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Optimal control problems with delays in state and control variables subject to mixed control–state constraints

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SUMMARY

Optimal control problems with delays in state and control variables are studied. Constraints are imposed as mixed control–state inequality constraints. Necessary optimality conditions in the form of Pontryagin's minimum principle are established. The proof proceeds by augmenting the delayed control problem to a nondelayed problem with mixed terminal boundary conditions to which Pontryagin's minimum principle is applicable. Discretization methods are discussed by which the delayed optimal control problem is transformed into a large-scale nonlinear programming problem. It is shown that the Lagrange multipliers associated with the programming problem provide a consistent discretization of the advanced adjoint equation for the delayed control problem. An analytical example and numerical examples from chemical engineering and economics illustrate the results. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Differential control systems with delays in state or control variables play an important role in the modelling of real-life phenomena in various fields of applications. Many papers have been devoted to delayed (other terminology: time lag, retarded, hereditary) optimal control problems and the derivation of necessary optimality conditions. Let us briefly review some papers concerning different classes of control problems. An introduction to time delay control problems can be found in Oğuztöreli [1]. Kharatishvili [2] was first to provide a maximum principle for optimal control

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problems with a constant state delay. In [3], he gave similar results for control problems with pure control delays. Halany [4] proves a maximum principle for optimal control problems with multiple constant delays in state and control variables that, however, are chosen to be equal for state and control. Similar results were obtained by Ray and Soliman [5]. Guinn [6] sketches a simple method for obtaining necessary conditions for control problems with a constant delay in the state variable. He suggests to augment the delayed control problem that yields a higher-dimensional undelayed control problem to which the standard maximum principle is applicable. Banks [7] derives a maximum principle for control systems with a time-dependent delay in the state variable. Delays in the control are admitted for systems linear in the control variable. Colonius and Hinrichsen [8] provide a unified approach to control problems with delays in the state variable by applying the theory of necessary conditions for optimization problems in function spaces. All articles mentioned so far do not consider general control or state inequality constraints.

Angell and Kirsch [9] treat functional differential equations with function-space state inequality constraints. However, they do not discuss the regularity of the multiplier associated with the state constraint and do not provide a numerical example with a pure state space constraint. To our knowledge, optimal control problems with constant delays in state and control variables and *mixed control–state inequality constraints* have not yet been considered in the literature. The first goal in this paper is to derive a Pontryagin-type minimum (maximum) principle for this class of delayed control problems. Concerning the development of numerical methods and the numerical treatment of practical examples, our impression is that this topic has not yet been adequately addressed in the literature. Bader [10] uses collocation methods to solve the boundary value problem for the retarded state variable and advanced adjoint variable. He successfully solves several academic examples, but his method does not give accurate results for the more difficult CSTR reactor problem described in Soliman and Ray [5, 11]. A similar CSTR reactor problem is considered in Oh and Luus [12] and Dadebo and Luus [13], who use the differential dynamic programming method with a moderate number of stages. Therefore, the second goal of this paper is the presentation of discretization and nonlinear programming methods that provide the optimal state, control and adjoint functions and allow for an accurate check of the necessary conditions.

The organization of this paper is as follows. Section 2 presents the statement of the delayed control problem with mixed state–control constraints. In Section 3, we recall the minimum principle for *nondelayed* control problems with control–state constraints. Here, a crucial feature is that the initial and the terminal boundary conditions must be considered in a general mixed form. Section 4 is devoted to the derivation of first-order necessary optimality conditions for the delayed optimal control problem given in Section 2. Essentially, the augmentation approach of Guinn [6] is generalized, which allows to use the minimum principle in Section 3. For technical reasons, we need the assumption that the ratio of the time delays in state and control is a rational number. The analysis in this section is based on the theses of Göllmann [14] and Kern [15]. In Section 5, the Euler discretization for the delayed control problem is discussed, which leads to a high-dimensional nonlinear programming problem. As in the nondelayed case, it can be shown that the Lagrange multipliers corresponding to the optimization problem constitute an Euler discretization for the advanced adjoint equations. In Section 6, we discuss an analytical example that allows to test the accuracy of the numerical solution for various step sizes. Sections 7 and 8 are devoted to the numerical solution and the verification of the minimum principle for two practical examples. The first example is taken from [5, 11] and describes the optimal control of a chemical tank reactor (CSTR reactor), whereas the second example arises in the optimal harvesting of a resource (optimal fishing).

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2. OPTIMAL CONTROL PROBLEMS WITH DELAYS IN STATE AND CONTROL

We consider retarded optimal control problems with constant delays $r \geqslant 0$ in the state variable $x(t) \in \mathbb{R}^n$ and $s \geqslant 0$ in the control variable $u(t) \in \mathbb{R}^m$. The following retarded control problem with mixed control–state inequality constraints will be referred to as problem (ROCP):

$$\text{Minimize} \quad J(u,x) = g(x(b)) + \int_a^b L(t, x(t), x(t-r), u(t), u(t-s))\,\mathrm{d}t \tag{1}$$

subject to the retarded differential equation, boundary conditions and mixed control–state inequality constraints

$$\dot{x}(t) = f(t, x(t), x(t-r), u(t), u(t-s)), \quad \text{a.e. } t \in [a,b] \tag{2}$$

$$x(t) = \varphi(t), \quad t \in [a-r, a] \tag{3}$$

$$u(t) = \psi(t), \quad t \in [a-s, a) \tag{4}$$

$$w(x(b)) = 0 \tag{5}$$

$$C(t, x(t), u(t)) \leqslant 0, \quad t \in [a,b] \tag{6}$$

For convenience, all functions

$$g : \mathbb{R}^n \to \mathbb{R}$$

$$L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

$$f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$$

$$w : \mathbb{R}^n \to \mathbb{R}^q, \quad 0 \leqslant q \leqslant n$$

$$C : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$$

are assumed to be twice continuously differentiable w.r.t. all arguments. A pair of functions $(u,x) \in L^\infty([a,b], \mathbb{R}^m) \times W^{1,\infty}([a,b], \mathbb{R}^n)$ is called an *admissible pair* for problem (ROCP), if the state x and control u satisfy restrictions (2)–(6). An admissible pair $(\hat{u}, \hat{x})$ is called a *locally optimal pair* or *weak minimum* for (ROCP), if

$$J(\hat{u}, \hat{x}) \leqslant J(u,x)$$

holds for all (u,x) admissible in a neighborhood of $(\hat{u}, \hat{x})$ with $\|u(t) - \hat{u}(t)\|$, $\|x(t) - \hat{x}(t)\| < \varepsilon$ for all $t \in [a,b]$ and $\varepsilon > 0$ sufficiently small. Instead of considering a *weak minimum*, we could use the more general notion of a *Pontryagin minimum*, thus admitting neighborhoods of $(\hat{u}, \hat{x})$ in the L^1-norm; cf. Milyutin and Osmolovskii [16].

3. FIRST-ORDER NECESSARY OPTIMALITY CONDITIONS FOR UNDELAYED OPTIMAL CONTROL PROBLEMS WITH MIXED CONSTRAINTS

Formally, any undelayed control problem is contained in the retarded problem (ROCP) by choosing $r = s = 0$. Owing to the absence of delays, the initial value profiles given by conditions (3) and (4)

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are omitted. However, the continuity of the state variables in the augmented problem necessitates to introduce a general boundary condition of mixed type,

$$w(x(a), x(b)) = 0 \tag{7}$$

which replaces the terminal boundary condition (5). This condition is indispensable in the proof of the necessary conditions presented in Section 4. The Hamiltonian or Pontryagin function for the nondelayed control problem without any constraints (6) is given by

$$H(t, x, u, \lambda) := L(t, x, u) + \lambda^* f(t, x, u) \tag{8}$$

The *augmented* Hamiltonian is defined by adjoining the mixed control–state constraint (6) by a multiplier $\mu \in \mathbb{R}^p$ to the Hamiltonian (8):

$$\mathscr{H}(t, x, u, \lambda, \mu) := L(t, x, u) + \lambda^* f(t, x, u) + \mu^* C(t, x, u) \tag{9}$$

Here and in the sequel, $*$ denotes the transposition. The extension of the classical Pontryagin's minimum principle to the mixed control–state constraints (6) requires a regularity condition or constraint qualification. For a locally optimal pair $(\hat{u}, \hat{x})$ and $t \in [a, b]$, let $J_0(t) := \{j \in \{1, \ldots, p\} | C_j(t, \hat{x}(t), \hat{u}(t)) = 0\}$ denote the set of active indices for the inequality constraint (6). Then, we assume the rank condition:

$$\operatorname{rank} \left(\frac{\partial C_j(t, \hat{x}(t), \hat{u}(t))}{\partial u} \right)_{j \in J_0(t)} = \# J_0(t) \tag{10}$$

The following necessary optimality conditions are to be found in Hestenes [17, Chapter 7, Theorem 3.1] and Neustadt [18, Chapter VI.3, p. 296].

