Dissipativity and Turnpike Property in a Controlled Stochastic Lotka-Volterra Model with Lvy Jumps.

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Abstract

We consider a controlled stochastic Lotka-Volterra model for a predator-two prey model with Lévy jumps. We study an optimal control problem for this model and the stability of its solutions. We discuss a type of stability for state trajectories, optimal control, and adjoint trajectories, called turnpike property. This property implies that an optimal trajectory, for most of the time, could remain in a neighborhood of a balanced equilibrium trajectory. Assuming linear growth and Lipschitz conditions on the drift and diffusion terms, applying algebraic Riccati theory and the Stochastic Maximum Principle, we express optimal control in terms of the Lvy process and the state and adjoint variables, and establish an exponential estimate of the convergence rate of the equilibrium trajectories. Furthermore, we introduce the definition of dissipativity for general stochastic systems and demonstrate the relationship between dissipativity and turnpike property for our controlled stochastic Lotka-Volterra model with Lévy jumps. We illustrate our results through a simulation with a numerical example.

Keywords: Lotka-Volterra stochastic model, maximum stochastic principle, Lévy jumps, dissipativity.

1 Introduction

In mathematical ecology, the Lotka-Volterra systems represent one of the most important models to analyze population dynamics, because they describe very well many aspects of interactions between species in competition, such as persistence, extinction and stability of its solutions, [1]. These models are more realistic when we consider natural random environmental variations, introducing Wiener processes, or even better, Lévy processes. Considering a dynamic with jumps in the model, which represent abrupt environmental changes in the ecosystem, allows for an adequate study of anomalous diffusion and nonlinearity in nature, such as volcanic eruptions, earthquakes, cyclonic storms, global warming and hazardous waste pollution, [2]. Beside, some nonlinear control systems have the following property called turnpike property: the optimal trajectory, the optimal control, and the corresponding adjoint vector remain exponentially close to a steady state. The Turnpike property of a solution of an optimal control problem means that an optimal trajectory for most of the time could stay in a neighborhood of a balanced equilibrium path, corresponding to the optimal steady-state solution. This property is a characteristic of the Turnpike theory which was introduced in 1958 in mathematical economics and recently has been applied in Control Theory in [3]. Many papers have been written about stochastic Lotka-Volterra models with jumps. In [4], the authors consider a competitive LotkaVolterra population dynamics with jumps without control functions and they investigate the sample Lyapunov exponent for each component of the solution and uniform boundedness of the pth moment with $p > 0$. Also, in [5] was studied the asymptotic convergence of a general stochastic population dynamics of the type Lotka-Volterra and driven by Lévy noise, given some important asymptotic pathwise estimation assuming different conditions over the Poisson's process coefficient, but they don't consider any control functions in the processes. In [6] we analyze the turnpike property of a stochastic controlled Lotka-Volterra system only with white noise.

Dissipativity theory is a system-theoretic concept that provides a powerful framework for the analysis and control design of open dynamical systems based on generalized system energy considerations. In particular, dissipativity theory deals with the study of the properties of dynamical systems related to the conservation, dissipation, and transport of mass and energy. For a dissipative dynamic system, the stored energy is at most equal to the sum of the initial energy stored in the system and the total energy supplied externally to the system.

The model to consider here is a controlled jump diffusion process given by the following non-linear stochastic ordinary differential equations system with initial and final conditions:

$$\begin{aligned} dx &= f(t,x(t),u(t))dt + g(t,x(t),u(t))dW(t) + x(t)u(t)\int_{\mathbb{R}^3} \gamma(t,x(t-),z)\tilde{N}(dt,dz), \\ x(t_0) &= (x_{10},x_{20},x_{30}), \end{aligned} \tag{1}$$

where $f : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is a measurable function called the drift, the process $u : \mathbb{R} \to \mathbb{R}^3$, $u(t) = (u_1(t), u_2(t), u_3(t))$, is a measurable and bounded function called the control, which belongs to a region control $U \in R^3$

