simplex and uses some clever arguments and constructions. The presented ideas show great potential for future work. However, small inaccuracies and gaps on top of the technical nature of the manuscript make the paper challenging to read. Furthemore, there seem to be an issue in the proof of Lemma 3.2 (see Comments 40 and 42).

Recommendation. *This work seems perfectly suited for publication in "Combinatorial Theory". I recommend to request a revision of the manuscript to fix the aforementioned issues and to improve the exposition. Hopefully, the comments below will help in this endeavour.*

2 Comments

If I were to choose three comments to address, I'd look at Comments 10, 40, and 42.

1. Page 1, line 13: this should read " algebraic geometry (projective normality of toric varieties)...".

2. Page 1, line -11 – -4: on page 5 the definition for Minkowski sum is given, however it's already used in this paragraph. I recommend moving the definition to the place where a notion/construction is used for the first time.

3. Page 1, line -5: in the (printed) version of the book that I have this is "Theorem 6.3.12".

4. Page 1, line -4: in the (printed) version of the book that I posess, it is called "$\mathbb{N}$-Minkowski summand".

5. Page 2, line 9: move the definition of parallelotopes from Question 1.5 up here.
6. Page 2, line -21: the Cayley sum $\mathrm{Cay}(P, Q)$ lies in $\mathbb{R}^{d+1}$ (as $P, Q \subset \mathbb{R}^d$).
7. Page 2, line -20 – -19: this should read "We usually require $\mathrm{Cay}(P, Q)$ to be full-dimensional (otherwise we can delete coordinates) but P or Q do not necessarily need to be full-dimensional."
8. Page 2, line -17 – -15: recall when a pair of lattice polytopes is said to have the IDP.
9. Page 3, line 2: explain the notation "$\dots i \in [4] \dots$".
10. Page 3, Lemma 2.2: why does this statement need a proof? Maybe I'm missing something, but shouldn't the q_i's be points in T_i distinct from p_1, p_2, p_3, p_4? Note that (in the notation of the proof of Lemma 2.2) for the special case $a = 1, b = 2$, one has $u = q_3$ and $v = q_4$.
11. Page 4, line 2: replace "parallelogram" with "parallelepiped".
12. Page 4, line 6: this should read $f \in (\mathbb{R}^3)^* \setminus \{0\}$.
13. Page 4, line 15: the equation should read

$$f(q_2) = f(q_1 + p_1 - p_2) = f(q_1) + (f(p_1) - f(p_2)) < \alpha + (\beta - \alpha) = \beta,$$

14. Page 4, line 19: this should read "The translation by the vector…".
15. Page 4, line 22: this should read "…This concludes the proof…".
16. Page 4, line -14 – -12: maybe it would be more straightforward to use the standard cube and study how the volume changes under the linear transformation that maps it onto the parallelepiped $C(T)$ circumscribed to T.
17. Page 4, line -10: this should read "…which by induction hypothesis can be covered unimodularly."
18. Page 5, line 1: this should read "two 3-polytopes with…".
19. Page 5, Example 2.5: list the corner tetrahedra, so that's easier to check the assertion.
20. Page 5, line 19: this should read "…covered, and hence Conjecture 1.1(1) in dimension three:"
21. Page 5, line -13: the notion "mixed IDP" is used without definition.
22. Page 5, line -4 – -3: the quantors in the definition of *unimodular* prod-simplex isn't clear. Should this be true for all vertices of T_1 resp. T_2?
23. Page 6, line 8: this should read "…closed and open line segments…".
24. Page 6, line 13: use the notion *prod-simplex* instead of parallelogram. Otherwise, prod-simplices are nowhere used.
25. Page 6, line -7: this should read "…That is, $\tilde{p}_i = p_i \times \{0\}$, $\tilde{q}_i = q_i \times \{1\} \dots$".
26. Page 6, line -6 – -5: this should read "…implies that one of the segments $[u, q_i]$ crosses one of the triangles…".