Theorem 3.1 (*Pontryagin's Minimum Principle*)
Let $(\hat{u}, \hat{x})$ be a locally optimal pair for the control problem (ROCP) without delays, i.e. $r = s = 0$, and the mixed boundary condition (7). Assume that the regularity condition (10) is satisfied. Then there exist a costate (adjoint) function $\hat{\lambda} \in W^{1,\infty}([a, b], \mathbb{R}^n)$, a multiplier function $\hat{\mu} \in L^\infty([a, b], \mathbb{R}^p)$ and a multiplier $\hat{\nu} \in \mathbb{R}^q$, such that the following conditions hold for a.e. $t \in [a, b]$:

(i) *adjoint differential equation*:

$$\dot{\hat{\lambda}}(t)^* = -\mathscr{H}_x(t, \hat{x}(t), \hat{u}(t), \hat{\lambda}(t), \hat{\mu}(t)) \tag{11}$$

(ii) *transversality conditions*:

$$\hat{\lambda}(a)^* = -\hat{\nu}^* w_{x^a}(\hat{x}(a), \hat{x}(b)) \tag{12}$$

$$\hat{\lambda}(b)^* = g_x(\hat{x}(b)) + \hat{\nu}^* w_{x^b}(\hat{x}(a), \hat{x}(b)) \tag{13}$$

(iii) *minimum condition for Hamiltonian*:

$$H(t, \hat{x}(t), \hat{u}(t), \hat{\lambda}(t)) \leqslant H(t, \hat{x}(t), u, \hat{\lambda}(t)) \tag{14}$$

for all $u \in \mathbb{R}^m$ satisfying $C(t, \hat{x}(t), u) \leqslant 0$;

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(iv) *local minimum condition for augmented Hamiltonian*:

$$\mathscr{H}_u(t,\hat{x}(t),\hat{u}(t),\hat{\lambda}(t),\hat{\mu}(t))=0 \tag{15}$$

(v) *nonnegativity of multiplier and complementarity condition*:

$$\hat{\mu}(t)\geqslant 0 \quad \text{and} \quad \hat{\mu}_i(t)C_i(t,\hat{x}(t),\hat{u}(t))=0, \quad i=1,\ldots,p \tag{16}$$

In (12) and (13), w_{x^a} and w_{x^b} denote partial derivatives of $w=w(x^a,x^b)$ with respect to its first and second arguments. In the following section, Theorem 3.1 will be used to derive necessary conditions for the retarded control problem (ROCP).

4. NECESSARY OPTIMALITY CONDITIONS FOR DELAYED OPTIMAL CONTROL PROBLEMS WITH MIXED CONTROL–STATE CONSTRAINTS

Now we study the retarded control problem (ROCP) with constant delays $r,s\geqslant 0$ and $(r,s)\neq(0,0)$. We shall use a transformation technique that requires the technical assumption that the ratio of the delays is a rational number.

Assumption 4.1 (*rationality assumption*)
Assume that $r,s\geqslant 0$, $(r,s)\neq(0,0)$ and

$$\frac{r}{s}\in\mathbb{Q} \quad \text{for } s>0 \quad \text{or} \quad \frac{s}{r}\in\mathbb{Q} \quad \text{for } r>0 \tag{17}$$

In particular, this assumption holds for any couple of rational numbers (r,s), where at least one number is nonzero.

The Hamiltonian H and the augmented Hamiltonian $\mathscr{H}$ for the delayed control problem (ROCP) are defined in analogy to nondelayed problems. However, in contrast to the nondelayed Hamiltonians, two additional arguments $y\in\mathbb{R}^n$ and $v\in\mathbb{R}^m$ denoting the delayed state and control variables are needed:

$$\begin{aligned} H(t,x,y,u,v,\lambda) &:= L(t,x,y,u,v)+\lambda^* f(t,x,y,u,v) \\ \mathscr{H}(t,x,y,u,v,\lambda,\mu) &:= L(t,x,y,u,v)+\lambda^* f(t,x,y,u,v)+\mu^* C(t,x,u) \end{aligned} \tag{18}$$

where $\lambda\in\mathbb{R}^n$, $\mu\in\mathbb{R}^p$.

We shall obtain necessary optimality conditions for the retarded control problem (ROCP) by first transforming (augmenting) problem (ROCP) to a higher-dimensional nondelayed control problem. To further study the augmented problem, we need Pontryagin's minimum principle for nondelayed control problems with mixed control–state constraints, which will be reviewed in the following section.

The following first-order necessary conditions can be found in Göllmann [14]; a precise proof under Assumption 4.1 has been given by Kern [15].

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Theorem 4.2 (*minimum principle for the retarded optimal control problem (ROCP)*)
Let $(\hat{u}, \hat{x})$ be locally optimal for (ROCP) with delays satisfying Assumption 4.1. Then there exist a costate (adjoint) function $\hat{\lambda} \in W^{1,\infty}([a,b], \mathbb{R}^n)$, a multiplier function $\hat{\mu} \in L^{\infty}([a,b], \mathbb{R}^p)$ and a multiplier $\hat{v} \in \mathbb{R}^q$, such that the following conditions hold for a.e. $t \in [a,b]$:

(i) *adjoint differential equation*:

$$\begin{aligned}\dot{\hat{\lambda}}(t)^* &= -\hat{\mathscr{H}}_x(t) - \chi_{[a,b-r]}(t)\hat{\mathscr{H}}_y(t+r)\\ &= -\mathscr{H}_x(t, \hat{x}(t), \hat{x}(t-r), \hat{u}(t), \hat{u}(t-s), \hat{\lambda}(t), \hat{\mu}(t))\\ &\quad -\chi_{[a,b-r]}(t)\mathscr{H}_y(t+r, \hat{x}(t+r), \hat{x}(t), \hat{u}(t+r), \hat{u}(t+r-s), \hat{\lambda}(t+r), \hat{\mu}(t+r))\end{aligned} \tag{19}$$

where $\hat{\mathscr{H}}_x(t)$ and $\hat{\mathscr{H}}_y(t)$ denote the evaluation of the partial derivatives $\mathscr{H}_x$ and $\mathscr{H}_y$ along $\hat{x}(t), \hat{x}(t-r), \hat{u}(t), \hat{u}(t-s), \hat{\lambda}(t), \hat{\mu}(t)$;

(ii) *transversality condition*:

$$\hat{\lambda}(b)^* = g_x(\hat{x}(b)) + \hat{v}^* w_x(\hat{x}(b)) \tag{20}$$

(iii) *minimum condition for Hamiltonian*:

$$\begin{aligned}&\hat{H}(t) + \chi_{[a,b-s]}(t)\hat{H}(t+s)\\ &\quad = H(t, \hat{x}(t), \hat{x}(t-r), \hat{u}(t), \hat{u}(t-s), \hat{\lambda}(t))\\ &\qquad + \chi_{[a,b-s]}(t) H(t+s, \hat{x}(t+s), \hat{x}(t+s-r), \hat{u}(t+s), \hat{u}(t), \hat{\lambda}(t+s))\\ &\quad \leqslant H(t, \hat{x}(t), \hat{x}(t-r), u, \hat{u}(t-s), \hat{\lambda}(t))\\ &\qquad + \chi_{[a,b-s]}(t) H(t+s, \hat{x}(t+s), \hat{x}(t+s-r), \hat{u}(t+s), u, \hat{\lambda}(t+s))\end{aligned} \tag{21}$$

for all $u \in \mathbb{R}^m$ satisfying $C(t, \hat{x}(t), u) \leqslant 0$;

(iv) *local minimum condition for augmented Hamiltonian*:

$$\hat{\mathscr{H}}_u(t) + \chi_{[a,b-s]}(t)\hat{\mathscr{H}}_v(t+s) = 0 \tag{22}$$

(v) *nonnegativity of multiplier and complementarity condition*:

$$\hat{\mu}(t) \geqslant 0 \quad \text{and} \quad \hat{\mu}_i(t) C_i(t, \hat{x}(t), \hat{u}(t)) = 0, \quad i = 1, \ldots, p \tag{23}$$

Proof
The proof uses a transformation technique suggested by Guinn [6] to derive first-order necessary conditions for unconstrained optimal control problems with pure state delays. In view of the rationality assumption (17), there exist integers $k, l \in \mathbb{N}$ with

$$\frac{r}{s} = \frac{k}{l} \quad \text{for } s \neq 0, \qquad \frac{s}{r} = \frac{l}{k} \quad \text{for } r \neq 0$$

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Without loss of generality, we may assume the first case. Then the delays r, s are integer multiples of the interval length $h := s/l$:

$$r = k \cdot h, \quad s = l \cdot h, \quad k, l \in \mathbb{N}$$

The time interval $[a, a+h]$ will be used below as the basis time interval for the augmented control problem. Without loss of generality, we may further assume that the interval length $b-a$ represents an integer multiple of h, i.e. we have $b-a = Nh$ with $N \in \mathbb{N}_+$.

Now we introduce the state variable $\Xi^* = (\xi_0^*, \ldots, \xi_{N-1}^*) \in \mathbb{R}^{Nn}$, $\xi_i \in \mathbb{R}^n$, and control variable $\Theta^* = (\theta_0^*, \ldots, \theta_{N-1}^*) \in \mathbb{R}^{Nm}$, $\theta_i \in \mathbb{R}^m$, which are defined by

$$\xi_i(t) := x(t+ih), \quad \theta_i(t) := u(t+ih) \quad \text{for } t \in [a, a+h], \quad i = 0, \ldots, N-1 \tag{24}$$

The continuity of the state $x(t)$ in $[a, b]$ implies the following boundary conditions for the augmented state $\Xi(t)$:

$$\xi_i(a+h) = \xi_{i+1}(a), \quad i = 0, \ldots, N-2$$

which can be expressed as

$$V_i(\xi_{i+1}(a), \xi_i(a+h)) := \xi_i(a+h) - \xi_{i+1}(a) = 0, \quad i = 0, \ldots, N-2 \tag{25}$$