and it is an adapted and cadlag function (continuous on the right and limit on the left), and $g(t,x,u)$, a measurable function defined also on $[0,T]\times\mathbb{R}^3\times\mathbb{R}^3$ and $\mathbb{R}^{3\times 3}$-valued, $g(t,x,u)=(g^1(t,x,u),g^2(t,x,u),g^3(t,x,u))$, where $g^j(t,x,u)=(g^{1j}(t,x,u),g^{2j}(t,x,u),g^{3j}(t,x,u))^\top$, $1\le j\le 3$ and for the compensated Poisson random measure $\tilde{N}(dt,dz)$, we write, according to Lévy decomposition theorem, [7], $\tilde{N}(dt,dz)=(\tilde{N}_1(dt,dz),\tilde{N}_2(dt,dz),\tilde{N}_3(dt,dz))$ and $\tilde{N}_j(dt,dz)=N_j(dt,dz)-\nu_j(dz_j)dt$, $1\le j\le 3$, with $N_j(dt,dz)$ Poisson counting measure and $x_i(t-)$ denotes the left hand limit of x at time t. Specifically, we consider the following functions $f(t,x(t),u(t))$, $g(t,x(t),u(t))$:

$$f=\begin{pmatrix} x_1(t)(1-\beta x_2(t)-\delta(t)x_3(t)-A_1u_1(t)) \\ x_2(t)(1-\beta x_1(t)-\epsilon x_3(t)-A_2u_2(t)) \\ x_3(t)(-1+\delta x_1(t)+\epsilon x_2(t)-A_3u_3(t)) \end{pmatrix}, g=\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \tag{2}$$

where $\eta,\omega,\kappa,\ \beta,\delta,\eta$ and ϵ in $(0,1]$. $u_1(t),u_2(t),u_3(t)$ are the controls with constants $A_1,A_2,A_3\in(0,1]$, standard independents Wiener processes $W_1(t),W_2(t),W_3(t)$ with parameters $\alpha_1,\alpha_2,\alpha_3\in(0,1]$, defined over a probability space $(\Omega,\mathcal{F},P)$, where $\mathcal{F}_s$ denotes the σ-algebra generated by all random variables X_i with $i\le s$; the collection of such σ-algebras forms a filter of the probability space. The class of admissible controls $\mathcal{U}$ is the set of $\mathcal{F}_s$-predictable processes with values in U. and, finally, $N(t)$ is a Poisson process independent of $W_i(t)$. In the above, as is conventional, P denotes a probability measure in the sample space Ω of the stochastic process $X:[0,T]\times\Omega\to[0,+\infty)$ and $E[X]$ denotes the expected value with respect to the probability measure P, that is, the integral $E[X_T]=\displaystyle\int_\Omega X_T(\omega)\,dP(\omega)$ in the sense of Lebesgue integration.

Considering the stochastic differential system (1), we assume the following hypothesis related with the Lipschitz and linear growth conditions in the x variable, for $f(x,t,u)$, $g(x,t,u)$ and $h:\mathbb{R}^3\to\mathbb{R}^3$ the jump coefficient or Poisson's process coefficient, defined by $h(x,t,u)=\displaystyle\int_{R^3}\gamma(t,x,z)\bar{N}(dt,dz)$.

(H1) There exist constants $\kappa_1<\infty$ and $\kappa_2<\infty$ such that $f(x,t,u)$, $g(x,t,u)$ and $h(x,t,u)$ satisfy:

a) $||f(x,t,u)||^2\le\kappa_1(1+||x||^2)$,
$||g(x,t,u)||^2\le\kappa_1(1+||x||^2)$, $\displaystyle\int_{R^3}||\gamma(t,x,z)||^2\bar{N}(dt,dz)\le\kappa_1(1+||x||^2)$.

b) $||f(x,t,u)-f(y,t,u)||^2\le\kappa_2||x-y||^2$,
$||g(x,t,u)-g(y,t,u)||^2\le\kappa_2||x-y||^2$, $\displaystyle\int_{R^3}||\gamma(t,x,z)-\gamma(t,y,z)||^2\bar{N}(dt,dz)\le\kappa_2||x-y||^2$. (H2) There exists $\kappa_3<\infty$, such that $\forall t\in R: ||u(t)||\le\kappa_3$
(H3) For $M\subset R^3$: $\displaystyle\sup_{x\in M}\sup_{0\le|z|\le c}|\gamma(s,x(s-),z)|<\infty$.
We are interested in the exponential stability of solutions to a stochastic optimal control problem, the Turnpike property for this problem, and its relationship to the dissipative property.