27. Page 6, line -2 – -1: this should read

$$\mathcal{T}^+ := \{\mathrm{Cay}(p, q), \mathrm{Cay}(t, \{q_j\})\},$$
$$\mathcal{T}^- := \{\mathrm{Cay}([p_1, u], q), \mathrm{Cay}([p_2, u], q), \mathrm{Cay}(t, \{q_i\})\}.$$

(That is, swap "i" and "j".)

28. Page 7, Figure 3: notice that the notation in this picture contradicts the notation from the text. For example, $\tilde{p}_i$ is assumed to lie at the "bottom" plane.

29. Page 7, line 4 – 5 below Figure 3: this should be "induction hypothesis".

30. Page 8, general comments about the proof of Lemma 3.2: it would help the reader to note that P and Q can be translated independently, so that we may assume $q_1 = 0$. Furthermore, up to a unimodular transformation, we may assume that $q_2 = (1, 0)$. However, note that the way p_1 is chosen doesn't guarantee that p_1 and p_2 lie on a common linear subspace (instead they lie on an affine line). In particular, the definition of the half-spaces V_1 and V_2 needs to be corrected.

31. Page 8, Figure 4: this picture is misleading. It's possible to assume that the line segment $[q_1, q_2]$ is horizontal, but the line segment $[p_1, p_2]$ is not guaranteed to be vertical.

32. Page 9, line 8 – 9: the last argument of the proof is more intricate than it seems. For example, I had to use Pick's theorem to see this.

33. Page 9, Proof of Claim 4.2: here we already need to use that both t and b intersect p. This is mentioned in the proof of Claim 4.3. Include this argument already in the proof of Claim 4.2.

34. Page 9, line 16: this should read "...weak Minkowski summand...".

35. Page 9, line 17: this should read "...edge parallel to r."

36. Page 9, line -19 – -1: I wouldn't use abbreviations like "w.l.o.g." or "w.r.t." in typed text.

37. Page 9, line -19 – -17: explain the two "w.l.o.g." assumptions. For example, the first "w.l.o.g." can be achieved by reflecting along the y-axis (under the above mentioned assumptions). The second "w.l.o.g." seems more in the sense of "the other case works similar".

38. Page 9, Proof of Claim 4.3(2): the punchline of the proof is that if $f_p(r_2) \leq f_p(r_1)$ then the side of Q that is parallel to r has to lie in V_2 which isn't possible by Claim 4.3(1) (note that this is "(1)" and not "(a)" as stated in the proof). I recommend revising the proof.

39. Page 9, line -3: I think it is a red herring to represent the line segment p vertically. It's not possible to assume both q horizontal and p vertical. I recommend to keep q horizontal and let p be "arbitrary".

40. Page 10, Claim 4.4: the statement seems incorrect: let us denote the line through q_1 and $q_1 + q_2 - p_1$ by d. By assumption, we have

$$g([p_1, q_2]) = g([q_1, q_1 + q_2 - p_1]) + 1 = g(d) + 1.$$

This, in particular, implies that for points x to the "left" of d (using the given illustrations), we have that $g(x) < g(d)$. Similarly for points y to the "right" of d we have $g(y) > g(d)$. As stated in the proof, t separates q_1 from $q_1 + q_2 - p_1$. From this it follows that

$$g(t_1) < g(d) < g(t_2).$$

Similarly, one can show that

$$g(b_1) < g(b_2) \qquad \text{and} \qquad g(r_1) > g(r_2).$$

41. Page 10, last line in the proof of Claim 4.4: it should read "conv$(p_1, q_2, p_2, p_1+p_2-q_2)\dots$". Furthermore, an inequality for ℓ is mentioned that doesn't appear in the statement.

42. Page 10, lines -8 – -11: if Comment 40 is correct, then the given argument needs to be revised appropriately.

43. Page 10, line -11: there is a superfluous ")", namely "...contradicts Claim 4.4)."

References

[BHH+19] Matthias Beck, Christian Haase, Akihiro Higashitani, Johannes Hofscheier, Katharina Jochemko, Lukas Katthän, and Mateusz Michał ek. Smooth centrally symmetric polytopes in dimension 3 are IDP. *Ann. Comb.*, 23(2):255–262, 2019.

[Oda08] Tadao Oda. Problems on minkowski sums of convex lattice polytopes. *arXiv:0812.1418*, 2008.