AN
INTRODUCTION
TO
APPLIED
MATHEMATICS

JAEGER

OXFORD

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BY

J. C. JAEGER
PROFESSOR OF APPLIED MATHEMATICS
IN THE UNIVERSITY OF TASMANIA

OXFORD
AT THE CLARENDON PRESS

Oxford University Press, Amen House, London E.C. 4

GLASGOW NEW YORK TORONTO MELBOURNE WELLINGTON
BOMBAY CALCUTTA MADRAS KARACHI CAPE TOWN IBADAN

Geoffrey Cumberlege, Publisher to the University

FIRST EDITION 1951
REPRINTED LITHOGRAPHICALLY IN GREAT BRITAIN
AT THE UNIVERSITY PRESS, OXFORD
FROM SHEETS OF THE FIRST EDITION
1956

PREFACE

This book consists of a short course of applied mathematics which assumes only a knowledge of the calculus and the fundamentals of statics and dynamics. It is based on lectures which I have given in the University of Tasmania, where all students, whether their main interests are in mathematics or not, are taught in the same classes. In it I have attempted, by choosing for detailed study the mathematics of problems of obvious practical importance, to provide a course which is more interesting and useful to physicists and engineers than the more academic ones, but which at the same time gives as good a training for mathematicians.

It is intended for students at the intermediate stage at which the normal practice is to develop mathematical skill by an intensive study of comparatively difficult problems in statics and dynamics. In place of this it offers a comprehensive course on ordinary differential equations and the subjects in which they occur, including the theory of vibrations, electric circuit theory, elasticity, and selected physical problems, as well as particle and rigid dynamics. In addition, it contains an introduction to partial differential equations and their applications, together with topics such as special functions, Fourier series, Fourier and Laplace transforms, and numerical methods, which are required for their solution. An introduction of this sort seems to me to make subsequent teaching at a higher level much easier, and it has also the great advantage of providing a proper mathematical background for students of physics and engineering who may study no more mathematics.

Applied mathematics at this level consists mainly of applied differential equations, and thus it seems proper that so much of the theory of differential equations as is needed should be regarded as a part of applied mathematics and taught in it with an eye to applications, rather than as academic pure mathematics. Accordingly, in the first three chapters I have given a short and self-contained, but fairly complete, treatment of ordinary differential equations, with many physical problems as examples. This is followed in Chapter IV by a long account of mechanical problems leading to ordinary linear differential equations; these are mostly vibration problems and are useful for developing manipulative skill, apart from their technical importance. In Chapter V similar problems in electric circuit theory are discussed and the analogy with mechanical problems developed; here, again, the object is to study the mathematics which arises in electrical problems rather than to develop the theory of the subject in detail, although in fact most of the important topics are covered, including the impedance of complicated circuits, servomechanisms, and triode oscillations as far as van der Pol's equation.

Chapter VII begins with non-linear problems in particle dynamics and electric circuit theory, including a discussion of the oscillations of non-linear systems by the method of Kryloff and Bogoliuboff: it concludes with the conventional problems of particle dynamics in two and three dimensions, most space being devoted to the motion of charged particles in electric and magnetic fields. Chapter VIII contains a relatively brief treatment of rigid dynamics as far as the gyroscope; this is included because of its increasing importance in control systems. An account of the elements of potential theory and the nature of the motion of a dynamical system is given in Chapter IX, which concludes with Lagrange's equations. Both vector and Cartesian methods are used in these chapters, a brief account of the elements of vector algebra having been given in Chapter VI.

Boundary value problems are considered in Chapter X, in particular those arising in the bending of beams. These provide many interesting examples and serve as an introduction to matters such as eigenvalue problems and the use of the delta function.

The last four chapters complete the design of giving an introduction to more advanced topics. Fourier series and integrals and their applications to initial and boundary value problems are treated in Chapter XI. Ordinary linear differential equations with variable coefficients are treated briefly in Chapter XII, where an outline of the properties of Bessel and Legendre functions is given and the occurrence of Mathieu's equation in vibration problems is sketched.

In Chapter XIII an introduction to partial differential equations is given. The important equations in two variables are derived and solved in interesting special cases, and the various methods available for their solution described. At the end of the chapter the corresponding equations in two and three dimensions are discussed, and the use of Bessel and Legendre functions in their solution indicated.

Finally, because of the growing importance of numerical mathematics, the principal interpolation formulae are given in Chapter XIV, together with a short account of step-by-step and relaxation methods for the solution of ordinary and partial differential equations.

The pace of most of the text is leisurely, but it is increased towards the ends of the chapters where more difficult matters, such as moving axes, Fourier integrals, etc., are introduced. These, of course, may be omitted if desired. As remarked earlier, a knowledge of the fundamental principles of statics and dynamics is taken for granted and these are not restated in the text. In the case of topics such as vectors, bending moments, and moments of inertia, which probably will already have been studied in statics, those parts of the theory which are needed are covered relatively briefly. No knowledge of electric circuit theory beyond the meanings of fundamental quantities such as capacitance and inductance is assumed. Other subjects are developed *ab initio* when required. A collection of some three hundred examples, containing answers, has been included. These have been constructed as far as possible to correspond to problems of practical interest and at the same time to give training in mathematical manipulation.

It is a pleasure to acknowledge the assistance of many friends who have discussed the project with me, and also that of the University of Tasmania in extending my leave of absence to enable me to complete the work.

J. C. J.

20 *May* 1950

CONTENTS

I

ORDINARY DIFFERENTIAL EQUATIONS AND THEIR OCCURRENCE

1. Introductory

MOST of the problems of applied mathematics with which we shall be concerned involve differential equations. The first step in the solution of such problems is the setting up of their differential equations together with any other conditions which may have to be satisfied; the second step consists of the solving of these differential equations; and the third and last step of expressing the solutions in a simple form from which physical conclusions can be drawn or numerical calculations made.

A working knowledge of the theory of differential equations is thus an essential preliminary, so in this chapter a brief account† of the common types and the way in which they arise is given, followed, in the next two chapters, by the standard methods for their solution.

2. Definitions

Any relation between the independent‡ variable x, the dependent variable y, and its successive derivatives dy/dx, d^2y/dx^2,..., etc., is called an *ordinary*§ *differential equation*. The *order* of a differential equation is the order of the highest differential coefficient occurring in it. Thus, for example,

$$\frac{d^2y}{dx^2}+xy = 1, \tag{1}$$

$$\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^3+y = 0, \tag{2}$$

† For fuller treatments see, for example, Piaggio, *Differential Equations* (Bell); Forsyth, *Treatise on Differential Equations* (Macmillan); Ince, *Ordinary Differential Equations* (Longmans).

‡ When discussing the *theory* of differential equations we shall take the independent variable to be x and the dependent variable y. In applications the symbols for dependent and independent variables are determined by the problems. It is assumed throughout that y has derivatives of all the orders involved for all values of x.

§ If there are two or more independent variables the equation is a partial differential equation: these will be discussed in Chapter XIII. Until then we shall usually omit the word 'ordinary'.

and $$\left(\frac{d^2y}{dx^2}\right)^2+y\left(\frac{dy}{dx}\right)^3+y=0, \tag{3}$$

are all second-order differential equations.

All the differential equations we shall need will contain only rational integral algebraical functions of the differential coefficients (fractional powers of y and x may sometimes appear), and in such cases the degree of the highest differential coefficient involved is called the *degree* of the equation. Thus (1) and (2) are both of the second order and the first degree, while (3) is of the second order and the second degree.

In the same way, we may have systems of simultaneous ordinary differential equations for several variables y, z,... in terms of a single independent variable x.

3. Ordinary linear differential equations

By far the most important type of differential equation with which we shall be concerned is that in which all terms are of at most the first degree in y and its derivatives. This is called an ordinary linear differential equation, and its general form for the nth order is

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=\phi(x). \tag{1}$$

The quantities $a_0(x)$, $a_1(x)$,..., $a_n(x)$ are called the coefficients; often they are constants, in which case the equation is referred to as an ordinary linear differential equation *with constant coefficients*; if they are functions of x it is an ordinary linear differential equation *with variable coefficients*.

Equation (1) will also often be described as an *inhomogeneous* linear differential equation to distinguish it from the corresponding equation with $\phi(x)=0$, namely

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=0, \tag{2}$$

which is called a *homogeneous* linear differential equation. The reason for this nomenclature† is that all terms of (2) are of the

† This usage is common in many branches of mathematics, but the term 'homogeneous' is often also used for certain special types of differential equation; cf. § 25 and Ex. 2 on Chapter II. These, however, are not of much importance in applied mathematics and no confusion will arise.

same kind, that is, containing y or its derivatives, while (1) contains the term $\phi(x)$ which is of a different kind.

Equations (1) and (2) have fundamental properties which distinguish them from all other types. Considering the homogeneous equation (2) first, suppose that y_1 and y_2 are two different solutions of it, so that

$$a_0(x)\frac{d^n y_1}{dx^n}+a_1(x)\frac{d^{n-1}y_1}{dx^{n-1}}+...+a_n(x)y_1 = 0, \tag{3}$$

$$a_0(x)\frac{d^n y_2}{dx^n}+a_1(x)\frac{d^{n-1}y_2}{dx^{n-1}}+...+a_n(x)y_2 = 0. \tag{4}$$

Then if c_1 and c_2 are any constants, it follows, by adding c_1 times (3) to c_2 times (4), that $c_1 y_1+c_2 y_2$ also satisfies (2). That is, if we know two solutions of (2), any linear combination of these is also a solution. And, similarly, if y_1, y_2,..., y_n are n different solutions of (2), the general linear combination of these, namely

$$c_1 y_1+c_2 y_2+...+c_n y_n,$$

where c_1, c_2,..., c_n are any constants, is also a solution. This result does not hold for the inhomogeneous equation (1).

The important result for the inhomogeneous linear equation (1) is that if y_1 is a solution of it with a function $\phi_1(x)$ on the right-hand side, y_2 a solution with $\phi_2(x)$ on the right-hand side, and so on, then $y_1+y_2+...+y_n$ satisfies

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_n(x)y = \phi_1(x)+...+\phi_n(x). \tag{5}$$

This follows by adding the equations of type (1) for y_1 to y_n. It will appear subsequently that, in many of the applications we shall make, $\phi(x)$ refers to the cause and y describes the effect, and the above result then implies that, *if the equations governing the problem are linear*, the effects of a number of superposed causes can be added. This result is often referred to as the Principle of Superposition.

If any powers or products of y and its derivatives occur in a differential equation it is described as *non-linear* and neither of the above results hold for it. For example, if y_1 and y_2 satisfy

$$\frac{d^2y}{dx^2}+y\frac{dy}{dx}+y = 0, \tag{6}$$

neither $c_1 y_1+c_2 y_2$, nor even $c_1 y_1$, satisfy (6) because of the product term $y(dy/dx)$.

The distinction between linear and non-linear differential equations is of fundamental importance. Broadly speaking, it will appear that the solution of the ordinary linear differential equation with constant coefficients (homogeneous or inhomogeneous) is a relatively easy matter; the ordinary linear differential equation with variable coefficients is more difficult, but its theory is well known and special equations of importance have been extensively studied; much less is known about non-linear differential equations, and there is no comprehensive general theory of them—a number of special types can be solved, but equations which arise in practice often do not belong to these and have to be solved by processes such as successive approximation. In any practical problem the first question to study is that of whether the differential equations involved are accurately linear (in which case a complete solution should be attainable), or are approximately linear (that is, if the terms which make them non-linear are small and can be neglected for a first approximation), or are non-linear. In the next sections we shall make a preliminary survey of the simplest equations which will arise later and discuss them from this point of view.

4. The differential equations of particle dynamics

The simplest equations arise in problems on motion of a particle of mass m in a straight line. Here, if y is the position of the particle at time t, Newton's second law gives

$$m\frac{d^2y}{dt^2} = \text{force}. \tag{1}$$

The force is in general a complicated function

$$F\left(y, t, \frac{dy}{dt}\right) \tag{2}$$

of position, time, and velocity. In many cases it simplifies further into an expression of type

$$f(y)+g\left(\frac{dy}{dt}\right)+h(t), \tag{3}$$

so that the differential equation (1) becomes

$$m\frac{d^2y}{dt^2} = f(y)+g\left(\frac{dy}{dt}\right)+h(t). \tag{4}$$

If $f(y) = ky$ and $g(dy/dt) = k'\,dy/dt$, where k and k' are constants, this becomes an inhomogeneous second-order linear differential equation and thus easy to solve. This is an important special case, since $f(y) = ky$ occurs in the theory of small oscillations, but usually

$$g\left(\frac{dy}{dt}\right) = k'\frac{dy}{dt}$$

is only a very rough approximation to the truth.

It appears that the problems of particle dynamics will usually be non-linear, but that linear equations will be encountered in approximations to systems of importance.

5. The differential equations of electric circuit theory

The charge Q on a condenser of capacitance C in a closed circuit containing inductance L and resistance R satisfies the differential equation [cf. Chap. V]

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q = 0, \tag{1}$$

where L, R, and C are constants. This is a linear second-order differential equation with constant coefficients. The same remark applies to more complicated circuits—the equations of electric circuit theory are (very nearly) accurately linear, in contrast to the equations of dynamics which are only linear in rather special cases.

Of course non-linear equations do arise in electric circuit theory (and, in fact, are so important that they have stimulated much modern theoretical work), for example, if the inductance has an iron core, L in (1) is not constant but a complicated function of dQ/dt. Again, if the circuits contain vacuum tubes the equations are only approximately linear, and in some problems, such as oscillator circuits, the non-linearity is of fundamental importance.

6. Differential equations appearing in other problems of applied mathematics

In the development of most branches of applied mathematics, differential equations are deduced from first principles at an early stage of the subject. For example, the fundamental differential equations of bending of beams are deduced in § 80 and those for steady flow of heat in § 16. We give below some examples of the method of writing down differential equations for practical problems. Others will be found in the examples on Chapters II and III.

Ex. 1. *The atoms of a radio-active substance A change into atoms of a substance B at a rate k times the number of atoms of A present.*

Let n be the number of atoms of A present at time t, then the statement above is equivalent to the differential equation

$$\frac{dn}{dt} = -kn. \tag{1}$$

Ex. 2. *The atoms of the substance B of Ex. 1 change into atoms of another substance C at a rate k_1 times the number n_1 of atoms of B present.*

This statement gives the differential equation

$$\frac{dn_1}{dt} = kn - k_1 n_1, \tag{2}$$

and the equations of the problem are the pair of equations (1) and (2). If the substance C also changes, another equation is added, and so on. More complicated equations of the same type appear in chemical kinetics, genetics, etc.

Ex. 3. *Mass M of water of specific heat unity is so well stirred that its temperature T is constant throughout the mass, it loses heat at a rate H times the difference between its temperature and the temperature T_0 of its surroundings.*

The differential equation is

$$M\frac{dT}{dt} = -H(T - T_0). \tag{3}$$

Ex. 4. *The rate of loss of heat in Ex. 3 is H' times the difference $T^4 - T_0^4$.*

Here in place of (3) the differential equation is

$$M\frac{dT}{dt} = -H'(T^4 - T_0^4) \tag{4}$$

and is non-linear.

Ex. 5. *Water of specific heat unity and temperature T_1 flows into a bath at the rate of m c.c. per sec. If T is the temperature of the water in the bath, and T_2 that of its surroundings, heat is lost from the bath at the rate $H(T-T_2)$.*

Suppose M is the mass of water in the bath at time t, then M satisfies

$$\frac{dM}{dt} = m. \tag{5}$$

Also the quantity of heat MT in the bath at time t satisfies

$$\frac{d}{dt}(MT) = mT_1 - H(T-T_2) \tag{6}$$

or, using (5),
$$M\frac{dT}{dt} + mT = mT_1 - H(T-T_2). \tag{7}$$

The pair of equations (5) and (7) are the differential equations of the problem.

7. The occurrence of differential equations in geometry

Differential equations often occur as the expression of some geometrical property of a curve.

For example, if we require the slope, dy/dx, at a point of a curve to be k times the ordinate at the point, we have

$$\frac{dy}{dx} = ky. \tag{1}$$

Or if we had required the curvature at a point to be k times the ordinate at the point we would have

$$\frac{d^2y}{dx^2} = ky\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}},$$

or, squaring,
$$\left(\frac{d^2y}{dx^2}\right)^2 = k^2y^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3, \tag{2}$$

an equation of the second order and second degree. Equations of this type appear in the theory of finite flexure of thin rods.

8. The elimination of arbitrary constants from a functional relation

Suppose we have a connexion between y and x involving a number of arbitrary constants, e.g.

$$y = c_1 e^{-x} + c_2 e^{-2x}, \tag{1}$$

where c_1 and c_2 are arbitrary constants, that is, are independent constants which may have any numerical values.

Differentiating, we have

$$\frac{dy}{dx} = -c_1 e^{-x} - 2c_2 e^{-2x}, \tag{2}$$

$$\frac{d^2y}{dx^2} = c_1 e^{-x} + 4c_2 e^{-2x}. \tag{3}$$

From (1), (2), and (3) we can eliminate c_1 and c_2, since adding (1) and (2) gives

$$y + \frac{dy}{dx} = -c_2 e^{-2x},$$

and adding (2) and (3) gives

$$\frac{dy}{dx} + \frac{d^2y}{dx^2} = 2c_2 e^{-2x}.$$

Thus
$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0. \tag{4}$$

It appears that a differential equation of the second order has been formed by eliminating the two arbitrary constants c_1 and c_2 from (1) and its first two derivatives. This differential equation is satisfied by all the functions (1), whatever values the constants c_1 and c_2 may be given.