2 Stochastic Optimal Control Problem

We establish the following Stochastic Optimal Control Problem (SOCP):
To find the controls $u_1(t)$, $u_2(t)$, $u_3(t)$ in system (1), which minimize the following expected cost functional in the Lagrange form:

$$J(u_1, u_2, u_3) = E\Big\{\frac{1}{2}\int_0^T \sum_{i=1}^{3}\Big(x_i^2(t) + u_i^2(t)\Big)\,dt\Big\},\ a.s. \tag{3}$$

Definition 1 The control $u^*(t) = (u_1^*(t), u_2^*(t), u_3^*(t))$ is said to be an optimal control if it satisfies $J(u^*(\cdot)) = \min_{u(\cdot)} J(u(\cdot))$. The corresponding state $x^*(t)$ is called optimal state and $(x^*(t), u^*(t))$ is called the pair optimal.

We define a generalized Hamiltonian function $H(x(t), p(t), q(t), u(t))$ associated to SOCP, [7]:

$$\begin{aligned} H \;&=\; \big\langle p(t), f(t,x,u)^\top \big\rangle + tr[q(t)g(t,x,u)^\top] - f_0(t,x,u) \\ &\quad + \sum_{i=1}^{3} x_i(t)p_i(t)u_i(t)\int_{\mathbb{R}} \gamma_i(t, x_i(t-), z)N(dt,dz). \end{aligned} \tag{4}$$

where $f_0(t,x,u) = \frac{1}{2}\sum_{i=1}^{3}\Big(x_i^2(t) + u_i^2(t)\Big)$. Then, the adjoint equations corresponding to the processes $p(t)$ are the followings:

$$\begin{aligned} dp(t) \;&=\; -\Big\{\big\langle f_x(t,x,u)^\top, p(t)\big\rangle + \sum_{j=1}^{3}\big\langle g_x^j(t,x(t),u(t))^\top, q_j(t)\big\rangle - (f_0(t,x(t),u(t)))_x\Big\}dt \\ &\quad + \langle q(t)\,dW(t)\rangle + u(t)p(t)\int_{\mathbb{R}^3}\gamma(t,x(t-),z)N(dt,dz), \\ p(T) \;&=\; (p_{11}, p_{21}, p_{31})^\top, \end{aligned} \tag{5}$$

and, according to the necessary conditions of stochastic maximum principle

$$\frac{\partial H}{\partial u_i}(x,p,q,u) = 0, \quad i = 1,2,3, \tag{6}$$

$$u_i(t) = -A_i p_i(t)x_i(t) + p_i(t)x_i(t)\int_{\mathbb{R}}\gamma_i(t, x_i(t-), z)N(dt,dz), \tag{7}$$

and we have the following sufficient conditions of optimality of SOCP in our model:

Theorem 1 *The control $u^*(t) = (u_1^*(t), u_2^*(t), u_3^*(t))$ given by (7) is the solution of the Stochastic Optimal Control Problem.*

Proof Let $u^*(t)$ be the control given by (7) and let $x^*(t)$ be the corresponding state of (1) associated to $u^*(t)$. Let $u(t) \in \mathcal{U}$ be any arbitrary control satisfying the system (1). We define the Hamiltonian corresponding to (4) by $H^*(t) = H(x^*(t), p(t), q(t), u^*(t))$. Now, we consider the product $\langle p(T), x(T) - x^*(T)\rangle$. By applying $It\hat{o}$ property with jumps, we have:

$$\begin{aligned} d\left\langle p(T), x(T) - x^*(T)\right\rangle &= \left\langle p(T), d(x(T) - x^*(T))\right\rangle + q(t)(g(x) - g(x^*))dt + q(t)(g(x) \\ &\quad - g(x^*))dt + p(t)u(t)\int_{R^3} \gamma(t, x(t-), z)\bar{N}(dt, dz)), \end{aligned} \tag{8}$$

which gives by integration, taking the expected value and using (**??**) and (3):

