Tensor Analysis

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Abstract

A physical quantity which is invariant under the coordinate transformation is known as Tensor.

Contravariant Tensor

If N quantities A^1,A^2,A^3.....A^N in one coordinate system (x^1,x^2,......x^N) are related with N other physical quantities $\overline{A}^1$,$\overline{A}^2$,.....$\overline{A}^N$ in the another coordinate system ($\overline{x}^1$,$\overline{x}^2$....$\overline{x}^N$) by the transformation relation.

$\overline{A}^q = \frac{\partial \overline{x}^q}{\partial x^p} A^p$

Then they are called components of a contravariant vector or tensor of the first rank or first order.

Covariant Tensor

If N quantities A_1,A_2,A_3.....A_N in one coordinate system(x^1,x^2,......x^N) are related with N other physical quantities $\overline{A}_1$,$\overline{A}_2$,.....$\overline{A}_N$ in the another coordinate system ($\overline{x}^1$,$\overline{x}^2$....$\overline{x}^N$) by the transformation rel^n.

$\overline{A}_j = \frac{\partial x^k}{\partial \overline{x}^j} A_k$

then they are called components of a covariant vector or tensor of the first rank or first order.

TU Question solution

Q.1.Suppose $ds^2 = g_{jk}dx^j dx^k$ is an invariant show that g_{jk} is symmetric covariant tensor of rank 2.
Solution: In (x^1,x^2,.......x^N) coordinate;

$$ds^2 = g_{jk}dx^j dx^k$$

In ($\overline{x}^1$,$\overline{x}^2$....$\overline{x}^N$) coordinate;

$$d\overline{s}^2 = \overline{g}_{pq}d\overline{x}^p d\overline{x}^q$$

As ds^2 is invariant ,we can write

$$ds^2 = d\overline{s}^2$$

$$g_{jk}dx^j dx^k = \overline{g}_{pq}d\overline{x}^p d\overline{x}^q$$

$$g_{jk}dx^j dx^k = \overline{g}_{pq}\frac{\partial \overline{x}^p}{\partial x^j}dx^j \frac{\partial \overline{x}^q}{\partial x^k}dx^k$$

$$g_{jk}dx^j dx^k = \overline{g}_{pq}\frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial \overline{x}^q}{\partial x^k}dx^j dx^k$$

i.e $g_{jk} = \frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial \overline{x}^q}{\partial x^k}\overline{g}_{pq}$
Again,
g$_{kj} = \frac{\partial \overline{x}^p}{\partial x^k}\frac{\partial \overline{x}^q}{\partial x^j}\overline{g}_{pq}$

$$=\frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial \overline{x}^q}{\partial x^k}\overline{g}_{pq}$$

i.e.$g_{kj} = g_{jk}$

g_{jk} is symmetric .
Hence, $g_{jk}$is a symmetric covariant tensor of rank two.

Q.2.Prove that if a tensor is symmetric in one system then it is symmetric in all system.
or
If $A^{pq} = A^{qp}$ then $\overline{A}^{lm} = \overline{A}^{ml}$

Solution:

$\overline{A}^{lm} = \frac{\partial \overline{x}^l}{\partial x^p} \frac{\partial \overline{x}^m}{\partial x^q} A^{pq}$

$\overline{A}^{lm} = \frac{\partial \overline{x}^m}{\partial x^q} \frac{\partial \overline{x}^l}{\partial x^p} A^{pq}$

$\overline{A}^{lm} = \frac{\partial \overline{x}^m}{\partial x^q} \frac{\partial \overline{x}^l}{\partial x^p} A^{qp}$

i.e.$\overline{A}^{lm} = \overline{A}^{ml}$
Hence Proved.

Q.3.Show that the velocity of a fluid at any point is a contravariant tensor of rank one.

Solution:
The velocity of fluid at any point has components $\frac{dx^k}{dt}$ in the coordinate systemx^k.

In coordinate system,$\overline{x}^j$,the velocity is $\frac{d\overline{x}^j}{dt}$.

Now,$\frac{d\overline{x}^j}{dt} = \frac{\partial \overline{x}^j}{\partial x^k} \frac{dx^k}{dt}$

i.e.$\overline{V}^j = \frac{\partial \overline{x}^j}{\partial x^k} V^k$
It shows that the velocity is a contravariant tensor of rank one or a contravariant vector.

Q.4.Show that every tensor can be expressed as the sum of two tensors,one of which is symmetric and the other skew-symmetric in a pair of covariant or contracariant indicies.

Solution: Let us consider a contracvariant tensor A^{pq} then we can write

$A^{pq} = \frac{1}{2}(A^{pq} + A^{qp}) + \frac{1}{2}(A^{pq} - A^{qp})$

Let $B^{pq} = \frac{1}{2}(A^{pq} + A^{qp})$,$C^{pq} = \frac{1}{2}(A^{pq} - A^{qp})$

i.e.$A^{pq} = B^{pq} + C^{pq}$..........(1)
Now, We have to proved equation (1) contains symmetric and skew symmetric tensors.
Now,As,

$B^{pq} = \frac{1}{2}(A^{pq} + A^{qp})$

Now, Exchange position of indices in B^{qp}

i.e. $B^{qp} = \frac{1}{2}(A^{qp} + A^{pq})$

$B^{qp} = \frac{1}{2}(A^{pq} + A^{qp})$

i.e.$B^{qp} = B^{pq}$
i.e.B^{pq} is symmetric contravariant tensor.
Again As,

$C^{pq} = \frac{1}{2}(A^{pq} - A^{qp})$

Now Exchange position of indices in C^{pq}

i.e.$C^{qp} = \frac{1}{2}(A^{qp} - A^{pq})$

$C^{qp} = -\frac{1}{2}(A^{pq} - A^{qp})$

$C^{pq} = -C^{qp}$

i.e.C^{pq} is skew symmetric contravariant tensor.
This shows that A^{pq} is sum of symmetric and skew symmetric tensors B^{pq} and C^{pq} respectively.