In general it can be shown [cf. Ince, loc. cit., p. 4] in much the same way that, if we are given a relation

$$f(x, y, c_1, \dots, c_n) = 0 \tag{5}$$

between x, y, and n arbitrary constants, differentiating successively n times gives n relations between x and y and its derivatives and the constants, and from these and the original equation $c_1, \dots, c_n$ can be eliminated, giving a differential equation of order n

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^ny}{dx^n}\right) = 0. \tag{6}$$

9. The complete primitive of a differential equation

In § 8 it was found that by eliminating the two arbitrary constants c_1 and c_2 from § 8 (1) we arrived at the differential equation § 8 (4). Looked at from another point of view this states that y given by (1) satisfies the differential equation (4) whatever values the constants c_1 and c_2 may have, or that § 8 (1) is a general solution of § 8 (4) containing two arbitrary constants.

In the same way § 8 (5) is a general solution of the nth-order differential equation § 8 (6) containing n arbitrary constants.

It can be shown† that the general solution of an nth-order differential equation of any degree contains n arbitrary constants; this solution is called the *complete primitive* of the differential equation. Any solution derived from it by giving the constants particular values is called a *particular integral.* In some types of non-linear equation it happens that a solution can be found which cannot be derived from the complete primitive in this way; such a solution is called a *singular solution.*

10. Determination of the arbitrary constants. Initial and boundary value problems

In § 9 it has been seen that the general solution of a differential equation of order n contains n arbitrary constants; this is all the information that can be extracted from the equation itself.

In order to specify a practical problem mathematically, additional information has to be provided. For example, considering the dynamical problem of § 4, we have the differential equation § 4 (4) which determines the way in which the motion is changing at time t; this is a second-order equation and, as in § 9, its complete primitive will contain two arbitrary constants. In the dynamical problem there will also be some information about the way in which the motion was started; for example, if the motion begins at $t = 0$ we will be given the position y and the velocity dy/dt at this time. These give two equations to determine the two arbitrary constants in the complete primitive.

For example, suppose we have to solve

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y = 0, \tag{1}$$

with $y = 1$ and $dy/dx = 0$ when $x = 0$. We have seen in § 8 that the complete primitive of this differential equation is

$$y = c_1 e^{-x}+c_2 e^{-2x}.$$

† The general theory of differential equations is a matter of some difficulty, the hardest points to discuss being the existence and nature of solutions. Cf. Ince, loc. cit., chap. iii.

The information that $y = 1$ and $dy/dx = 0$ at $x = 0$ gives

$$c_1+c_2 = 1,$$

$$c_1+2c_2 = 0.$$

Therefore $c_1 = 2$, $c_2 = -1$, and the required solution is

$$y = 2e^{-x}-e^{-2x}.$$

Problems such as these in which we have to find a solution of a differential equation for all positive values of x which satisfies certain conditions at $x = 0$ are called *initial value problems*. Clearly the problems of dynamics and electric circuit theory are of this type.

In another and equally important class of problem a solution of a differential equation is required in a restricted region, say $0 \leqslant x \leqslant a$, and the arbitrary constants in the complete primitive are to be found from a knowledge of y, dy/dx, etc., at the boundaries $x = 0$ and $x = a$ of the region. Such problems are called *boundary value problems*. For example, the solution of (1) with $y = 0$ at $x = 0$ and $y = 1$ at $x = a$ is

$$y = \frac{e^{-x}-e^{-2x}}{e^{-a}-e^{-2a}}.$$

II

II

ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

11. Introductory

It was remarked in Chapter I that many important problems of applied mathematics lead to ordinary linear differential equations with constant coefficients. Since explicit solutions of these are easily obtained, such problems form the core of the theory and are studied first in detail.

In this chapter we give first the classical methods of finding the general solution containing n arbitrary constants of the equation of order n. Then the case of simultaneous differential equations is discussed, and finally the determination of the arbitrary constants in initial and boundary value problems.

The chapter concludes with a brief account of the Laplace *transformation method* of solving linear differential equations with constant coefficients and given initial conditions. This is a completely alternative approach to that of §§ 12–15 and may be omitted if desired. It is perhaps simpler to learn and teach than the classical methods, but at the same time these are so much part of the common language of mathematics that it is impossible to omit them, and in the sequel they will usually be used. The advantages of the Laplace transformation method increase with the complexity of the problem, and it will occasionally be used to solve relatively complicated problems, in particular, transient problems involving several simultaneous differential equations.

12. The operator D

In the study of linear differential equations with constant coefficients it is convenient to use the symbol D for the operation of differentiation, so that

$$Dy = \frac{dy}{dx}. \tag{1}$$

Again, we write D^2y for $D(Dy)$, so that

$$D^2y = D(Dy) = D\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}, \tag{2}$$

and, similarly, when n is a positive integer,

$$D^n y = D(D^{n-1}y) = \frac{d^n y}{dx^n}. \tag{3}$$

It follows that when m and n are positive integers

$$D^m(D^n y) = D^{m+n}y, \tag{4}$$

so that D operating on y obeys the index law.

Also, if a and b are constants,

$$D(ay) = \frac{d}{dx}(ay) = a\frac{dy}{dx} = a\,Dy, \tag{5}$$

$$D(ay+bz) = a\frac{dy}{dx}+b\frac{dz}{dx} = a\,Dy+b\,Dz, \tag{6}$$

so that the orders of multiplication by a constant and the operation D may be interchanged.

Thus, provided we are considering only the operations of differentiation and multiplication by constants,† the operator D may be treated like an ordinary number. For example

$$\begin{aligned}\frac{d^2y}{dx^2}+2b\frac{dy}{dx}+b^2y &= (D^2+2bD+b^2)y\\ &= D(D+b)y+b(D+b)y\\ &= (D+b)^2y.\end{aligned}$$

In the same way, the so-called general differential expression of order n, namely

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\dots+a_n y, \tag{7}$$

where $a_1,\dots, a_n$ are constants, may be written

$$(D^n+a_1 D^{n-1}+\dots+a_{n-1}D+a_n)y, \tag{8}$$

and for shortness this will usually be written

$$f(D)y,$$

where
$$f(D) \equiv D^n+a_1 D^{n-1}+\dots+a_n. \tag{9}$$

† But if variables enter this is not the case, for example $D(xy) = xDy+y$.

Now suppose that $\alpha_1,..., \alpha_n$ are the roots (not necessarily all real or different) of the equation in α

$$f(\alpha) \equiv \alpha^n+a_1\alpha^{n-1}+...+a_n = 0. \tag{10}$$

Then $f(\alpha)$ can be factorized in the form

$$f(\alpha) \equiv (\alpha-\alpha_1)(\alpha-\alpha_2)...(\alpha-\alpha_n), \tag{11}$$

and the general differential expression (7) or (8) may be put in the form

$$f(D)y \equiv (D-\alpha_1)(D-\alpha_2)...(D-\alpha_n)y. \tag{12}$$

That this is equivalent to the form (7) follows on multiplying out and using the results (4) and (5).

As remarked above, the roots α_1, $\alpha_2,...$, α_n need not all be different: if α_1 is an r-ple root, α_2 an s-ple root, and α_p an m-ple root, (12) may be written

$$f(D)y \equiv (D-\alpha_1)^r(D-\alpha_2)^s...(D-\alpha_p)^m y. \tag{13}$$

The general homogeneous linear differential equation of order n may now be written

$$f(D)y = 0, \tag{14}$$

where $f(D)$ is defined by (7) and (9); the equation (10) is called the *auxiliary equation* for this differential equation, and the first step in the solution of the differential equation is to find the roots $\alpha_1,..., \alpha_n$ of (10), that is, in effect, to express the operator $f(D)$ in the form (13).

As remarked earlier, the simple results given above apply only to the operations of multiplication by constants and differentiation. If variables enter, the ordinary laws of differentiation have to be used, for example

$$D(xy) = xDy+y,$$

$$D^2(xy) = xD^2y+2Dy,$$

etc., and the simplicity is lost. General results are obtained by using Leibnitz's theorem, viz.

$$D^n(yz) = yD^nz+{}^nC_1\,DyD^{n-1}z+...+zD^ny. \tag{15}$$

We now prove three simple theorems which are of great importance in the sequel. These are all proved only for the case in which the function $f(D)$ is a polynomial in D.

THEOREM 1. *If $f(D)$ is a polynomial in D and k is a constant, then*

$$f(D)e^{kx} = f(k)e^{kx} \tag{16}$$

and

$$f(D)\left\{\frac{1}{f(k)}e^{kx}\right\} = e^{kx}. \tag{17}$$

For

$$De^{kx} = ke^{kx},$$
$$D^2e^{kx} = k^2e^{kx},$$

etc. Therefore

$$\begin{aligned} f(D)e^{kx} &= (D^n+a_1D^{n-1}+...+a_n)e^{kx} \\ &= (k^n+a_1k^{n-1}+...+a_n)e^{kx} \\ &= f(k)e^{kx}. \end{aligned}$$

(17) is, of course, equivalent to (16) and is stated here for reference.

THEOREM 2. *If $f(D^2)$ is a polynomial in D^2, then*

$$f(D^2)\frac{\sin}{\cos}\omega x = f(-\omega^2)\frac{\sin}{\cos}\omega x. \tag{18}$$

For

$$D^2\sin\omega x = -\omega^2\sin\omega x,$$
$$D^2\cos\omega x = -\omega^2\cos\omega x,$$

etc., and adding results of this type as in the proof of Theorem 1 gives (18).

THEOREM 3. *If $f(D)$ is a polynominal in D, and k is a constant, then*

$$f(D)\{e^{kx}y\} = e^{kx}f(D+k)y \tag{19}$$

and

$$f(D+k)y = e^{-kx}f(D)\{e^{kx}y\}. \tag{20}$$

It follows from Leibnitz's theorem (15) that

$$\begin{aligned} D^n(e^{kx}y) &= e^{kx}D^ny+{}^nC_1D(e^{kx})D^{n-1}y+...+yD^n(e^{kx}) \\ &= e^{kx}\{D^ny+{}^nC_1kD^{n-1}y+...+k^ny\} \\ &= e^{kx}\{(D^n+{}^nC_1kD^{n-1}+{}^nC_2k^2D^{n-2}+...+k^n)y\} \\ &= e^{kx}\{(D+k)^ny\}. \end{aligned}$$

Adding results of this type as in the proof of Theorem 1 we get

$$f(D)\{e^{kx}y\} = e^{kx}\{f(D+k)y\},$$

as required. The alternative form (20) which allows $f(D+k)y$ to be expressed in terms of $f(D)$ operating on $e^{kx}y$ is also often useful.

13. The homogeneous ordinary linear differential equation of order n

As in § 12 this may be written

$$f(D)y \equiv (D^n+a_1 D^{n-1}+...+a_n)y = 0. \qquad (1)$$

Also as in § 12, we call the equation

$$f(\alpha) \equiv \alpha^n+a_1 \alpha^{n-1}+...+a_n = 0 \qquad (2)$$

the auxiliary equation and find its roots. If it has an r-ple root α_1, an s-ple root α_2, etc., (1) may be written

$$(D-\alpha_1)^r(D-\alpha_2)^s...(D-\alpha_p)^m y = 0. \qquad (3)$$

We consider first the simple case of the first-order equation

$$\frac{dy}{dx}-\alpha_1 y = 0,$$

or

$$(D-\alpha_1)y = 0. \qquad (4)$$

Obviously†

$$y = c_1 e^{\alpha_1 x}, \qquad (5)$$

where c_1 is any constant, satisfies this, and since it contains one arbitrary constant is the general solution.

Next suppose that the roots $\alpha_1,..., \alpha_n$ of the auxiliary equation $f(\alpha) = 0$ are all different, so that (3) takes the form

$$(D-\alpha_1)(D-\alpha_2)...(D-\alpha_n)y = 0. \qquad (6)$$

Clearly if y satisfies

$$(D-\alpha_n)y = 0 \qquad (7)$$

it also satisfies (6). By (5) the solution of (7) is

$$c_n e^{\alpha_n x}. \qquad (8)$$

In the same way, since the order of the operations $(D-\alpha_1),..., (D-\alpha_n)$ in (6) may be interchanged in any way,

$$c_1 e^{\alpha_1 x},\ c_2 e^{\alpha_2 x},...,\ c_n e^{\alpha_n x}, \qquad (9)$$

where $c_1, c_2,..., c_n$ are any constants, are all independent solutions

† Alternatively, integrating with respect to x gives

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \alpha_1\,dx,$$

or

$$\ln y = \alpha_1 x+\text{constant}.$$

of (6). And, by the fundamental property of a linear differential equation derived in § 3, the sum of these solutions, namely

$$c_1 e^{\alpha_1 x}+c_2 e^{\alpha_2 x}+...+c_n e^{\alpha_n x}, \tag{10}$$

is also a solution of (6); since it contains n arbitrary constants it is the general solution.

Ex. 1. $$(D^2+3D+2)y = 0.$$

The auxiliary equation is

$$\alpha^2+3\alpha+2 = 0,$$

and its roots are -1 and -2. Thus the solution is

$$y = Ae^{-x}+Be^{-2x},$$

where A and B are arbitrary constants.

Ex. 2. $$(D^2+2D+5)y = 0.$$

The auxiliary equation is

$$\alpha^2+2\alpha+5 = 0,$$

and its roots are $-1\pm 2i$.

Thus the solution is

$$y = Ae^{-x+2ix}+Be^{-x-2ix}, \tag{11}$$

where A and B are arbitrary constants. This form of solution is not the most useful one and it is usually better to express it in real form. Using the result

$$e^{\pm ix} = \cos x \pm i \sin x,$$

(11) becomes

$$(A+B)e^{-x}\cos 2x+i(A-B)e^{-x}\sin 2x$$

and, since A and B are arbitrary, $A+B$ and $i(A-B)$ are also two arbitrary constants which may be taken as the fundamental constants in the general solution. Thus the solution may be written

$$y = Ce^{-x}\cos 2x+Ge^{-x}\sin 2x. \tag{12}$$

Alternatively, it may be put in the form

$$y = Ee^{-x}\cos(2x+F), \tag{13}$$

where E and F are arbitrary constants.

From the point of view of pure mathematics, that is, of finding a solution of the equation with two arbitrary constants, (11), (12), and (13) are all equally good answers. In the application to practical problems the subsequent work may be simplified by choosing the most suitable form to begin with, so it is desirable to bear all three types in mind and not to write down solutions always in one standard form. For examples of this see § 16 (11) and Exs. 13 and 14 on Chapter II.

Ex. 3.
$$(D^2-1)y = 0.$$
The roots of the auxiliary equation are ± 1, so the solution is
$$y = Ae^x+Be^{-x},$$
or, alternatively,
$$y = C\cosh x+E\sinh x.$$

Ex. 4.
$$(D^4-1)y = 0.$$
The auxiliary equation is $\alpha^4-1 = 0$, and its roots are ± 1, $\pm i$. The solution is
$$y = A\cosh x+B\sinh x+C\cos x+E\sin x,$$
or one of the other forms discussed above.

Ex. 5.
$$(D^4+4\omega^4)y = 0.$$
The auxiliary equation has roots $\omega(\pm 1\pm i)$, and the solution is
$$y = e^{\omega x}(A\cos\omega x+B\sin\omega x)+e^{-\omega x}(C\cos\omega x+E\sin\omega x),$$
or, alternatively,
$$y = A_1 e^{\omega x}\cos(\omega x+\theta_1)+B_1 e^{-\omega x}\cos(\omega x+\theta_2).$$

Ex. 6.
$$(D^3+3D^2+6D+6)y = 0.$$
The auxiliary equation is
$$\alpha^3+3\alpha^2+6\alpha+6 = 0,$$
and, like many equations that arise in practice, does not have simple roots and must be solved numerically. To two places of decimals the roots are
$$-1{\cdot}60 \quad\text{and}\quad -0{\cdot}70\pm 1{\cdot}81i.$$
Thus the solution is
$$y = Ae^{-1\cdot 60x}+Be^{-0\cdot 70x}\cos(1{\cdot}81x+C).$$

Passing now to the general case (3) in which the auxiliary equation has repeated roots, it appears that any solution of
$$(D-\alpha_1)^r y = 0 \tag{14}$$
will satisfy (3). Now, by § 12 (20), this can be written in the form
$$e^{\alpha_1 x}D^r\{e^{-\alpha_1 x}y\} = 0. \tag{15}$$

Thus $e^{-\alpha_1 x}y$ must be a function whose rth differential coefficient is zero, that is, a polynomial of degree $r-1$, so that

$$y = e^{\alpha_1 x}(A_0+A_1 x+...+A_{r-1}x^{r-1}), \tag{16}$$

and since this contains r arbitrary constants it is the general solution of (14).

Adding results of this type, it follows that the general solution of (3), containing $r+s+...+m = n$ arbitrary constants, is

$$y = (A_0+A_1 x+...+A_{r-1}x^{r-1})e^{\alpha_1 x}+...+ \\ +(K_0+K_1 x+...+K_{m-1}x^{m-1})e^{\alpha_p x}. \tag{17}$$

Ex. 7. $$(D-1)^3(D-2)y = 0.$$

The solution is $\quad y = (A+Bx+Cx^2)e^x+Ee^{2x}.$

Ex. 8. The general second-order equation

$$(D^2+2bD+c)y = 0.$$

The roots of the auxiliary equation are $-b\pm\sqrt{(b^2-c)}$. The solution is

$$\begin{aligned} e^{-bx}\{Ae^{x\sqrt{(b^2-c)}}+Be^{-x\sqrt{(b^2-c)}}\} &\quad \text{if } b^2 > c, \\ (A+Bx)e^{-bx} &\quad \text{if } b^2 = c, \\ Ae^{-bx}\cos\{x\sqrt{(c-b^2)}+B\} &\quad \text{if } b^2 < c. \end{aligned}$$

Finally, we note an alternative approach to the theory which will often be useful. Suppose we have to solve the equation

$$f(D)y = 0 \tag{18}$$

and we seek a solution in which y is proportional to $e^{\alpha x}$, say $y = Ae^{\alpha x}$. Substituting in (18) gives, using § 12, Theorem 1,

$$Ae^{\alpha x}f(\alpha) = 0.$$

Thus α must be a root of $f(\alpha) = 0$, which is just the auxiliary equation (2). If the roots of this equation are all different, combining all solutions of this type gives the general solution (10).