In terms of the new state and control variables Ξ and Θ, the retarded control problem (ROCP) is equivalent to the following undelayed optimal control problem on the time interval $[a, a+h]$:

$$\text{Minimize} \quad J(\Theta, \Xi) = g(\xi_{N-1}(a+h)) + \int_a^{a+h} \sum_{i=0}^{N-1} L(t+ih, \xi_i(t), \xi_{i-k}(t), \theta_i(t), \theta_{i-l}(t))\,\mathrm{d}t \tag{26}$$

subject to

$$\dot{\xi}_i(t) = f(t+ih, \xi_i(t), \xi_{i-k}(t), \theta_i(t), \theta_{i-l}(t)), \quad i = 0, \ldots, N-1, \ \ t \in [a, a+h] \tag{27}$$

$$\begin{aligned} V_i(\xi_{i+1}(a), \xi_i(a+h)) &= 0, \quad i = 0, \ldots, N-2 \\ V_{N-1}(\xi_{N-1}(a+h)) &:= w(\xi_{N-1}(a+h)) = 0 \end{aligned} \tag{28}$$

$$C(t+ih, \xi_i(t), \theta_i(t)) \leqslant 0, \quad i = 0, \ldots, N-1, \ \ t \in [a, a+h] \tag{29}$$

The fixed starting profiles (3) and (4) are included in this notation by considering the variables $\xi_{-k}, \ldots, \xi_{-1}$ and $\theta_{-l}, \ldots, \theta_{-l}$ defined by

$$\begin{aligned} \xi_i(t) &:= \varphi(t+ih), \quad i = -k, \ldots, -1 \\ \theta_i(t) &:= \psi(t+ih), \quad i = -l, \ldots, -1 \end{aligned}$$

However, note that $\xi_{-k}, \ldots, \xi_{-1}$ and $\theta_{-l}, \ldots, \theta_{-1}$ do not represent optimization variables. Introducing adjoint variables and multipliers for the augmented problem by (26)–(29) by

$$\Lambda = (\Lambda_0, \ldots, \Lambda_{N-1})^* \in \mathbb{R}^{N \cdot n}, \quad M = (M_0, \ldots, M_{N-1})^* \in \mathbb{R}^{N \cdot p}$$

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the Hamiltonian functions (8) and (9) for the nondelayed augmented control problem are given by

$$K(t,\Xi,\Theta,\Lambda,M)=\sum_{i=0}^{N-1}[L(t+ih,\xi_i,\xi_{i-k},\theta_i,\theta_{i-l})+\Lambda_i^* L(t+ih,\xi_i,\xi_{i-k},\theta_i,\theta_{i-l})] \quad (30)$$

$$\mathscr{K}(t,\Xi,\Theta,\Lambda,M)=\sum_{i=0}^{N-1}[L(t+ih,\xi_i,\xi_{i-k},\theta_i,\theta_{i-l})+\Lambda_i^* L(t+ih,\xi_i,\xi_{i-k},\theta_i,\theta_{i-l})]$$
$$+\sum_{i=0}^{N-1} M_i^* C(t+ih,\xi_i,\theta_i) \quad (31)$$

Every locally optimal pair $(\hat{u}(\cdot),\hat{x}(\cdot))$ for (ROCP) defines a pair $(\hat{\Theta}(\cdot),\hat{\Xi}(\cdot))$ that minimizes the augmented problem (26)–(29). Pontryagin's minimum principle for nondelayed problems (Theorem 3.1) assures the existence of a costate (adjoint) function $\hat{\Lambda}\in W^{1,\infty}([a,a+h],\mathbb{R}^{N\cdot n})$, a multiplier function $\hat{M}\in L^\infty([a,a+h],\mathbb{R}^{N\cdot p})$ and a vector $v\in\mathbb{R}^{(N-1)n+q}$, $\hat{v}=(\hat{v}_0^*,\ldots,\hat{v}_{N-2}^*,\hat{v}_{N-1}^*)^*$ where $\hat{v}_0,\ldots\hat{v}_{N-2}\in\mathbb{R}^n$ and $\hat{v}_{N-1}\in\mathbb{R}^q$, such that the following conditions hold for almost every $t\in[a,a+h]$:

1. *adjoint differential equation*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Lambda}(t)^*=-\mathscr{K}_\Xi(t,\hat{\Xi}(t),\hat{\Theta}(t),\hat{\Lambda}(t),\hat{M}(t)) \quad (32)$$

2. *transversality condition*:

$$\hat{\Lambda}_i(a)^*=-\hat{v}_i^*\frac{\partial}{\partial\xi_i}V_i(\hat{\xi}_{i+1}(a),\hat{\xi}_i(a+h)),\quad i=0,\ldots,N-2 \quad (33)$$

$$\hat{\Lambda}_i(a+h)^*=\hat{v}_i^*\frac{\partial}{\partial\xi_{i+1}}V_i(\hat{\xi}_{i+1}(a),\hat{\xi}_i(a+h)),\quad i=0,\ldots,N-2 \quad (34)$$

$$\hat{\Lambda}_{N-1}(a+h)^*=g_x(\hat{\xi}_{N-1}(a+h))+\hat{v}_{N-1}^* w_x(\hat{\xi}_{N-1}(a+h)) \quad (35)$$

3. *minimum condition for Hamiltonian*:

$$K(t,\hat{\Xi}(t),\hat{\Theta}(t),\hat{\Lambda}(t))\leqslant K(t,\hat{\Xi}(t),\Theta,\hat{\Lambda}(t)) \quad (36)$$

for all admissible $\Theta=(\theta_0^*,\ldots,\theta_{N-1}^*)^*\in\mathbb{R}^{Nm}$ satisfying $C(t+ih,\hat{\xi}_i(t),\theta_i)\leqslant 0$ for $i=0,\ldots,N-1$;

4. *local minimum condition for augmented Hamiltonian*:

$$\mathscr{K}_\Theta(t,\hat{\Xi}(t),\hat{\Theta}(t),\hat{\Lambda}(t),M(t))=0 \quad (37)$$

5. *nonnegativity of multiplier and complementarity condition*:

$$\hat{M}(t)\geqslant 0,\quad \hat{M}_i(t)^*C(t+ih,\hat{\xi}_i(t),\hat{\theta}_i(t))=0,\ \ i=0,\ldots,N-1 \quad (38)$$

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Evaluating the adjoint equation for the component $\hat{\Lambda}_j$ $(0\leqslant j\leqslant N-1)$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Lambda}_j(t)^* = & -L_x(t+jh,\hat{\xi}_j(t),\hat{\xi}_{j-k}(t),\hat{\theta}_j(t),\hat{\theta}_{j-l}(t)) \\ & -\chi_{\{0,\ldots,N-1-k\}}(j)L_y(t+(j+k)h,\hat{\xi}_{j+k}(t),\hat{\xi}_j(t),\hat{\theta}_{j+k}(t),\hat{\theta}_{j+k-l}(t)) \\ & -\hat{\Lambda}_j(t)^* f_x(t+jh,\hat{\xi}_j(t),\hat{\xi}_{j-k}(t),\hat{\theta}_j(t),\hat{\theta}_{j-l}(t)) \\ & -\chi_{\{0,\ldots,N-1-k\}}(j)\hat{\Lambda}_{j+k}(t)^* f_y(t+(j+k)h,\hat{\xi}_{j+k}(t),\hat{\xi}_j(t),\hat{\theta}_{j+k}(t),\hat{\theta}_{j+k-l}(t)) \\ & -\hat{M}_j(t)^* C_x(t+jh,\hat{\xi}_j(t),\hat{\theta}_j(t))\end{aligned}$$

Now we are able to define the adjoint function $\hat{\lambda}\in W^{1,\infty}([a,b],\mathbb{R}^n)$ and multiplier function $\hat{\mu}\in L^{\infty}([a,b],\mathbb{R}^p)$ for the retarded control problem (ROCP) in the following way. For $t\in[a,b]$, there exists $0\leqslant j\leqslant N-1$ with $a+jh\leqslant t\leqslant a+(j+1)h$. We substitute

$$\hat{\lambda}(t):=\hat{\Lambda}_j(t-jh),\quad \hat{\mu}(t):=\hat{M}(t-jh) \tag{39}$$

and obtain from the previous adjoint equation:

$$\begin{aligned}\dot{\hat{\lambda}}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\hat{\Lambda}_j(t-jh) \\ &= -L_x(t,\hat{x}(t),\hat{x}(t-kh),\hat{u}(t),\hat{u}(t-lh)) \\ &\quad -\chi_{\{0,\ldots,N-1-k\}}(j)L_y(t+kh,\hat{x}(t+kh),\hat{x}(t),\hat{u}(t+kh),\hat{u}(t+kh-lh)) \\ &\quad -\hat{\lambda}(t)^* f_x(t,\hat{x}(t),\hat{x}(t-kh),\hat{u}(t),\hat{u}(t-lh)) \\ &\quad -\chi_{\{1,\ldots,N-1-k\}}(j)\hat{\lambda}(t+kh)^* f_y(t+kh,\hat{x}(t+kh),\hat{x}(t),\hat{u}(t+kh),\hat{u}(t+kh-lh)) \\ &\quad -\hat{\mu}(t)^* C_x(t,\hat{x}(t),\hat{u}(t)) \\ &= -\mathscr{H}(t,\hat{x}(t),\hat{x}(t-r),\hat{u}(t),\hat{u}(t-s),\hat{\lambda}(t),\hat{\mu}(t)) \\ &\quad -\chi_{[a,b-r]}(t)\mathscr{H}(t+r,\hat{x}(t+r),\hat{x}(t),\hat{u}(t+r),\hat{u}(t+r-s),\hat{\lambda}(t+r)\hat{\mu}(t+r))\end{aligned}$$

Thus, we have found the adjoint equation (19). The transversality condition (34) for Λ_{N-1}

$$\hat{\Lambda}_{N-1}(a+h)^* = g_x(\hat{\xi}_{N-1}(a+h))+\hat{v}_{N-1}^* w_x(\hat{\xi}_{N-1}(a+h))$$

gives the desired transversality condition (20) for (ROCP) in view of $b=a+Nh$:

$$\hat{\lambda}(a+Nh)=g_x(\hat{x}(a+Nh))+\hat{v}^* w_x(\hat{x}(a+Nh)),\quad \hat{v}:=\hat{v}_{N-1}\in\mathbb{R}^q$$