$$\begin{aligned} &E\Big(\left\langle p(T), x(T) - x^*(T)\right\rangle - \left\langle p(0), x_i(0) - x^*(0)\right\rangle \Big) \\ &= E\Big(\int_0^T \left\langle p(t), d(x(t) - x^*(t))\right\rangle + \int_0^T \left\langle d(x(t) - x^*(t))dp(t)\right\rangle \\ &\quad + \int_0^T q(t)\Big(g(x(t)) - g((x^*(t))\Big)dt \\ &\quad + (x(t) - x^*(t))p(t)u(t)\int_{R^3} \gamma(t, x(t-), z)\bar{N}(dt, dz))\Big) \\ &= E\Big(\int_0^T \left\langle p(t), f(x(t)) - f(x^*(t))\right\rangle dt + \int_0^T q(t)\Big(g(x(t)) - g(x^*(t))\Big)dt \\ &\quad + (x(t) - x^*(t))p(t)u(t)\int_{R^3} \gamma(t, x(t-), z)\bar{N}(dt, dz)) \\ &\quad - \int_0^T \left\langle x(t) - x^*(t), \frac{\partial H^*}{\partial x}(t)\right\rangle dt\Big) \\ &= E\Big(\int_0^T \Big(H(x, p, q, u) - f_0(t, x, u) - (H^*(t) - f_0(t, x^*, u^*))\Big)dt\Big) \\ &\quad - E\Big(\int_0^T H(x, p, q, u) - H^*(t) - \left\langle (x(t) - x^*(t)), \frac{\partial H}{\partial x}(t)\right\rangle \Big)dt \\ &= E\Big(\int_0^T (H(x, p, q, u) - H^*(t) - \left\langle (x(t) - x^*(t)), \frac{\partial H}{\partial x}(t)\right\rangle)dt\Big) \\ &\quad - E\Big(\int_0^T f_0(t, x, u) - f_0(t, x^*, u^*)dt\Big) \\ &= E\Big(\int_0^T (H(x, p, q, u) - H^*(t) - \left\langle (x(t) - x^*(t)), \frac{\partial H}{\partial x}(t)\right\rangle)dt\Big) - \frac{J(u) - J(u^*)}{2} \end{aligned} \tag{9}$$

Transversality conditions in the stochastic maximum principle and equations (9) imply:

$$E\Big(\int_0^T H(x(t), p(t), q(t), u(t)) - H^*(t) - \left\langle x(t) - x^*(t), H_x(t)\right\rangle dt\Big) = J(u) - J(u^*) \tag{10}$$

and the convexity of $H(\cdot, p, q, \cdot)$ in $x(t), u(t)$, given by (4), implies:

$$\int_0^T (H(x(t), p(t), q(t), u(t)) - H^*(t))dt \leq \int_0^T \left\langle x(t) - x^*(t), H_x(t)\right\rangle dt. \tag{11}$$

Finally, from (10) and (11) we deduce: $J(u^*) \leq J(u) \quad \forall u(t)$, which proves that $u^*(t)$ is the optimal control and the proof is completed. □

Beside, consider the complete steady-state solution $\{\bar{x}(t), \bar{u}(t), \bar{p}(t), \bar{q}(t)\}$ of system (1) with the cost functional (3), adjoint system (5) and variable $q(t)$. We have the following theorem:

Theorem 2 *Assuming hypothesis (H1)-(H3) and given the steady-state solution* $(\bar{x}(t), \bar{u}(t), \bar{p}(t), \bar{q}(t))$ *corresponding to the solution* $\{x(t), u(t), p(t), q(t)\}$ *of system (1) with the cost functional (3), the solutions to SOCP satisfy the Turnpike property: there exit constants* C_1 *and* C_2 *such that:*

$$E||x_T(t) - \bar{x}(t)||^2 + E||u_T(t) - \bar{u}(t)||^2 + E||p_T(t) - \bar{p}(t)||^2 \leq C_2 e^{-2C_1(t-t_0)}. \tag{12}$$

Proof First, we will show that $E|x(t)|^2$ is bounded. The inequality $\left|\sum_{i=1}^n x_i\right|^2 \leq n\sum_{i=1}^n |x_i|^2$, $n \in N$ and the Ito-Lévy isometry property applied to expression:

$$\begin{aligned} x(t) &= x_0 + \int_0^T f(x,t,u)dt + \int_0^T g(x,t,u)dW_t \\ &\quad + x(t)u(t)\int_0^T \int_R \xi(t,x,z)\bar{N}(dt,dz), \end{aligned} \tag{13}$$

leads to:

$$\begin{aligned} E|x(t)|^2 &\leq 4(E|x(0)|^2 + T\int_0^T E|f(x,t,u)|^2 dt + \int_0^T E|g(x,t,u)dt|^2 \\ &\quad + E|x(t)u(t)|^2 \int_0^T E\Big|\int_R \nu(dz)dt)\Big|^2). \end{aligned}$$