If the roots are not all different, suppose that α is an r-ple root, so that the equation is of the form

$$(D-\alpha)^r F(D)y = 0. \tag{19}$$

We now seek a solution of the form $y = Y(x)e^{\alpha x}$, where $Y(x)$ is a function of x instead of a constant as before. If this is to satisfy (19) we must have

$$(D-\alpha)^r F(D)\{Y(x)e^{\alpha x}\} = 0,$$

or, by § 12, Theorem 3,

$$e^{\alpha x}F(D+\alpha)D^rY(x) = 0. \tag{20}$$

(20) will be satisfied if we choose $Y(x)$ so that $D^rY(x) = 0$, that is

$$Y(x) = A_0+A_1x+...+A_{r-1}x^{r-1},$$

where $A_0,..., A_{r-1}$ are constants.

Thus we have the solution

$$(A_0+A_1x+...+A_{r-1}x^{r-1})e^{\alpha x}$$

as in (16).

14. The inhomogeneous ordinary linear differential equation of order n

The general form of this is

$$f(D)y = \phi(x), \tag{1}$$

where $f(D)$ is a polynomial in D as in §§ 12, 13, and $\phi(x)$ is any function of x.

The first step in the solution is to find, by the methods of § 13, the general solution $y_1(x)$ containing n arbitrary constants of the homogeneous equation corresponding to (1), namely,

$$f(D)y = 0. \tag{2}$$

$y_1(x)$ is called the *Complementary Function* of (1).

Next we find a function of x, say $y_2(x)$, which satisfies (1); methods for doing this will be discussed below. This function $y_2(x)$ is called a *Particular Integral* of (1).

The sum of the complementary function and a particular integral, viz.

$$y_1(x)+y_2(x) \tag{3}$$

is the general solution of (1) containing n arbitrary constants. To prove this we notice that it contains n arbitrary constants since $y_1(x)$ does, also it satisfies (1) since, substituting,

$$f(D)\{y_1(x)+y_2(x)\} = f(D)y_1(x)+f(D)y_2(x) = \phi(x),$$

since $y_1(x)$ satisfies (2) and $y_2(x)$ satisfies (1).

It must be emphasized that any function whatever which satisfies (1) can be taken as a particular integral; many different particular integrals could be written down, but these differ by terms already included in the complementary function. Also if $\phi(x)$ is the sum of several functions, say,

$$\phi(x) = \phi_1(x)+...+\phi_r(x),$$

a particular integral of (1) is the sum of particular integrals of

$$f(D)y = \phi_r(x),$$

for $r = 1,..., n$.

We now have to study in detail methods for finding a particular integral of (1) when $\phi(x)$ has one of the common forms: a constant, a polynomial, a trigonometrical or exponential function, etc. There are many methods of doing this, but here we shall adopt the very simple one of seeking a particular integral similar in form to $\phi(x)$. Thus if $\phi(x)$ is a polynomial in x, we shall seek a polynomial for the particular integral; if $\phi(x)$ is e^{ax}, we shall seek a particular integral with e^{ax} as a factor, and so on.

(i) *$\phi(x)$ a constant c*

If the differential equation is

$$(D^n+a_1 D^{n-1}+...+a_n)y = c, \tag{4}$$

where $a_n \neq 0$, the constant

$$c/a_n \tag{5}$$

satisfies it and so is a particular integral.

If $f(D)$ has a factor D^r so that the differential equation is

$$(D^n+a_1 D^{n-1}+...+a_{n-r} D^r)y = c, \tag{6}$$

and we choose y so that

$$a_{n-r} D^r y = c \tag{7}$$

it will also satisfy (6). To satisfy (7), y has to be a function which, when differentiated r times, gives the constant c/a_{n-r}; the simplest such function is

$$y = \frac{cx^r}{a_{n-r}\, r!}. \tag{8}$$

Ex. 1. $$(D^2+3D+2)y = 1.$$

By (5), $\frac{1}{2}$ is a particular integral.

The complementary function is

$$Ae^{-2x}+Be^{-x}.$$

Thus the general solution is

$$\tfrac{1}{2}+Ae^{-2x}+Be^{-x}.$$

Ex. 2. $$D^2(D+1)y = 1.$$

The particular integral is to be a function which when differentiated twice gives 1, that is, $\frac{1}{2}x^2$.

The complementary function is

$$A+Bx+Ce^{-x}.$$

Thus the general solution is

$$A+Bx+\tfrac{1}{2}x^2+Ce^{-x}.$$

(ii) *$\phi(x)$ a polynomial $P_k(x)$ of degree k in x*

We seek a particular integral in the form of a polynomial. If the differential equation is

$$(D^n+a_1 D^{n-1}+...+a_n)y = P_k(x), \tag{9}$$

with $a_n \neq 0$, we assume for the particular integral a polynomial of degree k with unknown coefficients. The coefficients are found by substituting this in the left-hand side of (9) and comparing coefficients with $P_k(x)$.

If $a_n = ... = a_{n-r+1} = 0$, the differential equation has the form

$$(D^n+a_1 D^{n-1}+...+a_{n-r} D^r)y = P_k(x), \tag{10}$$

and we must assume a polynomial of degree $(k+r)$ for the particular integral, so that, when this is substituted in (10), there will be powers of x up to the kth on the left-hand side.

Ex. 3. $$(D^2+D+1)y = x+x^2+x^3. \tag{11}$$

We seek a polynomial particular integral

$$y = Ax^3+Bx^2+Cx+E.$$

If this is to satisfy (11) we must have

$$(6Ax+2B)+(3Ax^2+2Bx+C)+(Ax^3+Bx^2+Cx+E) \equiv x+x^2+x^3.$$

Equating coefficients of x^3, x^2, x, and the constant terms on the two sides gives

$$A = 1,$$
$$3A+B = 1,$$
$$6A+2B+C = 1,$$
$$2B+C+E = 0.$$

Solving, successively, gives $A = 1$, $B = -2$, $C = -1$, $E = 5$, and the required particular integral is

$$x^3-2x^2-x+5.$$

Adding the complementary function gives the general solution

$$x^3-2x^2-x+5+Fe^{-\frac{1}{2}x}\cos(\tfrac{1}{2}x\sqrt{3}+G).$$

Ex. 4. $$D^2(D+1)y = 1+x^3. \tag{12}$$

We must assume a polynomial of the fifth degree for y so that terms in x^3 will occur after it has been differentiated twice. Assume

$$y = Ax^5+Bx^4+Cx^3+Ex^2+Fx+G.$$

If this is to satisfy (12) we must have

$$(60Ax^2+24Bx+6C)+(20Ax^3+12Bx^2+6Cx+2E) \equiv 1+x^3.$$

Equating coefficients we find

$$A = \tfrac{1}{20}, \qquad B = -\tfrac{1}{4}, \qquad C = 1, \qquad E = -\tfrac{5}{2}.$$

There is no restriction on F and G, so we may take them to be zero. Thus a particular integral is

$$\tfrac{1}{20}x^5-\tfrac{1}{4}x^4+x^3-\tfrac{5}{2}x^2.$$

The complementary function is

$$Hx+J+Ke^{-x},$$

where H, J, K are arbitrary constants. If we had not taken $F = G = 0$ above, the terms so obtained would have been of the same type as $Hx+J$ in the complementary function.

The general solution is

$$\tfrac{1}{20}x^5-\tfrac{1}{4}x^4+x^3-\tfrac{5}{2}x^2+Hx+J+Ke^{-x}.$$

(iii) *$\phi(x)$ the exponential e^{ax} where $f(a) \neq 0$*

We require a particular integral of

$$f(D)y = e^{ax}. \tag{13}$$

As remarked earlier, we seek for a particular integral a solution of (13) of the same type as the right-hand side of (13), i.e. we seek a solution

$$y = Ye^{ax}, \tag{14}$$

where Y is a constant. Substituting in (13) we require

$$Yf(D)e^{ax} = e^{ax},$$

or, using § 12, Theorem 1,

$$Yf(a)e^{ax} = e^{ax}.$$

Thus $$Y = \frac{1}{f(a)},$$

and the required particular integral is

$$\frac{1}{f(a)}e^{ax}. \tag{15}$$

This argument breaks down if $f(a) = 0$: this case is discussed in (vi) below.

Ex. 5. $$(D^2+3D+2)y = \cosh 3x = \tfrac{1}{2}e^{3x}+\tfrac{1}{2}e^{-3x}.$$

Using (15), a particular integral is

$$\tfrac{1}{40}e^{3x}+\tfrac{1}{4}e^{-3x}.$$

The general solution is

$$\tfrac{1}{40}e^{3x}+\tfrac{1}{4}e^{-3x}+Ae^{-x}+Be^{-2x}.$$

(iv) $\phi(x)$ *the sine or cosine of* ωx, *provided* $f(i\omega) \neq 0$

Suppose that Y is a particular integral of

$$f(D)y = e^{i\omega x}, \tag{16}$$

found as in (iii): this will be a complex function of x, say $Y_1(x)+iY_2(x)$. Substituting this in the left-hand side of (16), and using the result

$$e^{i\omega x} = \cos\omega x+i\sin\omega x, \tag{17}$$

gives $$f(D)Y_1(x)+if(D)Y_2(x) = \cos\omega x+i\sin\omega x.$$

Equating real and imaginary parts on the two sides we see that the real and imaginary parts of Y, respectively, are particular integrals of

$$f(D)y = \cos\omega x, \tag{18}$$

and $$f(D)y = \sin\omega x. \tag{19}$$

Thus to find a particular integral either of (18) or (19) we find the particular integral

$$\frac{1}{f(i\omega)}\,e^{i\omega x} \tag{20}$$

of (16) as in (15), and pick out its real part if a particular integral of (18) is required, or its imaginary part for (19).

In the reduction it is usually best to use the modulus–argument notation for a complex number. If, as in Fig. 1,

$$z = x+iy \tag{21}$$

is a complex number, x is called the real part of z, written $\mathbf{R}(z)$; y is called the imaginary part of z, written $\mathbf{I}(z)$; and $z^* = x-iy$ is called the conjugate of z. The modulus of z, $|z|$, is defined by

$$|z| = \sqrt{(x^2+y^2)}, \tag{22}$$

and the argument of z, $\arg z$, is the angle θ defined by any two of

$$\sin\theta = \frac{y}{|z|}, \quad \cos\theta = \frac{x}{|z|}, \quad \tan\theta = \frac{y}{x}. \tag{23}$$

Since $y = |z|\sin\theta$ and $x = |z|\cos\theta$, the use of this notation enables us to write the complex number z in the form

$$z = x+iy = |z|(\cos\theta+i\sin\theta) = |z|e^{i\theta}. \tag{24}$$

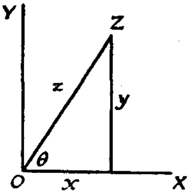


FIG. 1.

The specification of the angle θ in (23) needs some care; it is equivalent to the statement that†

$$\left.\begin{aligned} \arg z &= \tan^{-1}(y/x) && \text{if } x>0, \\ \arg z &= \pi+\tan^{-1}(y/x) && \text{if } x<0 \end{aligned}\right\}, \tag{25}$$

where $\tan^{-1}(y/x)$ is the angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose tangent is y/x.

Ex. 6. $$(D^3+4D^2+5D+2)y = \cos\omega x. \tag{26}$$

We find the particular integral of

$$(D^3+4D^2+5D+2)y = e^{i\omega x}.$$

By (15) this is

$$\frac{1}{(i\omega)^3+4(i\omega)^2+5i\omega+2}e^{i\omega x} = \frac{1}{(2-4\omega^2)+i(5\omega-\omega^3)}e^{i\omega x}. \tag{27}$$

Now let Z and θ be the modulus and argument of the complex number

$$(2-4\omega^2)+i(5\omega-\omega^3), \tag{28}$$

these may be written down by (22) and (25). Then (28) may be written

$$(2-4\omega^2)+i(5\omega-\omega^3) = Ze^{i\theta}.$$

Using this, (27) becomes

$$\frac{1}{Z}e^{i(\omega x-\theta)}.$$

For the particular integral of (26) we need the real part of this, namely

$$\frac{1}{Z}\cos(\omega x-\theta). \tag{29}$$

The form (29) of the particular integral is probably the most useful for applications. A form involving $\cos\omega x$ and $\sin\omega x$ may be obtained by putting in it the values of Z and θ obtained from (28); alternatively this form may be obtained from (27) by multiplying it above and below by the conjugate of the denominator; this gives

$$\frac{[(2-4\omega^2)-i(5\omega-\omega^3)][\cos\omega x+i\sin\omega x]}{(2-4\omega^2)^2+(5\omega-\omega^3)^2}.$$

† The student is warned against the common practice of writing $\tan^{-1}(y/x)$ in place of $\arg z$ in general formulae in which the sign of x is not known—this may lead to errors. The value of θ defined above is ambiguous by a multiple of 2π. The angle θ such that $-\pi < \theta \leqslant \pi$ is usually used, and is called the principal value of the argument.

The real part of this is

$$\frac{(2-4\omega^2)\cos\omega x+(5\omega-\omega^3)\sin\omega x}{(2-4\omega^2)^2+(5\omega-\omega^3)^2}.$$

(v) $\phi(x)$ *of the form* $e^{ax}\cos\omega x$ *or* $e^{ax}\sin\omega x$, *provided* $f(a+i\omega)\neq 0$

In this case we proceed as in (iv), finding a particular integral of

$$f(D)y = e^{(a+i\omega)x} \tag{30}$$

and taking its real or imaginary part as required.

Ex. 7. $$(D^2+D+1)y = e^{ax}\sin\omega x. \tag{31}$$

As in (iii) the particular integral of

$$(D^2+D+1)y = e^{(a+i\omega)x}$$

is

$$\frac{1}{(a+i\omega)^2+(a+i\omega)+1}e^{(a+i\omega)x}$$

$$=\frac{1}{(a^2+a+1-\omega^2)+i(\omega+2a\omega)}e^{(a+i\omega)x}$$

$$=\frac{1}{Z}e^{(a+i\omega)x-i\theta}, \tag{32}$$

where Z and θ are the modulus and argument of

$$(a^2+a+1-\omega^2)+i(\omega+2a\omega).$$

Taking the imaginary part of (32) gives as a particular integral of (31)

$$\frac{1}{Z}e^{ax}\sin(\omega x-\theta).$$

Adding the complementary function, the complete solution is

$$\frac{1}{Z}e^{ax}\sin(\omega x-\theta)+Ae^{-\frac{1}{2}x}\cos(\tfrac{1}{2}x\sqrt{3}+B).$$

(vi) $\phi(x)$ *of the form* e^{ax} *with* $f(a) = 0$. *This includes the case in which* $\phi(x)$ *is* $\sin\omega x$ *or* $\cos\omega x$ *with* $f(i\omega) = 0$

Since $f(a) = 0$, $f(D)$ must have $D-a$ as a factor so that the differential equation will be of the form

$$(D-a)^r F(D)y = e^{ax}, \tag{33}$$

where $F(a)\neq 0$.

For a particular integral we seek a solution of this of the form

$$y = Y(x)e^{ax} \tag{34}$$

in which the quantity $Y(x)$ multiplying e^{ax} is a function of x instead of a constant as in (14). If this is to satisfy (33) we must have

$$(D-a)^r F(D)\{Y(x)e^{ax}\} = e^{ax},$$

or, by § 12, Theorem 3,

$$e^{ax}D^r F(D+a)Y(x) = e^{ax}.$$

That is, $Y(x)$ is to be a function such that

$$D^r F(D+a)Y(x) = 1, \tag{35}$$

and the finding of such a function was discussed in (i).

Ex. 8. $$(D-a)^2(D-b)y = e^{ax}. \tag{36}$$

Assuming a particular integral

$$y = Y(x)e^{ax}$$

and substituting in (36) gives

$$(D-a)^2(D-b)\{Y(x)e^{ax}\} = e^{ax}.$$

Using § 12, Theorem 3, this becomes

$$D^2(D+a-b)Y(x) = 1,$$

and this is satisfied by $$Y(x) = \frac{x^2}{2(a-b)}.$$

Thus the required particular integral is

$$\frac{x^2}{2(a-b)}e^{ax}.$$

Adding the complementary function, the general solution of (36) is found to be

$$\left\{A+Bx+\frac{x^2}{2(a-b)}\right\}e^{ax}+Ce^{bx}.$$

Ex. 9. $$(D^2+n^2)y = \sin(nx+\beta). \tag{37}$$

We seek a particular integral of

$$(D^2+n^2)y = e^{i(nx+\beta)} \tag{38}$$

of the form $$y = Y(x)e^{inx}.$$

Substituting in (38) gives

$$(D+in)(D-in)\{Y(x)e^{inx}\} = e^{i(nx+\beta)},$$

or, using § 12, Theorem 3,

$$D(D+2in)Y(x) = e^{i\beta}.$$

This is satisfied by $$Y(x) = \frac{x}{2in}e^{i\beta},$$

so the required particular integral of (38) is

$$\frac{x}{2in}e^{i(nx+\beta)}.$$

The imaginary part of this, which is a particular integral of (37), is

$$-\frac{x}{2n}\cos(nx+\beta).$$

Adding the complementary function, the general solution of (37) is

$$A\cos nx+B\sin nx-\frac{x}{2n}\cos(nx+\beta).$$

(vii) *$\phi(x)$ of the form $e^{ax}P_k(x)$, where $P_k(x)$ is a polynomial in x*

The differential equation is

$$f(D)y = e^{ax}P_k(x). \tag{39}$$

For a particular integral we seek a solution of (39) of the form

$$y = Y(x)e^{ax}.$$

Substituting in (39) and using § 12, Theorem 3, as before, gives the equation for $Y(x)$

$$f(D+a)Y(x) = P_k(x). \tag{40}$$

The finding of $Y(x)$ from (40) was discussed in (ii).

