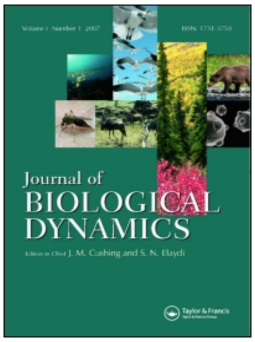

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Dynamical behaviour of a Lotka–Volterra competitive-competitive–cooperative model with feedback controls and time delays

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ABSTRACT

The aim of this paper is to investigate the dynamical behaviour of a class of three species Lotka–Volterra competitive-competitive–cooperative models with feedback controls and time delays. By developing a new analysis technique, we obtain some sufficient conditions that ensure these models have the dynamical property of permanence. We also give some sufficient conditions that guarantee the global attractivity of positive solutions for this system by constructing a new suitable Lyapunov function. Finally, we give some numerical simulations to illustrate our results in this paper.

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Lotka–Volterra model; feedback control; time delay; permanence; global attractive

1. Introduction

The modelling and analysis of the dynamics of biological populations by means of differential equations are of the primary concern in population growth problems. A well-known and extensively studied class of models in population dynamics is the Lotka–Volterra system which models certain types of interactions among various species. In the real world, the growth rate of a natural species will not often respond immediately to changes in its own species or that of an interacting species, but will rather do so after a time lag. Time delays are introduced to make the model respond better to impersonal law (see, [1–11]).

Lu et al. in [2] proposed and studied the following Lotka–Volterra system with discrete delays

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1 - a_1x_1(t) - a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12})], \\ \dot{x}_2(t) = x_2(t)[r_2 - a_2x_2(t) + a_{21}x_1(t-\tau_{21}) - a_{22}x_2(t-\tau_{22})], \end{cases} \tag{1}$$

with initial conditions

$$x_i(t) = \phi_i(t) \geq 0, t \in [-\tau_0, 0]; \phi_i(0) > 0, (i = 1, 2)$$

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where r_i, a_i, a_{ij} and τ_{ij} are constants with $a_i > 0, a_{ij} \geq 0 (i,j = 1,2)$ and $\tau_0 = \max\{\tau_{ij} : i, j = 1,2\}$, ϕ_{ij} is continuous on $[-\tau_0, 0]$. They show that delays can change the permanence for Lotka–Volterra cooperative systems. For certain delays with the same length, the delayed system has a similar property to the corresponding system without delays in the sense of permanence, but for a general delay case, the delays may destroy the permanence for the system. In 2010, Nakata and Muroya considered the following nonautonomous Lotka–Volterra cooperative systems with time delays (see, [3])

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1(t) - a_{11}^1(t)x_1(t-\tau) - a_{11}^2(t)x_1(t-2\tau) + a_{12}^1(t)x_2(t-\tau)], \\ \dot{x}_2(t) = x_2(t)[r_2(t) + a_{21}^0(t)x_1(t) + a_{21}^1(t)x_1(t-\tau) - a_{22}^0(t)x_2(t) - a_{22}^1(t)x_2(t-\tau)], \end{cases} \tag{2}$$

where $x_i(t) (i = 1,2)$ denote the density of i-species at time t, τ is a positive constant and $r_i(t), a_{ij}^l(t)$ $(1 \leq i,j \leq 2; 0 \leq l \leq 2)$ are continuous, bounded and strictly positive functions as $t \in [-\tau, +\infty)$. They obtained some sufficient conditions for the permanence of the system (2). Xu and Zu [4] investigated the following two-species delayed competitive model with stage structure and harvesting

$$\begin{cases} \dfrac{dx_1(t)}{dt} = \alpha(t)x_2(t) - \gamma x_1(t) - \alpha(t-\tau)e^{-\gamma\tau}x_2(t-\tau), \\ \dfrac{dx_2(t)}{dt} = \alpha(t-\tau)e^{-\gamma\tau}x_2(t-\tau) - \beta(t)x_2^2(t) - a_1(t)x_2(t)y(t) - E(t)x_2(t), \\ \dfrac{dy(t)}{dt} = y(t)(r_1(t) - a_2(t)x_2(t) - b(t)y(t)). \end{cases} \tag{3}$$

By using the differential inequality theory, some new sufficient conditions which ensure the permanence of the system are established. In [5], the authors considered the following competitor-competitor–mutualist Lotka–Volterra systems with discrete time delays

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1(t) - a_{11}^1(t)x_1(t-\tau) - a_{11}^2(t)x_1(t-2\tau) - a_{12}(t)x_2(t-2\tau) \\ \qquad\qquad + a_{13}(t)x_3(t-\tau)], \\ \dot{x}_2(t) = x_2(t)[r_2(t) - a_{21}(t)x_1(t-2\tau) - a_{22}^1(t)x_2(t-\tau) - a_{22}^2(t)x_2(t-2\tau) \\ \qquad\qquad + a_{23}(t)x_3(t-\tau)], \\ \dot{x}_3(t) = x_3(t)[r_3(t) + a_{31}(t)x_1(t-\tau) + a_{32}(t)x_2(t-\tau) - a_{33}^1(t)x_3(t) \\ \qquad\qquad - a_{33}^2(t)x_3(t-\tau)]. \end{cases} \tag{4}$$

And some sufficient conditions which guarantee the boundedness, permanence and global attraction for system (4) were obtained. In 2011, Xu et al. [6] studied the dynamical behaviours for the following Lokta–Volterra predator–prey model with two delays

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1 - a_{11}x_1(t-\tau_1) - a_{12}y(t-\tau_2)], \\ \dot{x}_2(t) = x_2(t)[-r_2 - a_{21}x_1(t-\tau_1) - a_{22}y(t-\tau_2)]. \end{cases} \tag{5}$$

Its linear stability and Hopf bifurcation are investigated by analysing the associated characteristic transcendental equation. Some explicit formulate for determining the stability and the direction of the Hopf bifurcation periodic solutions are obtained by using normal form theory and centre manifold theory.

One can find that an ecosystem in the real world is continuously distributed by some forces, which can result in changes in the biological parameters such as survival rates. The practical interest in ecology is the question of whether or not an ecosystem can withstand those disturbances which persist for a finite period of time. In the control systems, we regard the disturbance functions as control variables. These are of significance in the control of ecology balance. One of the methods to research it is to alter the system structurally by introducing feedback control variables. The feedback control mechanism might be implemented by means of some biological control schemes or harvesting procedure. In fact, during the last decade, the qualitative behaviour of the population dynamics with feedback control has been studied extensively. In 2009, Nie et al. [12] considered the following non-autonomous predator–prey Lotka–Volterra system with feedback controls

$$\begin{cases} \dot{x}_1(t) = x_1(t)[b_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) + c_1(t)u_1(t)], \\ \dot{x}_2(t) = x_2(t)[-b_2(t) + a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - c_2(t)u_2(t)], \\ \dot{u}_1(t) = f_1(t) - e_1(t)u_1(t) - d_1(t)x_1(t), \\ \dot{u}_2(t) = -e_2(t)u_2(t) + d_2(t)x_2(t), \end{cases} \tag{6}$$

where $x_1(t)$ is the prey population density and $x_2(t)$ is the predator population density, $b_1(t)$ and $a_{11}(t)$ are the intrinsic growth rate and density-dependent coefficient of the prey, respectively; $b_2(t)$ and $a_{22}(t)$ are the intrinsic growth rate and density-dependent coefficient of the predator, respectively; $a_{12}(t)$ is the capturing rate of the predator and $a_{21}(t)$ is the rate of conversion of nutrient into the reproduction of the predator; $u_i(t)(i = 1, 2)$ are control variables. They studied whether or not the feedback controls have an influence on the permanence of a positive solution of the general non-autonomous predator–prey Lotka–Volterra-type systems and establish the general criteria on the permanence of system (6), which is independent of some feedback controls. In addition, by constructing an appropriate Lyapunov function, some sufficient conditions are obtained for the global stability of any positive solution to system (6). In [13], Yang, Wang and Chen proposed and studied the following cooperation system with feedback controls

$$\begin{cases} \dot{x}_1(t) = x_1(b_1 - a_{11}x_1(t) + a_{12}x_2(t) - \alpha_1 u_1(t)), \\ \dot{x}_2(t) = x_2(b_2 + a_{21}x_1(t) - a_{22}x_2(t) - \alpha_2 u_2(t)), \\ \dot{u}_1(t) = -\eta_1 u_1(t) + a_1 x_1(t), \\ \dot{u}_2(t) = -\eta_2 u_2(t) + a_2 x_2(t), \end{cases} \tag{7}$$

where $b_i, a_{ij}, \alpha_i, \eta_i, a_i, i, j = 1, 2$ are positive constants. $x_i(t), (i = 1, 2)$ are the densities of the species at time t, $u_i(t), (i = 1, 2)$ denote feedback controls. They showed that if system (7) has a positive equilibrium, then feedback controls can only influence the position of the positive equilibrium, and have no influence on the stability. In 2018, Wang et al. [14] considered the following three-species Lokta–Volterra predator–prey system with feedback

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1(t) - a_{11}(t)x_1(t) - \dfrac{a_{12}(t)x_2(t)}{b_{12}(t)x_2(t) + x_1(t)} \\ \qquad -\dfrac{a_{13}(t)x_3(t)}{b_{13}(t)x_3(t) + x_1(t)} - d_1(t)u_1(t)], \\ \dot{x}_2(t) = x_2(t)[-r_2(t) + \dfrac{a_{21}(t)x_1(t)}{b_{12}(t)x_2(t) + x_1(t)} - a_{23}(t)x_3(t) + d_2(t)u_2(t)], \\ \dot{x}_3(t) = x_3(t)[-r_3(t) + \dfrac{a_{31}(t)x_1(t)}{b_{13}(t)x_3(t) + x_1(t)} - a_{32}(t)x_2(t) + d_3(t)u_3(t)], \\ \dot{u}_1(t) = e_1(t) - f_1(t)u_1(t) + q_1(t)x_1(t), \\ \dot{u}_2(t) = e_2(t) - f_2(t)u_2(t) - q_2(t)x_2(t), \\ \dot{u}_3(t) = e_3(t) - f_3(t)u_3(t) - q_3(t)x_3(t), \end{cases} \tag{8}$$

By using a comparison theorem and constructing a suitable Lyapunov function as well as developing some new analysis techniques, the authors established a set of easily verifiable sufficient conditions which guarantee the permanence of the system and the global attractivity of positive solution for the predator–prey system (8). Furthermore, some conditions for the existence, uniqueness and stability of a positive periodic solution for the corresponding periodic system were obtained by using the fixed point theory and some new analysis method. More work on feedback controls can be found in (cf. [15–21] and the references cited therein). As is known to all, the Lotka–Volterra system with time delay and feedback control can respond better to impersonal law. In recent years, more and more attention has been paid to some ecosystem models with both feedback control and time delay (see, [22–27]). In 1993, Gopalsamy et al. [22] studied a class of autonomous single-species ecosystem with feedback control and time delay

$$\begin{cases} \dfrac{dn(t)}{dt} = rn(t)[1 - (\dfrac{a_1 n(t) + a_2 n(t-\tau)}{K}) - cu(t)], \\ \dfrac{du(t)}{dt} = -au(t) + bn(n-\tau), \end{cases} \tag{9}$$

where $u(t)$ denotes an indirect control variable, $\tau, a_2, a, b, c, r \in (0, \infty)$ and $a_1 \in [0, \infty)$. Some sufficient conditions were obtained for the global asymptotic stability of the positive equilibrium for the system (9). In order to show that whether the feedback control variables play an essential role on the persistent property of Lotka–Volterra cooperative systems or not, Xu and Chen [26] established and studied the following system with time delay and feedback control

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1(t) - a_1(t)x_1(t) - a_{11}(t)x_1(t-\tau) + a_{12}(t)x_2(t-\tau) - b_1(t)u_1(t-\sigma_1)], \\ \dot{x}_2(t) = x_2(t)[r_2(t) - a_2(t)x_2(t) + a_{21}(t)x_1(t) - a_{22}(t)x_2(t-\tau) - b_2(t)u_2(t-\sigma_2)], \\ \dot{u}_1(t) = -c_1(t)u_1(t) + d_1(t)x_1(t-\eta_1), \\ \dot{u}_2(t) = -c_2(t)u_2(t) + d_2(t)x_2(t-\eta_2). \end{cases} \tag{10}$$

They obtained some new sufficient conditions which ensured the system to be permanent, and showed that feedback control variables had no influence on the permanence of the system. In 2017, Xu and Li [27] considered the following competition and cooperation model of two enterprises with multiple delays and feedback controls

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1(t) - a_1(t)x_1(t) - b_1(t)(x_2(t) - c_2(t))^2 - e_1(t)u_1(t - \tau_1(t))], \\ \dfrac{du_1(t)}{dt} = -\alpha_1(t)u_1(t) + \beta_1(t)x_1(t - \sigma_1(t)), \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2(t) - a_2(t)x_2(t) + b_2(t)(x_1(t) - c_1(t))^2 - e_2(t)u_2(t - \tau_2(t))], \\ \dfrac{du_2(t)}{dt} = -\alpha_2(t)u_2(t) + \beta_2(t)x_2(t - \sigma_2(t)). \end{cases} \tag{11}$$

Some sufficient conditions that guarantee the existence of a unique globally asymptotically stable nonnegative almost periodic solution for the system (11) were obtained by constructing a suitable Lyapunov functional and using the comparison theorem of differential equations.

