# Python 2.*

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**Resumen**

Este documento es una pequeña guía de Python

# Índice

## 1. Sobre el lenguaje

Differential Geometry

Preface Differential geometry studies properties of curves, surfaces, and smooth manifolds by methods of mathematical analysis. Riemannian geometry is a section of differential geometry which studies smooth manifolds with an additional structure, Riemannian metric. The main part of this book is devoted to exactly Riemannian geometry. The exception is affine differential geometry, projective differential geometry, and connections more general than the Levi-Civitá connection, which originates from a Riemannian metric. The book begins with the simplest object of differential geometry, curves in the plane. The most important characteristic of a curve at a given point is curvature. The first chapter considers both local properties of curvature (the Frenet–Serret formula and osculating circles) and global ones (the total curvature of a closed curve and the four-vertex theorem). The total oriented curvature of a closed curve is invariant with respect to a regular homotopy, and vice versa: if the total oriented curvatures of two curves are equal, then these curves are regularly homotopic (this is the Whitney–Graustein theorem). To each curve corresponds its evolute, that is, the locus of centers of all osculating circles of this curve, that is, the envelope of the family of normals.With respect to its evolute, the given curve is an involute. But the inverse operation of assigning an involute to a curve is ambiguous: every curve has a whole family of involutes (a curve orthogonal to a family of normals can be drawn through every point of a normal).We give two proofs of the isoperimetric inequality between the length of a closed curve without self-intersections and the area which it bounds. A part of the differential geometry of plane curves is not related to a Riemannian metric, that is, remains beyond the scope of Riemannian geometry. It includes enveloping families of curves, affine unimodular differential geometry, and projective differential geometry. The chapter about plane curves is concluded by elements of integral geometry: we derive a formula expressing the measure of a set of straight lines intersecting a given curve. The second chapter studies curves in spaces, first in three-dimensional space and then in many-dimensional ones. Given a curve in three-dimensional space, we define curvature and torsion, derive the Frenet–Serret formula, and define osculating planes and spheres. We also define the total curvature of a closed curve and prove v

Fenchel's theorem (that the total curvature of a closed curve is at least 2) and the Fáry-Milnor theorem (that the total curvature of a knotted closed curve is at least 4). At the end of the chapter, we consider curves in many-dimensional space, define quantities generalizing curvature and torsion for such curves, and derive the generalized Frenet–Serret formulas. The third chapter is devoted to surfaces in three-dimensional space. On a surface in R3, the first quadratic form is introduced; this is the inner product of tangent vectors to the surface. With a curve on a surface, we associate a Darboux frame and use it to define the geodesic curvature, the normal curvature, and the geodesic torsion of a curve on a surface. A geodesic on a surface is a curve with zero geodesic curvature; any shortest curve on a surface joining two given points is geodesic. On a surface in R3, the second quadratic form is also defined. In a basis with respect to which the matrix of the first quadratic form is the identity matrix and the matrix of the second quadratic form is diagonal, the diagonal elements of the second quadratic form are the principal curvatures. The Gaussian curvature of a surface is the product of principal curvatures. The principal curvatures cannot be expressed only in terms of the first quadratic form, but the Gaussian curvature can. The Gaussian curvature of a surface can also be defined in a different way, in terms of differential forms on the surface. The integral of the Gaussian curvature over a polygon on the surface can be related to the integral of the geodesic curvature over the boundary of this polygon and the sum of exterior angles of the polygon (the Gauss–Bonnet formula). We define the parallel transport of a vector along a curve and the covariant differentiation of vector fields and introduce the Riemannian curvature tensor. Using geodesics, we define the exponentialmap of a tangent space to a surface. To study properties of geodesics, we derive the first and the second variation formula. Using Jacobi vector fields and conjugate points, we find out when the length of a geodesic is not globally minimal.We prove the theorem on the local isometry of surfaces of constant Gaussian curvature. At the end of the chapter, we introduce the Laplace–Beltrami operator, which is a generalization to surfaces of the Laplace operator in the plane. In the fourth chapter we discuss two topics, hypersurfaces in many-dimensional space and connections on vector bundles. The study of connections on vector bundles is based on certain prerequisites concerning manifolds and vector bundles over manifolds. Thus, beginning in Chap. 4, the reader is supposed to have background knowledge of manifolds, tangent vectors, differential forms, vector bundles over manifolds, sections of bundles, and the inverse function theorem; all the necessary information can be found in the books [Pr2] and [Pr3]. For a hypersurface in Euclidean space, we define the Weingarten operator and use it to introduce the second, third, etc. quadratic forms. We define connections first on hypersurfaces and then on any manifolds and vector bundles over manifolds. In parallel, we introduce geodesics with respect to a given connection.We also define the curvature tensor and the torsion tensor of a given connection and introduce the curvature matrix of a connection. The fifth chapter is concerned with the general theory of Riemannian manifolds. A Riemannian manifold is a manifold whose every tangent space is equipped with