To verify the minimum condition for the Hamiltonian H, we consider $t\in[a,b]$ and the corresponding index $j\in\{0,\ldots,N-1\}$ with $a+jh\leqslant t\leqslant a+(j+1)h$. Substituting $t':=t-jh\in[a,a+h]$, the minimum condition (36) gives

$$K(t',\hat{\Xi}(t'),\hat{\Theta}(t'),\hat{\Lambda}(t'))\leqslant K(t',\hat{\Xi}(t'),\Theta,\hat{\Lambda}(t')) \tag{40}$$

for all admissible $\Theta\in\mathbb{R}^{Nm}$ satisfying (29). The local minimum condition (37) yields $\mathscr{K}_{\Theta}(t')=0$. Now we define an admissible control policy $\Theta(\cdot)=(\theta_0^*,\ldots,\theta_{N-1}^*)^*\in\mathbb{R}^{Nm}$ by

$$\theta_i:=\begin{cases}\hat{u}(t'+ih), & i\neq j\\ u, & i=j\end{cases},\quad i=0,\ldots,N-1$$

where the control vector $u\in\mathbb{R}^m$ is admissible for (ROCP), i.e. $C(t,\hat{x}(t),u)\leqslant 0$. Evaluating inequality (40) for this vector Θ and removing equal expressions on both sides, we obtain for the remaining terms associated with j and $j+l$:

$$\begin{aligned}
&L(t'+jh,\hat{\xi}_j(t'),\hat{\xi}_{j-k}(t'),\hat{u}(t'+jh),\hat{u}(t'+(j-l)h))\\
&\quad+\hat{\Lambda}_j(t')^* f(t'+jh,\hat{\xi}_j(t'),\hat{\xi}_{j-k}(t'),\hat{u}(t'+jh),\hat{u}(t'+(j-l)h))\\
&\quad+\chi_{\{0,\ldots,N-1-l\}}(j)L(t'+(j+l)h,\hat{\xi}_{j+l}(t'),\hat{\xi}_{j+l-k}(t'),\hat{u}(t'+(j+l)h),\hat{u}(t'+jh))\\
&\quad+\chi_{\{0,\ldots,N-1-l\}}(j)\hat{\Lambda}_{j+l}(t')^* f(t'+(j+l)h,\hat{\xi}_{j+l}(t'),\hat{\xi}_{j+l-k}(t'),\hat{u}(t'+(j+l)h),\hat{u}(t'+jh))\\
&\qquad\leqslant L(t'+jh,\hat{\xi}_j(t'),\hat{\xi}_{j-k}(t'),u,\hat{u}(t'+(j-l)h))\\
&\qquad\quad+\hat{\Lambda}_j(t')^* f(t'+jh,\hat{\xi}_j(t'),\hat{\xi}_{j-k}(t'),u,\hat{u}(t'+(j-l)h))\\
&\qquad\quad+\chi_{\{0,\ldots,N-1-l\}}(j)L(t'+(j+l)h,\hat{\xi}_{j+l}(t'),\hat{\xi}_{j+l-k}(t'),\hat{u}(t'+(j+l)h),u)\\
&\qquad\quad+\chi_{\{0,\ldots,N-1-l\}}(j)\hat{\Lambda}_{j+l}(t')^* f(t'+(j+l)h,\hat{\xi}_{j+l}(t'),\hat{\xi}_{j+l-k}(t'),\hat{u}(t'+(j+l)h),u)
\end{aligned}$$

Redefining the adjoint and multiplier function in (39) with $t'=t-jh$, we obtain the desired minimum condition (21) for the Hamiltonian H, respectively, the local minimum condition (22) for the augmented Hamiltonian. Condition (38) immediately implies the multiplier and complementarity condition (23) in view of (39). □

Remark

Soliman, Ray [5] have discussed bang-bang and singular controls that appear in control problems, where the control $u\in\mathbb{R}^m$ is partitioned into controls $u_1\in\mathbb{R}^{m_1}$ and $u_2\in\mathbb{R}^{m_2}$ with control u_1 appearing *linearly* in the system. The control–state constraint (6) then reduces to bounds for u_1:

$$u_{1,\min}\leqslant u_1(t)\leqslant u_{1,\max}\quad\text{for } t\in[a,b],\quad u_{1,\min},u_{1,\max}\in\mathbb{R}^{m_1}$$

The minimum condition (21) shows that the control $u_1(t)$ is determined by the sign of the components of the *switching vector function*

$$\sigma(t)=H_{u_1}(t)+\chi_{[a,b]}(t+s)H_{v_1}(t+s)\tag{41}$$

whereas the control u_2 satisfies the local minimum condition (22)

$$\mathscr{H}_{u_2}(t)+\chi_{[a,b]}(t+s)\mathscr{H}_{v_2}(t+s)=0\tag{42}$$

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The CSTR control problem in Section 6 provides an example with such a partitioning of the control vector. Soliman and Ray [5] study junction phenomena for bang-bang and singular arcs. They not only give conditions under which junction results for control systems without delay carry over to delayed systems, but also give examples for delayed systems that exhibit unusual features. Some examples illustrating these unusual features have been worked out by Kern [15]. Further work is needed to fully develop the theory.

5. DISCRETIZATION, OPTIMIZATION AND CONSISTENCY OF ADJOINT EQUATIONS

Without restrictions, we may assume that the cost functional for the retarded control problem (ROCP) is given in Mayer form:

$$J(u,x)=g(x(b))$$

The reduction of the more general cost functional (1) to Mayer form proceeds as for undelayed control systems by the introduction of the additional state variable x_0 through the retarded equation:

$$\dot{x}_0(\tau)=L(t,x(t),x(t-r),u(t),u(t-s)), \quad x_0(a)=0$$

Then the cost functional (1) is rewritten in Mayer form $J(u,\tilde{x})=g(x(b))+x_0(b)$ with the new state variable $\tilde{x}=(x_0,x^*)\in\mathbb{R}^{n+1}$.

As for undelayed differential equations, there exist standard integration schemes of Euler or Runge–Kutta type for the retarded differential equation $\dot{x}(t)=f(t,x(t),x(t-r),u(t),u(t-s))$. Using a uniform step size $h>0$, it is crucial to match the delays r and s to the step size h by the following assumption:

$$\frac{r}{h}=k\in\mathbb{N}, \quad \frac{s}{h}=l\in\mathbb{N} \tag{43}$$

Note that, if h satisfies (43), any fraction h/v with $v\in\mathbb{N}$ also does. Therefore, restriction (43) is satisfied for all finer grids. The rationality assumption (17) regarding the delays r and s imply the existence of positive and coprime integers p and q satisfying

$$r=p\frac{s}{q}$$

Defining $h_{\max}:=s/q$ gives the maximum interval length for an elementary transformation interval that satisfies

$$\frac{r}{h_{\max}}=k\in\mathbb{N}, \quad \frac{s}{h_{\max}}=l\in\mathbb{N}$$

We have to assume that $b-a$ represents an integer multiple of $h_{\max}$:

$$b-a=N_{\min}h_{\max}=N_{\min}\frac{s}{q}$$

This gives a condition for the minimum grid point number for an equidistant discretization mesh

$$N_{\min}:=\frac{(b-a)q}{s}$$

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which can be multiplied by an arbitrary integer in order to refine the grid:

$$N = j \cdot N_{\min}, \quad j \in \mathbb{N}, \quad j \geqslant 1$$

Defining

$$h = \frac{b-a}{N} = \frac{b-a}{jN_{\min}} = \frac{h_{\max}}{j}$$

gives an arbitrary small and appropriate step size h for the mesh that matches condition (43).

For simplicity, we discuss Euler's integration method with step size $h=(b-a)/N$ for $N \in \mathbb{N}_+$ and grid points $t_i = a + ih$, $i = 0, 1, \ldots, N$. Using the approximations $x(t_i) \approx x_i \in \mathbb{R}^n$ and $u(t_i) \approx u_i \in \mathbb{R}^m$, we obtain the following nonlinear programming problem (NLP):

$$\text{Minimize} \quad J(u, x) = g(x_N) \tag{44}$$

subject to

$$x_i - x_{i+1} + hf(t_i, x_i, x_{i-k}, u_i, u_{i-l}) = 0, \quad i = 0, \ldots, N-1 \tag{45}$$

$$w(x_N) = 0 \tag{46}$$

$$C(t_i, x_i, u_i) \leqslant 0, \quad i = 0, \ldots, N-1 \tag{47}$$

Herein, the initial value profiles φ and ψ provide the values:

$$x_{-i} := \varphi(a - ih), \quad i = 0, \ldots, k \tag{48}$$

$$u_{-i} := \psi(a - ih), \quad i = 1, \ldots, l \tag{49}$$

The optimization variable in (NLP) is represented by the vector:

$$z := (u_0, x_1, u_1, x_2, \ldots, u_{N-1}, x_N) \in \mathbb{R}^{N(m+n)}$$

The Lagrangian function for (NLP) is given by

$$\begin{aligned}\mathscr{L}(z, \lambda, \mu, v_N) := g(x_N) + v_N^* w(x_N) + \sum_{i=0}^{N-1} \lambda_i^* (x_i - x_{i+1} + hf(t_i, x_i, x_{i-k}, u_i, u_{i-l})) \\ + \sum_{i=0}^{N-1} \mu_i^* C(t_i, x_i, u_i)\end{aligned} \tag{50}$$

with Lagrange multipliers $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) \in \mathbb{R}^{n \cdot N}$, $\lambda_i \in \mathbb{R}^n$ $(i = 0, \ldots, N-1)$, for Equations (45), $\mu = (\mu_0, \mu_1, \ldots, \mu_{N-1}) \in \mathbb{R}^{p \cdot N}$, $\mu_i \in \mathbb{R}^p$ $(i = 0, \ldots, N-1)$, for the inequality constraints (47), and $v_N \in \mathbb{R}^q$ for the boundary condition (46). Note that the variables x_{-i} $(i = 0, \ldots, k)$ and u_{-i} $(i = 1, \ldots, l)$ appearing in the Lagrangian have fixed values in view of the initial conditions (48) and (49). The Karush–Kuhn–Tucker (KKT) necessary optimality conditions for (NLP) then imply the existence of Lagrange multipliers $(\bar{\lambda}, \bar{\mu}, \bar{v}_N)$ satisfying the following equations:

$$\frac{\partial \mathscr{L}}{\partial x_i}(\bar{\lambda}, \bar{\mu}, \bar{v}_N) = 0 \quad (i = 1, \ldots, N), \quad \frac{\partial \mathscr{L}}{\partial u_i}(\bar{\lambda}, \bar{\mu}, \bar{v}_N) = 0 \quad (i = 0, \ldots, N-1)$$