However, as far as we know, no work has been done until now for the three-species Lotka–Volterra system with feedback control and time delay. Motivated by the above work, we propose and investigate the following three species Lotka–Volterra competitive–cooperative model with feedback controls and time delays

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1(t) - a_{11}^1(t)x_1(t-\tau) - a_{11}^2(t)x_1(t-2\tau) - a_{12}(t)x_2(t-2\tau) \\ \qquad + a_{13}(t)x_3(t-\tau) - d_1(t)u_1(t)], \\ \dot{x}_2(t) = x_2(t)[r_2(t) - a_{21}(t)x_1(t-2\tau) - a_{22}^1(t)x_2(t-\tau) - a_{22}^2(t)x_2(t-2\tau) \\ \qquad + a_{23}(t)x_3(t-\tau) + d_2(t)u_2(t)], \\ \dot{x}_3(t) = x_3(t)[r_3(t) + a_{31}(t)x_1(t-\tau) + a_{32}(t)x_2(t-\tau) - a_{33}^1(t)x_3(t) \\ \qquad - a_{33}^2(t)x_3(t-\tau) + d_3(t)u_3(t)], \\ \dot{u}_1(t) = e_1(t) - f_1(t)u_1(t) + q_1(t)x_1(t), \\ \dot{u}_2(t) = e_2(t) - f_2(t)u_2(t) - q_2(t)x_2(t), \\ \dot{u}_3(t) = e_3(t) - f_3(t)u_3(t) - q_3(t)x_3(t), \end{cases} \tag{12}$$

where $x_i(t), i = 1, 2, 3$ stands for the densities of the species at time t, and $u_i(t), i = 1, 2, 3$ are the indirect control variables. The given coefficients $a_{12}(t), a_{13}(t), a_{21}(t), a_{23}(t), a_{31}(t), a_{32}(t), r_i(t), d_i(t), e_i(t), f_i(t), q_i(t) (i = 1, 2, 3), a_{11}^l(t), a_{22}^l(t), a_{33}^l(t), (l = 1, 2)$ are continuous, bounded and strictly positive functions on $[0, \infty)$. $r_i(t), (i = 1, 2, 3)$ denote the intrinsic growth rate of the i-th species at time t. Especially, $a_{ii}^l(t), (i = 1, 2, 3,\ l = 1, 2)$ denote the internal competitive coefficient of the three species at time t. $a_{12}(t), a_{21}(t)$ are the competitive coefficient of species $x_1(t)$ and $x_2(t)$ at time t, respectively. $a_{13}(t), a_{23}(t), a_{31}(t), a_{32}(t)$ are the cooperative coefficient of species $x_i(t), (i = 1, 2, 3)$ at time t, respectively. τ is a positive constant.

Due to the biological interpretation of the system (8), it is reasonable to consider only positive solution of the system (8), in other words, take admissible initial conditions

$$\begin{aligned} x_i(t) &= \phi_i(t), i = 1, 2, 3 \text{ for } t \in [-2\tau, 0) \text{ and } \phi_1(0) > 0, \\ u_i(t) &= \varphi_i(t), i = 1, 2, 3 \text{ for } t > 0 \text{ and } \varphi_1(0) > 0. \end{aligned} \tag{13}$$

Obviously, the solutions of system (12) with the initial values (13) are positive for all $t \geq 0$.

Comparing the systems (4) and (12), one could see that we introduce the control variables $u_i(t)$ $(i = 1, 2, 3)$ so as to implement a feedback control mechanism. Our main purpose in this paper is to establish some sufficient conditions which ensure the system to be permanence and global attractivity by constructing a new appropriate Lyapunov function and developing a new analysis technique. This paper is organized as follows: In Section 2, we provide the conditions for the permanence to system (12). In Section 3, by constructing a nonnegative Lyapunov function, we shall derive sufficient conditions for the global attractivity of positive solution for the Lotka–Volterra Competitive-Competitive–Cooperative model (12). Some numerical simulations to the system are given in Section 4.

2. Permanence

In order to establish a permanence result for the system (12), we need some preparations. Firstly, we introduce the following notations and definitions. Given a function $g(t)$ defined on $[t_0 + \infty)$, we set

$$g^m = \sup\{g(t)|t_0 < t < +\infty\},\ g^l = \inf\{g(t)|t_0 < t < +\infty\}.$$

Definition 2.1: System (12) is called permanent, if there exist positive constants $M_i, N_i, m_i, n_i (i = 1, 2, 3)$, and T, such that $m_i \le x_i(t) \le M_i, n_i \le u_i(t) \le N_i$ for any positive solution $Z(t) = (x_1(t), x_2(t), x_3(t), u_1(t), u_2(t), u_3(t))$ of (8) as $t > T$.

As a direct corollary of Lemma 2.1 of Chen [1], we have.

Lemma 2.1: *If $a > 0, b > 0$ and $\dot{x} \ge b - ax$, when $t \ge 0$ and $x(0) > 0$, we have*

$$\liminf_{t\to+\infty} x(t) \ge b/a.$$

If $a > 0, b > 0$ and $\dot{x} \le b - ax$, when $t \ge 0$ and $x(0) > 0$, we have

$$\limsup_{t\to+\infty} x(t) \le b/a.$$

Lemma 2.2 (see [3], Lemma 2.2): *Assume that for $y(t) > 0$, it holds that*

$$\dot{y}(t) \le y(t)(\lambda - \sum_{l=0}^{m} \mu^l y(t - l\tau)) + D,$$

with initial conditions $y(t) = \phi(t) \ge 0$ for $t \in [-m\tau, 0)$ and $\phi(0) > 0$, where

$$\lambda > 0, \mu^l \ge 0 (l = 0, 1, 2, \cdots, m), \mu = \sum_{l=0}^{m} \mu^l > 0 \text{ and } D \ge 0,$$

are constants. Then there exist a positive constant $M_y < +\infty$ such that

$$\limsup_{t\to+\infty} y(t) \le M_y = -\frac{D}{\lambda} + (\frac{D}{\lambda} + y^*) \exp(\lambda m\tau) < +\infty, \tag{14}$$

where $y = y^*$ is the unique solution of equation $y(\lambda - \mu y) + D = 0$.

Lemma 2.3 (see [3], Lemma 2.3): *Assume that for $y(t) > 0$, it holds that*

$$\dot{y}(t) \geq y(t)[\lambda - \sum_{l=0}^{m} \mu^l y(t - l\tau)].$$

If the system (14) holds, then, there exists a positive constant $m_y > 0$ such that for $\mu = \sum_{l=0}^{m} \mu^l > 0$

$$\liminf_{t\to+\infty} y(t) \geq m_y = \frac{\lambda}{\mu} \exp\{(\lambda - \mu M_y)m\tau\} > 0.$$

For the system (12), let

$$N_2 = \frac{e_2^m}{f_2^l}, N_3 = \frac{e_3^m}{f_3^l},$$

$$P_1 = \frac{(r_1^m + r_3^m + d_3^m N_3)^2}{a_{11}^{1l} a_{33}^{2l}} \exp\{(r_1^m + r_3^m + d_3^m N_3)\tau\},$$

$$P_2 = \frac{(r_2^m + r_3^m + d_2^m N_2 + d_3^m N_3)^2}{a_{22}^{1l} a_{33}^{2l}} \exp\{(r_2^m + r_3^m + d_2^m N_2 + d_3^m N_3)\tau\},$$

$$M_1 = -\frac{a_{13}^m P_1}{r_1^m} + (\frac{a_{13}^m P_1}{r_1^m} + x_1^*) \exp(2r_1^m \tau), N_1 = \frac{e_1^m + q_1^m M_1}{f_1^l},$$

$$M_2 = -\frac{a_{23}^m P_2}{r_2^m + d_2^m N_2} + (\frac{a_{23}^m P_2}{r_2^m + d_2^m N_2} + x_2^*) \exp(2(r_2^m + d_2^m N_2)\tau),$$

$$M_3 = x_3^* \exp((r_3^m + (a_{31}^m + a_{32}^m + d_3^m) \max\{M_1, M_2, N_3\})\tau),$$

$$m_1 = \frac{r_1^l - a_{12}^m M_2 - d_1^m N_1}{a_{11}^{1m} + a_{11}^{2m}} \exp[(r_1^l - a_{12}^m M_2 - d_1^m N_1 - (a_{11}^{1m} + a_{11}^{2m})M_1)2\tau],$$

$$m_2 = \frac{r_2^l - a_{21}^m M_1}{a_{22}^{1m} + a_{22}^{2m}} \exp[(r_2^l - a_{21}^m M_1 - (a_{22}^{1m} + a_{22}^{2m})M_2)2\tau],$$

$$m_3 = \frac{r_3^l + a_{31}^l m_1 + a_{32}^l m_2}{a_{33}^{1m} + a_{33}^{2m}} \exp[(r_3^l + a_{31}^l m_1 + a_{32}^l m_2 - (a_{33}^{1m} + a_{33}^{2m})M_3)\tau],$$

$$n_1 = \frac{e_1^l + q_1^l m_1}{f_1^m}, n_2 = \frac{e_2^l - q_2^m M_2}{f_2^m}, n_3 = \frac{e_3^l - q_3^m M_3}{f_3^m},$$

where x_1^* is the unique positive solution of equation $x_1[r_1^m - (a_{11}^{1l} + a_{11}^{2l})x_1] + a_{13}^m P_1 = 0$, x_2^* is the unique positive solution of equation $x_2[r_2^m + d_2^m N_2 - (a_{22}^{1l} + a_{22}^{2l})x_2] + a_{23}^m P_2 = 0$, and x_3^* is the unique positive solution of equation $x_3(t)[r_3^m + (a_{31}^m + a_{32}^m + d_3^m) \max\{M_1, M_2, N_3\} - (a_{33}^{1l} + a_{33}^{2l})x_3] = 0$.

Theorem 2.1: *Assume the following conditions satisfy*

$$(H_1)\ a_{11}^{2l} > a_{31}^m,\ (H_2)\ a_{12}^l > a_{32}^m,\ (H_3)\ a_{33}^{1l} > a_{13}^m,\ (H_4)\ a_{21}^l > a_{31}^m$$
$$(H_5)\ a_{22}^{2l} > a_{32}^m,\ (H_6)\ a_{33}^{1l} > a_{23}^m,\ (H_7)\ r_1^l > a_{12}^m M_2 + d_1^m N_1,$$
$$(H_8)\ r_2^l > a_{21}^m M_1,\ (H_9)\ e_2^l > q_2^m M_2,\ (H_{10})\ e_3^l > q_3^m M_3.$$

Then the system (12) is permanent.