An inner product (Riemannian metric). On a Riemannian manifold, there exists a unique torsion-free connection compatible with the Riemannian metric (the Levi- Cività connection). The Riemann tensor of this connection has several symmetries (satisfies several identities). Geodesics on Riemannianmanifolds have some specific features in comparison with geodesics for arbitrary connections; in particular, any such geodesic is locally a shortest curve. A Riemannian manifold is said to be geodesically complete if all geodesics on this manifold can be extended without bound. According to the Hopf–Rinowtheorem, geodesic completeness is equivalent to the completeness of the Riemannian manifold as a metric space. The Riemann tensor can be described by using the sectional curvatures corresponding to twodimensional subspaces. For Riemannian submanifolds, as well as for hypersurfaces, we can introduce the second quadratic form and prove generalizations of Gauss' and Weingarten's formulas. An important class of Riemannian submanifolds is formed by totally geodesic submanifolds (a submanifold M is totally geodesic if each geodesic on M is also a geodesic on the ambient manifold). In the manydimensional case, just as in the case of surfaces, we obtain the first and the second variation formulas and use them to introduce Jacobi fields and define conjugate points. The chapter is concluded by a discussion of the holonomy (transformations of the tangent space obtained by the parallel transport of vectors along closed curves) and an interpretation of curvature as infinitesimal holonomy. The sixth chapter discusses the differential geometry of Lie groups, that is, manifolds endowed with a group structure consistent with the smooth structure. With each Lie group, its Lie algebra is associated, which is the tangent space at the identity element in which the multiplication of elements is defined as taking the commutator of the left-invariant vector fields corresponding to tangent vectors. The Lie algebra of a Lie group is mapped to this Lie group by the exponential map. For a Lie group and a Lie algebra, adjoint representations are defined; the adjoint representation of a Lie algebra is used to define the Killing form. On a Lie group, there exist various connections and metrics related to the group structure in various ways. On a compact Lie group, invariant integration can be defined. Some properties of Lie groups are possessed by more general spaces, namely, by homogeneous and symmetric ones. The last (seventh) chapter is devoted to some of the applications of differential geometry: comparison theorems, relationship between curvature and topological properties of manifolds, and the Laplace operator on Riemannian manifolds.

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Chapter 1 Curves in the Plane

The simplest object of differential geometry is a curve in the plane. The definition of a curve varies between different areas of mathematics. In many cases, it is natural to represent a curve as the trajectory of a moving point. In doing so, one should distinguish between a parameterized curve (t) = (x(t), y(t)) and a nonparameterized curve, which is the image of a parameterized curve, i.e., a set in the plane. The functions x(t) and y(t) are not arbitrary. They are usually assumed to be smooth. But even under this assumption, a nonparameterized curve may have corners. This can be avoided by requiring the derivatives x'(t) and y'(t) not to vanish simultaneously. In that case, the parameterized curve is said to be smooth. A smooth nonparameterized curve is the image of a smooth parameterized curve.

Geometry deals with both closed and nonclosed (that is, joining two different points) parameterized curves. A smooth (not necessarily closed) parameterized curve in the plane is a map : $[a, b]ßR^2$ such that (t) =(x(t), y(t)), where x and y are smooth functions, and $v(t) =$

racddt (t) eq $0forallt[a, b](weassumethatthederivativehavefinitelimitsatt = aandt = b).Asmoothparameterizedcurve$ : [a, b] → $R^2$ is said to be closed if (a) = (b) and v(a) = v(b).