We use (H1), (H2) and (H3) to get

$$\begin{aligned} E|x(t)|^2 &\leq 4(E|x(0)|^2 + T\kappa_1 \int_0^T E|x(t)|^2 dt + \kappa_1 \int_0^T E|x(t)|^2 + \kappa_1\kappa_3 \int_0^T E|x(t)|^2 dt) \\ &\leq 4\Big(E|x(0)|^2 + \kappa_1(T+1+\kappa_3)\int_0^T E|x(t)|^2 dt\Big). \end{aligned}$$

By using Gronwall's inequality we obtain $E|x(t)|^2 \leq K$, taking $K = 4e^{\kappa_1(T+1+\kappa_3)}|x(0)|^2$. Now, we consider a perturbation of variables $x(t), u(t), p(t), q(t)$ as follow:

$$\begin{aligned} x_T(t) &= \bar{x}(t) + \delta x(t), \\ u_T(t) &= \bar{u}(t) + \delta u(t), \\ p_T(t) &= \bar{p}(t) + \delta p(t), \\ q_T(t) &= \bar{q}(t) + \psi(t)\delta q(t) + x(t)u(t)\int_{R^3} \gamma(t,x,z)\bar{N}(dt,dz)\,\delta q(t). \end{aligned} \tag{14}$$

We denote $\dfrac{\partial H}{\partial u}$ by H_u, $\dfrac{\partial^2 H}{\partial u^2}$ by H_{uu}, $\cdots$, etc. and, by using the Hamiltonian perturbed and the stochastic Maximum Principle with jumps, we obtain:

$$\begin{aligned} \delta u(t) &= -H_{uu}^{-1}\Big(H_{ux}\delta x(t) + H_{up}\delta p(t) + H_{uq}\delta q(t)\Big(\psi(t) + x(t)u(t)\int_{R^3} \gamma(t,x,z)\bar{N}(dt,dz)\Big)\Big), \\ \delta \dot{x} &= \delta H_p + \delta H_q = H_{px}\delta x(t) + H_{pu}\delta u(t) + H_{pp}\delta p(t) + H_{pq}\delta q(t). \end{aligned} \tag{15}$$

Observing that $H_{pp} = H_{pq} = 0$ and $\delta H_q = 0$, we obtain:

$$\delta \dot{x}(t) = \big(H_{px} - H_{pu}H_{uu}^{-1}H_{ux}\big)\delta x(t) - H_{pu}H_{uu}^{-1}H_{up}\delta p(t)$$

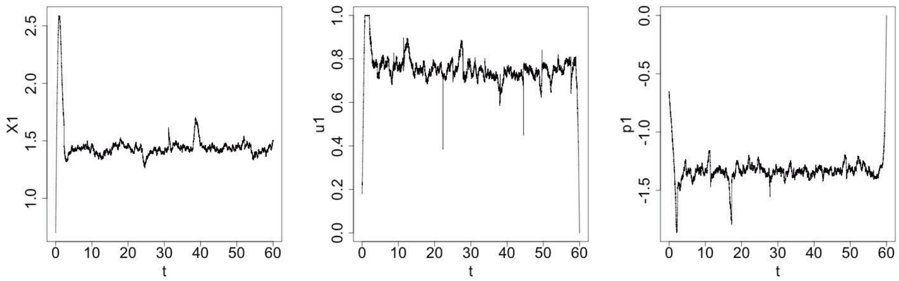

Fig. 1 Stochastic states limit trajectory $x_1(t)$, $u_1(t)$ and $p_1(t)$, using the Euler-Maruyama scheme up to the 13th iteration.

$$-H_{pu}H_{uu}^{-1}H_{uq}\delta q(t)\Big(\psi(t+x(t)u(t)\int_{R^3}\gamma(t,x,z)\bar{N}(dt,dz)\Big). \tag{16}$$