(viii) *$\phi(x)$ any function of x*

We consider first the first-order equation

$$(D-\alpha_1)y = \phi(x), \tag{41}$$

and seek a particular integral

$$y = Y(x)e^{\alpha_1 x}. \tag{42}$$

Substituting (42) in (41) and using § 12, Theorem 3 gives

$$e^{\alpha_1 x}DY(x) = \phi(x).$$

That is

$$Y(x) = \int^x e^{-\alpha_1\xi}\phi(\xi)\,d\xi, \tag{43}$$

and the required particular integral of (41) is

$$e^{\alpha_1 x}\int^x e^{-\alpha_1\xi}\phi(\xi)\,d\xi. \tag{44}$$

In the same way, a particular integral of

$$(D-\alpha_1)^2 y = \phi(x) \tag{45}$$

is found to be the repeated integral

$$e^{\alpha_1 x}\int^x d\eta\int^\eta e^{-\alpha_1\xi}\phi(\xi)\,d\xi. \tag{46}$$

Next consider the equation

$$(D-\alpha_1)(D-\alpha_2)y = \phi(x). \tag{47}$$

As before, we seek a particular integral of type

$$y = Y_1(x)e^{\alpha_1 x}. \tag{48}$$

Substituting in (47) and using § 12, Theorem 3, as before, gives

$$D(D+\alpha_1-\alpha_2)Y_1(x) = e^{-\alpha_1 x}\phi(x),$$

or
$$(D+\alpha_1-\alpha_2)Y_1(x) = \int^x e^{-\alpha_1\xi}\phi(\xi)\,d\xi. \tag{49}$$

(49) is of the same type as (41), so that, by (44),

$$Y_1(x) = e^{(\alpha_2-\alpha_1)x}\int^x e^{(\alpha_1-\alpha_2)\eta}\,d\eta\int^\eta e^{-\alpha_1\xi}\phi(\xi)\,d\xi. \tag{50}$$

From (48) and (50) the required particular integral of (47) is

$$e^{\alpha_2 x}\int^x e^{(\alpha_1-\alpha_2)\eta}\,d\eta\int^\eta e^{-\alpha_1\xi}\phi(\xi)\,d\xi \tag{51}$$

$$= \frac{e^{\alpha_1 x}}{\alpha_1-\alpha_2}\int^x e^{-\alpha_1\xi}\phi(\xi)\,d\xi-\frac{e^{\alpha_2 x}}{\alpha_1-\alpha_2}\int^x e^{-\alpha_2\xi}\phi(\xi)\,d\xi, \tag{52}$$

on integrating by parts.

For the equation
$$(D^2+n^2)y = \phi(x), \tag{53}$$

the particular integral (52) becomes

$$\frac{1}{n}\int^x \sin n(x-\xi)\phi(\xi)\,d\xi. \tag{54}$$

The method leading to (52) may be extended step by step, and it is found that the particular integral of

$$(D-\alpha_1)(D-\alpha_2)...(D-\alpha_n)y = \phi(x) \tag{55}$$

is
$$\sum_{r=1}^{n}\frac{1}{\beta_r}e^{\alpha_r x}\int^x e^{-\alpha_r\xi}\phi(\xi)\,d\xi, \tag{56}$$

where
$$\beta_r = (\alpha_r-\alpha_1)...(\alpha_r-\alpha_{r-1})(\alpha_r-\alpha_{r+1})...(\alpha_r-\alpha_n), \tag{57}$$

provided, of course, that $\alpha_1,..., \alpha_n$ are all different.

These results allow particular integrals to be written down for any form of $\phi(x)$. For the simple explicit functions discussed earlier this method is slower than those already given, but the general formulae are occasionally useful.

15. Simultaneous ordinary linear differential equations with constant coefficients

The solution of these follows from the theory of §§ 12–14, but with minor complications. To illustrate, we discuss the system of three equations for u, v, w in terms of x

$$F_1(D)u+G_1(D)v+H_1(D)w = \phi_1(x), \tag{1}$$

$$F_2(D)u+G_2(D)v+H_2(D)w = 0, \tag{2}$$

$$F_3(D)u+G_3(D)v+H_3(D)w = 0, \tag{3}$$

where $F_1(D)$, etc., are polynomials in D. We write

$$\Delta(D) = \begin{vmatrix} F_1(D) & G_1(D) & H_1(D) \\ F_2(D) & G_2(D) & H_2(D) \\ F_3(D) & G_3(D) & H_3(D) \end{vmatrix} \tag{4}$$

and use small letters $f_1(D), f_2(D)$, etc., for the cofactors of $F_1(D)$, $F_2(D)$, etc., in this determinant. Because of the properties of D proved in § 12 there is an important (but not a complete) analogy between the solution of the system (1) to (3) and that of the corresponding system of algebraic equations in which D is replaced by a pure number.

In particular, if we assume

$$u = f_1(D)V, \qquad v = g_1(D)V, \qquad w = h_1(D)V, \tag{5}$$

where V is a function of x to be determined, (2) and (3) are satisfied automatically since, from the properties of the determinant $\Delta(D)$,

$$f_1(D)F_2(D)+g_1(D)G_2(D)+h_1(D)H_2(D)$$
$$= f_1(D)F_3(D)+g_1(D)G_3(D)+h_1(D)H_3(D) = 0.$$

Also (1) becomes $$\Delta(D)V = \phi_1(x), \tag{6}$$

which is an ordinary inhomogeneous equation for V and can be solved as in § 14. When V has been found, u, v, and w follow immediately from (5). Before discussing this procedure further we give an example.

Ex. 1.
$$(D+1)u-v = e^{-2x},$$
$$Dv+(D+1)w = 0,$$
$$-2u+(D-3)w = 0.$$

For these, (6) becomes

$$(D^2-1)(D-2)V = e^{-2x},$$

and it follows that

$$V = Ae^{-x}+Be^{x}+Ce^{2x}-\tfrac{1}{12}e^{-2x}.$$

Then from (5)

$$u = (D^2-3D)V = 4Ae^{-x}-2Be^{x}-2Ce^{2x}-\tfrac{5}{6}e^{-2x}, \tag{7}$$
$$v = -(2D+2)V = -4Be^{x}-6Ce^{2x}-\tfrac{1}{6}e^{-2x}, \tag{8}$$
$$w = 2DV = -2Ae^{-x}+2Be^{x}+4Ce^{2x}+\tfrac{1}{3}e^{-2x}. \tag{9}$$

All of u, v, w are expressed in terms of the three arbitrary constants in the solution for V. This raises the question of the number of arbitrary constants to be expected in the solution of a set of simultaneous differential equations, and the result† is that this number is in all cases the degree, N, of the highest power of D in the expansion of $\Delta(D)$, that is, just the number‡ which appears in V. In general, the solution of n first-order equations will contain n arbitrary constants, the solution of n second-order equations $2n$, and so on, but in special cases the number may be less.

Ex. 2.
$$(L_1 D+R_1)u+MDv = 1, \tag{10}$$
$$MDu+(L_2 D+R_2)v = 0, \tag{11}$$
with $L_1 L_2 = M^2$. Here, using this condition,
$$\Delta(D) = (L_1 R_2+R_1 L_2)D+R_1 R_2,$$
and, writing $\alpha = R_1 R_2/(L_1 R_2+R_1 L_2)$,
$$V = \frac{1}{R_1 R_2}+Ae^{-\alpha x}.$$
Then by (5)
$$u = (L_2 D+R_2)V = (1/R_1)+(R_2-L_2\alpha)Ae^{-\alpha x},$$
$$v = -MDV = MA\alpha e^{-\alpha x}.$$
Because of the relation§ $L_1 L_2 = M^2$, $\Delta(D)$ is of the first order in D and the solution contains only the one arbitrary constant A. If $L_1 L_2 \neq M^2$ there will be two arbitrary constants.

The method described above is due to Routh.|| If $\phi_1(x) = 0$ it determines the complementary function of the set of equations, and in addition, as set out above, it determines the particular integral for the important case in which one of the

† The proof is complicated, cf. Ince, loc. cit., § 6.4.

‡ It is essential that this number should appear also in the final solution. If it happens that $f_1(D)$, $g_1(D)$, $h_1(D)$, and $\Delta(D)$ all have a common factor, say $(D-\alpha)$, a term $Ae^{\alpha x}$ will appear in V but not in u, v, or w, and the solution so obtained is not the most general one. The general solution in such cases can usually be found by seeking a solution of the homogeneous equations in terms of the cofactors of another row, e.g. $u = f_2(D)V$, etc., and combining the results. See Ex. 4 at the end of the chapter.

§ This is the case of 'perfect coupling' in a transformer. If L_2 times (10) is subtracted from M times (11), an algebraic equation is obtained. In cases of this type a differential equation of lower order than those given can always be found in this way.

|| *Advanced Rigid Dynamics* (ed. 4, 1884, p. 156). An alternative method of finding particular integrals is given by Goodstein (*Math. Gazette*, **33** (1949), 30) together with some discussion of simultaneous equations in general.

equations is inhomogeneous. If all are inhomogeneous, that is, if (2) and (3) have terms $\phi_2(x)$ and $\phi_3(x)$, respectively, on their right-hand sides, a particular integral of the whole set is the sum of the particular integrals for the three cases

$$\phi_1(x) \neq 0, \quad \phi_2(x) = \phi_3(x) = 0; \qquad \phi_2(x) \neq 0, \quad \phi_1(x) = \phi_3(x) = 0;$$

and

$$\phi_3(x) \neq 0, \quad \phi_1(x) = \phi_2(x) = 0.$$

The first of these is that found above, the others are found in the same way using cofactors of the second and third rows, respectively, in (5).

Ex. 3.

$$(D+1)u-v = e^{-2x}, \tag{12}$$

$$Dv+(D+1)w = 4x, \tag{13}$$

$$-2u+(D-3)w = 0. \tag{14}$$

The complementary function and the particular integral for the system with $4x$ replaced by zero have been found in Ex. 1. For the present example we have to add to this the particular integral of (12) to (14) with e^{-2x} replaced by zero. To find this, we assume

$$u = (D-3)V, \qquad v = (D+1)(D-3)V, \qquad w = 2V, \tag{15}$$

and we require the particular integral of

$$(D^2-1)(D-2)V = 4x,$$

which is $V = (2x+1)$. Then by (15) the required particular integral is $u = -6x-1$, $v = -6x-7$, $w = 4x+2$, and the complete solution of (12) to (14) is obtained by adding these values to (7) to (9), respectively.

The more usual method of solving simultaneous equations is by elimination: it may be used with advantage in simple cases such as the following.

Ex. 4.

$$(D+5)u+2v = x, \tag{16}$$

$$(D-1)u+Dv = 1. \tag{17}$$

Here, if u is known, v can be written down from (16). To find u, we eliminate v by differentiating (16) which gives

$$D(D+5)u+2Dv = Dx = 1,$$

and subtracting twice (17) from it. This gives

$$(D^2+3D+2)u = -1.$$

The solution of this is

$$u = Ae^{-x}+Be^{-2x}-\tfrac{1}{2}.$$

Then from (16)

$$v = \tfrac{1}{2}x-\tfrac{1}{2}(D+5)u$$
$$= \tfrac{1}{4}(2x+5)-2Ae^{-x}-\tfrac{3}{2}Be^{-2x}.$$

In all cases a process of elimination can be performed to give a differential equation for any one of the variables, say u, which will be of the form

$$\Delta(D)u = \psi(x), \tag{18}$$

where $\psi(x)$ is a known function of x and $\Delta(D)$ is the determinant (4) of the system. The solution of (18) will involve N arbitrary constants, and as remarked on p. 30 this is the total number of arbitrary constants in the solution. Now in general the other variables will not be expressible directly in terms of u as in Ex. 4 and have to be solved for separately, each having its own equation of type (18), and in solving these equations additional constants will be introduced. These cannot be independent of the earlier ones, and connexions between them have to be found by substituting the solutions in the original equations.

16. Problems leading to ordinary linear differential equations

Problems on dynamics and electric circuit theory will be given in subsequent chapters; here we give a few in other fields.

Ex. 1. *The problem of* § 6, *Ex. 2.*

Writing D for d/dt, the equations to be solved are

$$(D+k)n = 0, \tag{1}$$

$$(D+k_1)n_1 - kn = 0. \tag{2}$$

The solution of (1) is

$$n = Ae^{-kt}, \tag{3}$$

where A is an arbitrary constant. Putting (3) in (2) gives

$$(D+k_1)n_1 = kAe^{-kt},$$

of which the solution is

$$n_1 = \frac{kA}{k_1-k}e^{-kt} + Be^{-k_1 t}, \tag{4}$$

provided $k_1 \neq k$. The constants A and B can be determined from the initial conditions. Suppose that at $t = 0$ there were N atoms of A and none of B. That is, $n = N$, $n_1 = 0$, when $t = 0$. Substituting these values in (3) and (4) gives $A = N$, $B = -kN/(k_1-k)$.

Ex. 2. *The problem of* § 6, *Ex. 3. The initial temperature of the water* T_1.

We have to solve

$$M\frac{dT}{dt} + HT = HT_0 \tag{5}$$

with $T = T_1$ when $t = 0$.

The general solution of (5) is

$$T = T_0 + Ae^{-Ht/M},$$

and to make $T = T_1$ when $t = 0$ we must have

$$A = T_1 - T_0.$$

Ex. 3. *Steady flow of heat in a uniform rod.*

We take the x-axis in the direction of the rod, and suppose it to be so thin that its temperature T at x is constant across its cross-section. If a and K are the area and thermal conductivity of the rod, it is known from the theory of conduction of heat that the rate of flow of heat along it at the point x is

$$-Ka\frac{dT}{dx} \tag{6}$$

per unit time. Also we assume that the rod loses heat from its surface to its surroundings at the rate HT per unit time per unit surface area at any point.

Now consider the element of the rod between x and $x+\delta x$. Heat flows into this across the face x at the rate (6); it flows out across the face $x+\delta x$ at the rate

$$-Ka\frac{dT}{dx} - Ka\frac{d^2T}{dx^2}\delta x, \tag{7}$$

neglecting terms in δx^2; and if p is the perimeter of the rod, heat is lost from the surface at the rate

$$HpT\,\delta x. \tag{8}$$

Since the temperature in the rod is steady, the amount of heat flowing into the element must be equal to the amount flowing out, that is, by (6), (7), and (8),

$$\frac{d^2T}{dx^2} - \frac{Hp}{Ka}T = 0. \tag{9}$$

The differential equation (9) has to be solved with given boundary conditions. Suppose, for example, that the end $x = 0$ of the rod is at temperature T_1 and the end $x = l$ at T_2. The general solution of (9) may be written

$$T = A\sinh\mu x + B\sinh\mu(l-x),$$

where

$$\mu^2 = Hp/Ka, \tag{10}$$

and A and B are arbitrary constants. The conditions at the ends give

$$T_1 = B\sinh\mu l, \qquad T_2 = A\sinh\mu l,$$

and we get finally

$$T = \frac{T_1\sinh\mu(l-x) + T_2\sinh\mu x}{\sinh\mu l}. \tag{11}$$

Ex. 4. *Heat is generated within a slab $0 < x < l$ of solid at the rate $a+bT$ per unit time per unit volume, where a and b are constants and T is the temperature.*

We consider the case in which there is no flow of heat over the plane

$x = 0$, and flow at a rate H times the temperature over the plane $x = l$. That is, as in (6)

$$\frac{dT}{dx} = 0, \qquad x = 0, \tag{12}$$

$$-K\frac{dT}{dx} = HT, \qquad x = l. \tag{13}$$

We calculate the steady temperature under these conditions. The differential equation for T is

$$\frac{d^2T}{dx^2}+\frac{b}{K}T = -\frac{a}{K}; \tag{14}$$

this may be obtained by an argument similar to that of Ex. 3 or by putting $\partial T/\partial t = 0$ in § 106 (21).

The general solution of (14) is

$$T = A\sin\omega x+B\cos\omega x-\frac{a}{b}, \tag{15}$$

where $$\omega^2 = b/K.$$

The boundary conditions (12) and (13) give

$$A = 0,$$

$$\omega KB\sin\omega l = HB\cos\omega l-Ha/b.$$

Thus, finally, $$T = \frac{Ha\cos\omega x}{b(H\cos\omega l-\omega K\sin\omega l)}-\frac{a}{b}. \tag{16}$$

If b is small, ω is small and the denominator in (16) is positive. But the denominator decreases as b increases, and is zero when ω is the smallest root of

$$K\omega\tan\omega l = H. \tag{17}$$

Thus if b, H, l, etc., are connected by this critical relation the steady temperature becomes infinite. Physically this means that heat is being produced more rapidly than it can be conducted away through the solid. An effect of this sort always appears in questions involving explosions or spontaneous combustion, but in fact heat is usually generated at rates such as ae^{kT} or $ae^{-k/T}$ which lead to more difficult non-linear equations.

Ex. 5. *Parallel flow in a heat interchanger.*

Suppose that two fluids are flowing steadily in the direction of the x-axis on either side of a thin partition. The fluids are supposed to be so well stirred that their temperatures are constant in any plane $x =$ constant. Let M_1 be the mass of the first fluid in contact with unit area of the partition, c_1 its specific heat, u_1 its velocity, and T_1 its temperature at x; let M_2, c_2, u_2, T_2 be the corresponding quantities for the second fluid.

The partition is supposed to be such that the rate of flow of heat across it at any point, in the direction from the first fluid to the second, is

$$b(T_1-T_2) \tag{18}$$

per unit time per unit area, where b is a constant.