The length of a curve : $[a, b]ßR^2$ can be defined as the limit of the lengths of polygonal chains with vertices on the curve. In more detail, we choose a partition $a = t_0 < t_1 < \ < t_n = b$ of the interval [a, b], consider the polygonal chain $P_0P_1...P_n$, where $P_i = (x(t_i), y(t_i))$, and find the limit of the lengths of these polygonal chains as the maximum of the numbers $\Delta t_i = t_it_{i1}$ tends to zero.

It is easy to show that the length of a curve : $[a, b]ßR^2$ is equal to

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

Indeed, the length of a polygonal chain $P_0P_1...P_n$ equals n

$$\sum_{i=1}^{n} \sqrt{(x_ix_{i1})^2 + (y_iy_{i1})^2}$$

where $x_i = x(t_i)$ and $y_i = y(t_i)$. Using the mean value theorem, we can rewrite this sum in the form$\sum_{i=1}^{n} \Delta t_i \sqrt{(x'(_i))^2 + (y(_i))^2}$, where the $_i$ and $_i$ are some numbers between $t_{i1}$ and $t_i.Thelimitofthissumhastherequiredform.$

The area of the figure bounded by a closed curve : $[a, b]ßR^2$ without selfintersections can be defined as the least upper bound for the areas of polygons contained in it and the greatest lower bound for the areas of polygons containing it. A figure for which these two numbers are equal is said to be squarable. The area of nonsquarable figures is not defined. Any figure bounded by a smooth closed curve is squarable.

The oriented area of a figure bounded by a parameterized closed curve without self-intersections equals the area of this figure in absolute value. When the curve is traversed counterclockwise, the oriented area is positive, and when it is traversed clockwise, the oriented area is negative.

A formula for the oriented area of a figure bounded by a parameterized (possibly self-intersecting) curve can be obtained from the following formula for the oriented area A of the triangle with vertices (0, 0), $(x_1, y_1)$, and $(x_2, y_2)$:

$$A =$$

rac12(x$_1 y_2 x_2 y_1$).

By analogy with this formula, the oriented area of a polygon with consecutive vertices $(x_i, y_i)$, i = 0, 1, . . . ,n, can be defined as

rac12 $\sum_{i=0}^{n} (x_i y_{i+1} x_{i+1} y_i)$,

where $(x_{n+1}, y_{n+1}) = (x_0, y_0)$. The polygon may have self-intersections.

The formula for the oriented area of a figure bounded by a closed curve : [a, b] → R2 can now be obtained by choosing a partition a = t0 <t1 <· · · <tn = b of the interval [a, b], considering a polygon P0P1 . . .Pn, where Pi = (x(ti), y(ti )), and passing to the limit of the oriented areas of such polygons as the maximum of the numbers ti = ti ti1 tends to zero. We set xi = x(ti) and yi = y(ti) and use the mean value theorem: xi+1 = xi +x (i)ti , yi+1 = yi +y (i)ti . As a result, we obtain the following formula for the oriented area of a polygon P0P1 . . .Pn: 1 2 n1 i=0 (x(ti)y (i ) y(ti)x (i))ti ,

where ti = ti+1 ti and i, i [ti, ti+1]. If the numbers ti are positive and the maximum among them tends to zero, then we obtain the integral expression 1 2 b a (xy yx )dt for the oriented area of the figure bounded by the curve . Integrating by parts, it is easy to showthat b a yx dt = b a xy dt. Indeed, b a yx dt + b a xy dt = (xy)|b a = 0, because the curve is closed. Thus, the oriented area A of a figure bounded by a smooth closed curve can be calculated by any of the following three equivalent formulas: A = b a yx dt = b a xy dt = 1 2 b a (xy yx )dt . (1.1) Problem 1.1 A closed curve bounds a convex figure. The endpoints of a chord of length a + b move on the curve . A point M of this chord divides it in the ratio a : b. As the chord moves, M traces a closed curve . Prove that the area of the figure bounded by the curves and equals ab. 1.1 Curvature and the Frenet–Serret Formulas Let (t) = x(t), y(t) be a smooth curve. It is often convenient to replace the parameter t by the arc length parameter s = s(t) = t 0 v( ) d, where v( ) = d () d . The arc length parameter is the length of the arc of enclosed between the points (0) and (t). For the arc length parameter, we have ds dt = v(t). Therefore, d ds = d dt · dt ds = v(t) v(t), whence