Analogous for $p(t)$ we get:

$$\begin{aligned}\delta\dot{p}(t) &= -H_{xx}\delta x(t) - H_{xu}\delta u(t) - H_{xp}\delta p(t) - H_{xq}\delta q(t)\\ &= (-H_{xx}+H_{xu}H_{uu}^{-1}H_{ux})\delta x(t) + (-H_{xp}+H_{xu}H_{uu}^{-1}H_{up})\delta p(t)\\ &\quad+\Big(-H_{xq}+H_{xu}H_{uu}^{-1}H_{uq}\Big(\psi(t)+x(t)u(t)\int_{R^3}\gamma(t,x,z)\bar{N}(dt,dz)\Big)\Big)\,\delta q(t)\end{aligned} \tag{17}$$

Now, following [3] for our stochastic extended system, we define $A = H_{px} - H_{pu}H_{uu}^{-1}H_{ux}$, $R = -H_{xx} + H_{xu}H_{uu}^{-1}H_{ux}$, $B = H_{pu}$, $Z(t) = (\delta x(t), \delta p(t))^\top$, $d\hat{W}(t) = (dW(t), dW(t))^\top$,
$\hat{x}(t)\hat{u}(t)\int_{R^3}\gamma(t,x,z)\bar{N}(dt,dz) = \Big(x(t)u(t)\int_{R^3}\gamma(t,x,z)\bar{N}(dt,dz), x(t)u(t)\int_{R^3}\gamma(t,x,z)\bar{N}(dt,dz)\Big)$,
where $dW(t) = (dW_1(t), dW_2(t), dW_3(t))$, $dW_i = \psi(t)\delta q_i(t)$, $(A)^*$ denotes transposition of A and
$\hat{M} = \begin{pmatrix} A & -BH_{uu}^{-1}B^* \\ R & -A^* \end{pmatrix}$, $Q = \begin{pmatrix} -BH_{uu}^{-1}B^* \\ -H_{px} + H_{xu}H_{uu}^{-1}H_{uq} \end{pmatrix}$, $Z_0 = \begin{pmatrix} x_T(0) - \bar{x} \\ p_T(0) - \bar{p} \end{pmatrix}$.
So, we may write the systems (1) and (5) as follows:

$$dZ(t) = \hat{M}Z(t)dt + Qd\hat{W}(t) + Q\,\hat{x}(t)\hat{u}(t)\int_{R^3}\gamma(t,x,z)\bar{N}(dt,dz), \quad Z(0) = Z_0. \tag{18}$$

Existence-and-uniqueness of solution of equation (18) is guaranteed by assumption (H1) and [7]. Besides, since $\hat{M}$ is a matrix time-independent [3], the solution of system (18), is given by:

$$Z(t) = e^{\hat{M}(t-t_0)}Z_0 + \int_{t_0}^{t} e^{\hat{M}(s-s_0)}Q\,d\hat{W}(s) + \hat{x}(t)\hat{u}(t)\int_{t_0}^{t} e^{\hat{M}(s-s_0)}Qh(x,t,u)\,dN(s). \tag{19}$$

Now, we focus the matrix $\hat{M}$ in the deterministic part of equation (19) to apply the methods of the Riccati theory used in [3] to found the constant C_1: considering the algebraic Riccati equation:

$$XA + A^*X - XBH_{uu}^{-1}B^*X - R = 0,$$

and its minimal symmetric negative definite matrix solution, E_-, whose existence and uniqueness is guaranteed in [3], the Riccati theory allows to obtain a diagonal matrix equivalent to $\hat{M}$ that satisfies equation (19), whose upper diagonal is $A + BH_{uu}^{-1}B^*E_-$ and its eigenvalues have negative real parts. So, defining

$$C_1 = -\max\{\mu | \mu \in Spec(A + BH_{uu}^{-1}B^*E_-)\} > 0, \tag{20}$$

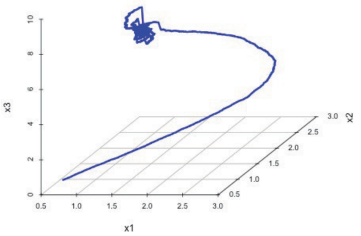

Fig. 2 (x1; x2; x3)-limit trajectory in the phase space, using the Euler- Maruyama scheme up to the 13th iteration with white noise and Lévy jumps

we have, [8]:

$$||e^{\hat{M}(t-t_0)}|| \leq e^{-C_1(t-t_0)}. \tag{21}$$

By [7], from (19), we apply the Schwarz inequality and Ito's isometry to obtain:

$$\begin{aligned} E||Z(t)||^2 \leq\ & 3(E||e^{-C_1(t-t_0)}Z_0||^2 + E||\int_{t_0}^{t} e^{-C_1(s-s_0)}Qd\hat{W}(s)||^2 \\ & +E||\hat{x}(t)\hat{u}\int_{t_0}^{t} e^{-C_1(s-s_0)}Q\gamma(t,x,z)\bar{N}(dt,dz)||^2). \end{aligned} \tag{22}$$