To find the differential equations satisfied by T_1 and T_2 consider the first fluid in the region between x and $x+\delta x$. Heat is carried into this region at the rate

$$M_1 u_1 c_1 T_1, \tag{19}$$

per unit time, per unit width normal to the direction of flow, and is carried out of it over the plane $x+\delta x$ at the rate

$$M_1 u_1 c_1 T_1 + M_1 u_1 c_1 \frac{dT_1}{dx}\,\delta x. \tag{20}$$

Also heat flows through the partition at the rate

$$b(T_1 - T_2)\,\delta x. \tag{21}$$

Since the flow of heat is assumed to be steady, the rate of flow of heat into the region must be equal to the rate of flow out, that is, by (19), (20), (21)

$$M_1 u_1 c_1 \frac{dT_1}{dx}\,\delta x + b(T_1 - T_2)\,\delta x = 0.$$

Therefore
$$k_1 \frac{dT_1}{dx} + T_1 - T_2 = 0, \tag{22}$$

where
$$k_1 = M_1 u_1 c_1 / b.$$

In the same way, considering the second fluid we should have

$$k_2 \frac{dT_2}{dx} - (T_1 - T_2) = 0, \tag{23}$$

where
$$k_2 = M_2 u_2 c_2 / b.$$

Adding (22) and (23) gives

$$k_1 \frac{dT_1}{dx} + k_2 \frac{dT_2}{dx} = 0. \tag{24}$$

Any two of (22)–(24) may be taken as the differential equations of the problem. Suppose that the 'hot fluid' enters at temperature T and the 'cold fluid' at zero. Then we have to solve (22) and (23) with

$$T_1 = T, \qquad T_2 = 0 \qquad \text{when } x = 0. \tag{25}$$

(24) and (25) give immediately

$$k_1 T_1 + k_2 T_2 = k_1 T, \tag{26}$$

and substituting (26) in (22) gives

$$\frac{dT_1}{dx} + \frac{k_1 + k_2}{k_1 k_2} T_1 = \frac{T}{k_2}. \tag{27}$$

The solution of this with $T_1 = T$ when $x = 0$ is

$$T_1 = \frac{T}{k_1 + k_2}\{k_1 + k_2 e^{-(k_1+k_2)x/k_1 k_2}\}, \tag{28}$$

and T_2 follows from (26). The temperatures of both fluids tend to $k_1 T/(k_1 + k_2)$ for large values of x.

17. Heaviside's unit function and Dirac's delta function

Heaviside's unit function $H(x)$ is defined by

$$\left.\begin{aligned} H(x) &= 0, \quad x \leqslant 0 \\ H(x) &= 1, \quad x > 0 \end{aligned}\right\}. \tag{1}$$

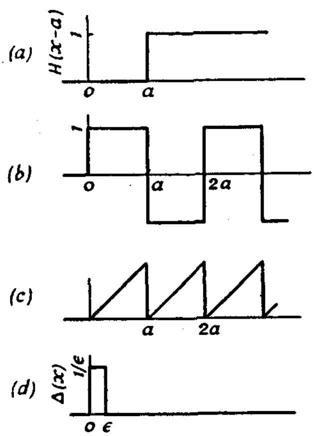


FIG. 2

It has an ordinary discontinuity at the point $x = 0$ and was defined to facilitate the representation of functions which have such discontinuities. The graph of $H(x-a)$ is shown in Fig. 2 (*a*).

For example, the function

$$H(x)-H(x-a)$$

is unity in $0 < x \leqslant a$, and zero for $x \leqslant 0$ and $x > a$.

The function

$$H(x)\sin x+H(x-\pi)\sin(x-\pi)$$

has the value $\sin x$ for $0 \leqslant x \leqslant \pi$ and is zero for $x < 0$ and $x > \pi$.

The 'square wave' of Fig. 2 (*b*) is represented by

$$H(x)+2\sum_{r=1}^{\infty}(-1)^r H(x-ra), \tag{2}$$

and the saw-tooth wave of Fig. 2 (*c*) by

$$x-\sum_{r=1}^{\infty} H(x-ra). \tag{3}$$

Periodic functions of forms such as these are of considerable importance in modern technical practice.

Since $H(x)$ has only an ordinary discontinuity it is integrable and may be used in formulae involving definite integrals—the results again may often be expressed simply in terms of unit functions. Thus

$$\left.\begin{aligned} \int_0^x H(\xi-a)\,d\xi &= 0, & x < a \\ &= (x-a), & x > a \end{aligned}\right\},$$

which may be written

$$\int_0^x H(\xi-a)\,d\xi = (x-a)H(x-a). \tag{4}$$

Similarly

$$\int_0^x (\xi-a)^n H(\xi-a)\,d\xi = \frac{1}{n+1}(x-a)^{n+1}H(x-a). \tag{5}$$

Particular integrals of differential equations of type

$$f(D)y = \phi(x),$$

where $\phi(x)$ contains Heaviside functions, may be written down from the integral formulae of § 14 (viii). For example, a particular integral of

$$(D^2+n^2)y = H(x-a)$$

is, by § 14 (54),

$$\frac{1}{n}\int_0^x \sin n(x-\xi)H(\xi-a)\,d\xi = \frac{1}{n^2}\{1-\cos n(x-a)\}H(x-a). \tag{6}$$

The particular integral (6) is available for all values of x. Using the ordinary method we would have to treat the regions $x < a$ and $x > a$ separately.

In all the above formulae, only integration of $H(x)$ has been in question and offers no difficulty. But care is needed in formulae involving differentiation: the differential coefficient of $H(x)$ is zero except at $x = 0$, where it is not defined. It may be regarded as being the function $\delta(x)$ defined below.

The Dirac delta function. This is not a mathematical function at all in the usual sense of the term. Its use is to represent symbolically an ideally concentrated quantity (such as a concentrated load on a beam, or an impulsive force in dynamics) in the same way that Heaviside's unit function was used to represent a discontinuous quantity.

We define the delta function $\delta(x)$ as the limit as $\epsilon \to 0$ of the function† $\Delta(x)$ shown in Fig. 2 (*d*) defined by

$$\left.\begin{array}{ll}\Delta(x) = 0, & x \leqslant 0 \\ \Delta(x) = 1/\epsilon, & 0 < x \leqslant \epsilon \\ \Delta(x) = 0, & x > \epsilon\end{array}\right\}, \tag{7}$$

or, in terms of the unit function,

$$\delta(x) = \lim_{\epsilon \to 0} \frac{H(x)-H(x-\epsilon)}{\epsilon}. \tag{8}$$

Thus $\delta(x)$ is very large in a vanishingly small region to the right of $x = 0$ and is zero elsewhere. Also from (7)

$$\int\limits_{-\infty}^{\infty} \Delta(x)\, dx = \frac{1}{\epsilon}\int\limits_{0}^{\epsilon} dx = 1,$$

for all ϵ however small. As $\epsilon \to 0$ this becomes

$$\int\limits_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{9}$$

In the same way

$$\left.\begin{array}{ll}\int\limits_{-\infty}^{x} \Delta(\xi-a)\, d\xi = 0, & x \leqslant a \\ \qquad\qquad = 1, & x > a+\epsilon\end{array}\right\},$$

so that, in the limit as $\epsilon \to 0$,

$$\int\limits_{-\infty}^{x} \delta(\xi-a)\, d\xi = H(x-a). \tag{10}$$

Also, if $f(x)$ is continuous in the range $a \leqslant x \leqslant a+\epsilon$,

$$\int\limits_{-\infty}^{\infty} \Delta(x-a)f(x)\, dx = \frac{1}{\epsilon}\int\limits_{a}^{a+\epsilon} f(x)\, dx$$

$$= f(a+\theta\epsilon), \quad 0 < \theta < 1, \tag{11}$$

by the first mean value theorem for integrals. In the limit as $\epsilon \to 0$, (11) becomes

$$\int\limits_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a). \tag{12}$$

† There are many other functions which possess the properties of $\delta(x)$ in the limit as a parameter tends to zero. For example, the continuous function $(1/\epsilon\pi^{\frac{1}{2}})\exp[-x^2/\epsilon^2]$ has often been used.

Since the δ function is defined by a limiting process, all operations on it except the very simplest will involve the interchange of order of limiting processes and be difficult to justify rigorously. This justification can be supplied by comparatively advanced mathematics, but here we shall simply use the function as a convenient short notation and regard results obtained by its use as suggested rather than proved. Many interesting results may be obtained by treating $\delta(x)$ as an ordinary function with the properties (9), (10), and (12), and these can usually be verified by more conventional analysis. For example, a particular integral of

$$(D^2+n^2)y = \delta(x-a) \tag{13}$$

is, by § 14 (54)

$$\left.\begin{aligned} \frac{1}{n}\int\limits_0^x \sin n(x-\xi)\delta(\xi-a)\,d\xi &= 0, & x<a \\ &= \frac{1}{n}\sin n(x-a), & x>a \end{aligned}\right\}.$$

This may be written

$$\frac{1}{n}\sin n(x-a)H(x-a). \tag{14}$$

An important use of the δ function is in the passage from a continuously varying quantity to a discrete or concentrated one. Suppose, for example, that formulae have been derived for the behaviour of a beam with a continuously varying load $w(x)$ per unit length. A concentrated load W at the point $x = a$ may be treated by putting

$$w(x) = W\,\delta(x-a)$$

in the formulae.

18. The Laplace transformation method

This is a method for the solution of an ordinary linear differential equation with constant coefficients (or of systems of such equations) with given values of the function and its derivatives at $x = 0$.

We define the *Laplace transform* $\bar{y}$ of a function y of x as

$$\bar{y} = \int\limits_0^\infty e^{-px}y\,dx, \tag{1}$$

where p is supposed to be a real positive number sufficiently large to make the integral (1) convergent. $\bar{y}$ is a function of p, and to emphasize this we shall sometimes write it as $\bar{y}(p)$.

We begin by calculating the Laplace transforms of the common functions that will be needed.

If

$$y = 1, \qquad \bar{y} = \int_0^\infty e^{-px}\,dx = \frac{1}{p}; \tag{2}$$

$$y = H(x-a), \quad \bar{y} = \int_a^\infty e^{-px}\,dx = \frac{1}{p}\,e^{-ap}; \tag{3}$$

$$y = \delta(x-a), \quad \bar{y} = \int_0^\infty e^{-px}\,\delta(x-a)\,dx = e^{-ap}; \tag{4}$$

$$y = e^{\alpha x}, \qquad \bar{y} = \int_0^\infty e^{-(p-\alpha)x}\,dx = \frac{1}{p-\alpha}. \tag{5}$$

(5), in which α may be real or complex, may be used to give the transforms of $\cos\omega x$, etc. Thus if

$$y = \cos\omega x = \tfrac{1}{2}e^{i\omega x}+\tfrac{1}{2}e^{-i\omega x},$$

$$\bar{y} = \frac{1}{2(p-i\omega)}+\frac{1}{2(p+i\omega)} = \frac{p}{p^2+\omega^2}. \tag{6}$$

Proceeding in this way, or quoting the results as known definite integrals, we can construct a table of Laplace transforms containing all those needed for the solution of differential equations of the types discussed in this chapter.

This table can be extended by a simple theorem.

THEOREM 1. *If $\bar{y}(p)$ is the Laplace transform of y, then $\bar{y}(p+a)$ is the Laplace transform of $e^{-ax}y$.*

This follows immediately since

$$\int_0^\infty e^{-px}e^{-ax}y\,dx = \int_0^\infty e^{-(p+a)x}y\,dx = \bar{y}(p+a).$$

y	$\bar{y}$	
1	$1/p$	(7)
x^n	$\dfrac{n!}{p^{n+1}},\quad n = 0, 1, 2,...$	(8)
$H(x-a)$	$\dfrac{1}{p}e^{-ap}$	(9)
$\delta(x-a)$	e^{-ap}	(10)
$e^{\alpha x}$	$\dfrac{1}{p-\alpha}$	(11)
$x^n e^{\alpha x}$	$\dfrac{n!}{(p-\alpha)^{n+1}},\quad n = 0, 1, 2,...$	(12)
$\sin\omega x$	$\dfrac{\omega}{p^2+\omega^2}$	(13)
$\cos\omega x$	$\dfrac{p}{p^2+\omega^2}$	(14)
$\sinh\alpha x$	$\dfrac{\alpha}{p^2-\alpha^2}$	(15)
$\cosh\alpha x$	$\dfrac{p}{p^2-\alpha^2}$	(16)
$\dfrac{x}{2\omega}\sin\omega x$	$\dfrac{p}{(p^2+\omega^2)^2}$	(17)
$\dfrac{1}{2\omega^3}(\sin\omega x-\omega x\cos\omega x)$	$\dfrac{1}{(p^2+\omega^2)^2}$	(18)

As an example, (12) above follows immediately from (8) by this theorem. Also from (13) and (14), respectively, it follows that

$$\frac{\omega}{(p+\alpha)^2+\omega^2} \text{ is the transform of } e^{-\alpha x}\sin\omega x, \tag{19}$$

and
$$\frac{p+\alpha}{(p+\alpha)^2+\omega^2} \text{ is the transform of } e^{-\alpha x}\cos\omega x. \tag{20}$$

Next we need a theorem on the Laplace transforms of the derivatives of a function.

THEOREM 2. *If $y, Dy,..., D^{n-1}y$ are continuous functions of x, and $y_0, y_1,..., y_{n-1}$ are their values when $x = 0$, then*

$$p\bar{y}-y_0 \quad \textit{is the transform of } Dy, \tag{21}$$
$$p^2\bar{y}-py_0-y_1 \quad ,, \quad ,, \quad D^2y, \tag{22}$$
$$p^3\bar{y}-p^2y_0-py_1-y_2 \quad ,, \quad ,, \quad D^3y, \tag{23}$$
$$p^n\bar{y}-p^{n-1}y_0-p^{n-2}y_1...-y_{n-1} \quad ,, \quad ,, \quad D^ny. \tag{24}$$

These results follow immediately by integration by parts. Thus the Laplace transform of Dy is

$$\int_0^\infty e^{-px}\frac{dy}{dx}\,dx = [ye^{-px}]_0^\infty + p\int_0^\infty e^{-px}y\,dx = p\bar{y}-y_0,$$

since y_0 is the value of y when $x = 0$. Again

$$\int_0^\infty e^{-px}\frac{d^2y}{dx^2}\,dx = \left[e^{-px}\frac{dy}{dx}\right]_0^\infty + p\int_0^\infty e^{-px}\frac{dy}{dx}\,dx = p^2\bar{y}-py_0-y_1.$$

(23) and (24) follow in the same way.

Suppose, now, that we wish to find the solution of

$$(D^n+a_1D^{n-1}+\ldots+a_n)y = \phi(x) \tag{25}$$

which has the values y_0, y_1,..., y_{n-1} of y, Dy,..., $D^{n-1}y$ when $x = 0$. We take the Laplace transform of both sides of (25), using (21) to (24) in the left-hand side, and writing down the Laplace transform of $\phi(x)$ from the table. We thus obtain

$$(p^n+a_1p^{n-1}+\ldots+a_n)\bar{y} = \bar{\phi}+a_{n-1}y_0+a_{n-2}(py_0+y_1)+\ldots+$$
$$+(p^{n-1}y_0+p^{n-2}y_1+\ldots+y_{n-1}). \tag{26}$$

The equation (26) is called the *subsidiary equation* corresponding to the differential equation (25) and its initial conditions. With a little practice, particularly for first- and second-order equations, subsidiary equations can be written down immediately.

(26) gives $\bar{y}$, the Laplace transform of the solution y. If $\phi(x)$ is one of the common functions appearing in the table, $\bar{y}$ is a quotient of polynomials in p, the degree of whose numerator is less than that of its denominator. Thus if the roots of the equation

$$p^n+a_1p^{n-1}+\ldots+a_n = 0 \tag{27}$$

are known, $\bar{y}$ can be expressed in partial fractions. When this has been done, y can be found from $\bar{y}$ by looking up each fraction in the table of transforms (using Theorem 1 if any of the fractions have general quadratic denominators). The result found in this way is certainly the unique solution of the problem since there is a theorem (Lerch's theorem) which states that if two continuous functions have the same Laplace transform they must be identical.

The equation (27) is the auxiliary equation § 12 (10) in the present notation: naturally, whatever method of solving the differential equation is used this equation will appear.

It will be noticed that the whole of the algebra of the solution consists of the expressing of $\bar{y}$ in partial fractions: if the roots of (27) are all different this may be done by the formula†

$$\frac{f(p)}{g(p)} = \sum_{r=1}^{n} \frac{f(\alpha_r)}{g'(\alpha_r)(p-\alpha_r)} \tag{28}$$

$$= \sum_{r=1}^{n} \frac{1}{p-\alpha_r}\left[\frac{(p-\alpha_r)f(p)}{g(p)}\right]_{p=\alpha_r}, \tag{29}$$

where $\alpha_1,\ldots, \alpha_n$ are the roots of

$$g(p) = 0, \tag{30}$$

provided these are all different (but not necessarily real).

Systems of simultaneous linear differential equations with constant coefficients may be treated in the same way.

Ex. 1. $$(D^2+n^2)y = \sin \omega x, \quad \omega \neq n,$$
with $y = y_0$, $Dy = y_1$, when $x = 0$.

The subsidiary equation is, by (13) and (22),

$$(p^2+n^2)\bar{y} = \frac{\omega}{p^2+\omega^2}+py_0+y_1.$$

Thus $$\bar{y} = \frac{\omega}{\omega^2-n^2}\left\{\frac{1}{p^2+n^2}-\frac{1}{p^2+\omega^2}\right\}+\frac{py_0+y_1}{p^2+n^2}.$$

Therefore, by (13) and (14),

$$y = \frac{1}{n(\omega^2-n^2)}\{\omega \sin nx - n \sin \omega x\}+y_0 \cos nx+\frac{y_1}{n}\sin nx.$$

Ex. 2. $$(D^2+2\kappa D+n^2)y = 0$$
with $y = y_0$, $Dy = y_1$, when $x = 0$.