Q.4.Show that the geodesics in a Riemannian space are given by
$\frac{d^2x^r}{ds^2}+\left[\begin{array}{c} r \\ pq \end{array}\right]\frac{dx^p}{ds}\frac{dx^q}{ds}=0$

Solution: We must determine the extremum of $\int_{t_1}^{t_2}\sqrt{g_{pq}\dot{x}^p\dot{x}^q}dt$
Using Euler's equation with
$F=\sqrt{g_{pq}\dot{x}^p\dot{x}^q}$
We have
$\frac{\partial F}{\partial x^k}=\frac{1}{2}(g_{pq}\dot{x}^p\dot{x}^q)^{-1/2}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q$

$\frac{\partial F}{\partial \dot{x}^k}=\frac{1}{2}(g_{pq}\dot{x}^p\dot{x}^q)^{-1/2}2g_{pk}\dot{x}^p$

Using $\frac{ds}{dt}=\sqrt{g_{pq}\dot{x}^p\dot{x}^q}$,Euler's equation can be written as

$\frac{d}{dt}\frac{g_{pk}\dot{x}^p}{\dot{s}}-\frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q$=0

$g_{pk}\ddot{x}^p+\frac{\partial g_{pk}}{\partial x^q}\dot{x}^p\dot{x}^q-\frac{1}{2}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q=\frac{g_{pk}\dot{x}^p\ddot{s}}{\dot{s}}$

Writing $\frac{\partial g_{pk}}{\partial x^q}\dot{x}^p\dot{x}^q=\frac{1}{2}\left[\frac{\partial g_{pk}}{\partial x^q}+\frac{\partial g_{qk}}{\partial x^p}\right]\dot{x}^p\dot{x}^q$ This equation becomes

$g_{pk}\ddot{x}^p+[pq,k]\dot{x}^p\dot{x}^q=\frac{g_{pk}\dot{x}^p\ddot{s}}{\dot{s}}$

If we use arc length as parameters $\dot{s}=1$,$\ddot{s}=0$ and the equation becomes
$g_{pk}\frac{d^2x^p}{ds^2}+[pq,k]\frac{dx^p}{ds}\frac{dx^q}{ds}=0$
Multiplying by g^{rk} ,We obtain

$\frac{d^2x^r}{ds^2}+\left[\begin{array}{c} r \\ pq \end{array}\right]\frac{dx^p}{ds}\frac{dx^q}{ds}=0$

Q.5.A quantity A(p,q,r) is such that in the coordinate system x^i ,$A(p,q,r)B_r^{qs} = c_p^s$ where B_r^{qs} is an arbitrary tensor and c_p^s is a tensor.Prove that A(p,q,r) is a tensor.

Solution: In the transformed coordinate $\overline{x}^i$,
$\overline{A}(j,k,l)\overline{B}_l^{km} = \overline{c}_j^m$
Then,$\overline{A}(j,k,l)\frac{\partial\overline{x}^k}{\partial x^q}\frac{\partial\overline{x}^m}{\partial x^s}\frac{\partial x^r}{\partial\overline{x}^l}B_r^{qs} = \frac{\partial\overline{x}^m}{\partial x^s}\frac{\partial x^p}{\partial\overline{x}^j}C_p^s$

$=\frac{\partial\overline{x}^m}{\partial x^s}\frac{\partial x^p}{\partial\overline{x}^j}A(p,q,r)B_r^{qs}$

$\frac{\partial\overline{x}^m}{\partial x^s}[\frac{\partial\overline{x}^k}{\partial x^q}\frac{\partial x^r}{\partial\overline{x}^l}\overline{A}(j,k,l) - \frac{\partial x^p}{\partial\overline{x}^j}A(p,q,r)]B_r^{qs} = 0$

Inner multiplication by $\frac{\partial x^n}{\partial\overline{x}^m}$

$\delta_s^n[\frac{\partial\overline{x}^k}{\partial x^q}\frac{\partial x^r}{\partial\overline{x}^l}\overline{A}(j,k,l) - \frac{\partial x^p}{\partial\overline{x}^j}A(p,q,r)]B_r^{qs} = 0$

$[\frac{\partial\overline{x}^k}{\partial x^q}\frac{\partial x^r}{\partial\overline{x}^l}\overline{A}(j,k,l) - \frac{\partial x^p}{\partial\overline{x}^j}A(p,q,r)]B_r^{qs} = 0$

Since B_r^{qn} is an arbitrary tensor, we have

$[\frac{\partial\overline{x}^k}{\partial x^q}\frac{\partial x^r}{\partial\overline{x}^l}\overline{A}(j,k,l) - \frac{\partial x^p}{\partial\overline{x}^j}A(p,q,r)] = 0$

Inner multiplication by $\frac{\partial x^q}{\partial\overline{x}^m}\frac{\partial\overline{x}^n}{\partial x^r}$ yields

$\delta_m^k\delta_l^n\overline{A}(j,k,l) - \frac{\partial x^p}{\partial\overline{x}^j}\frac{\partial x^q}{\partial\overline{x}^m}\frac{\partial\overline{x}^n}{\partial x^r}A(p,q,r) = 0$

$\overline{A}(j,m,n) = \frac{\partial x^p}{\partial\overline{x}^j}\frac{\partial x^q}{\partial\overline{x}^m}\frac{\partial\overline{x}^n}{\partial x^r}A(p,q,r)$

Which shows that A(p,q,r) is a tensor and justifies use of notation A_{pq}^r.