The previous inequality yields:

$$\begin{aligned} E||Z(t)||^2 \leq\ & 3\Big(e^{-2C_1(t-t_0)}||Z_0||^2 + e^{-2C_1(t-t_0)}\int_{t_0}^{t} E||Q||^2 ds \\ & +e^{-2C_1(t-t_0)}E||\hat{x}(t)\hat{u}(t)||^2\int_{t_0}^{t} E||Q\int_{R^3}\gamma(t,x,z)\bar{N}(dt,dz)||^2\Big), \end{aligned}$$

and, by assumptions $(H1a)$, $(H2)$, and the fact showed that $E|x(t)|^2$ is bounded and, in our model, the matrix Q is bounded, we deduce:

$$\begin{aligned} E||Z(t)||^2 \leq\ & 3e^{-2C_1(t-t_0)}\Big(||Z_0||^2 + \int_{t_0}^{t} E||Q||^2 ds \\ & +E||\hat{x}(t)||^2||\hat{u}(t)||^2\int_{t_0}^{t} E||Q||^2\kappa_1||Z(s)||^2 ds\Big) \\ \leq\ & 3e^{-2C_1(t-t_0)}\Big(||Z_0||^2 + ||Q||^2 + \kappa_3^2 K\kappa_1\int_{t_0}^{t} E||Z(s)||^2 ds\Big). \end{aligned} \tag{23}$$

Finally, considering $||Z_0|| < \infty$ and $||Q|| < \infty$, from Gronwall's inequality and setting $C_2 = 3(||Z_0||^2 + ||Q||^2 + \kappa_3^2K^2\kappa_1)$, inequality (23) can be rewritten in the form:

$$E||Z(t)||^2 \leq C_2 e^{-2C_1(t-t_0)}, \tag{24}$$

from which the proof is complete. □

Now, we introduce a Lyapunov function, which is a Ito-Lévy process $V(x,t) \in C^{2,1}(\mathbb{R}^3 \times \mathbb{R}^+; \mathbb{R}^+)$ and also we introduce the linear operator or diffusion operator $L : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, acting on $V(x,t)$, defined as following:

$$\begin{aligned} LV(t,x) \;=\; & \frac{\partial V}{\partial t}(t,x) + \left\langle \frac{\partial V}{\partial x}(t,x), f(t,x,u) \right\rangle + \frac{1}{2} trace\Big(g^\top(t,x) \frac{\partial^2 V}{\partial x^2}(t,x) g(t,x) \Big) \\ & + x(t)u(t) \int_{|z|<c} \big\{ V\big(t, x(t-) + \gamma(x,t)\big) - V(t, x(t-) - \gamma_i(x,t)) \frac{\partial V}{\partial x}(x,t) \big\} \nu(dz), \end{aligned}$$

where $c \in (0,\infty)$ is the maximum jump size. Also, we define the following controlled processes [9], which will appear in the proof of the next result:

$$\bar{I}_1(t) \;=\; x(t)u(t) \int_0^t \int_{|z|<c} \frac{V(x(s) + \xi(x,s,z)) - V(x(s))}{V(x(s))} - \frac{\xi_i(x,s,z)}{V(x(s))} V_x(x(s))\, \nu(dz)ds,$$

$$\bar{I}_2(t) \;=\; x(t)u(t) \int_0^t \int_{|z|<c} (log \frac{V(x(s) + \xi(x,s,z))}{V(x(s))} + 1 - \frac{V(x(s) + \xi(x,s,z))}{V(x(s))} \nu(dz)ds),$$