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For indices $i=1,\ldots,N-k-1$, the first set of equations yields the relations

$$\begin{aligned}0=&-\bar{\lambda}^*_{i-1}+\bar{\lambda}^*_i+h\bar{\lambda}^*_i f_x(t_i,\bar{x}_i,\bar{x}_{i-k},\bar{u}_i,\bar{u}_{i-l})+h\bar{\lambda}^*_{i+k}f_y(t_{i+k},\bar{x}_{i+k},\bar{x}_i,\bar{u}_{i+k},\bar{u}_{i+k-l})\\&+\bar{\mu}^*_i C_x(t_i,\bar{x}_i,\bar{u}_i)\end{aligned} \tag{51}$$

for indices $i=N-k,\ldots,N-1$ we obtain

$$0=-\bar{\lambda}^*_{i-1}+\bar{\lambda}^*_i+h\bar{\lambda}^*_i f_x(t_i,\bar{x}_i,\bar{x}_{i-k},\bar{u}_i,\bar{u}_{i-l})+\bar{\mu}^*_i C_x(t_i,\bar{x}_i,\bar{u}_i) \tag{52}$$

and, finally, for the index $i=N$ we obtain the boundary condition:

$$0=-\bar{\lambda}_{N-1}+g_x(\bar{x}_N)+\bar{v}^*_N w_x(\bar{x}_N) \tag{53}$$

Rearranging the preceding equations, we find

$$\begin{aligned}\bar{\lambda}^*_i-\bar{\lambda}^*_{i-1}&=-h[\bar{\lambda}^*_i f_x(t_i,\bar{x}_i,\bar{x}_{i-k},\bar{u}_i,\bar{u}_{i-l})+\bar{\lambda}^*_{i+k}f_y(t_{i+k},\bar{x}_{i+k},\bar{x}_i,\bar{u}_{i+k},\bar{u}_{i+k-l})\\&\qquad+\frac{1}{h}\bar{\mu}^*_i C_x(t_i,\bar{x}_i,\bar{u}_i)],\quad i=1,\ldots,N-k-1\\ \bar{\lambda}^*_i-\bar{\lambda}^*_{i-1}&=-h[\bar{\lambda}^*_i f_x(t_i,\bar{x}_i,\bar{x}_{i-k},\bar{u}_i,\bar{u}_{i-l})+\frac{1}{h}\bar{\mu}^*_i C_x(t_i,\bar{x}_i,\bar{u}_i)],\quad i=N-k,\ldots,N-1\\ \bar{\lambda}_{N-1}&=g_x(\bar{x}_N)+\bar{v}^*_N w_x(\bar{x}_N)\end{aligned} \tag{54}$$

Furthermore, the KKT conditions imply the nonnegativity of the multipliers $\bar{\mu}_i$ and the complementarity conditions:

$$\bar{\mu}_i\geqslant 0,\quad \bar{\mu}^*_i C(t_i,\bar{x}_i,\bar{u}_i)=0\quad \text{for } i=0,\ldots,N-1 \tag{55}$$

Introducing the *scaled* multipliers μ_i/h, Equations (54) can be identified as the Euler discretization of the advanced adjoint equation (19) with boundary condition (20). Hence, the Lagrange multipliers provide the following approximations:

$$\hat{\lambda}(t_i)\approx\bar{\lambda}_i\in\mathbb{R}^n,\quad \hat{\mu}(t_i)\approx\bar{\mu}_i/h\in\mathbb{R}^p\ \ (i=0,\ldots,N-1)$$

Then the first-order conditions $\partial\mathscr{L}/\partial u_i=0, i=0,\ldots,N-1$, represent the discretized local minimum condition (22) for the augmented Hamiltonian.

To solve the optimization problem (NLP) in (44)–(47) numerically, we employ the programming language AMPL developed by Fourer *et al.* [19] in conjunction with the optimization solvers LOQO by Vanderbei [20] or IPOPT by Wächter *et al.* [21, 22]. Both solvers also provide the Lagrange multipliers and thus yield the discretized adjoint variables for the control problem (ROCP). Alternatively, the optimization problem (NLP) can be solved using the code NUDOCCCS developed by Büskens [23]. However, a comparison analysis is not an issue of this paper and should be subject of a more detailed study. The considered numerical method illustrates that a rather straightforward approach can easily be implemented by different optimizers within the AMPL code. Instead of Euler's discretization scheme, we may also use Runge–Kutta integration schemes of higher order such as the Heun approach (cf. Kern [15]). For undelayed optimal control problems, Hager [24] has given a detailed consistency analysis for higher-order Runge–Kutta schemes. A similar study should be undertaken for retarded control problems, which, however, is beyond the scope of this paper.

For notational ease in the following examples, we suppress the 'hat' to denote optimal solutions.

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6. AN ANALYTICAL EXAMPLE

We consider the following optimal control problem with the delay $r=1$ in the state and $s=2$ in the control:

$$\text{Minimize} \quad \int_0^3 (x^2(t)+u^2(t))\,\mathrm{d}t \tag{56}$$

subject to

$$\dot{x}(t)=x(t-1)u(t-2), \quad t\in[0,3] \tag{57}$$

$$x(t)=1, \quad t\in[-1,0] \tag{58}$$

$$u(t)=0, \quad t\in[-2,0] \tag{59}$$

A control–state constraint will be imposed later. The Hamiltonians (18) for this problem are identical:

$$\mathscr{H}(t,x,y,u,v)=H(t,x,y,u,v)=x^2+u^2+\lambda yv \tag{60}$$

For an optimal pair (u,x), the adjoint equations (19) in Theorem 4.2 yield

$$\begin{aligned}\dot{\lambda}(t) &= -\mathscr{H}_x(t,x(t),x(t-1),u(t),u(t-2),\lambda(t))\\ &\quad -\chi_{[0,2]}(t)\mathscr{H}_y(t+1,x(t+1),x(t),u(t+1),u(t-1),\lambda(t+1))\\ &= -2x(t)-\chi_{[0,2]}(t)\lambda(t+1)u(t-1)\end{aligned}$$

It immediately follows from (57)–(59) that

$$x(t)=1 \quad \text{for } t\in[0,2]$$

The state variable can only be influenced by the control $u(t-2)$ on the terminal interval $[2,3]$. Hence, it suffices to determine the optimal control $u(t)$ on the interval $[0,1]$. The minimum condition (21) requires the minimization of the expression

$$H(t,x(t),x(t-1),u,u(t-2))+\chi_{[0,1]}(t)H(t+2,x(t+2),x(t+1),u(t+2),u)$$

w.r.t. the control variable u for $t\in[0,3]$. For $t\in[0,1]$, we obtain $2u(t)+\lambda(t+2)x(t+1)=0$, which yields the control:

$$u(t)=-\tfrac{1}{2}\lambda(t+2)x(t+1)=-\tfrac{1}{2}\lambda(t+2), \quad t\in[0,1]$$

On the interval $[1,3]$, we immediately obtain

$$u(t)=0 \quad \text{for } t\in[1,3]$$

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Then on $[2,3]$, the adjoint and the state equations become

$$\dot{\lambda}(t) = -2x(t), \quad \dot{x}(t) = u(t-2) = -\tfrac{1}{2}\lambda(t-2+2) = -\tfrac{1}{2}\lambda(t)$$

This yields a second-order differential equation for λ

$$\ddot{\lambda}(t) = -2\dot{x}(t) = \lambda(t) \quad \text{for } t \in [2,3]$$

which has the general solution:

$$\lambda(t) = Ae^{t} + Be^{-t}, \quad x(t) = -\tfrac{1}{2}(Ae^{t} - Be^{-t})$$

The constants A and B can be determined from the transversality condition (20) and the continuity of the state $x(t)$ at $t=2$

$$\lambda(3) = 0, \quad x(2) = 1$$

from which we find

$$A = \frac{-2e^{-2}}{e^{2}+1}, \quad B = \frac{2e^{4}}{e^{2}+1}$$

Then the control u on the first segment $[0,1]$ is given by

$$u(t) = \frac{e^{-2}}{e^{2}+1}e^{t+2} - \frac{e^{4}}{e^{2}+1}e^{-(t+2)} \quad \text{for } t \in [0,1]$$