Proof. By the fifth equation of system (8), we have

$$\dot{u}_2(t) \le e_2(t) - f_2(t)u_2(t) \le e_2^m - f_2^l u_2(t).$$

$$\limsup_{t\to+\infty} u_2(t) \le \frac{e_2^m}{f_2^l} = N_2. \tag{15}$$

Moreover, similar to the above discussion of the fifth equation of system (12), from the sixth of system (12), we have

$$\limsup_{t\to+\infty} u_3(t) \le \frac{e_3^m}{f_3^l} = N_3. \tag{16}$$

Next, suppose that $\limsup\limits_{t\to+\infty} x_1(t)x_3(t-\tau) = +\infty$, then there exists a time sequence $\{t_k\}_{k=1}^{\infty}$ such that

$$\limsup_{k\to+\infty} x_1(t_k)x_3(t_k-\tau) = +\infty, \tag{17}$$

and

$$\frac{d}{dt}(x_1(t)x_3(t-\tau))|_{t=t_k} \ge 0, \quad k = 1, 2, \cdots. \tag{18}$$

From system (12), one has

$$\begin{aligned}\frac{d}{dt}(x_1(t)x_3(t-\tau)) &= x_1(t)x_3(t-\tau)[r_1(t) + r_3(t-\tau) - a_{11}^1(t)x_1(t-\tau)\\
&\quad - a_{11}^2(t)x_1(t-2\tau) - a_{12}(t)x_2(t-2\tau) + a_{13}(t)x_3(t-\tau)\\
&\quad - d_1(t)u_1(t) + a_{31}(t-\tau)x_1(t-2\tau) + a_{32}(t-\tau)x_2(t-2\tau)\\
&\quad - a_{33}^1(t-\tau)x_3(t-\tau) - a_{33}^2(t-\tau)x_3(t-2\tau) + d_3(t-\tau)u_3(t-\tau)]\\
&\le x_1(t)x_3(t-\tau)[r_1^m + r_3^m + d_3^m N_3 - (a_{11}^{2l} - a_{31}^m)x_1(t-2\tau)\\
&\quad - (a_{12}^l - a_{32}^m)x_2(t-2\tau) - (a_{33}^{1l} - a_{13}^m)x_3(t-\tau)\\
&\quad - a_{11}^{1l}x_1(t-\tau) - a_{33}^{2l}x_3(t-2\tau) - d_1^l u_1(t)].\end{aligned} \tag{19}$$

From (18), (19), we can obtain

$$\begin{aligned}&(a_{11}^{2l} - a_{31}^m)x_1(t_k-2\tau) + (a_{12}^l - a_{32}^m)x_2(t_k-2\tau) + (a_{33}^{1l} - a_{13}^m)x_3(t_k-\tau)\\
&+d_1^l u_1(t_k) + a_{11}^{1l}x_1(t_k-\tau) + a_{33}^{2l}x_3(t_k-2\tau) \le r_1^m + r_3^m + d_3^m N_3.\end{aligned} \tag{20}$$

Thus, by the assumption of the Theorem 2.1 and (20), it holds that

$$x_1(t_k-\tau) \le \frac{r_1^m + r_3^m + d_3^m N_3}{a_{11}^{1l}}, \quad x_3(t_k-2\tau) \le \frac{r_1^m + r_3^m + d_3^m N_3}{a_{33}^{2l}}.$$

Moreover, by (19) and the assumption of the Theorem 2.1, it follows that

$$\frac{d}{dt}(x_1(t)x_3(t-\tau)) \le x_1(t)x_3(t-\tau)[r_1^m + r_3^m + d_3^m N_3]. \tag{21}$$

By integrating both sides of (21) from $t_k-\tau$ to t_k further, we have

$$x_1(t_k)x_3(t_k-\tau) \le x_1(t_k-\tau)x_3(t_k-2\tau)\exp\{(r_1^m + r_3^m + d_3^m N_3)\tau\}.$$

Therefore

$$x_1(t_k)x_3(t_k-\tau) \le \frac{(r_1^m + r_3^m + d_3^m N_3)^2}{a_{11}^{1l}a_{33}^{2l}}\exp\{(r_1^m + r_3^m + d_3^m N_3)\tau\}.$$

However, it leads to a contradiction with (17). Thus, we have

$$\limsup_{t\to+\infty}(x_1(t)x_3(t-\tau)) \le P_1 = \frac{(r_1^m + r_3^m + d_3^m N_3)^2}{a_{11}^{1l}a_{33}^{2l}}\exp\{(r_1^m + r_3^m + d_3^m N_3)\tau\}. \tag{22}$$

Moreover, similar to the above discussion, we can also obtain that

$$\begin{aligned}\limsup_{t\to+\infty}&(x_2(t)x_3(t-\tau)) \le P_2 \\ &= \frac{(r_2^m + r_3^m + d_2^m N_2 + d_3^m N_3)^2}{a_{22}^{1l}a_{33}^{2l}}\exp\{(r_2^m + r_3^m + d_2^m N_2 + d_3^m N_3)\tau\}.\end{aligned} \tag{23}$$

According to the first equation of system (21) and (22), it follows that

$$\begin{aligned}\dot{x}_1(t) &\le x_1(t)[r_1 - a_{11}^1 x_1(t-\tau) - a_{11}^2 x_1(t-2\tau) + a_{13}x_3(t-\tau)] \\ &\le x_1(t)[r_1^m - a_{11}^{1l}x_1(t-\tau) - a_{11}^{2l}x_1(t-2\tau)] + a_{13}^m P_1.\end{aligned}$$

From Lemma 2.2, we have

$$\limsup_{t\to+\infty} x_1(t) \le -\frac{a_{13}^m P_1}{r_1^m} + \left(\frac{a_{13}^m P_1}{r_1^m} + x_1^*\right)\exp(2r_1^m\tau) = M_1, \tag{24}$$

where x_1^* is the unique positive solution of equation $x_1[r_1^m - (a_{11}^{1l} + a_{11}^{2l})x_1] + a_{13}^m P_1 = 0$.

Similar to the above discussion of the first equation of system (12), from (23) and the second equation of system (12), we obtain

$$\begin{aligned}\dot{x}_2(t) &\leq x_2(t)[r_2 - a_{22}^1 x_2(t-\tau) - a_{22}^2 x_2(t-2\tau) + a_{23}x_3(t-\tau) + d_2u_2(t)]\\ &\leq x_2(t)[r_2^m - a_{22}^{1l}x_2(t-\tau) - a_{22}^{2l}x_2(t-2\tau) + d_2^m N_2] + a_{23}^m P_2.\end{aligned}$$

So, we have

$$\limsup_{t\to+\infty} x_2(t) \leq -\frac{a_{23}^m P_2}{r_2^m + d_2^m N_2} + (\frac{a_{23}^m P_2}{r_2^m + d_2^m N_2} + x_2^*)\exp((r_2^m + d_2^m N_2)2\tau) = M_2. \quad (25)$$

where x_2^* is the unique positive solution of the following equation

$$x_2[r_2^m + d_2^m N_2 - (a_{22}^{1l} + a_{22}^{2l})x_2] + a_{23}^m P_2 = 0.$$

From the third equation of system (12), we obtain

$$\begin{aligned}\dot{x}_3(t) &= x_3(t)[r_3 + a_{31}x_1(t-\tau) + a_{32}x_2(t-\tau) - a_{33}^1 x_3(t) - a_{33}^2 x_3(t-\tau) + d_3u_3(t)]\\ &\leq x_3(t)[r_3^m + a_{31}^m M_1 + a_{32}^m M_2 + d_3^m N_3 - a_{33}^{1l}x_3(t) - a_{33}^{2l}x_3(t-\tau)]\\ &\leq x_3(t)[r_3^m + (a_{31}^m + a_{32}^m + d_3^m)\max\{M_1, M_2, N_3\} - a_{33}^{1l}x_3(t) - a_{33}^{2l}x_3(t-\tau)].\end{aligned}$$

By Lemma 2.2, it holds that

$$\limsup_{t\to+\infty} x_3(t) \leq x_3^* \exp((r_3^m + (a_{31}^m + a_{32}^m + d_3^m)\max\{M_1, M_2, N_3\})\tau) = M_3. \quad (26)$$

where x_3^* is the unique positive solution of the following equation

$$x_3(t)[r_3^m + (a_{31}^m + a_{32}^m + d_3^m)\max\{M_1, M_2, N_3\} - (a_{33}^{1l} + a_{33}^{2l})x_3] = 0.$$

From the fourth equation of system (12), one has

$$\dot{u}_1(t) \leq e_1^m - f_1^l u_1(t) + q_1^m M_1.$$

Thus, according to Lemma 2.1, it follows that

$$\limsup_{t\to+\infty} u_1(t) \leq \frac{e_1^m + q_1^m M_1}{f_1^l} = N_1. \quad (27)$$

On the contrary, from the first equation of system (12), we have

$$\begin{aligned}\dot{x}_1(t) &\geq x_1(t)[r_1 - a_{11}^1 x_1(t-\tau) - a_{11}^2 x_1(t-2\tau) - a_{12}x_2(t-2\tau) - d_1(t)u_1(t)]\\ &\geq x_1(t)[r_1^l - a_{11}^{1m}x_1(t-\tau) - a_{11}^{2m}x_1(t-2\tau) - a_{12}^m M_2 - d_1^m N_1].\end{aligned}$$

By Lemma 2.3, one easily verifies that

$$\begin{aligned}&\liminf_{t\to+\infty} x_1(t)\\ &\geq \frac{r_1^l - a_{12}^m M_2 - d_1^m N_1}{a_{11}^{1m} + a_{11}^{2m}}\exp[2(r_1^l - a_{12}^m M_2 - d_1^m N_1 - (a_{11}^{1m} + a_{11}^{2m})M_1)\tau] = m_1.\end{aligned} \quad (28)$$

By the same way, from the second and third equations of system (12), we deduce

$$\begin{aligned}\dot{x}_2(t) &\geq x_2(t)[r_2 - a_{21}x_1(t-2\tau) - a_{22}^1 x_2(t-\tau) - a_{22}^2 x_2(t-2\tau)]\\ &\geq x_2(t)[r_2^l - a_{21}^m M_1 - a_{22}^{1m} x_2(t-\tau) - a_{22}^{2m} x_2(t-2\tau)],\\ \dot{x}_3(t) &\geq x_3(t)[r_3 + a_{31}x_1(t-\tau) + a_{32}x_2(t-\tau) - a_{33}^1 x_3(t) - a_{33}^2 x_3(t-\tau)]\\ &\geq x_3(t)[r_3^l + a_{31}^l m_1 + a_{32}^l m_2 - a_{33}^{1m} x_3(t) - a_{33}^{2m} x_3(t-\tau)].\end{aligned}$$

Thus, by Lemma 2.3, we have

$$\liminf_{t\to+\infty} x_2(t) \geq \frac{r_2^l - a_{21}^m M_1}{a_{22}^{1m} + a_{22}^{2m}} \exp[(r_2^l - a_{21}^m M_1 - (a_{22}^{1m} + a_{22}^{2m})M_2)2\tau] = m_2, \tag{29}$$

and

$$\begin{aligned}&\liminf_{t\to+\infty} x_3(t)\\ &\geq \frac{r_3^l + a_{31}^l m_1 + a_{32}^l m_2}{a_{33}^{1m} + a_{33}^{2m}} \exp[(r_3^l + a_{31}^l m_1 + a_{32}^l m_2 - (a_{33}^{1m} + a_{33}^{2m})M_3)\tau] = m_3.\end{aligned} \tag{30}$$

According to the fourth equation of system (12), we have

$$\dot{u}_1(t) \geq e_1^l - f_1^m u_1(t) + q_1^l m_1.$$

From Lemma 2.1, we can obtain

$$\liminf_{t\to+\infty} u_1(t) \geq \frac{e_1^l + q_1^l m_1}{f_1^m} = n_1. \tag{31}$$

Similarly, from the fifth and sixth equations of system (12), it follows that

$$\dot{u}_2(t) \geq e_2^l - f_2^m u_2(t) - q_2^m M_2, \dot{u}_3(t) \geq e_3^l - f_3^m u_3(t) - q_3^m M_3.$$

Moreover, by Lemma 2.1, it follows that

$$\liminf_{t\to+\infty} u_2(t) \geq \frac{e_2^l - q_2^m M_2}{f_2^m} = n_2, \tag{32}$$

and

$$\liminf_{t\to+\infty} u_3(t) \geq \frac{e_3^l - q_3^m M_3}{f_3^m} = n_3. \tag{33}$$

From (15), (16), and (24)–(33), this completes the proof of Theorem 2.1.