The subsidiary equation is

$$(p^2+2\kappa p+n^2)\bar{y} = py_0+y_1+2\kappa y_0.$$

Thus $$\bar{y} = \frac{(p+\kappa)y_0}{(p+\kappa)^2+(n^2-\kappa^2)}+\frac{y_1+\kappa y_0}{(p+\kappa)^2+(n^2-\kappa^2)}.$$

† Gibson, *Treatise on the Calculus* (1906), § 120.

Therefore, by (13), (14), and Theorem 1,

$$y = e^{-\kappa x}\left\{y_0 \cos x(n^2-\kappa^2)^{\frac{1}{2}}+\frac{(y_1+\kappa y_0)}{(n^2-\kappa^2)^{\frac{1}{2}}}\sin x(n^2-\kappa^2)^{\frac{1}{2}}\right\},$$

provided $n^2 > \kappa^2$.

Ex. 3.
$$\left.\begin{aligned}(D+1)y+Dz &= 0\\ (D-1)y+2Dz &= e^{-x}\end{aligned}\right\},$$

with $y = y_0$, $z = 0$, when $x = 0$.

The subsidiary equations are

$$(p+1)\bar{y}+p\bar{z} = y_0,$$

$$(p-1)\bar{y}+2p\bar{z} = \frac{1}{p+1}+y_0.$$

Solving, for example, for $\bar{y}$, we get

$$\bar{y} = \frac{y_0}{p+3}-\frac{1}{(p+1)(p+3)}$$

$$= \frac{y_0}{p+3}-\frac{1}{2(p+1)}+\frac{1}{2(p+3)}.$$

Therefore
$$y = (y_0+\tfrac{1}{2})e^{-3x}-\tfrac{1}{2}e^{-x}.$$

EXAMPLES ON CHAPTER II

1. Solve the following differential equations

(i) $(D^2+2D+4)y = 8$.
(ii) $D^3(D^2+n^2)y = 1$.
(iii) $(D^3+4D^2+5D+2)y = 4+2x+2x^2$.
(iv) $D^2(D^2-a^2)y = x-x^2$.
(v) $(D^3+3D^2+3D+2)y = 1+e^{-x}$.
(vi) $(D^3+2D^2+D+1)y = e^{2x}$.
(vii) $(D^2-D-2)y = \sin \omega x$.
(viii) $(D^3+D^2-D-1)y = \cos 2x$.
(ix) $(D^2+n^2)y = e^{-\alpha x}\sin \beta x$.
(x) $(D+1)y = e^{-x}\cos x$.
(xi) $(D+1)^2 y = 1-e^{-x}$.
(xii) $(D^2-4)y = \cosh 2x$.
(xiii) $(D+1)(D^2+n^2)y = \sin nx$.
(xiv) $(D^2+2D+2)y = e^{-x}\sin x$.
(xv) $(D+1)(D+2)y = (1+x)e^{-x}$.
(xvi) $(D^2+4)y = x\sin x$.
(xvii) $(D^2+n^2)y = x-2(x-a)H(x-a)$.
(xviii) $(D-1)u-2v = e^{-x}$; $-2u+(D-1)v = 1$.
(xix) $Du-v = 0$; $Dv-w = 0$; $Dw-u = 0$.
(xx) $(D^2-4D)u-(D-1)v = e^{4x}$; $(D+6)u+(D^2-D)v = 0$.

The solutions are as follows:

(i) $Ae^{-x}\cos(x\sqrt{3}+B)+2$.

(ii) $A+Bx+Cx^2+x^3/6n^2+E\cos(nx+F)$.

(iii) $x^2-4x+8+(A+Bx)e^{-x}+Ce^{-2x}$.

(iv) $\dfrac{1}{12a^2}x^4-\dfrac{1}{6a^2}x^3+\dfrac{1}{a^4}x^2+A+Bx+Ce^{ax}+Ee^{-ax}$.

(v) $Ae^{-2x}+Be^{-\frac{1}{2}x}\cos(\frac{1}{2}x\sqrt{3}+C)+\frac{1}{2}+e^{-x}$.

(vi) $Ae^{-1\cdot 75x}+Be^{-0\cdot 12x}\cos(0\cdot 74x+C)+\frac{1}{19}e^{2x}$.

(vii) $Ae^{2x}+Be^{-x}-\dfrac{(\omega^2+2)\sin\omega x-\omega\cos\omega x}{\omega^4+5\omega^2+4}$.

(viii) $Ae^{x}+(B+Cx)e^{-x}-(\cos 2x+2\sin 2x)/25$.

(ix) $A\cos(nx+B)+\dfrac{(\alpha^2+n^2-\beta^2)\sin\beta x+2\alpha\beta\cos\beta x}{(\alpha^2+n^2-\beta^2)^2+4\alpha^2\beta^2}$.

(x) $Ae^{-x}+e^{-x}\sin x$.

(xi) $1+(A+Bx-\frac{1}{2}x^2)e^{-x}$.

(xii) $A\cosh 2x+(B+\frac{1}{4}x)\sinh 2x$.

(xiii) $Ae^{-x}+\left(B-\dfrac{x}{2(1+n^2)}\right)\sin nx+\left(C-\dfrac{x}{2n(1+n^2)}\right)\cos nx$.

(xiv) $(A-\frac{1}{2}x)e^{-x}\cos x+Be^{-x}\sin x$.

(xv) $(A+\frac{1}{2}x^2)e^{-x}+Be^{-2x}$.

(xvi) $A\sin 2x+B\cos 2x+\frac{1}{3}x\sin x-\frac{2}{9}\cos x$.

(xvii) $A\sin nx+B\cos nx+\dfrac{x}{n^2}-\dfrac{2}{n^3}\{n(x-a)-\sin n(x-a)\}H(x-a)$.

(xviii) $u = -\frac{2}{3}-e^{-x}(2A+\frac{1}{4}-\frac{1}{2}x)+2Be^{3x}$,
$v = \frac{1}{3}+2(A-\frac{1}{4}x)e^{-x}+2Be^{3x}$.

(xix) $u = Ae^{x}-\frac{1}{2}(B+C\sqrt{3})e^{-\frac{1}{2}x}\cos\frac{1}{2}x\sqrt{3}-\frac{1}{2}(C-B\sqrt{3})e^{-\frac{1}{2}x}\sin\frac{1}{2}x\sqrt{3}$,
$v = Ae^{x}+(B\cos\frac{1}{2}x\sqrt{3}+C\sin\frac{1}{2}x\sqrt{3})e^{-\frac{1}{2}x}$,
$w = Ae^{x}-\frac{1}{2}(B-C\sqrt{3})e^{-\frac{1}{2}x}\cos\frac{1}{2}x\sqrt{3}-\frac{1}{2}(C+B\sqrt{3})e^{-\frac{1}{2}x}\sin\frac{1}{2}x\sqrt{3}$.

(xx) $u = 2Ae^{-x}+2Ce^{2x}+6Ee^{3x}+\frac{2}{5}e^{4x}$,
$v = -5Ae^{-x}-7Be^{x}-8Ce^{2x}-9Ee^{3x}-\frac{1}{3}e^{4x}$.

2. Solve $x^2D^2y+4xDy+2y = x$. Equations such as this in which all terms are of the type x^rD^ry are known as 'homogeneous' or 'Euler' equations and can be reduced to linear equations, and thus solved, by the substitution $x = e^t$. The solution is $y = Ax^{-1}+Bx^{-2}+x/6$.

3. Solve the following with initial values $y_0, y_1,\ldots$ of $y, Dy,\ldots$, etc.

(i) $(D+1)(D+2)y = e^{-x}$.

(ii) $(D^2+n^2)y = \sin nx$.

(iii) $(D+1)u-(5D+7)v = 1;\; u+(D-1)v = 0$.

The solutions are

(i) $(y_1+2y_0+x-1)e^{-x}-(y_1+y_0-1)e^{-2x}$.

(ii) $\left(\dfrac{1}{2n^2}+\dfrac{y_1}{n}\right)\sin nx+\left(y_0-\dfrac{x}{2n}\right)\cos nx$.

(iii) $u = -\frac{1}{6}+(12v_0-3u_0+\frac{3}{2})e^{-2x}+(4u_0-12v_0-\frac{4}{3})e^{-3x}$,
$v = -\frac{1}{6}+(4v_0-u_0+\frac{1}{2})e^{-2x}+(u_0-3v_0-\frac{1}{3})e^{-3x}$.

4. Solve the equations

$$(D+1)u+2Dv+4D^2w = 0,$$
$$(2D+1)u+Dv+D^2w = 0,$$
$$3Du+v+w = 0,$$

with $u = 1$, $v = w = Dw = 0$, when $x = 0$. If the method of § 15 is used note that this is the exceptional case referred to in a footnote. The solution is $10u = 9e^{-\frac{1}{3}x}-4e^{\frac{1}{2}x}+5e^x$, $10v = 9e^{-\frac{1}{3}x}+26e^{\frac{1}{2}x}-25e^x-10$, $w = 1+e^x-2e^{\frac{1}{2}x}$.

5. (i) Show that if $\bar{y}$ is the Laplace transform of y, $\bar{y}/p$ is the transform of

$$\int_0^x y(\xi)\,d\xi.$$

(ii) Show that if $\bar{y}(p)$ is the Laplace transform of $y(x)$, $\bar{y}(p/\omega)$ is the transform of $\omega\, y(\omega x)$.

(iii) Show that if $\bar{y}$ and $\bar{z}$ are the Laplace transforms of y and z, $\bar{y}\bar{z}$ is the Laplace transform of

$$\int_0^x y(x-\xi)z(\xi)\,d\xi = \int_0^x y(\xi)z(x-\xi)\,d\xi.$$

The proof depends on a change of variable in the double integrals.

(iv) Using (iii) and § 18 (28), deduce § 14 (56).

6. A tank contains volume V of water, initially at zero temperature. Water is run off from it at a constant rate of volume v per second, and replaced at the same rate by water at temperature T_1. Show that the temperature at time t is $$T_1(1-e^{-vt/V}).$$

7. Two tanks A, B, each of volume V, contain water at time $t = 0$. For $t > 0$, volume v of solution containing mass m of solute flows into A per second; mixture flows from A to B at the same rate; and mixture flows away from B at the same rate. Show that the mass of solute in B at any time is $$(mV/v)(1-e^{-vt/V})-mte^{-vt/V}.$$

8. A substance A changes into a substance B at a rate α times the amount of A present; B changes into C at a rate β times the amount of B present. If initially only A is present and its amount is a, show that the amount of C at time t is

$$a+a(\beta e^{-\alpha t}-\alpha e^{-\beta t})/(\alpha-\beta).$$

9. If the probability of an event happening in the interval $(t, t+\delta t)$ is $\lambda\,\delta t$, independent of t, and $P_n(t)$ is written for the probability of n events happening in time t, it is known that $P_0(0) = 1, P_n(0) = 0$, and

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t); \qquad \frac{dP_n(t)}{dt} = \lambda P_{n-1}(t)-\lambda P_n(t), \quad n = 1, 2, 3,\ldots.$$

Show that

$$P_0(t) = e^{-\lambda t}, \qquad P_1(t) = \lambda te^{-\lambda t}, \qquad P_n(t) = (\lambda t)^n e^{-\lambda t}/n!.$$

10. $y_1, y_2, y_3, \ldots$ are the number of atoms present at time t of the elements of a radioactive series in which the rth element decays into the $(r+1)$th at a rate λ_r times the number of atoms y_r present. If $y_1 = N$, $y_2 = y_3 = \ldots = 0$ when $t = 0$, show (preferably by using § 18 (29)) that at time t

$$y_r = N\lambda_1\lambda_2\ldots\lambda_{r-1}\sum_{s=1}^{r}\frac{1}{\beta_s}e^{-\lambda_s t},$$

where $$\beta_s = (\lambda_1-\lambda_s)\ldots(\lambda_{s-1}-\lambda_s)(\lambda_{s+1}-\lambda_s)\ldots(\lambda_r-\lambda_s).$$

11. In an epidemic the rate at which healthy people become infected is α times their number, the rates of recovery and death of sick people are, respectively, β and γ times their number. If initially there are N healthy people and no sick ones, show that the number of deaths up to time t is

$$\alpha\gamma N(c-d+de^{ct}-ce^{dt})/\{cd(c-d)\},$$

where c and d are the roots of $(z+\alpha)(z+\beta+\gamma)-\alpha\beta = 0$.

12. Mass M_1 of a perfect conductor of specific heat c_1, initially at temperature T, is placed at $t = 0$ in a calorimeter containing mass M_2 of water, of specific heat unity, at zero temperature. Heat is exchanged between the mass M_1 and the water at a rate k_1 times their temperature difference, and heat is lost by the water at a rate k_2 times its temperature. If T_1 and T_2 are the temperatures of the mass M_1 and the water, show that

$$(D+a)T_1-aT_2 = 0, \qquad (D+b+c)T_2-bT_1 = 0,$$

where $b = k_1/M_2$, $c = k_2/M_2$, $a = k_1/M_1c_1$. If λ_1 and λ_2 are the roots of $\alpha^2+(a+b+c)\alpha+ac = 0$, show that these are both real and negative, and that

$$T_2 = bT(e^{\lambda_1 t}-e^{\lambda_2 t})/(\lambda_1-\lambda_2).$$

13. If the end $x = 0$ of a uniform rod of length l, which loses heat from its surface at a rate H times its temperature, is held at temperature T_1 and there is no loss of heat at the end $x = l$, show that the temperature at the point x is $T_1\cosh\mu(l-x)\operatorname{sech}\mu l$, in the notation of § 16, Ex. 3. Deduce that the ratio of the heat removed by a long thin cooling fin of thickness d on a surface to that which would be removed from the area at its base if it were not present is $(2/\mu d)\tanh\mu l$.

14. If a thin wire is heated by electric current the differential equation for its temperature T is

$$\frac{d^2T}{dx^2}+b^2T = -k,$$

where k is a positive constant depending on the current, and $b^2 = \alpha k-\mu^2$, where α is the temperature coefficient of resistance and μ^2 is defined in § 16 (10). If the ends $x = 0$ and $x = 2l$ of the wire are at zero temperature, show that

$$T = \frac{k}{b^2}\left\{\frac{\cos b(l-x)}{\cos bl}-1\right\}$$

if $b^2 > 0$, and find the corresponding solutions for $b^2 = 0$ and $b^2 < 0$.

15. In a counterflow heat exchanger the only changes from the conditions of § 16, Ex. 5, are that the partition extends from $x = 0$ to $x = a$, and that while the first fluid is admitted at $x = 0$ at temperature T and flows in the direction of x increasing as before, the second fluid is admitted at $x = a$ at zero temperature and flows with velocity u_2 in the direction of x decreasing. Show that, with the notation of § 16, Ex. 5, and writing $\alpha = (k_2-k_1)/k_1 k_2$,

$$T_1 = T\{k_1 e^{-\alpha a}-k_2 e^{-\alpha x}\}\{k_1 e^{-\alpha a}-k_2\}^{-1}.$$

16. (i) Defining D^{-1} as the operation of indefinite integration, show that $D^r D^{-r} y = y$ but that this is not true of $D^{-r}D^r y$.

(ii) Let $f(D)$ be the polynomial $D^r(a_0 D^n+a_1 D^{n-1}+...+a_n)$, where $a_n \neq 0$, and let $(b_0+b_1 D+...+b_k D^k)$ and $R_k(D)$ be the quotient and remainder obtained by expanding $(a_0 D^n+...+a_n)^{-1}$ in ascending powers of D as if D were an ordinary algebraic variable. Show that if $P_k(x)$ is a polynomial in x of degree k

$$f(D)\{D^{-r}(b_0+b_1 D+...+b_k D^k)P_k(x)\} = P_k(x).$$

(iii) If $f(D)$ and $P_k(x)$ are defined in (ii), show that a particular integral of $f(D)y = P_k(x)$ can be found by writing this symbolically as

$$\frac{1}{f(D)} P_k(x),$$

expanding $1/f(D)$ in ascending powers of D, and operating on $P_k(x)$ with this series. Find particular integrals of Ex. 1, (i)–(iv) in this way.

17. The conventional method of finding a particular integral of $f(D)y = e^{ax}P_k(x)$, where $P_k(x)$ is a polynomial in x, is to write this symbolically in the form $\{1/f(D)\}e^{ax}P_k(x)$ and to proceed as if § 12, Theorem 3 held for this expression. If $f(D)$ has the form $(D-a)^r\phi(D)$ where $\phi(a) \neq 0$ this procedure gives

$$\frac{1}{(D-a)^r\phi(D)}\{e^{ax}P_k(x)\} = e^{ax}\left\{\frac{1}{D^r\phi(D+a)} P_k(x)\right\},$$

and the latter is evaluated as in Ex. 16.

Verify, by using Ex. 16 and § 12, Theorem 3, that the result obtained by this formal procedure is in fact a particular integral. Discuss the special cases $k = 0$ and $r = 0$ independently. Solve Ex. 1, (v)–(xvi) in this way.

18. Deduce § 14 (56) formally by expanding $1/f(D)$ in partial fractions by the formula § 18 (29). Verify, by operating on it with $f(D)$, that the result so obtained is a particular integral.

III

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

19. Introductory

In Chapter I the importance of the fact that the ordinary linear differential equation of any order with constant coefficients was easy to solve was stressed, but the fact that many of the equations arising in applied mathematics are non-linear was noted. Equations of the first order occupy an important position because a number of non-linear equations, as well as the general linear equation of this order, are easily soluble.

In this chapter the commonest types are considered briefly and a few examples from fields other than dynamics are studied.