Definition 2 A Control Problem is called dissipative stochastically with respect to a function called dissipation rate $r(x,u)$, if there exists a function $\nu(x,u)$ called storage such that the stochastic process $\{\nu(x,u) - \int_0^{t_0} r(x,u)ds : t \geq 0\}$ to be un martingale such that:

$$E\Big(\nu(x(t),u(t)) - \int_0^{t_1} r(x,u)ds | F_{t_1}\Big) \leq \nu(x_{t_1}, u_{t_2}) - \int_0^{t_0} r(x,u)ds \tag{25}$$

Theorem 3 *Assuming hypothesis (H1)-(H3), if there exist a C^2-function $\mathcal{V}(x_t,u_t)$, a continuous and increasing function $r(x)$ and positive constants c_2, c_3, such that:*

i) $L\mathcal{V}(x_t,u_t) \leq -c_2 r(||x_t - \bar{x}_t||)$
ii) $\mathcal{V}(x_t,u_t) \leq f_0(x_t,u_t) - f_0(x_0,u_0)$,

then the system (1) is dissipative stochastically.

Proof We consider the process $e^{c_3 t}\mathcal{V}(x_t,u_t)$. Applying Ito's isometry property:

$$\begin{aligned} e^{c_3 t}\mathcal{V}(x_t,u_t) \;=\; & \mathcal{V}(x_0,u_0) + \int_0^t c_3 e^{c_3 s}\mathcal{V}(x_s,u_s)ds + \int_0^t e^{c_3 s}\mathcal{V}_x(x_s,u_s)(f(x_s,u_s)ds + g(x_s,u_s)dW_s), \\ & + x_t u_t \int_0^t \int_{y\leq c} e^{c_3 s}(V(x_x,u_s) + \xi(x_s,y) - V(x_s,u_s)\bar{N}(ds,dy), \\ & + x_t u_t \int_0^t \int_{y\leq c} e^{c_3 s}(V(x_s,u_s) + \xi - V(x_s,u_s) - V_x(x_s,u_s)\xi)\nu(dy)ds. \end{aligned}$$

Using definition of LV, I_1, I_2 y $(H3)$ we have:

$$Ee^{c_3 t}\mathcal{V}(x_t,u_t) = \mathcal{V}(x_0,u_0) + E(\int_0^t c_3 e^{c_3 s}(\mathcal{V}(x_s,u_s)ds + L\mathcal{V}(x_s,u_s)ds)) \tag{26}$$

Considering hypothesis $i)$, $ii)$ and $V(x_0, u_0) \leq V(x_0, u_0)e^{c_3 t}$:

$$E\mathcal{V}(x_t, u_t) \leq \mathcal{V}(x_0, u_0) + E(\int_0^t c_3(f_0(x_s, u_s) - f_0(x_0, u_o))ds - c_2 r(||x_t, \bar{x}_s||)ds)) \quad (27)$$

then, since $x_t, u_t, r(x_t, G_t)$, $\mathcal{F}_t$ are super martingale, the process $\{\mathcal{V}(x_t, u_t) - \int_0^{t_1} r(x_s, G_s)ds : t \geq 0\}$ is also a super martingale and (27) implies the conclusion. □

Example 1 Let consider the Poisson process given by $\xi(t, x(t-), z) = sign(z)\frac{z^2}{1+z^2}$ and define $V(x,t) = \frac{1}{u^2(t)}||x(t) - \bar{x}||^2$. The corresponding diffusion operator $LV(x,t)$ is:

$$\begin{aligned} LV(x,t) \; &= -\Big(\frac{||x(t) - \bar{x}||^2}{u^3(t)} + 2\frac{||x(t) - \bar{x}||}{u^2(t)}\Big) f(t,x,u) + trace(g^\top g(t,x,u)) \\ &\quad + x(t)u(t)\int_{|z|<c}\Big(\frac{||x(t) - \bar{x}||^2}{u^2(t)} sign(z)\frac{z^2}{1+z^2} + \frac{z^4}{(1+z^2)^2}\Big)\nu dz. \end{aligned}$$

The conditions of Theorem 3 are satisfied, obtaining the stochastic dissipation of (1).

3 Conclusion

In this paper, we study an optimal control problem applied to a Lotka-Volterra controlled model with L'evy jumps. We show that its solutions exhibit the Turnpike property via the stochastic maximum principle. We show that, under the existence of a Lyapunov function satisfying the boundedness conditions, the studied model is stochastically dissipative. We perform a simulation of the results.

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