3. Globally attractive

In this section, we shall prove that the system (12) is globally attractive. To get the sufficient conditions for globally attractive of system (12), we give firstly the following definition and Lemma.

Definition 3.1: System (12) is said to be globally attractive, if there exists a positive solution $X(t) = (x_1(t), x_2(t), x_3(t), u_1(t), u_2(t), u_3(t))$ of the system (12) such that

$$\lim_{t\to+\infty} |x_i(t) - y_i(t)| = 0, \qquad \lim_{t\to+\infty} |u_i(t) - v_i(t)| = 0,$$

for any other positive solution $Y(t) = (Y_1(t), Y_2(t), Y_3(t), v_1(t), v_2(t), v_3(t))$ of the system (12).

Lemma 3.1 (See [28], Lemma 8.2): *If the function $f(t) : R^+ \to R$ is uniformly continuous, and the limit $\lim\limits_{t\to+\infty} \int_0^t f(s)ds$ exists and is finite, then $\lim\limits_{t\to+\infty} f(t) = 0$.*

Theorem 3.1: *Let $a_{11}^M = \max\{a_{11}^{1m}, a_{11}^{2m}\}$, $\quad a_{22}^M = \max\{a_{22}^{1m}, a_{22}^{2m}\}$. Assume that the system (12) satisfies $(H_1) - (H_{10})$ and the following conditions satisfy*

$$(H_{11}) \liminf_{t\to+\infty} A_i(t) > 0, \quad B_i > 0, \quad (i = 1, 2, 3)$$

where

$$\begin{aligned} A_1(t) = {}& a_{11}^{1l} + a_{11}^{2l} - \sum_{k=1}^{2} \int_{t-k\tau}^{t} a_{11}^{\mathrm{k}}(s+k\tau)ds[r_1^m + (a_{11}^{1m} + a_{11}^{2m})M_1 \\ & + a_{12}^m M_2 + a_{13}^m M_3 + d_1^m N_1] - M_1 \sum_{k=1}^{2} \left(\int_t^{t+k\tau} a_{11}^{\mathrm{k}}(s+k\tau)ds \times a_{11}^{\mathrm{k}}(t+k\tau) \right) \\ & - 2\tau M_1 a_{11}^M \sum_{k=1}^{2} a_{11}^{\mathrm{k}}(t+k\tau) - (1 + \tau M_2(a_{22}^{1m} + 2a_{22}^{2m}))a_{21}(t+2\tau) \\ & - (1 + \tau M_3 a_{33}^{2m})a_{31}(t+\tau) - q_1^m, \end{aligned}$$

$$\begin{aligned} A_2(t) = {}& a_{22}^{1l} + a_{22}^{2l} - \sum_{k=1}^{2} \int_{t-k\tau}^{t} a_{22}^{\mathrm{k}}(s+k\tau)ds[r_2^m + a_{21}^m M_1 + (a_{22}^{1m} + a_{22}^{2m})M_2 \\ & + a_{23}^m M_3 + d_2^m N_2] - M_2 \sum_{k=1}^{2} \left(\int_t^{t+k\tau} a_{22}^{\mathrm{k}}(s+k\tau)ds \times a_{22}^{\mathrm{k}}(t+k\tau) \right) \\ & - 2\tau M_2 a_{22}^M \sum_{k=1}^{2} a_{22}^{\mathrm{k}}(t+k\tau) \\ & - (1 + \tau M_1(a_{11}^{1m} + 2a_{11}^{2m}))a_{12}(t+2\tau) - (1 + \tau M_3 a_{33}^{2m})a_{32}(t+\tau) - q_2^m, \end{aligned}$$

$$\begin{aligned} A_3(t) = {}& a_{33}^{1l} + a_{33}^{2l} - (\tau M_3 a_{33}^{2m})(a_{33}^{1m} + a_{33}^2(t+\tau)) - \int_{t-\tau}^{t} a_{33}^2(s+\tau)ds[r_3^m + a_{31}^m M_1 \\ & + a_{32}^m M_2 + (a_{33}^{1m} + a_{33}^{2m})M_3 + d_3^m N_3] - (1 + \tau M_1(a_{11}^{1m} + 2a_{11}^{2m}))a_{13}(t+\tau) \\ & - (1 + \tau M_2(a_{22}^{1m} + 2a_{22}^{2m}))a_{23}(t+\tau) - q_3^m, \end{aligned}$$

$$\begin{aligned}
B_1 &= f_1^l - (1+\tau M_1(a_{11}^{1m} + 2a_{11}^{2m}))d_1^m,\\
B_2 &= f_2^l - (1+\tau M_2(a_{22}^{1m} + 2a_{22}^{2m}))d_2^m,\\
B_3 &= f_3^l - (1+\tau M_3 a_{33}^{2m})d_3^m.
\end{aligned}$$

Then system (12) is globally attractive.

Proof. Suppose that $(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t), u_3(t))$ and $(y_1(t), y_2(t), y_3(t), v_1(t), v_2(t), v_3(t))$ are any two different positive solutions of the system (12). Then from Theorem 2.1, there exist positive constants M_i, N_i, m_i, n_i, $i = 1, 2, 3$ and T, such that

$$m_i \le x_i(t), y_i(t) \le M_i,\ n_i \le u_i(t), v_i(t) \le N_i, i = 1, 2, 3,\ \text{for all } t \ge T.$$

Define

$$V_{11}(t) = |\ln x_1(t) - \ln y_1(t)|.$$

Calculating the upper right derivative of $V_{11}(t)$ along the solution of system (12), we obtain

$$\begin{aligned}
D^+V_{11}(t) &= \operatorname{sgn}(x_1(t) - y_1(t))[-a_{11}^1(t)(x_1(t-\tau) - y_1(t-\tau))\\
&\quad - a_{11}^2(t)(x_1(t-2\tau) - y_1(t-2\tau))\\
&\quad - a_{12}(t)(x_2(t-2\tau) - y_2(t-2\tau)) + a_{13}(t)(x_3(t-\tau) - y_3(t-\tau))\\
&\quad - d_1(t)(u_1(t) - v_1(t))]\\
&= \operatorname{sgn}(x_1(t) - y_1(t))[-(a_{11}^1(t) + a_{11}^2(t))(x_1(t) - y_1(t)) - d_1(t)(u_1(t) - v_1(t))\\
&\quad - a_{12}(t)(x_2(t-2\tau) - y_2(t-2\tau)) + a_{13}(t)(x_3(t-\tau) - y_3(t-\tau))\\
&\quad + a_{11}^1(t)\int_{t-\tau}^{t} (\dot{x}_1(\theta) - \dot{y}_1(\theta)d\theta + a_{11}^2(t)\int_{t-2\tau}^{t} (\dot{x}_1(\theta) - \dot{y}_1(\theta)d\theta]\\
&= \operatorname{sgn}(x_1(t) - y_1(t))[-(a_{11}^1(t) + a_{11}^2(t))(x_1(t) - y_1(t)) - d_1(t)(u_1(t) - v_1(t))\\
&\quad - a_{12}(t)(x_2(t-2\tau) - y_2(t-2\tau)) + a_{13}(t)(x_3(t-\tau) - y_3(t-\tau))\\
&\quad + \sum_{k=1}^{2} a_{11}^k(t)\int_{t-k\tau}^{t} (x_1(\theta)[r_1(\theta) - a_{11}^1(\theta)x_1(\theta-\tau)\\
&\quad - a_{11}^2(\theta)x_1(\theta-2\tau) - a_{12}(\theta)x_2(\theta-2\tau)\\
&\quad + a_{13}(\theta)x_3(\theta-\tau) - d_1(\theta)u_1(\theta)] - y_1(\theta)[r_1(\theta) - a_{11}^1(\theta)y_1(\theta-\tau)\\
&\quad - a_{11}^2(\theta)y_1(\theta-2\tau)\\
&\quad -a_{12}(\theta)y_2(\theta-2\tau) + a_{13}(\theta)y_3(\theta-\tau) - d_1(\theta)v_1(\theta)])d\theta]\\
&= \operatorname{sgn}(x_1(t) - y_1(t))[-(a_{11}^1(t) + a_{11}^2(t))(x_1(t) - y_1(t)) - d_1(t)(u_1(t) - v_1(t))\\
&\quad - a_{12}(t)(x_2(t-2\tau) - y_2(t-2\tau)) + a_{13}(t)(x_3(t-\tau) - y_3(t-\tau))\\
&\quad + \sum_{k=1}^{2} a_{11}^k(t)\int_{t-k\tau}^{t} ((x_1(\theta) - y_1(\theta))[r_1(\theta) - a_{11}^1(\theta)y_1(\theta-\tau)
\end{aligned}$$

$$
\begin{aligned}
&-a_{11}^2(\theta)y_1(\theta-2\tau)\\
&-a_{12}(\theta)y_2(\theta-2\tau)+a_{13}(\theta)y_3(\theta-\tau)-d_1(\theta)v_1(\theta)]\\
&+x_1(\theta)[-a_{11}^1(\theta)(x_1(\theta-\tau)\\
&-y_1(\theta-\tau))-a_{11}^2(\theta)(x_1(\theta-2\tau)-y_1(\theta-2\tau))-a_{12}(\theta)(x_2(\theta-2\tau)\\
&-y_2(\theta-2\tau))\\
&+a_{13}(\theta)(x_3(\theta-\tau)-y_3(\theta-\tau))-d_1(\theta)(u_1(\theta)-v_1(\theta))d\theta]\\
\le{}&-(a_{11}^1(t)+a_{11}^2(t))|x_1(t)-y_1(t)|+d_1(t)|u_1(t)\\
&-v_1(t)|+a_{12}(t)|x_2(t-2\tau)-y_2(t-2\tau)|\\
&+a_{13}(t)|x_3(t-\tau)-y_3(t-\tau)|\\
&+\sum_{k=1}^{2}(a_{11}^k(t)\int_{t-k\tau}^{t}([r_1(\theta)+a_{11}^1(\theta)y_1(\theta-\tau)+a_{11}^2(\theta)y_1(\theta-2\tau)\\
&+a_{12}(\theta)y_2(\theta-2\tau)+a_{13}(\theta)y_3(\theta-\tau)+d_1(\theta)v_1(\theta)]|x_1(\theta)-y_1(\theta)|\\
&+x_1(\theta)[a_{11}^1(\theta)\\
&\times|x_1(\theta-\tau)-y_1(\theta-\tau)|+a_{11}^2(\theta)|x_1(\theta-2\tau)-y_1(\theta-2\tau)|\\
&+a_{12}(\theta)|x_2(\theta-2\tau)-y_2(\theta-2\tau)|\\
&+a_{13}(\theta)|x_3(\theta-\tau)-y_3(\theta-\tau)|+d_1(\theta)|u_1(\theta)-v_1(\theta)|])d\theta). \qquad (34)
\end{aligned}
$$

Next, we define that

$$
\begin{aligned}
V_{12}(t)={}&\sum_{k=1}^{2}\int_{t-k\tau}^{t}\int_{s}^{t}a_{11}^k(s+k\tau)([r_1(\theta)+a_{11}^1(\theta)y_1(\theta-\tau)+a_{11}^2(\theta)y_1(\theta-2\tau)\\
&+a_{12}(\theta)y_2(\theta-2\tau)+a_{13}(\theta)y_3(\theta-\tau)+d_1(\theta)v_1(\theta)]|x_1(\theta)-y_1(\theta)|\\
&+x_1(\theta)[a_{11}^1(\theta)|x_1(\theta-\tau)-y_1(\theta-\tau)|+a_{11}^2(\theta)|x_1(\theta-2\tau)-y_1(\theta-2\tau)|\\
&+a_{12}(\theta)|x_2(\theta-2\tau)-y_2(\theta-2\tau)|+a_{13}(\theta)|x_3(\theta-\tau)-y_3(\theta-\tau)|\\
&+d_1(\theta)|u_1(\theta)-v_1(\theta)|)d\theta ds. \qquad (35)
\end{aligned}
$$