The general equation of the first order and degree may be written

$$f(x,y)\frac{dy}{dx}+g(x,y) = 0, \tag{1}$$

and its solution will contain one arbitrary constant. We shall usually write C for an arbitrary constant when it occurs.

20. Equations in which the variables are separable

If $f(x,y)$ and $g(x,y)$ in the general equation § 19 (1) are both of the form $\phi(x)\psi(y)$, the equation may be written

$$F(y)\frac{dy}{dx} = G(x). \tag{1}$$

Integrating this gives the solution

$$\int F(y)\,dy = \int G(x)\,dx+C. \tag{2}$$

Ex. $$x\tan y\frac{dy}{dx}-1 = 0.$$

This may be written $$\tan y\frac{dy}{dx} = \frac{1}{x}.$$

Integrating we have $$\int \tan y\,dy = \int\frac{dx}{x}.$$

That is,† $$-\ln\cos y = \ln x+C.$$

Or $$x\cos y = A.$$

† The notation $\ln x$ for $\log_e x$ will always be used.

21. Problems leading to first-order equations with the variables separable

Many of the problems of particle dynamics are of this type. Here we give some other examples.

Ex. 1. *The problem of § 6, Ex. 4.*

The differential equation to be solved is

$$M\frac{dT}{dt} = -H'(T^4-T_0^4).$$

The solution is

$$\begin{aligned}
-\frac{H't}{M}+C &= \int \frac{dT}{T^4-T_0^4} \\
&= \frac{1}{2T_0^2}\int \left\{\frac{1}{2T_0(T-T_0)}-\frac{1}{2T_0(T+T_0)}-\frac{1}{T^2+T_0^2}\right\}dT \\
&= \frac{1}{4T_0^3}\ln\frac{T-T_0}{T+T_0}-\frac{1}{2T_0^3}\tan^{-1}\frac{T}{T_0}.
\end{aligned}$$

Ex. 2. *The law of mass action.*

This states that, if in a chemical reaction n_1 molecules of a substance A, n_2 molecules of a substance B, n_3 of C, and so on, combine to form any number of resultants according to the formula

$$n_1 A+n_2 B+n_3 C+\ldots = \text{any number of resultants,} \tag{1}$$

then the rate of the reaction, that is the rate of increase of the amount transformed, is proportional to

$$(a-x)^{n_1}(b-y)^{n_2}(c-z)^{n_3}\ldots, \tag{2}$$

where a, b, c,... are the amounts of A, B, C,... initially present, and x, y, z,... are the amounts of A, B, C,... transformed up to time t. The quantities x, y, z,... are connected by (1), thus, for example,

$$x/n_1 = y/n_2 = z/n_3 = \ldots. \tag{3}$$

For the *second-order reaction*

$$A+B \rightarrow \text{any resultants,}$$

(2) gives
$$\frac{dx}{dt} = k(a-x)(b-x),$$

where k is a known constant. Therefore

$$\int \frac{dx}{(a-x)(b-x)} = kt+C.$$

The solution of this for which $x = 0$ when $t = 0$ is

$$\frac{1}{b-a}\ln\frac{a(b-x)}{b(a-x)} = kt. \tag{4}$$

For the *third-order reaction*

$$2A+B \to \text{any resultants},$$

(2) becomes
$$\frac{dx}{dt} = k'(a-2x)^2(b-x).$$

The solution of this for which $x = 0$ when $t = 0$ is

$$\frac{1}{(a-2b)^2}\left\{\frac{2x(2b-a)}{a(a-2x)}+\ln\frac{b(a-2x)}{a(b-x)}\right\} = k't.$$

Finally it should be remarked that a unimolecular reaction leads to a linear equation with constant coefficients, and that a chain of such reactions leads to a chain of such equations similar to those of § 6, Ex. 2.

Ex. 3. *Ionization and recombination.*

If a gas is ionized so that the number n of electrons per unit volume is equal to the number of positive ions, the rate at which electrons and positive ions recombine to form neutral molecules is

$$\alpha n^2,$$

where α is a constant called the coefficient of recombination.

Suppose that ions are produced at a constant rate I per unit volume for $t > 0$ in an initially un-ionized gas. Then n satisfies

$$\frac{dn}{dt} = I-\alpha n^2.$$

The solution of this is

$$t = \int \frac{dn}{I-\alpha n^2}+C.$$

That is
$$t = \frac{1}{(\alpha I)^{\frac{1}{2}}}\tanh^{-1}n\left(\frac{\alpha}{I}\right)^{\frac{1}{2}}+C,$$

and since $n = 0$ when $t = 0$ gives $C = 0$, we get finally

$$n = \left(\frac{I}{\alpha}\right)^{\frac{1}{2}}\tanh t(\alpha I)^{\frac{1}{2}}.$$

Ex. 4. *Flow of liquid in an open channel.*

If liquid is flowing along a channel in the direction of the x-axis, its depth h at x satisfies the differential equation

$$\frac{dh}{dx} = i\frac{h^3-H^3}{h^3-\alpha H^3}, \tag{5}$$

where i is the (small and constant) slope of the channel, α is a constant, and H is a constant depending on the quantity of water flowing (it is the depth of a rectangular channel which gives the same flow). All the cases $h \gtrless H$ and $\alpha \gtrless 1$ may occur, so there are many different possibilities.

Putting $h = mH$, the solution of (5) is

$$\frac{ix}{H} = \int \frac{m^3-\alpha}{m^3-1}\,dm+C,$$

$$= m-(1-\alpha)\phi(m)+C,$$

where

$$\phi(m) = -\int \frac{dm}{m^3-1} = \frac{1}{6}\ln\frac{m^2+m+1}{(m-1)^2}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2m+1}{\sqrt{3}}. \tag{6}$$

The function $\phi(m)$ is called the 'backwater function'.

22. The first-order linear equation

The most general first-order linear equation is

$$\frac{dy}{dx}+Py = Q, \tag{1}$$

where P and Q are functions of x.

In the homogeneous case, $Q = 0$, this becomes the separable equation

$$\frac{1}{y}\frac{dy}{dx} = -P, \tag{2}$$

the solution of which is

$$\ln y = -\int P\,dx+C,$$

or

$$y = Ae^{-\int P\,dx}, \tag{3}$$

where A is a constant.

The equation (1) may now be solved by a general method, 'variation of parameters', which allows the solution of an inhomogeneous linear differential equation to be deduced from the solution of the corresponding homogeneous equation.

We seek a solution of (1) of the form

$$y = ze^{-\int P\,dx}, \tag{4}$$

where z is a function of x. This form is suggested by (3), the constant A being replaced by the function z. Substituting (4) in (1) gives

$$\frac{dz}{dx}e^{-\int P\,dx}-Pze^{-\int P\,dx}+Pze^{-\int P\,dx} = Q.$$

Thus

$$\frac{dz}{dx} = Qe^{\int P\,dx},$$

and

$$z = \int Qe^{\int P\,dx}\,dx+C. \tag{5}$$

Hence, finally, the solution of (1) is

$$y = e^{-\int P\,dx}\Big\{\int Qe^{\int P\,dx}\,dx+C\Big\}, \tag{6}$$

where C is an arbitrary constant.

Another method of solving this equation will be given in § 26. The way in which (6) generalizes the results of Chapter II for the case in which P is a constant, say α, is worth noting explicitly; αx is replaced by $\int P\,dx$.

Ex.
$$x\frac{dy}{dx}+y = xe^{-x}.$$

Here, in the above notation, $P = 1/x$ and

$$\int P\,dx = \ln x.$$

Thus (6) gives

$$\begin{aligned} y &= e^{-\ln x}\Big\{\int e^{-x+\ln x}\,dx+C\Big\} \\ &= \frac{1}{x}\Big\{\int xe^{-x}\,dx+C\Big\} \\ &= \frac{C}{x}-e^{-x}\Big(1+\frac{1}{x}\Big). \end{aligned}$$

23. Equations reducible to the linear form

It is often possible to reduce an equation to the linear form by a suitable substitution. The following are two important types.

(i) *The equation*

$$\frac{dy}{dx}+Pe^{y} = Q, \tag{1}$$

where P and Q are functions of x only. Putting

$$z = e^{-y}, \qquad \frac{dz}{dx} = -e^{-y}\frac{dy}{dx},$$

the equation (1) becomes

$$\frac{dz}{dx}+Qz = P, \tag{2}$$

which is linear in z. Equations containing a term in e^{y} are of considerable importance in chemical kinetics and similar problems.

(ii) *Bernoulli's equation,*

$$\frac{dy}{dx}+Py = Qy^n, \tag{3}$$

where P and Q are functions of x only.

Put $$z = y^{1-n},$$

$$\frac{dz}{dx} = \frac{1-n}{y^n}\frac{dy}{dx},$$

and (1) becomes

$$\frac{dz}{dx}+(1-n)Pz = (1-n)Q, \tag{4}$$

which is a linear equation in z.

Riccati's equation,

$$\frac{dy}{dx} = Py^2+Qy+R, \tag{5}$$

where P, Q, and R are functions of x only, which is a little more general than (3) and of considerable importance in dynamics, is an example of an equation with no simple method of solution (that is, in the general case).

Making the substitution

$$y = -\frac{1}{Pz}\frac{dz}{dx}$$

it transforms into

$$\frac{d^2z}{dx^2}-\left(Q+\frac{1}{P}\frac{dP}{dx}\right)\frac{dz}{dx}+PRz = 0, \tag{6}$$

which is a linear second-order equation in z with variable coefficients.

24. Problems leading to first-order linear equations

Ex. 1. *The problem of § 6, Ex. 5, with $M = M_0$ and $T = T_0$ when $t = 0$.*

From § 6 (5) we have $$M = mt+M_0, \tag{1}$$

and substituting this in § 6 (6) gives the linear equation for T

$$\frac{dT}{dt}+\frac{m+H}{mt+M_0}T = \frac{mT_1+HT_2}{mt+M_0}. \tag{2}$$

Using § 22 (6), and writing $\mu = (m+H)/m$, $t_0 = M_0/m$, the solution of (2) is seen to be

$$T = (t+t_0)^{-\mu}\left\{\int \frac{mT_1+HT_2}{m(t+t_0)}(t+t_0)^{\mu}\,dt+C\right\}$$

$$= \frac{mT_1+HT_2}{m\mu}+C(t+t_0)^{-\mu}. \tag{3}$$

The requirement that $T = T_0$ when $t = 0$ gives

$$T_0 = \frac{mT_1+HT_2}{m\mu}+Ct_0^{-\mu},$$

and substituting this value of C in (3) gives finally

$$T = \frac{mT_1+HT_2}{m\mu}+\left\{\frac{m\mu T_0-mT_1-HT_2}{m\mu}\right\}\left(1+\frac{t}{t_0}\right)^{-\mu}.$$

Ex. 2. *Fungus spores are destroyed by heat at a rate proportional to the product of the number present and an exponential function of the temperature T.*

That is, if n is the number of spores per unit volume at time t, n satisfies the differential equation

$$\frac{dn}{dt} = -kne^{cT}, \tag{4}$$

where k and c are constants, and the temperature T is a given function of the time. By § 22 (6) the solution of (4) is

$$n = A\exp\left\{-k\int e^{cT}\,dt\right\}. \tag{5}$$

If n_0 is the number of spores per unit volume when $t = 0$, this becomes

$$n = n_0\exp\left\{-k\int\limits_0^t e^{cT}\,dt\right\}. \tag{6}$$

Ex. 3. *The equation of radiative transfer.*

Suppose radiation is being propagated in the direction of the x-axis in a medium which absorbs the radiation at a rate ρkI per unit volume and also emits radiation at the rate ρkB per unit volume, where ρ, k, and B are known functions of x, and I is the intensity of the radiation at x. Then I satisfies the equation

$$\frac{dI}{dx} = -k\rho(I-B).$$

If I_0 is the intensity at $x = 0$, the solution of this is

$$I = \exp\left[-\int\limits_0^x \rho k\,dx\right]\left\{\int\limits_0^x k\rho B\exp\left[\int\limits_0^x \rho k\,dx\right]dx+I_0\right\}.$$

25. Homogeneous equations

These are of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right). \tag{1}$$

It follows that by the substitution

$$y = vx \tag{2}$$

the function $f(y/x)$ is reduced to a function of v only.

Also from (2) $$\frac{dy}{dx} = v + x\frac{dv}{dx}, \tag{3}$$

so that (1) becomes $$x\frac{dv}{dx} = f(v) - v. \tag{4}$$

(4) is an equation with the variables separable, and its solution is

$$\int \frac{dv}{f(v)-v} = \ln x + C. \tag{5}$$

Ex. 1. $$\frac{dy}{dx} = \frac{x^2+y^2}{xy}.$$

Making the substitutions (2) and (3), this becomes

$$x\frac{dv}{dx} + v = \frac{1+v^2}{v},$$

or $$x\frac{dv}{dx} = \frac{1}{v}.$$

Thus the solution is $$\tfrac{1}{2}v^2 = \ln x + C,$$

or $$\frac{y^2}{2x^2} = \ln x + C.$$

Ex. 2. $$\frac{dy}{dx} + \frac{x-3y+2}{3x-y+6} = 0.$$

This equation is not homogeneous as it stands, but if the variables are changed to Y and X defined by

$$X - 3Y = x - 3y + 2,$$

$$3X - Y = 3x - y + 6,$$

it becomes $$\frac{dY}{dX} + \frac{X-3Y}{3X-Y} = 0,$$

which is homogeneous. Substituting $Y = vX$, this gives

$$X\frac{dv}{dX} + \frac{1-v^2}{3-v} = 0.$$

The solution of this is $$\frac{(v+1)^2X}{v-1} = C,$$

or $$\frac{(X+Y)^2}{(Y-X)} = C,$$

or, finally, reverting to the original notation,

$$\frac{(x+y+2)^2}{y-x-2} = C.$$

26. Exact equations

The equation $$f(x,y)\frac{dy}{dx}+g(x,y) = 0 \tag{1}$$

is called an exact equation if the left-hand side, as it stands, is the differential coefficient of a function $\phi(x,y)$ of x and y. That is, if (1) has the form

$$\frac{d}{dx}\phi(x,y) = 0, \tag{2}$$

so that its solution is $$\phi(x,y) = C. \tag{3}$$

For example, the equation

$$x\frac{dy}{dx}+y = 0.$$

is exact, and its solution is

$$xy = C.$$

In the notation of differentials the definition requires that

$$f(x,y)\,dy+g(x,y)\,dx$$

is to be an exact differential $d\phi$.

The condition that (1) be exact is

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}, \tag{4}$$

and this condition is both necessary and sufficient, that is, if the equation is exact (4) must hold; and if (4) holds, (1) must be exact. We prove these statements in order. First suppose that (1) is exact so that it can be expressed in the form (2); this may be written

$$\frac{\partial\phi}{\partial y}\frac{dy}{dx}+\frac{\partial\phi}{\partial x} = 0. \tag{5}$$

Comparing (1) and (5) we must have

$$f(x,y) = k\frac{\partial\phi}{\partial y}, \qquad g(x,y) = k\frac{\partial\phi}{\partial x}, \tag{6}$$

where k is a constant. Then from (6)

$$\frac{\partial f}{\partial x} = k\frac{\partial^2 \phi}{\partial x \partial y} = k\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial g}{\partial y},$$

so that (4) is satisfied as required.

Next we prove that if (4) is satisfied we can find a function ϕ such that (1) can be put in the form (2). Write

$$G(x,y) = \int g(x,y)\,dx, \tag{7}$$

the integral being evaluated as if y were constant. Then

$$\frac{\partial G(x,y)}{\partial x} = g(x,y),$$

$$\frac{\partial^2 G(x,y)}{\partial y \partial x} = \frac{\partial g(x,y)}{\partial y} = \frac{\partial f(x,y)}{\partial x}, \tag{8}$$

using (4). (8) may be written

$$\frac{\partial}{\partial x}\left\{\frac{\partial G(x,y)}{\partial y} - f(x,y)\right\} = 0,$$

so that

$$\frac{\partial G(x,y)}{\partial y} - f(x,y) = \psi(y), \tag{9}$$

where $\psi(y)$ is a function of y only.

We now choose for the required function $\phi(x,y)$,

$$\phi(x,y) = G(x,y) - \int \psi(y)\,dy, \tag{10}$$

and show that this has the required properties. We have

$$\frac{\partial \phi(x,y)}{\partial x} = \frac{\partial G(x,y)}{\partial x} = g(x,y),$$

$$\frac{\partial \phi(x,y)}{\partial y} = \frac{\partial G(x,y)}{\partial y} - \psi(y) = f(x,y),$$

by (9), and thus (1) is put in the form (2). This process may be used to find $\phi(x,y)$ if this is not immediately obvious.

There are many equations which are not exact as they stand but can be made so by multiplying by a suitable function of x and y. Such a function is called an *integrating factor*. For example, the equation

$$x\frac{dy}{dx} - y = 0, \tag{11}$$

is not exact, but if multiplied by x^{-2} it becomes

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0, \tag{12}$$

which is exact and can be written

$$\frac{d}{dx}\left(\frac{y}{x}\right) = 0,$$

so that the solution of (11) is

$$y = Cx.$$

One method of finding an integrating factor is to use the condition (4). We illustrate this by considering the important case of the linear equation

$$\frac{dy}{dx}+Py = Q, \tag{13}$$

where P and Q are functions of x. Suppose ψ is an integrating factor of (13), then

$$\psi\frac{dy}{dx}+\psi(Py-Q) = 0$$

must be exact. By (4) this requires

$$\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial y}\{\psi Py-\psi Q\}$$

$$= (Py-Q)\frac{\partial\psi}{\partial y}+\psi P. \tag{14}$$

Any function which satisfies (14) will be suitable, in particular a function $\psi(x)$ of x only chosen so that

$$\frac{d\psi}{dx}-\psi P = 0. \tag{15}$$

A solution of (15) is

$$\psi = e^{\int P\,dx} \tag{16}$$

so that this is an integrating factor of (13). Multiplying (13) by it gives

$$\left\{\frac{dy}{dx}+Py\right\}e^{\int P\,dx} = Qe^{\int P\,dx},$$

or

$$\frac{d}{dx}\{ye^{\int P\,dx}\} = Qe^{\int P\,dx},$$

so that, integrating,

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx+C, \tag{17}$$

which is the solution already found in § 22.