Now we evaluate the costate on the second interval $[1,2]$. The advanced differential equation

$$\begin{aligned}\dot{\lambda}(t) &= -2x(t) - \lambda(t+1)u(t-1) = -2 + \frac{1}{2}(\lambda(t+1))^{2} \\ &= -2 + \frac{1}{2}\left(\frac{-2e^{-2}}{e^{2}+1}e^{t+1} + \frac{2e^{4}}{e^{2}+1}e^{-(t+1)}\right)^{2}\end{aligned}$$

and the continuity of the costate, $\lambda(2^{-}) = \lambda(2^{+}) = 2(e^{2}-1)/e^{2}+1 \approx 1.523188311$, yield the explicit solution:

$$\begin{aligned}\lambda(t) &= \lambda(2^{+}) + \int_{2}^{t}\left(-2 + \frac{1}{2}(\lambda(\tau+1))^{2}\right) \mathrm{d}\tau \\ &= \frac{e^{2t-2} - e^{6-2t}}{(e^{2}+1)^{2}} - t \cdot \left(\frac{4e^{2}}{(e^{2}+1)^{2}} + 2\right) + \frac{4(e^{2}-1)}{(e^{2}+1)^{2}} + 6 \quad \text{for } t \in [1,2]\end{aligned}$$

Similarly, we can compute $\lambda(t)$ on $[0,1]$. Since $x(t)=1$ and $u(t)=0$ on $[0,1]$, the adjoint equation reduces to

$$\dot{\lambda}(t) = -2x(t) - \lambda(t+1)u(t-1) = -2$$

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Then the continuity of λ in $t=1$, $\lambda(1^-)=\lambda(1^+)=2(e^2-1)/(e^2+1)^2+3\approx 3.181568497$, leads to the following representation:

$$\lambda(t)=\lambda(1^+)+2-2t=-2t+\frac{2(e^2-1)}{(e^2+1)^2}+5 \quad \text{for } t\in[0,1]$$

Summing up our findings, we have obtained the optimal solution (x,u,λ):

$$\text{for } t\in[0,1]: x(t)=1, \quad u(t)=\frac{e^{-2}}{e^2+1}e^{t+2}-\frac{e^4}{e^2+1}e^{-(t+2)}$$

$$\lambda(t)=-2t+\frac{2(e^2-1)}{(e^2+1)^2}+5$$

$$\text{for } t\in[1,2]: x(t)=1, \quad u(t)=0$$

$$\lambda(t)=\frac{e^{2t-2}-e^{6-2t}}{(e^2+1)^2}-t\cdot\left(\frac{4e^2}{(e^2+1)^2}+2\right)+\frac{4(e^2-1)}{(e^2+1)^2}+6$$

$$\text{for } t\in[2,3]: x(t)=\frac{e^{-2}}{e^2+1}e^t+\frac{e^4}{e^2+1}e^{-t}, \quad u(t)=0, \quad \lambda(t)=\frac{-2e^{-2}}{e^2+1}e^t+\frac{2e^4}{e^2+1}e^{-t}$$

The analytical optimal solution allows us to determine the optimal performance index explicitly after some lengthy computations:

$$J=\int_0^3 (x^2(t)+u^2(t))\,\mathrm{d}t=\frac{3}{2}-\frac{3e^2+1}{(e^2+1)^2}+1+\frac{e^4+4e^2-1}{2(e^2+1)^2}=3-\frac{2}{e^2+1}\approx 2.761594156$$

Let us now compare the analytical solution with the numerical results that are obtained by applying the discretization and optimization methods in Section 5. We solve the Euler-discretized nonlinear optimization problem (44)–(49) using the interior point code IPOPT developed by Wächter *et al.* [21, 22] with error tolerance $\text{tol}=10^{-10}$. The starting solution is $x(t)\equiv 1$ and $u(t)\equiv 0$. Using a coarse discretization with $N=600$ grid points, we find the performance index $J(x,u)=2.765928244$ in 0.0127 central processing unit (CPU) seconds. This value means a deviation of about 0.16% from the analytical value $J=2.761594156$. Increasing the discretization by a factor 100, i.e. using $N=60\,000$ gridpoints, we obtain $J(x,u)=2.761638$ in 2.5 CPU seconds. The extremely fine discretization with $N=480\,000$ gridpoints yields after 476.2 s of CPU time an objective functional of $J(x,u)=2.761599$, which is correct in 5 decimals. In Figure 1, the numerical solution trajectories for a mesh of $N=600$ grid points are presented.

Next, we impose the mixed control–state constraint:

$$u(t)+x(t)\geqslant 0.3 \quad \text{for } t\in[0,6] \tag{61}$$

We have doubled the length of the time interval to obtain a more interesting structure of boundary arcs for the mixed control–state constraint. Here, it is not possible any more to determine an optimal solution analytically. Again, we use an Euler discretization with $N=600$ or $60\,000$ grid points. The numerical results for the optimal state, the optimal control and the adjoint variable arising from a mesh size of $N=600$ points are displayed in Figure 2. The constraint function $x(t)+u(t)$ and the corresponding multiplier $\mu(t)$ are presented in Figure 3.

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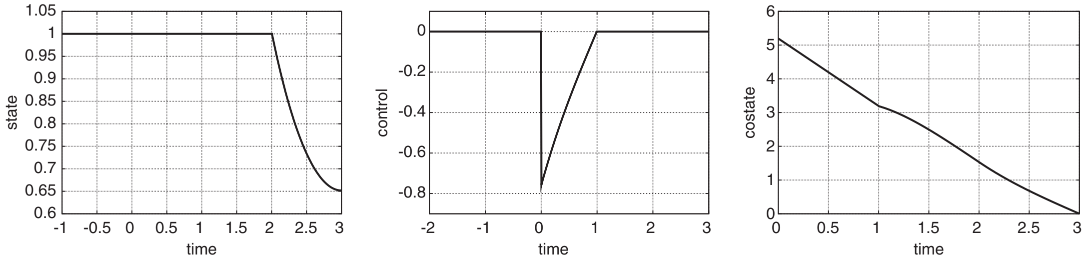

Figure 1. Optimal state $x(t)$, control $u(t)$ and adjoint $\lambda(t)$ determined numerically ($N=600$).

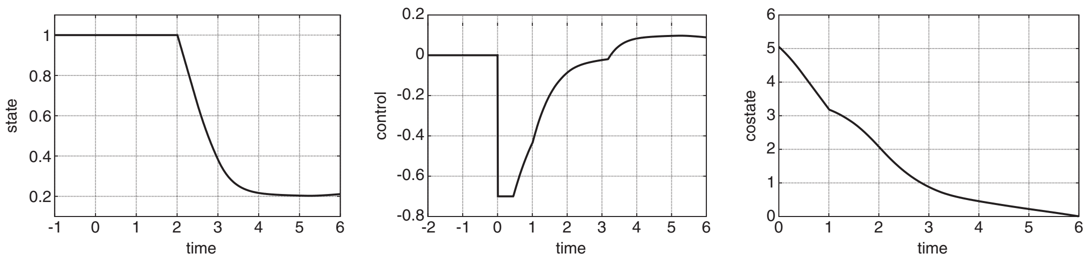

Figure 2. Constrained problem: optimal state $x(t)$, control $u(t)$ and adjoint $\lambda(t)$ (obey different time scales as initial value profiles are depicted for x and u).

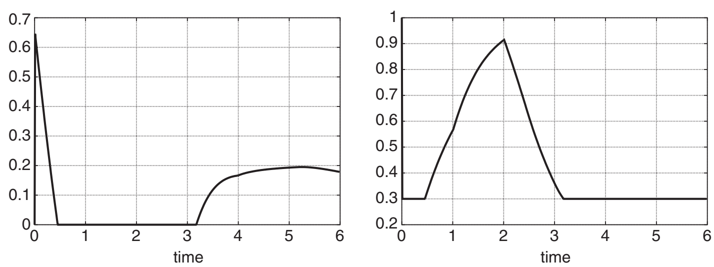

Figure 3. Constrained problem: multiplier $\mu(t)$ and function $x(t)+u(t)$.

The performance index for $N=600$ is $J(x,u)=3.121827278$ with a CPU time of 0.32 s, whereas $N=60000$ gives $J(x,u)=3.108259352$ with a CPU time of 65.8 s. The necessary optimality conditions in Theorem 4.2 provide the existence of a multiplier function $\hat{\mu}$ satisfying

$$\mu(t)\geqslant 0, \quad \mu(t)(0.3-u(t)-x(t))=0 \quad \text{for } t\in[0,6] \tag{62}$$

Figure 3 clearly shows that the computed multiplier does satisfy this condition. We have two boundary arcs $[0,t_1], t_1\approx 0.46$, and $[t_2,6], t_2\approx 3.18$, where the control–state constraint becomes active. In the interior of the boundary arcs, the multiplier $\mu(t)$ is strictly positive, whereas $\mu(t)$ vanishes on the interior arc.

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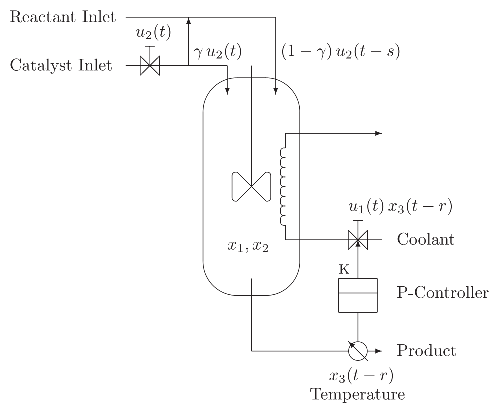

Figure 4. Continuous stirred tank reactor (CSTR) after Soliman and Ray.