Then, from (34) and (35), we have

$$
\begin{aligned}
\sum_{i=1}^{2}V_{1i}(t)\le{}&-(a_{11}^1(t)+a_{11}^2(t))|x_1(t)-y_1(t)|+d_1(t)|u_1(t)-v_1(t)|\\
&+a_{12}(t)|x_2(t-2\tau)-y_2(t-2\tau)|+a_{13}(t)|x_3(t-\tau)-y_3(t-\tau)|\\
&+\sum_{k=1}^{2}\int_{t-k\tau}^{t}a_{11}^k(s+k\tau)ds[r_1(t)+(a_{11}^1(t)+a_{11}^2(t))M_1+a_{12}(t)M_2\\
&+a_{13}(t)M_3+d_1(t)N_1]|x_1(t)-y_1(t)|+M_1\sum_{k=1}^{2}\int_{t-k\tau}^{t}a_{11}^k(s+k\tau)ds
\end{aligned}
$$

$$
\begin{aligned}
&\times [a_{11}^1(t)|x_1(t-\tau)-y_1(t-\tau)| + a_{11}^2(t)|x_1(t-2\tau)-y_1(t-2\tau)| \\
&+ a_{12}(t)|x_2(t-2\tau)-y_2(t-2\tau)| + a_{13}(t)|x_3(t-\tau)-y_3(t-\tau)| \\
&+ d_1(t)|u_1(t)-v_1(t)|] \\
\le\ & -(a_{11}^1(t)+a_{11}^2(t))|x_1(t)-y_1(t)| + (1+M_1\tau(a_{11}^{1m}+2a_{11}^{2m}))d_1(t)|u_1(t)-v_1(t)| \\
&+ a_{12}(t)|x_2(t-2\tau)-y_2(t-2\tau)| + a_{13}(t)|x_3(t-\tau)-y_3(t-\tau)| \\
&+ \sum_{k=1}^{2}\int_{t-k\tau}^{t} a_{11}^k(s+k\tau)ds[r_1(t)+(a_{11}^1(t)+a_{11}^2(t))M_1 + a_{12}(t)M_2 \\
&+ a_{13}(t)M_3 + d_1(t)N_1]|x_1(t)-y_1(t)| + M_1\sum_{k=1}^{2}(\int_{t-k\tau}^{t} a_{11}^k(s+k\tau)ds \\
&\times a_{11}^k(t)|x_1(t-k\tau)-y_1(t-k\tau)|) + 2\tau M_1 a_{11}^M\sum_{k=1}^{2}(a_{11}^k(t)|x_1(t-k\tau) \\
&-y_1(t-k\tau)|) \\
&+ M_1\tau(a_{11}^{1m}+2a_{11}^{2m})[a_{12}(t)|x_2(t-2\tau)-y_2(t-2\tau)| + a_{13}(t)|x_3(t-\tau) \\
&-y_3(t-\tau)|].
\end{aligned} \tag{36}
$$

Define

$$
\begin{aligned}
V_{13}(t) = {} & M_1\sum_{k=1}^{2}\int_{t-k\tau}^{t}\int_{w}^{w+k\tau} a_{11}^k(s+k\tau)a_{11}^k(w+k\tau)|x_1(w)-y_1(w)|dsdw \\
&+ 2\tau M_1 a_{11}^M\sum_{k=1}^{2}\int_{t-k\tau}^{t} a_{11}^k(w+k\tau)|(x_1(w)-y_1(w)|dw \\
&+ (1+\tau M_1(a_{11}^{1m}+2a_{11}^{2m}))\int_{t-2\tau}^{t} a_{12}(w+2\tau)|x_2(w)-y_2(w)|dw \\
&+ (1+\tau M_1(a_{11}^{1m}+2a_{11}^{2m}))\int_{t-\tau}^{t} a_{13}(w+\tau)|x_3(w)-y_3(w)|dw.
\end{aligned} \tag{37}
$$

Let

$$
V_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t). \tag{38}
$$

According to (36) and (37), calculating the upper right derivative of $V_1(t)$, we have

$$
\begin{aligned}
D^+V_1(t) \le {} & -\{a_{11}^{1l} + a_{11}^{2l} - \sum_{k=1}^{2}\int_{t-k\tau}^{t} a_{11}^{\mathrm{k}}(s+k\tau)ds[r_1^m + (a_{11}^{1m}+a_{11}^{2m})M_1 \\
&+ a_{12}^m M_2 + a_{13}^m M_3 + d_1^m N_1] - M_1\sum_{k=1}^{2}(\int_{t}^{t+k\tau} a_{11}^{\mathrm{k}}(s+k\tau)ds \times a_{11}^{\mathrm{k}}(t+k\tau)) \\
&- 2\tau M_1 a_{11}^M\sum_{k=1}^{2} a_{11}^k(t+k\tau)\}|x_1(t)-y_1(t)| + (1+\tau M_1(a_{11}^{1m}+2a_{11}^{2m}))a_{12}(t+2\tau)
\end{aligned}
$$

$$\times |x_2(t) - y_2(t)| + (1 + \tau M_1(a_{11}^{1m} + 2a_{11}^{2m}))a_{13}(t+\tau)|x_3(t) - y_3(t)|$$
$$+ (1 + M_1\tau(a_{11}^{1m} + 2a_{11}^{2m}))d_1^m|u_1(t) - v_1(t)|. \tag{39}$$

Similarly, we define $V_{21}(t) = |\ln x_2(t) - \ln y_2(t)|$, then one obtain

$$\begin{aligned}
D^+V_{21}(t) &= \operatorname{sgn}(x_2(t) - y_2(t))[-a_{21}(t)(x_1(t-2\tau) - y_1(t-2\tau)) - a_{22}^1(t)(x_2(t-\tau) - y_2(t-\tau)) \\
&\quad - a_{22}^2(t)(x_2(t-2\tau) - y_2(t-2\tau)) + a_{23}(t)(x_3(t-\tau) - y_3(t-\tau)) + d_2(t)(u_2(t) - v_2(t)] \\
&= \operatorname{sgn}(x_2(t) - y_2(t))[-(a_{22}^1(t) + a_{22}^2(t))(x_2(t) - y_2(t)) - a_{21}(t)(x_1(t-2\tau) - y_1(t-2\tau)) \\
&\quad + a_{23}(t)(x_3(t-\tau) - y_3(t-\tau)) + d_2(t)(u_2(t) - v_2(t)) + a_{22}^1(t)\int_{t-\tau}^t (\dot{x}_2(\theta) - \dot{y}_2(\theta))d\theta \\
&\quad + a_{22}^2(t)\int_{t-2\tau}^t (\dot{x}_2(\theta) - \dot{y}_2(\theta))d\theta] \\
&= \operatorname{sgn}(x_2(t) - y_2(t))[-(a_{22}^1(t) + a_{22}^2(t))(x_2(t) - y_2(t)) - a_{21}(t)(x_1(t-2\tau) - y_1(t-2\tau)) \\
&\quad + a_{23}(t)(x_3(t-\tau) - y_3(t-\tau)) + d_2(t)(u_2(t) - v_2(t)) + \sum_{k=1}^2 a_{22}^k(t)\int_{t-k\tau}^t (x_2(\theta)[r_2(\theta) \\
&\quad - a_{21}(\theta)x_1(\theta-2\tau) - a_{22}^1(\theta)x_2(\theta-\tau) - a_{22}^2(\theta)x_2(\theta-2\tau) + a_{23}(\theta)x_3(\theta-\tau) + d_2(\theta)u_2(\theta)] \\
&\quad - y_2(\theta)[r_2(\theta) - a_{21}(\theta)y_1(\theta-2\tau) - a_{22}^1(\theta)y_2(\theta-\tau) - a_{22}^2(\theta)y_2(\theta-2\tau) + a_{23}(\theta)y_3(\theta-\tau) \\
&\quad + d_2(\theta)v_2(\theta)])d\theta] \\
&= \operatorname{sgn}(x_2(t) - y_2(t))[-(a_{22}^1(t) + a_{22}^2(t))(x_2(t) - y_2(t)) - a_{21}(t)(x_1(t-2\tau) - y_1(t-2\tau)) \\
&\quad + a_{23}(t)(x_3(t-\tau) - y_3(t-\tau)) + d_2(t)(u_2(t) - v_2(t)) + \sum_{k=1}^2 a_{22}^k(t)\int_{t-k\tau}^t ((x_2(\theta) - y_2(\theta)) \\
&\quad \times [r_2(\theta) - a_{21}(\theta)y_1(\theta-2\tau) - a_{22}^1(\theta)y_2(\theta-\tau) - a_{22}^2(\theta)y_2(\theta-2\tau) + a_{23}(\theta)y_3(\theta-\tau) \\
&\quad + d_2(\theta)v_2(\theta)] + x_2(\theta)[-a_{21}(\theta)(x_1(\theta-2\tau) - y_1(\theta-2\tau)) - a_{22}^1(\theta)(x_2(\theta-\tau) - y_2(\theta-\tau)) \\
&\quad - a_{22}^2(\theta)(x_2(\theta-2\tau) - y_2(\theta-2\tau)) + a_{23}(\theta)(x_3(\theta-\tau) - y_3(\theta-\tau)) + d_2(\theta)(u_2(\theta) - v_2(\theta))])d\theta] \\
&\le -(a_{22}^1(t) + a_{22}^2(t))|x_2(t) - y_2(t)| + d_2(t)|u_2(t) - v_2(t)| + a_{21}(t)|x_1(t-2\tau) - y_1(t-2\tau)| \\
&\quad + a_{23}(t)|x_3(t-\tau) - y_3(t-\tau)| + \sum_{k=1}^2 (a_{22}^k(t)\int_{t-k\tau}^t ([r_2(\theta) + a_{21}(\theta)y_1(\theta-2\tau) + a_{22}^1(\theta)y_2(\theta-\tau) \\
&\quad + a_{22}^2(\theta)y_2(\theta-2\tau) + a_{23}(\theta)y_3(\theta-\tau) + d_2(\theta)v_2(\theta)]|x_2(\theta) - y_2(\theta)| \\
&\quad + x_2(\theta)[a_{21}(\theta)|x_1(\theta-2\tau) - y_1(\theta-2\tau)| + a_{22}^1(\theta)|x_2(\theta-\tau) - y_2(\theta-\tau)| \\
&\quad + a_{22}^2(\theta)|x_2(\theta-2\tau) - y_2(\theta-2\tau)| + a_{23}(\theta)|x_3(\theta-\tau) - y_3(\theta-\tau)| + d_2(\theta)|u_2(\theta) - v_2(\theta)|])d\theta).
\end{aligned} \tag{40}$$

On the other hand, define

$$V_{22}(t) = \sum_{k=1}^2 \int_{t-k\tau}^t \int_s^t a_{22}^k(s+k\tau)([r_2(\theta) + a_{21}(\theta)y_1(\theta-2\tau) + a_{22}^1(\theta)y_2(\theta-\tau)$$
$$+ a_{22}^2(\theta)y_2(\theta-2\tau) + a_{23}(\theta)y_3(\theta-\tau) + d_2(\theta)v_2(\theta)]|x_2(\theta) - y_2(\theta)|$$

$$
\begin{aligned}
&+ x_2(\theta)[a_{21}(\theta)|x_1(\theta-2\tau)-y_1(\theta-2\tau)| + a_{22}^1(\theta)|x_2(\theta-\tau)-y_2(\theta-\tau)| \\
&+ a_{22}^2(\theta)|x_2(\theta-2\tau)-y_2(\theta-2\tau)| + a_{23}(\theta)|x_3(\theta-\tau)-y_3(\theta-\tau)| \\
&+ d_2(\theta)|u_2(\theta)-v_2(\theta)|])d\theta ds.
\end{aligned}
\tag{41}
$$

