~~Stationary Contact~~

Ans 12) Convergence :

To prove: Rate of convergence of Newton's method is quadratic.

Proof: let r be the root of $f(x)=0$

let x_n be the numerical root that we get numerically.

$$f(r) = 0 = f(x_n+\delta) = f(x_n) + f'(x_n)(r-x_n) + O(x_n^2)$$

In Newton-Raphson we ignore the higher order term:

$$\therefore \quad x_{n+1} = x_n - f(x_n)/f'(x_n)$$

$$\Rightarrow \quad f(x_n) = f'(x_n)(x_n - x_{n+1})$$

$\therefore$ Using the above two eq^n we get

$$f'(x_n)(r-x_{n+1}) + \frac{f''(\xi)}{2}(r-x_n)^2 = 0 \text{ where } \xi \text{ is : } 0<\xi<|r-x_n|$$

Now let $\lambda_n = (r-x_n)$

$$\therefore \quad \lambda_{n+1} = r - x_{n+1}$$

It's quite clear that λ_n is the error in the sol^n after n^{th} iteration.

$$\therefore \quad \lambda_{n+1} = \frac{-f''(\xi)}{2f'(x_n)} \sim \lambda_n^2 \Rightarrow \lambda_{n+1} \propto \lambda_n^2.$$

Thus we see that N-R convergence quadratically.

Scanned by CamScanner

(b) Rate of convergence for secant method

I use the same notation as in previous problem.

Now if $\lambda_{n+1} = C\lambda_n^a \Rightarrow a$ is the rate of convergence.

$$x_{n+1} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{---(1)}$$

Now $\lambda_{n+1} = x_{n+1} - r$

$\lambda_n = x_n - r$

Using the above eq^n in the main eq^n (1)

$$\lambda_{n+1} = \frac{\lambda_{n-1}f(x_n) - f(x_{n-1})\lambda_n}{f(x_n) - f(x_{n-1})}$$

Using mean value theorem,

$$f'(\eta_n) = \frac{f(x_n) - f(r)}{x_n - r} \qquad \text{where} \quad 0 < \eta_n < |x_n - r|$$

$$\therefore \quad f'(\eta_n) = \frac{f(x_n)}{\lambda_n}$$

$$\therefore \quad f(x_{n-1}) = \lambda_{n-1} f'(\eta_{n-1})$$

$$\lambda_{n+1} = \frac{\lambda_n \lambda_{n-1} \, f'(\eta_n) - f'(\eta_{n-1})}{f(x_n) - f(x_{n-1})}$$

ie., $\lambda_{n+1} \propto \lambda_n \lambda_{n-1}$

For rate of convergence $\quad \lambda_n \propto \lambda_{n-1}^{a}$

i.e., $\quad \lambda_{n+1} \propto \lambda_n^{a}$

$\therefore \quad \lambda_n^{a} \propto \lambda_{n-1} \lambda_{n-1}$

i.e, $\lambda_n \propto \lambda_{n-1}^{(a+1)/a}$

$\Rightarrow \quad a = (a+1)/p$

$\Rightarrow \quad p = \frac{1 \pm \sqrt{5}}{2} \simeq 1.6$

$\therefore \quad \boxed{\lambda_{n+1} \propto \lambda_n^{1.6}}$ This gives the late of convergence