7. A NONLINEAR CHEMICAL TANK REACTOR MODEL

We consider a continuous nonlinear stirred tank reactor system (CSTR) that runs an irreversible chemical reaction (Figure 4). The model is taken from Soliman and Ray [11, 25]; cf. also Bader [10]. The process is described by the relative concentration x_1 of the product, the relative concentration x_2 of the catalyst and the relative temperature in the reaction vessel. All these quantities represent the relative deviation to an equilibrium and thus held completely dimensionless. This model is based upon earlier work by Soliman and Ray [25] and has been slightly modified. The chemical agents in the vessel are stirred by an agitator and thus kept in a permanent movement. The reaction is steered by two control functions. The catalyst feed is split into a fraction $\gamma u_2(t)$ entering the vessel directly and a remaining fraction $(1-\gamma)u_2(t-s)$ entering the vessel with a time delay r due to prior mixing with the reactant feed. The temperature inside the vessel is controlled by a function $u_1(t)$ representing a time-dependent proportional gain of a heat exchanger device. The adjustment of the temperature depends on a feedback p-controller that depends on the outlet temperature $x_3(t-r)$.

Our goal is to transfer the system in a balance within a fixed time interval optimally. The objective functional essentially represents the deviation of the state to its equilibrium.

Problem (*CSTR*)
Minimize

$$J(u,x) = \int_0^{0.2} (\|x(t)\|_2^2 + 0.01u_2^2(t))\,\mathrm{d}t$$

subject to

$$\dot{x}_1(t) = -x_1(t) - R(t)$$

$$\dot{x}_2(t) = -x_2(t) + 0.9u_2(t-s) + 0.1u_2(t)$$

$$\dot{x}_3(t) = -2x_3(t) + 0.25R(t) - 1.05u_1(t)x_3(t-r)$$

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for a.e. $t \in [0, 0.2]$, where

$$R(t) = R(t, x_1(t), x_2(t), x_3(t)) := (1 + x_1(t))(1 + x_2(t)) \exp\left(\frac{25x_3(t)}{1 + x_3(t)}\right)$$

and the initial and terminal conditions, respectively, control constraint

$$x_3(t) = -0.02, \quad t \in [-r, 0)$$

$$u_2(t) = 1, \quad t \in [-s, 0)$$

$$x(0) = (0.49, -0.0002, 0.02)^*$$

$$x(0.2) = (0, 0, 0)^*$$

$$|u_1(t)| \leqslant 500, \quad t \in [0, 0.2]$$

We choose the state delay $r = 0.015$ and control delay $s = 0.02$. Bader [10] attempted to solve this CSTR problem by using shooting methods. However, due to the complicated structure of the control, Bader could only obtain a coarse approximation of the optimal solution. We solve the discretized control problem (NLP) in Section 5 by utilizing the interior point code IPOPT. The numerical computations have been carried out with $N = 16\,000$ grid points. We obtain an optimized performance index of $J = 0.011970541$ with the very large CPU time of 63 932 s. A possible explanation for this fact will be given at the end of the section. The computed optimal solution and the adjoint variables λ_1, λ_2 and λ_3 are shown in Figures 5–7.

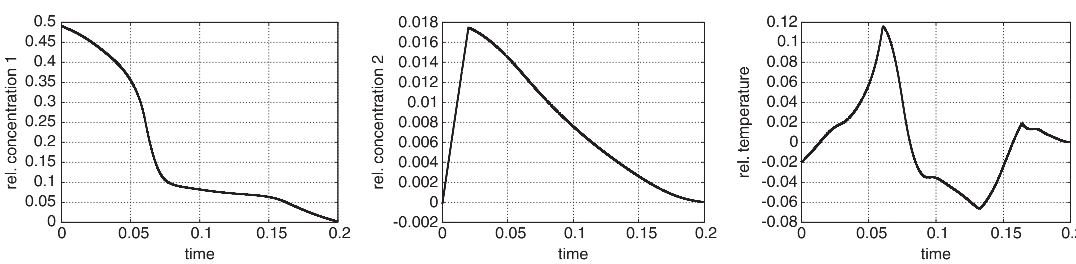

Figure 5. Optimal concentrations x_1, x_2 and optimal temperature x_3.

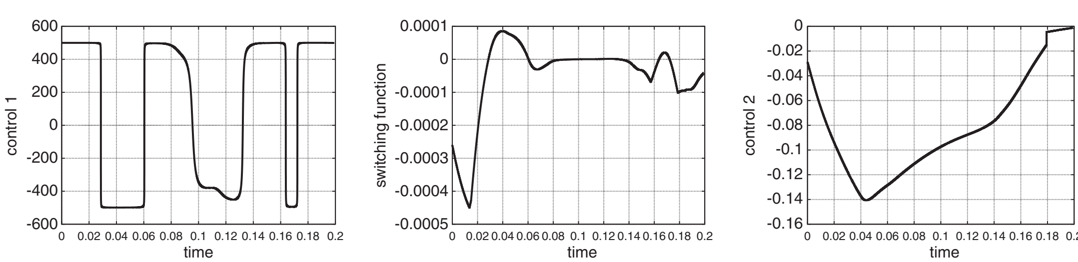

Figure 6. Optimal control function u_1, switching function σ_1 and optimal control u_2.

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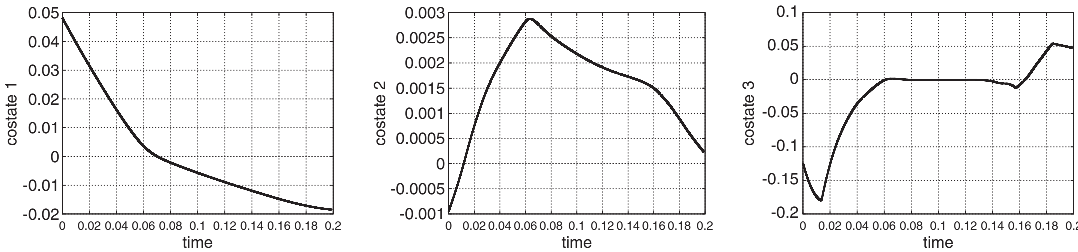

Figure 7. Adjoint variables λ_1, λ_2 and λ_3.

Let us discuss the minimum principle in Theorem 4.2 in greater detail. Since there are no mixed control–state constraints, the Hamiltonian functions (18) are given by

$$
\begin{aligned}
H(t, x_1, x_2, x_3, y_3, \lambda, u_1, u_2, v_2) &= \mathscr{H}(t, x_1, x_2, x_3, y_3, \lambda, u_1, u_2, v_2) \\
&= \|x\|_2^2 + 0.01u_2^2 \\
&\quad + \lambda_1(-x_1 - R(x)) + \lambda_2(-x_2 + 0.9v_2 + 0.1u_2) \\
&\quad + \lambda_3(-2x_3 + 0.25R(x) - u_1 y_3(x_3 + 0.125))
\end{aligned} \tag{63}
$$

The adjoint advanced ordinary differential equation (19) becomes

$$
\begin{aligned}
\dot{\lambda}_1 &= -2x_1 + \lambda_1 + (\lambda_1 - 0.25\lambda_3)\frac{\partial R(x)}{\partial x_1} \\
\dot{\lambda}_2 &= -2x_2 + \lambda_2 + (\lambda_1 - 0.25\lambda_3)\frac{\partial R(x)}{\partial x_2} \\
\dot{\lambda}_3 &= -2x_3 + 2\lambda_3 + (\lambda_1 - 0.25\lambda_3)\frac{\partial R(x)}{\partial x_3} + \lambda_3 u_1 y_3 + \chi_{[0,\,0.2-r]}(t)\lambda_3^+ u_1^+ (x_3^+ + 0.125)
\end{aligned}
$$

where $y_3 = x_3(t-r)$, $x_3^+ = x_3(t+r)$, $u_1^+ = u_1(t+r)$ and $\lambda_3^+ = \lambda_3(t+r)$. Since the terminal state $x(0.2)$ is fixed, no boundary conditions are prescribed for $\lambda(0.2)$. The computed initial value is $\lambda(0) = (0.048251674, -0.000949667, -0.123610902)$. The evaluation of the minimum condition (14) is as follows. The control component u_1 appears linearly in the system and is not delayed. Then the switching function (41) is given by

$$
\sigma_1(t) = \frac{\partial \mathscr{H}}{\partial u_1}(t) = -\lambda_3(t)x_3(t-r)(x_3(t) + 0.125) \tag{64}
$$

Bang-bang arcs of u_1 are determined by the control law:

$$
u_1(t) = \begin{Bmatrix} -500 \text{ if } \sigma(t) > 0 \\ +500 \text{ if } \sigma(t) < 0 \end{Bmatrix} \tag{65}
$$

A *singular arc* of u_1 is characterized by the property that $\sigma(t) \equiv 0$ holds on a nontrivial subinterval. However, in contrast to undelayed control problems, it is not possible to find a closed expression

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for a singular control u_1 by differentiating the switching function. Figure 6 shows that the control u_1 has six bang-bang arcs and one intermediate singular arc. We have jointly plotted $u_1(t)$ and the adequately scaled $\sigma(t)$ to demonstrate that the behavior of the switching function perfectly matches the control law (65).

As the control component u_2 appears quadratically in the cost functional and is unconstrained, it is determined uniquely by minimum condition (22), which yields

$$\frac{\partial \mathscr{H}}{\partial u_2}(t)+\chi_{[0,0.2]}(t+s)\frac{\partial \mathscr{H}}{\partial v_2}(t+s)=0 \quad \text{for } t\in[0,\ 0.2]$$

Thus, we have

$$0.02u_2(t)+0.1\lambda_2(t)+\chi_{[0,0.2]}(t+s)0.9\lambda_2(t+s)=0 \quad \text{for } t\in[0,\ 0.2]$$

which in view of $0.2-s=0.18$ determines the control u_2 by

$$u_2(t)=\left\{\begin{array}{l} -5\lambda_2(t)-\chi_{[0,0.2]}(t+s)\cdot 45\lambda_2(t+s) \text{ for } t\in[0,\ 0.18] \\ -5\lambda_2(t) \text{ for } t\in[0.18, 0.2] \end{array}\right\} \tag{66}$$

The control law (66) shows that control u_2 exhibits here a single discontinuity at $t=0.18$, provided $\lambda_2(0.18)\neq 0$ holds. This behavior is different from the undelayed case where the regularity condition of the Hamiltonian function implies the continuity of control.