From (40), (41), we have

$$
\begin{aligned}
\sum_{i=1}^{2} D^+ V_{2i} \le & -(a_{22}^1(t)+a_{22}^2(t))|x_2(t)-y_2(t)| + d_2(t)|u_2(t)-v_2(t)| \\
& + a_{21}(t)|x_1(t-2\tau)-y_1(t-2\tau)| \\
& + a_{23}(t)|x_3(t-\tau)-y_3(t-\tau)| + \sum_{k=1}^{2}\int_{t-k\tau}^{t} a_{22}^k(s+k\tau)ds[r_2(t)+a_{21}(t)M_1 \\
& + (a_{22}^1(t)+a_{22}^2(t))M_2 + a_{23}(t)M_3 + d_2(t)N_2]|x_2(t)-y_2(t)| \\
& + M_2\sum_{k=1}^{2}\int_{t-k\tau}^{t} a_{22}^k(s+k\tau)ds[a_{21}(t)|x_1(t-2\tau)-y_1(t-2\tau)| \\
& + a_{22}^1(t)|x_2(t-\tau)-y_2(t-\tau)| \\
& + a_{22}^2(t)|x_2(t-2\tau)-y_2(t-2\tau)| + a_{23}(t)|x_3(t-\tau)-y_3(t-\tau)| \\
& + d_2(t)|u_2(t)-v_2(t)|] \\
\le & -(a_{22}^1(t)+a_{22}^2(t))|x_2(t)-y_2(t)| \\
& + (1+\tau M_2(a_{22}^{1m}+2a_{22}^{2m}))d_2(t)|u_2(t)-v_2(t)| \\
& + a_{21}(t)|x_1(t-2\tau)-y_1(t-2\tau)| + a_{23}(t)|x_3(t-\tau)-y_3(t-\tau)| \\
& + \sum_{k=1}^{2}\int_{t-k\tau}^{t} a_{22}^k(s+k\tau)ds[r_2(t)+a_{21}(t)M_1+(a_{22}^1(t)+a_{22}^2(t))M_2 \\
& + a_{23}(t)M_3 + d_2(t)N_2]|x_2(t)-y_2(t)| + M_2\sum_{k=1}^{2}(\int_{t-k\tau}^{t} a_{22}^k(s+k\tau)ds \\
& \times a_{22}^k(t)|x_2(t-k\tau)-y_2(t-k\tau)|) + 2\tau M_2 a_{22}^M \sum_{k=1}^{2}(a_{22}^k(t)|x_2(t-k\tau) \\
& -y_2(t-k\tau)|) \\
& + \tau M_2(a_{22}^{1m}+2a_{22}^{2m})[a_{21}(t)|x_1(t-2\tau)-y_1(t-2\tau)| + a_{23}(t)|x_3(t-\tau) \\
& -y_3(t-\tau)|].
\end{aligned}
\tag{42}
$$

Furthermore, define

$$
V_{23}(t) = M_2\sum_{k=1}^{2}\int_{t-k\tau}^{t}\int_{w}^{w+k\tau} a_{22}^k(s+k\tau)a_{22}^k(w+k\tau)|x_2(w)-y_2(w)|dsdw
$$

$$+ 2\tau M_2 a_{22}^M \sum_{k=1}^{2} \int_{t-k\tau}^{t} a_{22}^k(w+k\tau)|x_2(w)-y_2(w)|dw$$

$$+ (1+\tau M_2(a_{22}^{1m}+2a_{22}^{2m})) \int_{t-2\tau}^{t} a_{21}(w+2\tau)|x_1(w)-y_1(w)|dw$$

$$+ (1+\tau M_2(a_{22}^{1m}+2a_{22}^{2m})) \int_{t-\tau}^{t} a_{23}(w+\tau)|x_3(w)-y_3(w)|dw. \tag{43}$$

Let

$$V_2(t) = V_{21}(t) + V_{22}(t) + V_{23}(t). \tag{44}$$

From (42) and (43), we can get the upper right derivative of $V_2(t)$

$$\begin{aligned}
D^+V_2(t) \le -\{&a_{22}^{1l} + a_{22}^{2l} - \sum_{k=1}^{2} \int_{t-k\tau}^{t} a_{22}^{\mathrm{k}}(s+k\tau)ds[r_2^m + a_{21}^m M_1 + (a_{22}^{1m}+a_{22}^{2m})M_2 \\
&+ a_{23}^m M_3 + d_2^m N_2] - M_2 \sum_{k=1}^{2} (\int_{t}^{t+k\tau} a_{22}^{\mathrm{k}}(s+k\tau)ds \times a_{22}^{\mathrm{k}}(t+k\tau)) \\
&- 2\tau M_2 a_{22}^M \sum_{k=1}^{2} a_{22}^{\mathrm{k}}(t+k\tau)\} \\
&\times |x_2(t)-y_2(t)| + (1+\tau M_2(a_{22}^{1m}+2a_{22}^{2m})a_{21}(t+2\tau)|x_1(t)-y_1(t)| \\
&+ (1+\tau M_2(a_{22}^{1m}+2a_{22}^{2m})a_{23}(t+\tau)|x_3(t)-y_3(t)| \\
&+ (1+\tau M_2(a_{22}^{1m}+2a_{22}^{2m}))d_2^m|u_2(t)-v_2(t)|.
\end{aligned} \tag{45}$$

Similarly, we define

$$V_{31}(t) = |\ln x_3(t) - \ln y_3(t)|.$$

Then, it follows that

$$\begin{aligned}
D^+V_{31} = {} & \mathrm{sgn}(x_3(t)-y_3(t))[a_{31}(t)(x_1(t-\tau)-y_1(t-\tau)) + a_{32}(t)(x_2(t-\tau) \\
& - y_2(t-\tau)) - a_{33}^1(t)(x_3(t)-y_3(t)) - a_{33}^2(t)(x_3(t-\tau)-y_3(t-\tau)) \\
& + d_3(t)(u_3(t)-v_3(t)] \\
= {} & \mathrm{sgn}(x_3(t)-y_3(t))[a_{31}(t)(x_1(t-\tau)-y_1(t-\tau)) + a_{32}(t)(x_2(t-\tau) \\
& - y_2(t-\tau)) - (a_{33}^1(t)+a_{33}^2(t))(x_3(t)-y_3(t)) + a_{33}^2(t) \\
& \times \int_{t-\tau}^{t} (\dot{x}_3(\theta)-\dot{y}_3(\theta))d\theta + d_3(t)(u_3(t)-v_3(t))] \\
= {} & \mathrm{sgn}(x_3(t)-y_3(t))[-(a_{33}^1(t)+a_{33}^2(t))(x_3(t)-y_3(t)) + d_3(t)(u_3(t)-v_3(t)) \\
& + a_{31}(t)(x_1(t-\tau)-y_1(t-\tau)) + a_{32}(t)(x_2(t-\tau)-y_2(t-\tau)) \\
& + a_{33}^2(t) \int_{t-\tau}^{t} x_3(\theta)[r_3(\theta) + a_{31}(\theta)x_1(\theta-\tau) + a_{32}(\theta)x_2(\theta-\tau)
\end{aligned}$$

$$
\begin{aligned}
&- a^1_{33}(\theta)x_3(\theta) - a^2_{33}(\theta)x_3(\theta-\tau) + d_3(\theta)u_3(\theta)] \\
&- y_3(\theta)[r_3(\theta) + a_{31}(\theta)y_1(\theta-\tau) + a_{32}(\theta)y_2(\theta-\tau) \\
&- a^1_{33}(\theta)y_3(\theta) - a^2_{33}(\theta)y_3(\theta-\tau) + d_3(\theta)v_3(\theta)]d\theta \\
= {} & \operatorname{sgn}(x_3(t) - y_3(t))[-(a^1_{33}(t) + a^2_{33}(t))(x_3(t) - y_3(t)) + d_3(t)(u_3(t) - v_3(t)) \\
&+ a_{31}(t)(x_1(t-\tau) - y_1(t-\tau)) + a_{32}(t)(x_2(t-\tau) - y_2(t-\tau)) + a^2_{33}(t) \\
&\times \int_{t-\tau}^{t} ((x_3(\theta) - y_3(\theta))[r_3(\theta) + a_{31}(\theta)y_1(\theta-\tau) + a_{32}(\theta)y_2(\theta-\tau) \\
&- a^1_{33}(\theta)y_3(\theta) - a^2_{33}(\theta)y_3(\theta-\tau) + d_3(\theta)v_3(\theta)] + x_3(\theta)[a_{31}(\theta)(x_1(\theta-\tau) \\
&- y_1(\theta-\tau)) + a_{32}(\theta)(x_2(\theta-\tau) - y_2(\theta-\tau)) - a^1_{33}(\theta)(x_3(\theta) \\
&- y_3(\theta)) - a^2_{33}(\theta)(x_3(\theta-\tau) - y_3(\theta-\tau)) + d_3(\theta)(u_3(\theta) - v_3(\theta))])d\theta] \\
\le {} & -(a^1_{33}(t) + a^2_{33}(t))|x_3(t) - y_3(t)| + d_3(t)|u_3(t) - v_3(t)| + a_{31}(t)|x_1(t-\tau) \\
&- y_1(t-\tau)| + a_{32}(t)|(x_2(t-\tau) - y_2(t-\tau)| + a^2_{33}(t) \\
&\times \int_{t-\tau}^{t} ([r_3(\theta) + a_{31}(\theta)y_1(\theta-\tau) + a_{32}(\theta)y_2(\theta-\tau) + a^1_{33}(\theta)y_3(\theta) \\
&+ a^2_{33}(\theta)y_3(\theta-\tau) + d_3(\theta)v_3(\theta)]|x_3(\theta) - y_3(\theta)| + x_3(\theta)[a_{31}(\theta)|x_1(\theta-\tau) \\
&- y_1(\theta-\tau)| + a_{32}(\theta)|x_2(\theta-\tau) - y_2(\theta-\tau)| + a^1_{33}(\theta)|x_3(\theta) - y_3(\theta)| \\
&+ a^2_{33}(\theta)|x_3(\theta-\tau) - y_3(\theta-\tau)| + d_3(\theta)|u_3(\theta) - v_3(\theta)|])d\theta. \qquad (46)
\end{aligned}
$$

Let

$$
\begin{aligned}
V_{32}(t) = {} & \int_{t-\tau}^{t}\int_{s}^{t} a^2_{33}(s+\tau)([r_3(\theta) + a_{31}(\theta)y_1(\theta-\tau) + a_{32}(\theta)y_2(\theta-\tau) \\
&+ a^1_{33}(\theta)y_3(\theta) + a^2_{33}(\theta)y_3(\theta-\tau) + d_3(\theta)v_3(\theta)]|x_3(\theta) - y_3(\theta)| \\
&+ x_3(\theta)[a_{31}(\theta)|x_1(\theta-\tau) - y_1(\theta-\tau)| + a_{32}(\theta)|x_2(\theta-\tau) - y_2(\theta-\tau)| \\
&+ a^1_{33}(\theta)|x_3(\theta) - y_3(\theta)| + a^2_{33}(\theta)|x_3(\theta-\tau) - y_3(\theta-\tau)| \\
&+ d_3(\theta)|u_3(\theta) - v_3(\theta)|])d\theta ds. \qquad (47)
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\sum_{i=1}^{2} D^+V_{3i} \le {} & -(a^1_{33}(t) + a^2_{33}(t))|x_3(t) - y_3(t)| + d_3(t)|u_3(t) - v_3(t)| \\
&+ a_{31}(t)|x_1(t-\tau) - y_1(t-\tau)| + a_{32}(t)|(x_2(t-\tau) - y_2(t-\tau)| \\
&+ \int_{t-\tau}^{t} a^2_{33}(s+\tau)ds[r_3(t) + a_{31}(t)M_1 + a_{32}(t)M_2 \\
&+ (a^1_{33}(t) + a^2_{33}(t))M_3 + d_3(t)N_3]|x_3(t) - y_3(t)| \\
&+ M_3\int_{t-\tau}^{t} a^2_{33}(s+\tau)ds[a_{31}(t)|x_1(t-\tau) - y_1(t-\tau)|
\end{aligned}
$$

$$
\begin{aligned}
&+ a_{32}(t)|x_2(t-\tau) - y_2(t-\tau)| + a_{33}^1(t)|x_3(t) - y_3(t)| \\
&+ a_{33}^2(t)|x_3(t-\tau) - y_3(t-\tau)| + d_3(t)|u_3(t) - v_3(t)|] \\
\le & -(a_{33}^1(t) + a_{33}^2(t) - \tau M_3 a_{33}^{2m} a_{33}^1(t))|x_3(t) - y_3(t)| \\
&+ (1 + \tau M_3 a_{33}^{2m}) d_3(t)|u_3(t) - v_3(t)| + a_{31}(t)|x_1(t-\tau) - y_1(t-\tau)| \\
&+ a_{32}(t)|(x_2(t-\tau) - y_2(t-\tau)| + \int_{t-\tau}^{t} a_{33}^2(s+\tau)ds[r_3(t) \\
&+ a_{31}(t)M_1 + a_{32}(t)M_2 + (a_{33}^1(t) + a_{33}^2(t))M_3 + d_3(t)N_3]|x_3(t) - y_3(t)| \\
&+ \tau M_3 a_{33}^{2m}[a_{31}(t)|x_1(t-\tau) - y_1(t-\tau)| + a_{32}(t)|x_2(t-\tau) - y_2(t-\tau)| \\
&+ a_{33}^2(t)|x_3(t-\tau) - y_3(t-\tau)|]. \qquad (48)
\end{aligned}
$$

Take

$$
\begin{aligned}
V_{33}(t) = {} & (1 + \tau M_3 a_{33}^{2m}) \int_{t-\tau}^{t} a_{31}(w+\tau)|x_1(w) - y_1(w)|dw \\
&+ (1 + \tau M_3 a_{33}^{2m}) \int_{t-\tau}^{t} a_{32}(w+\tau)|x_2(w) - y_2(w)|dw \\
&+ \tau M_3 a_{33}^{2m} \int_{t-\tau}^{t} a_{33}^2(w+\tau)|x_3(w) - y_3(w)|dw. \qquad (49)
\end{aligned}
$$

Moreover, we take

$$
V_3(t) = V_{31}(t) + V_{32}(t) + V_{33}(t). \qquad (50)
$$

Then, we have

$$
\begin{aligned}
D^+V_3(t) \le & -\{a_{33}^{1l} + a_{33}^{2l} - \tau M_3 a_{33}^{2m}[a_{33}^{1m} + a_{33}^2(t+\tau)] - \int_{t-\tau}^{t} a_{33}^2(s+\tau)ds \\
& \times [r_3^m + a_{31}^m M_1 + a_{32}^m M_2 + (a_{33}^{1m} + a_{33}^{2m})M_3 + d_3^m N_3]\} \\
& \times |x_3(t) - y_3(t)| + (1 + \tau M_3 a_{33}^{2m}) a_{31}(t+\tau)|x_1(t) - y_1(t)| \\
& + (1 + \tau M_3 a_{33}^{2m}) a_{32}(t+\tau)|(x_2(t) - y_2(t)| + (1 + \tau M_3 a_{33}^{2m}) d_3^m |u_3(t) - v_3(t)|. \qquad (51)
\end{aligned}
$$

Take $V_4(t) = |\ln u_1(t) - \ln v_1(t)|$, $V_5(t) = |\ln u_2(t) - \ln v_2(t)|$, $V_6(t) = |\ln u_3(t) - \ln v_3(t)|$, and calculate the upper right derivative of $V_4(t), V_5(t), V_6(t)$, we have

$$
\begin{aligned}
D^+V_4(t) &\le \operatorname{sgn}(u_1(t) - v_1(t))[-f_1(t)(u_1(t) - v_1(t)) + q_1(t)(x_1(t) - y_1(t)) \\
&\le -f_1^l|u_1(t) - v_1(t)| + q_1^m|x_1(t) - y_1(t)|, \qquad (52)
\end{aligned}
$$

$$
\begin{aligned}
D^+V_5(t) &\le \operatorname{sgn}(u_2(t) - v_2(t))[-f_2(t)(u_2(t) - v_2(t)) + q_2(t)(x_2(t) - y_2(t)) \\
&\le -f_2^l|u_2(t) - v_2(t)| + q_2^m|x_2(t) - y_2(t)|, \qquad (53)
\end{aligned}
$$

and

$$
\begin{aligned}
D^+V_6(t) &\le \operatorname{sgn}(u_3(t) - v_3(t))[-f_3(t)(u_3(t) - v_3(t)) + q_3(t)(x_3(t) - y_3(t)) \\
&\le -f_3^l|u_3(t) - v_3(t)| + q_3^m|x_3(t) - y_3(t)|. \qquad (54)
\end{aligned}
$$

Moreover, we give a Lyapunov function as follows

$$V(t)=\sum_{i=1}^{6}V_i(t),$$

from (39), (45), (51)–(54), we can obtain

$$D^+V(t)\le-\sum_{i=1}^{3}(A_i(t)|x_i(t)-y_i(t)|+B_i|u_i(t)-v_i(t)|) \tag{55}$$

for all $t\ge t+\tau$.

In view of the conditions of Theorem 3.1, there exist a constant $\alpha>0$ and $T^*\ge T+\tau$ such that for all $t\ge T^*$, it holds that

$$A_i(t)\ge\alpha>0,\ B_i\ge\alpha>0,\ (i=1,2,3). \tag{56}$$

Integrating from $T^*$to t on both sides of (55) and by (56), we have

$$V(t)+\alpha\int_{T^*}^{t}(\sum_{i=1}^{3}[|x_i(s)-y_i(s)|+|u_i(s)-v_i(s)|])ds\le V(T^*)<+\infty. \tag{57}$$

Therefore, $V(t)$ is bounded on $[T^*,+\infty)$, and we have

$$\int_{T^*}^{\infty}(\sum_{i=1}^{3}[|x_i(t)-y_i(t)|+|u_i(t)-v_i(t)|])ds\le\frac{V(T)}{\alpha}<+\infty. \tag{58}$$

By (58), we have

$$\sum_{i=1}^{3}(|x_i(t)-y_i(t)|+|u_i(t)-v_i(t)|)\in L^1(T,+\infty). \tag{59}$$

From the uniformity permanence of the system (12), $\sum_{i=1}^{3}[|x_i(t)-y_i(t)|+|u_i(t)-v_i(t)|]$ is uniformity continuous on $[T^*,+\infty)$. By Lemma 3.1, we can obtain

$$\lim_{t\to+\infty}|x_i(t)-y_i(t)|=0,\ \lim_{t\to+\infty}|u_i(t)-v_i(t)|=0,\ (i=1,2,3).$$

This completes the proof of Theorem 3.1.

4. Numerical simulation

In this section, we give some numerical simulations to support our theoretical analysis. As an example, we consider the following Lotka–Volterra competitive-competitive–cooperative model with feedback controls and time delays and choose the

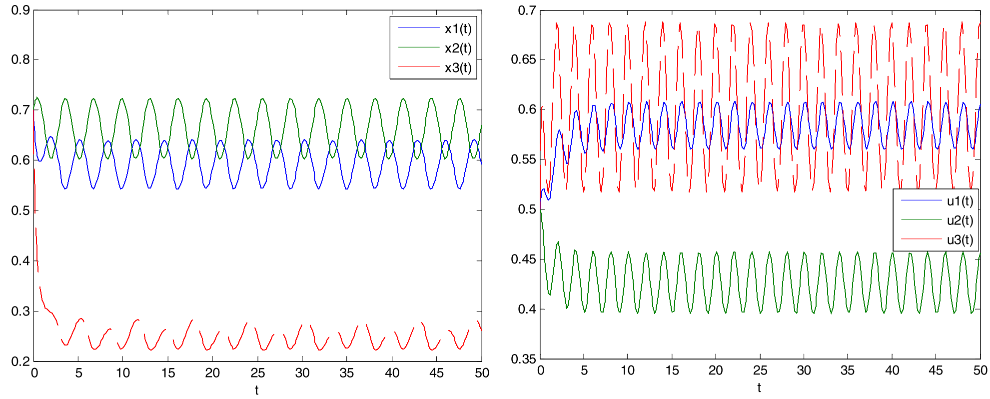

Figure 1. The numerical solution of systems (60) with the initial conditions (61) and $\tau = 0.075$.

periodic function as the coefficients of the model

$$\begin{cases} \dot{x}_1(t) = x_1(t)[\frac{2+|\sin(t)|}{2} - \frac{5+|\sin(t)|}{3}x_1(t-\tau) - \frac{1+|\cos(t)|}{5}x_1(t-2\tau) - \frac{2+\sin(t)}{550}x_2(t-2\tau) \\ +\frac{2+\cos(t)}{300}x_3(t-\tau) - (0.015+0.001\cos\pi t)u_1(t)], \\ \dot{x}_2(t) = x_2(t)[\frac{2+|\cos(t)|}{2} - \frac{2+\cos(t)}{550}x_1(t-2\tau) - \frac{6+|\cos(t)|}{4}x_2(t-\tau) - \frac{1+|\sin(t)|}{5}x_2(t-2\tau) \\ +\frac{2+\cos(t)}{300}x_3(t-\tau) + (0.0075+0.0005\sin\pi t)u_2(t)], \\ \dot{x}_3(t) = x_3(t)[\frac{1+|\sin(t)|}{2} + \frac{1.5+\cos(t)}{1500}x_1(t-\tau) + \frac{1.5+\sin(t)}{1500}x_2(t-\tau) - \frac{5+\cos(t)}{2}x_3(t) \\ -\frac{5+|\sin(t)|}{6}x_3(t-\tau) + (0.016+0.004\sin\pi t)u_3(t)], \\ \dot{u}_1(t) = (0.35+0.05\sin\pi t) - (0.6+0.1\cos\pi t)u_1(t) + (0.0016+0.0003\sin\pi t)x_1(t), \\ \dot{u}_2(t) = (0.3+0.05\cos\pi t) - (0.7+0.2\cos\pi t)u_2(t) - (0.0015+0.0005\sin\pi t)x_2(t), \\ \dot{u}_3(t) = (3+0.4\sin\pi t) - (5+0.5\sin\pi t)u_3(t) - (0.0005+0.00015\sin\pi t)x_3(t). \end{cases} \tag{60}$$

By calculating, we have

$$\begin{aligned} &P_1 \approx 5.50,\ P_2 \approx 6.14,\ x_1^* \approx 1.22,\ x_2^* \approx 1.09,\ x_3^* \approx 0.37, \\ &M_1 \approx 1.38,\ M_2 \approx 1.54,\ M_3 \approx 0.4, \\ &N_1 \approx 0.81,\ N_2 \approx 0.7, N_3 \approx 0.76,\ m_1 \approx 0.43,\ m_2 \approx 0.51,\ m_3 \approx 0.21, \\ &n_1 \approx 0.52,\ n_2 \approx 0.36, \\ &n_3 \approx 0.47,\ r_1^l - a_{12}^m M_2 - d_1^m N_1 \approx 0.98, r_2^l - a_{21}^m \\ &M_1 \approx 0.99,\ e_2^l - q_2^m M_2 \approx 0.25,\ e_3^l - q_3^m M_3 \approx 2.60, \\ &\liminf_{t\to+\infty} A_1(t) \approx 0.35, \liminf_{t\to+\infty} A_2(t) \approx 0.40,\ \liminf_{t\to+\infty} A_3(t) \approx 0.48, \\ &B_1 \approx 0.48, B_2 \approx 0.49,\ B_3 \approx 4.48. \end{aligned}$$

It is easy to show that the system (60) satisfies the conditions of Theorem 3.1. It follows from Theorem 3.1 that the Lotka–Volterra competitive-competitive–cooperative model (60) is permanent and globally attractive. By employing the software package MATLAB 7.1, we can solve the numerical solutions of the system (60) which are shown in Figures 1–3. Figure 1 shows that the permanence of the systems (60) with time delay $\tau = 0.075$ and the

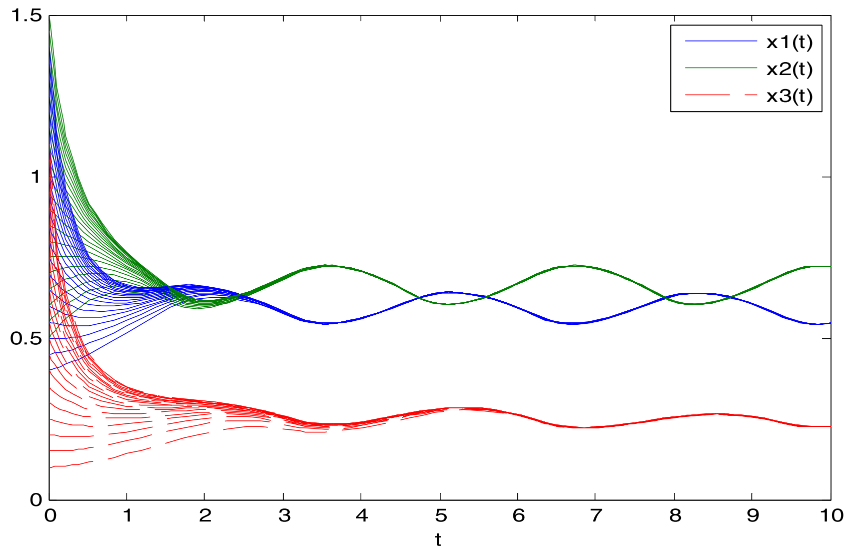

Figure 2. The numerical solution of systems (60) with the different initial conditions.

initial conditions

$$x_1(t_0) = 0.7, x_2(t_0) = 0.7, x_3(t_0) = 0.7, u_1(t_0) = 0.5, u_2(t_0) = 0.5, u_3(t_0) = 0.5. \quad (61)$$

From Figure 2, it is not difficult to find that the system (60) is globally attractive. Figure 3 shows the dynamical behaviour of the systems (60).