Remark
The very large CPU time computing time for this CSTR problem is caused by three factors: (1) the control variable u_2 is delayed, (2) the control u_1 is not penalized in the cost functional and has a singular arc and (3) terminal conditions $x(0.2)=(0,0,0)$ are prescribed. The convergence is speeded up considerably by introducing the penalty term $0.01u_1(t)^2$ in the cost functional and deleting the terminal conditions. Dadebo and Luus [13] treat a similar CSTR with $n=4$ state variables, but no delay in the control variable and no terminal conditions. Using a fine grid with $N=20\,000$, the CPU time for computing the optimal solution of this CSTR problem is in the range of a minute.

8. OPTIMAL CONTROL OF A RENEWABLE RESOURCE

In this section, we discuss the optimal control of a logistic growth process. Such a model can be used in biology to describe pathogenic cell growth in inflammatory processes, whereas in economy it describes the interaction between production and consumption or the harvesting of a renewable resource.

A well-known example is optimal fishing, where the fact that overfishing reduces the profit for the fishing industry in the long run indicates the importance of developing of a long-time fishing strategy.

The following model is based on models developed by May [26, 27] and has been studied by Feddermann [28]. Let $x(t)$ denote the biomass population and $u(t)$ the harvesting effort. In the following control model with fixed final time $t_f>0$, only the state variable $x(t)$ has a delay $r\geqslant 0$:

$$\text{Maximize} \quad J(u,x)=\int_0^{t_f} e^{-\mathrm{d}t}(pu(t)-c_E x(t)^{-1}u(t)^3)\,\mathrm{d}t \tag{67}$$

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subject to

$$\dot{x}(t) = ax(t)\left(1 - \frac{x(t-r)}{b}\right) - u(t) \tag{68}$$

$$x(t) \equiv x_0, \quad t \in [-r, 0] \tag{69}$$

$$x(t) \geqslant x_0, \quad t \in [0, t_\mathrm{f}] \tag{70}$$

$$u(t) \geqslant 0, \quad t \in [0, t_\mathrm{f}] \tag{71}$$

A similar model with a linear cost functional was considered in Clarke and Wolenski [29] as an illustrative example to compute the sensitivity of the value function with respect to the time-lag r. The data are chosen as follows: market price $p=2$, discount rate $d=0.05$, harvesting cost $c_E=0.2$, growth rates $a=3$ and $b=5$, initial value $x_0=2$ and final time $t_\mathrm{f}=20$.

For these data, our computations show the state and control inequality constraints (70) and (71) do not become active. Hence, we do need to take into account the multiplier μ in the Hamiltonian (18) which is given here by (note that we are minimizing):

$$H(t,x,y,u,\lambda) = e^{-dt}(-pu + c_E x^{-1}u^3) + \lambda\left(ax\left(1-\frac{y}{b}\right) - u\right)$$

The adjoint equation (19) and transversality condition (20) yield

$$\dot{\lambda}(t) = c_E e^{-dt} x^{-2}(t)u^3(t) - a\lambda(t)\left(1 - \frac{x(t-r)}{b}\right) + \chi_{[0,t_\mathrm{f}]}(t+r)\lambda(t+r)\frac{a}{b}x(t+r), \quad \lambda(t_\mathrm{f}) = 0 \tag{72}$$

The minimum condition (21) implies

$$0 = \frac{\partial H}{\partial u}(t) = e^{-dt}(-p + 3c_e x^{-1}(t)u^2(t)) - \lambda(t)$$

which gives the control relation using the above data:

$$u(t) = \sqrt{\tfrac{5}{3}\exp(0.05t)\ x(t)\ \lambda(t) + \tfrac{10}{3}x(t)} \tag{73}$$

We apply the discretization methods in Section 5 and solve the resulting NLP with a mesh size of $N=40\,000$ grid points by the interior point code LOQO developed by Vanderbei [20, 30]. For different delays $r \geqslant 0$, the uncontrolled state trajectories $x(t)$ with $u(t)=0$ are shown in Figure 8(a) and are contrasted in Figure 8(b) with the *optimal* state trajectories. Optimal controls and the associated adjoint functions are depicted in Figure 9. Feddermann [28] has obtained similar results using the optimal control package NUDOCCCS developed by Büskens [23]. We conclude this section by listing the computed values of the (maximized) objective functional (67) and the CPU times for different delays:

$r=0.0$:	$J=56.290449$,	CPU: 518.442 s
$r=0.1$:	$J=56.416287$,	CPU: 1148.58 s
$r=0.2$:	$J=56.542214$,	CPU: 2656.41 s
$r=0.3$:	$J=56.662908$,	CPU: 774.235 s
$r=0.4$:	$J=56.780054$,	CPU: 13809.8 s
$r=0.5$:	$J=56.876896$,	CPU: 2688.78 s

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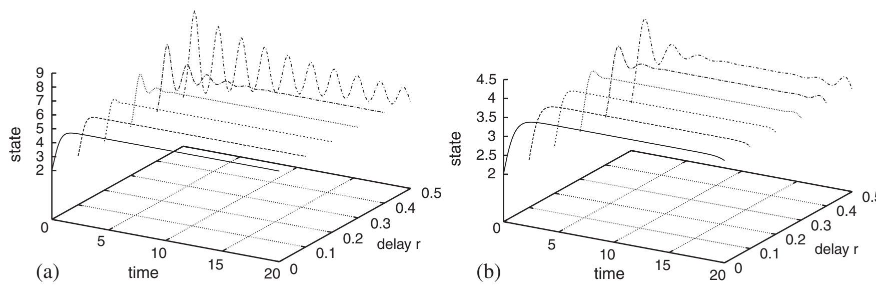

Figure 8. Delays $r=0, 0.1, 0.2, 0.3, 0.4, 0.5$: (a) uncontrolled state trajectories $x(t)$ and (b) optimal state trajectories $x(t)$.

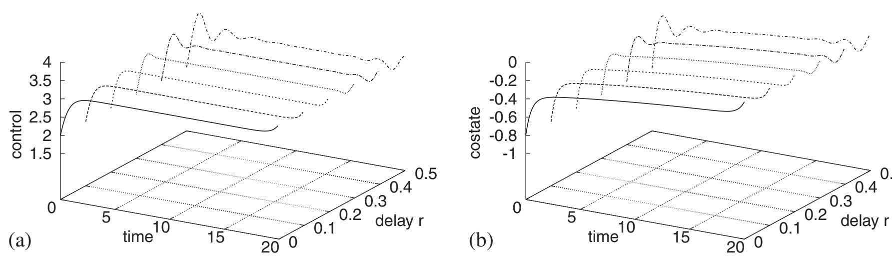

Figure 9. Delays $r=0, 0.1, 0.2, 0.3, 0.4, 0.5$: (a) optimal control $u(t)$ and (b) adjoint variable $\lambda(t)$.

Clarke, Wolenski [29] has presented conditions under which the optimal value function $V=V(r)$ is differentiable w.r.t. the delay r. It would be of interest to verify their explicit formula numerically for the derivative $\mathrm{d}V/\mathrm{d}r$ of the value function at $r=0$. The above results yield the crude approximation $\mathrm{d}V(r)/\mathrm{d}r \approx 1.2$ at $r=0$.

9. SOFTWARE VERSIONS AND CPU TYPE USED

For the numerical results presented in the preceding sections, all computations have been performed on a SUN Sparc V9 CPU under Sun Solaris 2.7 operating system. The versions of the software applied are IPOPT Version 2.2.1e, LOQO Version 6.06 and AMPL Version 20010613.

10. CONCLUSION

The purpose of this paper is twofold. Firstly, a Pontryagin-type minimum principle is derived for retarded optimal control problems with delays in the state and control variable when the control system is subject to a mixed control–state constraint. Under the assumption that the ratio of state and control delay is a rational number (this is not a restriction for numerical computation), the retarded control system was transformed to an augmented nondelayed control problem, to which the classical Pontryagin's minimum principle is applicable. Then a suitable retransformation of state, control and adjoint variables yields the minimum principle for the retarded control

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problem. The second goal is to develop efficient numerical methods for computing the optimal state, control and adjoint variables. In particular, the adjoint variables enable us to check the necessary optimality conditions with high accuracy. We present a discretization method (for simplicity only Euler's method) whereby the control problem is transcribed into a high-dimensional nonlinear programming problem. Excellent results are being obtained using the optimization solvers LOQO by Vanderbei [20] and IPOPT by Wächter *et al.* [21, 22].

Several issues for retarded control problems, which could not adequately be addressed in this paper, require further work. The theory of bang–bang and singular control problems initiated by Soliman and Ray [5] should be studied in more detail; cf. also Kern [15]. The transformation techniques in Section 4 can also be applied to retarded control problems with *pure* state inequality constraints. This approach will eventually lead to conditions, under which the multipliers associated with state constraints (cf. Angell and Kirsch [9]) are sufficiently regular. Finally, the theory of second-order sufficient conditions (cf. Chan and Yung [31] for unconstrained control problems) should be generalized to control problems with constraints and must be made amenable to numerical verification. The proof presented in this paper can be extended to problems with multiple time lags in state and control. This issue is subject to current research.

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