5. Conclusion

This paper presents the use of Lyapunov stability theorem for system of nonlinear differential equations. This method is a powerful tool for solving nonlinear differential equations in mathematical physics, chemistry and engineering etc. The technique constructing an appropriate Lyapunov function provides a new efficient method to handle the nonlinear structure with time delay and feedback control.

We have dealt with the problem of positive solution for a class of three-species Lotka–Volterra competitive- competitive–cooperative with feedback controls and time delays. By developing some new analysis techniques and constructing a new suitable Lyapunov function, we obtain some sufficient conditions which ensure the system to be permanent and globally attractive. Our results show that feedback control variables and time delay terms have influence both the persistent property and global attractive of system (12). Moreover, some numerical simulations to the system (60) are given to illustrate our results obtained in this paper. In particular, the sufficient conditions that we obtained are very simple and practical, which provide flexibility for the application and analysis of the Lotka–Volterra models with feedback controls and time delays.

Remark: The main contribution and innovation of this paper are as follows: (1) The control variables are introduced to the known model (4) to implement a feedback control mechanism, and the new model can better describe the interactions among multi-species. Obviously, system (4) is the special case of the new system (12). To the best of the author's

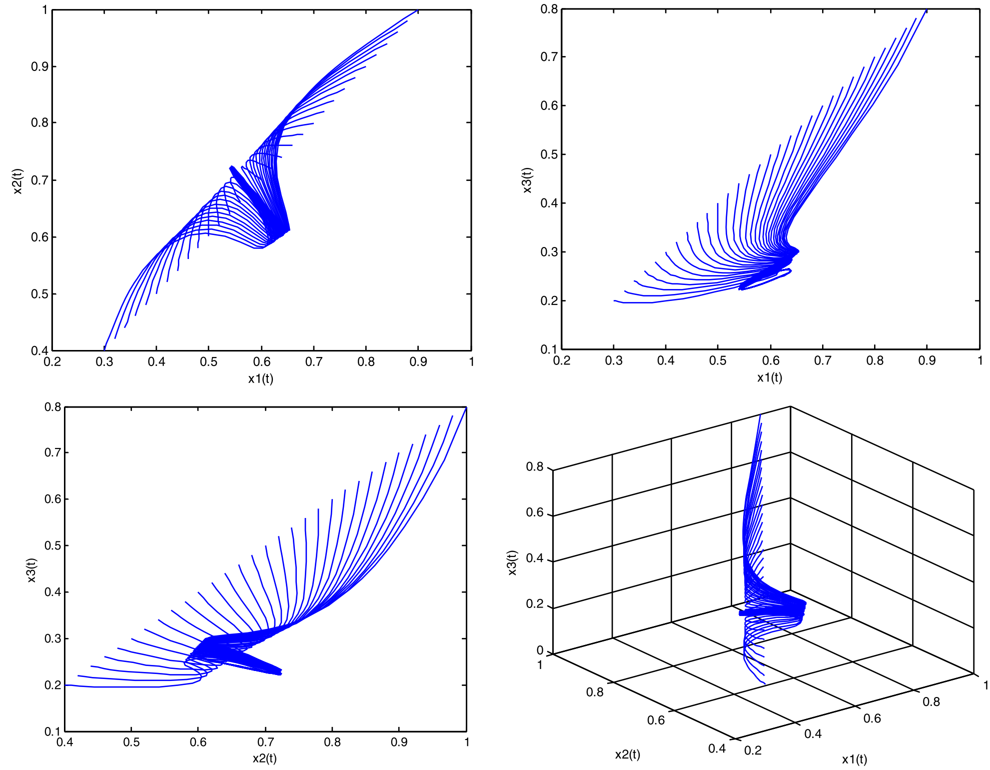

Figure 3. The dynamical behaviour of systems (60).

knowledge, this is the first time such a system is proposed. (2) To study the new model, we obtain some new methods and skills (such as the new structure method of the Lyapunov function, the applications of delay differential inequalities) that can also be used to research other related models with multi-delays and feedback controls. Because of the complexity of the new system, the structure of the Lyapunov function is completed step by step to overcome the difficulties brought about by the multiple time delays and feedback controls, please see pages 10–17. (3) In this paper, the research contents are richer than the related references. We study not only the permanence and global attractivity of the new system but also some numerical simulations to the new system (12) are given to illustrate our results obtained in this paper. (4) The sufficient conditions obtained herein are new, general, and easily verifiable, which provide flexibility for the application and analysis of three-species multi-delays Lotka–Volterra predator–prey model with feedback controls.

Disclosure statement

No potential conflict of interest was reported by the authors.

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References

[1] F.D. Chen, *On a nonlinear nonautonomous predator-prey model with diffusion and distributed delay*, J. Comput. Appl. Math. 180 (1) (2005), pp. 33–49.

[2] G.C. Lu, Z.Y. Lu, and X.Z. Lian, *Delay effect on the permanence for Lotka-Volterra cooperative systems*, Nonlinear Anal. Real World Appl. 11 (4) (2010), pp. 2810–2816.

[3] Y. Nakata and Y. Muroya, *Permanence for nonautonomous Lotka-Volterra cooperative systems with delays*, Nonlinear Anal. Real World Appl. 11 (1) (2010), pp. 528–534.

[4] C.J. Xu and Y.S. Zu, *Permanence of a two species delayed competitive model with stage structure and harvesting*, Bull. Korean Math. Soc. 52 (4) (2015), pp. 1069–1076.

[5] G.C. Lu and Z.Y. Lu, *Permanence for two-species Lotka-Volterra cooperative systems with delays*, Math. Biosci. Eng. 5 (3) (2008), pp. 477–484.

[6] C.J. Xu, X.H. Tang, M.X. Liao, and X.F. He, *Bifurcation analysis in a delayed Lokta-Volterra predator-prey model with two delays*, Nonlinear Dyn. 66 (1-2) (2011), pp. 169–183.

[7] A. Muhammadhaji, Z.D. Teng, and M. Rehim, *Dynamical behavior for a class of delayed competitive- mutualism systems*, Diff. Equ. Dyn. Sys. 23 (3) (2015), pp. 281–301.

[8] C.J. Xu, X.H. Tang, and M.X. Liao, *Stability and bifurcation analysis of a delayed predator-prey model of prey dispersal in two-patch environments*, Appl. Math. Comput. 216 (10) (2010), pp. 2920–2936.

[9] C.J. Xu and M.X. Liao, *Bifurcation behaviours in a delayed three-species food-chain model with Holling type-II functional response*, Appl. Anal. 92 (12) (2013), pp. 2468–2486.

[10] C.J. Xu and M.X. Liao, *Bifurcation analysis of an autonomous epidemic predator-prey model with delay*, Annali di Matematica Pura ed Applicata 193 (1) (2014), pp. 23–28.

[11] C.J. Xu and P.L. Li, *Oscillations for a delayed predator-prey model with Hassell-Varley-type functional response*, C. R. Biol. 338 (4) (2015), pp. 227–240.

[12] L. Nie, Z. Teng, L. Hu, and J. Peng, *Permanence and stability in non-autonomous predator-prey Lotka-Volterra systems with feedback controls*, Comput. Math. Appl. 58 (2009), pp. 436–448.

[13] K. Yang, H.N. Wang, and F.D. Chen, *On the stability property of a Lotka-Volterra cooperation system with feedback controls*, Mathematica Applicata 27 (2) (2014), pp. 243–247. (in Chinese)

[14] C.Y. Wang, Y.Q. Zhou, Y.H. Li, and R. Li, *Well-posedness of a ratio-dependent Lotka-Volterra system with feedback control*, Bound. Value Probl. 2018 (2018), p. ID: 117.

[15] Y. Fan and L. Wang, *Global asymptotical stability of a Logistic model with feedback control*, Nonlinear Anal. Real World Appl. 11 (2010), pp. 2686–2697.

[16] K. Gopalsamy and P. Weng, *Global attractivity in a competition system with feedback controls*, Comput. Math. Appl. 45 (2003), pp. 665–676.

[17] F. Chen, *The permanence and global attractivity of Lotka–Volterra competition system with feedback controls*, Nonlinear Anal. Real World Appl. 7 (2006), pp. 133–143.

[18] J. Li, A. Zhao, and J. Yan, *The permanence and global attractivity of a Kolmogorov system with feedback controls*, Nonlinear Anal Real World Appl. 10 (2009), pp. 506–518.

[19] C.Y. Wang, X.W. Li, and H. Yuan, *The permanence of a ratio-dependent Lotka-Volterra predator-prey model with feedback control*, Adv. Mat. Res. 765–767 (2013), pp. 327–330.

[20] R.Y. Han, F.D. Chen, and X.D. Xie, *Stability of Lotka-Volterra cooperation system with single feedback control*, Ann. Appl. Math. 31 (3) (2015), pp. 287–296.

[21] C.Y. Wang, H. Liu, and S. Pan, *Globally attractive of a ratio-dependent Lotka-Volterra predator-prey model with feedback control*, Adv. Biosci. Bioeng. 4 (5) (2016), pp. 59–66.

[22] K. Gopalsamy and P.X. Weng, *Feedback regulation of logistic growth*, Int. J. Math. Math. Sci. 16 (1) (1993), pp. 177–192.

[23] F.D. Chen, Z. Li, and Y.J. Huang, *Note on the permanence of a competitive system with infinite delay and feedback controls*, Nonlinear Anal. Real World Appl. 8 (2) (2007), pp. 680–687.

[24] F.D. Chen, J.H. Yang, and L.J. Chen, *Note on the persistent property of a feedback control system with delays*, Nonlinear Anal. Real World Appl. 11 (2) (2010), pp. 1061–1066.

[25] Z. Li, M.A. Han, and F.D. Chen, *Influence of feedback controls on an autonomous Lotka-Volterra competitive system with infinite delays*, Nonlinear Anal. Real World Appl. 14 (1) (2013), pp. 402–413.

[26] J.Y. Xu and F.D. Chen, *Permanence of a Lotka-Volterra cooperative system with time delays and feedback controls*, Commun. Math. Biol. Neurosci. 2015 (2015), p. Article ID: 18.
[27] C.J. Xu and P.L. Li, *Almost periodic solutions for a competition and cooperation model of two enterprises with time-varying delays and feedback controls*, J. Appl. Math. Comput. 53 (1–2) (2017), p. 397411.
[28] H.K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice-Hall, Englewood Cliffs, 2002.