27. Equations of the first order but of higher degree than the first

The equation of the nth degree will be

$$\phi_0(x,y)\left(\frac{dy}{dx}\right)^n+\phi_1(x,y)\left(\frac{dy}{dx}\right)^{n-1}+...+\phi_n(x,y) = 0. \qquad (1)$$

This may be factorized into the form

$$\left\{f_1(x,y)\frac{dy}{dx}+g_1(x,y)\right\}...\left\{f_n(x,y)\frac{dy}{dx}+g_n(x,y)\right\} = 0. \qquad (2)$$

Therefore if y satisfies any one of the n equations

$$\left.\begin{array}{r} f_1(x,y)\dfrac{dy}{dx}+g_1(x,y) = 0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f_n(x,y)\dfrac{dy}{dx}+g_n(x,y) = 0 \end{array}\right\} \qquad (3)$$

it will satisfy (2). The solution of (1) consists of the collection of all the solutions of (3). Suppose $\psi_1(x,y,c_1) = 0$, ..., $\psi_n(x,y,c_n) = 0$ are the solutions of these found by the methods of the previous sections, where $c_1,..., c_n$ are arbitrary constants, then any of these satisfies (1) and its general solution may, if desired, be written

$$\psi_1(x,y,c)\psi_2(x,y,c)...\psi_n(x,y,c) = 0, \qquad (4)$$

where c is an arbitrary constant. One arbitrary constant c is sufficient in (4), since, as it varies, the individual solutions $\psi_1(x,y,c)$, etc., run through all possible values.

As an example consider the equation

$$\left(\frac{dy}{dx}\right)^2+3\left(\frac{dy}{dx}\right)+2 = 0, \qquad (5)$$

that is, factorizing, $\left(\dfrac{dy}{dx}+2\right)\left(\dfrac{dy}{dx}+1\right) = 0.$

The solution of $\dfrac{dy}{dx}+2 = 0$

is the family of straight lines

$$y+2x = c_1, \qquad (6)$$

and that of $\dfrac{dy}{dx}+1 = 0$

is the family of straight lines

$$y+x = c_2. \qquad (7)$$

(5) is satisfied by all the lines (6) and (7): its general solution may be written

$$(y+2x+c)(y+x+c) = 0.$$

A phenomenon which appears in certain equations of this type is that they may have additional solutions which are not comprised in the general solution. Such solutions are called singular solutions. For example, for the equation

$$\left(\frac{dy}{dx}\right)^2 - y = 0, \tag{8}$$

or

$$\left(\frac{dy}{dx} - y^{\frac{1}{2}}\right)\left(\frac{dy}{dx} + y^{\frac{1}{2}}\right) = 0,$$

the general solution, found as above, is

$$\{2y^{\frac{1}{2}} - (x+c)\}\{2y^{\frac{1}{2}} + (x+c)\} = 0,$$

or

$$4y = (x+c)^2, \tag{9}$$

which is a family of parabolas. But (8) is also satisfied by

$$y = 0, \tag{10}$$

which is not included in the general solution (9) and is a singular solution. The line (10) is in fact the envelope of the parabolas (9).

EXAMPLES ON CHAPTER III

1. Solve the following differential equations:

(i) $\dfrac{dy}{dx} = 1 + y^2.$

(ii) $\dfrac{dy}{dx} - (1+e^y)\sin x = 0.$

(iii) $x\dfrac{dy}{dx} - y = x^3.$

(iv) $(1+x^2)\dfrac{dy}{dx} + 4xy = 1.$

(v) $\dfrac{dy}{dx} + y = xy^3.$

(vi) $(x^3 - y^3)\dfrac{dy}{dx} + 3x^2y = 0.$

(vii) $\dfrac{dy}{dx} - \dfrac{x+y}{3x-y-1} = 0.$

(viii) $(\sin x + x\cos y)\dfrac{dy}{dx} + (\sin y + y\cos x) = 0.$

(ix) $3xy^2\dfrac{dy}{dx} + 2y^3 + 3x = 0.$

The solutions are

(i) $x = \tan^{-1}y + C.$ (ii) $\cos x - \ln(1+e^{-y}) = C.$

(iii) $y = \frac{1}{2}x^3 + Cx.$ (iv) $y = (x + \frac{1}{3}x^3 + C)/(1+x^2)^2.$

(v) $y = \{\frac{1}{2} + x + Ce^{2x}\}^{-\frac{1}{2}}.$ (vi) $4x^3y - y^4 = C.$

(vii) $\ln(y - x + \frac{1}{2}) + (4x-1)/(2y-2x+1) = C.$

(viii) $x\sin y + y\sin x = C.$ (ix) $x^3 + x^2y^3 = C.$

2. (i) Show that in the second-order reaction $2A \to$ any resultants (cf. § 21) if x is the amount of A transformed in time t, and a is the amount of A originally present,

$$\frac{x}{a(a-x)} = kt.$$

(ii) Show that if a, b, c are the amounts of A, B, and C originally present, and x is the amount transformed in the third-order reaction $A+B+C \to$ any resultants,

$$\frac{1}{(b-a)(c-a)}\ln\frac{a}{a-x}+\frac{1}{(a-b)(c-b)}\ln\frac{b}{b-x}+\frac{1}{(a-c)(b-c)}\ln\frac{c}{c-x} = kt.$$

3. The differential equation for a reversible second-order reaction in which the second substance is not present initially is

$$\frac{dx}{dt} = k_1(a-x)^2-k_2x^2.$$

Putting $y = x/(a-x)$ show that

$$\frac{1}{2a\sqrt{(k_1k_2)}}\ln\frac{(a-x)\sqrt{k_1}+x\sqrt{k_2}}{(a-x)\sqrt{k_1}-x\sqrt{k_2}} = t.$$

4. Molten metal in a cylindrical container is cooling by conduction of heat radially outwards. At the instant of solidification the volume of the metal solidifying is reduced by a fraction α. If h is the height of the free surface of the liquid when the radius of the surface at which solidification is taking place is r, show that

$$\frac{dh}{dr} = \frac{2\alpha h}{r}.$$

If $h = h_0$ when $r = r_0$ show that the shape of the upper surface when the metal is all solid is $h = h_0(r/r_0)^{2\alpha}$.

5. A substance decomposes according to the unimolecular law

$$\frac{dw}{dt} = -ke^{-E/RT}w,$$

where w is the amount of the substance present, k, E, and R are constants, and T is the absolute temperature. At time $t = 0$, $w = w_0$ and $T = T_0$. Heat is given off during the reaction at a rate proportional to the reaction velocity (no heat being lost to the surroundings) so that $dT/dt = -q\,dw/dt$, where q is a constant. Show that the temperature at any time is given by

$$kt = \int_{T_0}^{T}\frac{e^{E/RT}\,dT}{T_0+qw_0-T}.$$

6. If there are initially N electrons per unit volume and they disappear by recombination (cf. § 21, Ex. 3) show that the number present at time t is $N/(1+N\alpha t)$.

7. The density ρ in the earth's atmosphere is assumed to vary with height h according to the law $\rho = \rho_0 \exp(-h/H)$, where ρ_0 and H are constants. Radiation is incident on it vertically from outside, its intensity at infinity being I_0. If the radiation is absorbed at the rate $k\rho I$ per unit volume, where I is its intensity [cf. § 24, Ex. 3], show that

$$I = I_0 \exp\{-Hk\rho_0 e^{-h/H}\}.$$

Ions are produced by the absorbed radiation at a rate β times the rate of absorption. Show that the rate of ion production is

$$\beta I_0 k\rho_0 \exp\{-h/H - Hk\rho_0 e^{-h/H}\}.$$

Show that this has a maximum at the height $H\ln(Hk\rho_0)$. This is Chapman's theory of the formation of the ionosphere.

8. Show that if h_1 and h_2 are the depths of a stream at distances x_1 and x_2 from an origin,

$$i(x_1 - x_2) = (h_1 - h_2) - (1-\alpha)H\{\phi(h_1/H) - \phi(h_2/H)\},$$

in the notation of § 21, Ex. 4.

9. In a sterilizing process the temperature T is raised linearly from zero to T_0 in time t_0, and subsequently decreases exponentially according to the law $T = T_0 \exp[-\alpha(t-t_0)]$. If n_0 is the number of spores initially present (cf. § 24 (4)), show that the number n_1 at time $t_1 > t_0$ is given by

$$\ln(n_0/n_1) = (kt_0/cT_0)\{e^{cT_0} - 1\} + k\{\mathrm{Ei}(cT_0) - \mathrm{Ei}(cT_0 e^{-\alpha(t_1 - t_0)})\}/\alpha,$$

where $\mathrm{Ei}(x)$ denotes the tabulated function

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \eta^{-1} e^{\eta}\, d\eta.$$

10. The path of a ray in a spherically symmetrical refracting medium is determined by Snell's law, $\mu r \sin i = \text{constant}$, where μ is a given function of r, and i is the angle between the path and the radius vector to the origin. Show that if $\mu \to 1$ as $r \to \infty$, the equation of the path of a ray which for large values of r is parallel to the axis of polar coordinates and distant p from it is

$$\theta = p \int_{r}^{\infty} \frac{dr}{r(\mu^2 r^2 - p^2)^{\frac{1}{2}}}.$$

11. Ions are produced for $t > 0$ in a medium at the rate kt per unit time per unit volume, and they disappear by recombination (cf. § 21, Ex. 3) at the rate αn^2. Making the substitution $n = (1/\alpha z)\, dz/dt$, show that z must satisfy

$$\frac{d^2 z}{dt^2} - \alpha k t z = 0.$$

For the solution of this see § 98. The problem corresponds to production of ions in the ionosphere near sunrise.

12. The bottoms of two equal tanks are on the same level and are connected by a pipe which is such that the rate of flow of water is proportional to $\sqrt{(h_1-h_2)}$, where h_1 and h_2 are the heights of water in the two tanks. If water is pumped into the first tank at a constant rate, show that h_1 and h_2 satisfy

$$\frac{dh_1}{dt} = k-k'(h_1-h_2)^{\frac{1}{2}}, \qquad \frac{dh_2}{dt} = k'(h_1-h_2)^{\frac{1}{2}},$$

where k and k' are constants.

If both tanks are empty at $t = 0$, show that

$$\frac{k}{2k'^2}\ln\left\{\frac{k-2k'(kt-2h_2)^{\frac{1}{2}}}{k}\right\}+\frac{1}{k'}(kt-2h_2)^{\frac{1}{2}}+t = 0.$$

IV

DYNAMICAL PROBLEMS LEADING TO ORDINARY LINEAR DIFFERENTIAL EQUATIONS

28. Introductory

In this chapter we shall consider problems which lead to linear differential equations. These will usually be 'vibration problems' on the motion of several masses constrained by springs, or the rotation of a number of wheels on a shaft. As remarked in § 4, quite complicated problems of these types can be solved, provided they are idealized in such a way that the equations of motion become linear.

Newton's second law for the motion of a (constant) mass m along the x-axis is

$$m\frac{d^2x}{dt^2} = \text{force}, \tag{1}$$

and, as in § 4, we assume that the force is a sum of terms depending on position, velocity, and time: i.e.

$$m\ddot{x} = f(x)+g(\dot{x})+h(t), \tag{2}$$

where, as is usual in dynamics, a dot is used to denote differentiation with respect to the time.

If the equation (2) is to be linear, $f(x)$ must be proportional to x, say $f(x) = -\lambda x$, and $g(\dot{x})$ must be proportional to $\dot{x}$, say $g(\dot{x}) = -k\dot{x}$.

Restoring force proportional to displacement is provided by the strain of an ideally elastic body or a perfect spring; we use the term 'stiffness' for the constant of proportionality λ, so that the restoring force exerted is the stiffness times the relative displacement of the ends.

Resistance to motion proportional to velocity is provided by the shearing of an ideal viscous fluid.

If $f(x)$ and $g(\dot{x})$ have these forms, (2) becomes

$$m\ddot{x} = -\lambda x-k\dot{x}+h(t), \tag{3}$$

which is an inhomogeneous second-order linear differential equation.

We also get a linear equation if the resistance to motion is due to the 'coulomb' or 'solid' friction between two solids which slide on one another. In this case there is a constant force μR opposite to the direction of motion, R being the normal reaction at the contact, and μ the coefficient of dynamic friction.

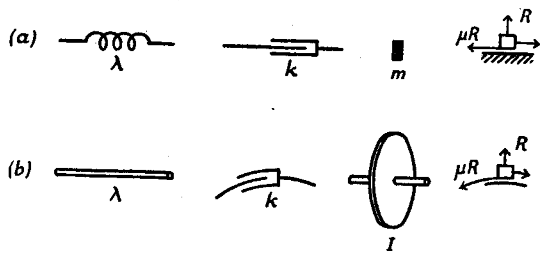


FIG. 3.

In this case, however, different equations of motion must be used for the two directions of motion since the sign of the constant force μR must be changed. This was not necessary in the case of resistance to motion $-k\dot{x}$, since $(-k\dot{x})$ changes sign automatically with $\dot{x}$ and thus is always in the opposite direction to it.

The fundamental elements considered above may be represented diagrammatically as in Fig. 3 (a). The first represents an ideal spring of stiffness λ, the second a dash-pot containing ideal viscous liquid giving resistance to motion k times the relative velocity of sliding, the third a mass m, and the fourth coulomb friction μ times the normal reaction R and in a direction opposite to an assumed direction of sliding. Complicated systems can be built up by combining these elements, and in all cases the equations of motion will be linear.

Similar equations arise for the angular motion of wheels on elastic shafts. Here the angular displacement θ of a marked line on a wheel from a fixed reference direction is the dependent variable corresponding to x, and the equation of motion of a wheel of moment of inertia I is

$$I\ddot{\theta} = \text{torque}. \tag{4}$$

An ideally elastic shaft of 'stiffness' λ provides a restoring torque λ times the relative angular displacement of its ends, and resistance to motion $k\dot{\theta}$ is provided by shearing of an ideal viscous liquid. Thus there are elements for rotational motion as in Fig. 3 (*b*) similar to those of Fig. 3 (*a*) for linear motion; also there will be an analogy between linear and rotational motion in the sense that certain systems will have the same differential equations, except for a change of notation, θ for x, I for m, etc. In Chapter V this analogy will be developed and extended to include electrical systems. In this chapter problems will be stated for linear motion, but the equivalent diagrams of types Fig. 3 (*a*) and (*b*) for linear and rotational motion will both be given.

Finally it should be stated that all the forces acting on the mass are supposed to be specified, and that unless gravity is mentioned explicitly (e.g. by saying that the mass moves vertically) it is supposed not to be effective, for example, the motion might be regarded as taking place on a smooth horizontal plane. There is no loss of generality in omitting gravity forces in these *linear* problems: if they are included there will be a position of static equilibrium calculable from the weights of the masses and the linear forces, and the equations of motion about this equilibrium position will be the same as those we discuss.

29. The damped harmonic oscillator: free vibrations

We consider the motion of a particle of mass m along the x-axis, Fig. 4; the particle is supposed to be constrained by a spring of stiffness λ, the displacement x of the particle being measured from the position in which the spring is unstrained;† there is resistance to motion k times the velocity, and the particle is acted on by an external force $F(t)$.

The equation of motion, § 28 (3) is

$$m\ddot{x} = -\lambda x - k\dot{x} + F(t), \tag{1}$$

† If the spring hangs vertically so that gravity acts on the mass m we may *either* include mg in the force $F(t)$, taking the origin in the unstrained position, cf. § 30 (i), *or* we may measure x from the position of static equilibrium (a deflexion of g/n^2), in which case the same equations result except that $F(t)$ is to be the external force other than gravity.

or $$(D^2+2\kappa D+n^2)x = \frac{1}{m}F(t), \tag{2}$$

where, here and subsequently, we use the notation

$$\kappa = \frac{k}{2m}, \qquad n^2 = \frac{\lambda}{m}, \tag{3}$$

and write D for d/dt.

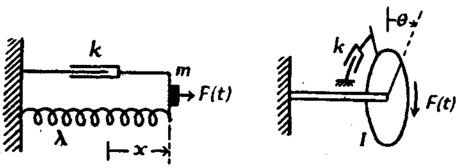


FIG. 4.

In this section we shall study the case $F(t) = 0$ in which there is no external force; this gives the 'free oscillations' of the system when disturbed in any way. This solution will also be needed in discussing the general equation (2) of which it is the complementary function.

Putting $F(t) = 0$ in (2) we have to solve

$$(D^2+2\kappa D+n^2)x = 0. \tag{4}$$

The auxiliary equation § 13 (2) is

$$\alpha^2+2\kappa\alpha+n^2 = 0,$$

and its roots are $$\alpha = -\kappa\pm\sqrt{(\kappa^2-n^2)}. \tag{5}$$

Thus the solution takes three distinct forms according as $\kappa^2 \gtreqless n^2$.

(i) *The case $\kappa < n$*

Writing $$n' = \sqrt{(n^2-\kappa^2)}, \tag{6}$$

the values (5) of α become $-\kappa\pm in'$ and the general solution of (4) is
$$x = Ae^{-\kappa t}\sin(n't+B), \tag{7}$$

where A and B are arbitrary constants determined by the way in which the motion was started when $t = 0$.

In the case $\kappa = 0$, no resistance to motion, the solution is simply $$A\sin(nt+B), \tag{8}$$

a harmonic oscillation of frequency $n/2\pi$: this will be referred to as the *undamped natural frequency.*

The effect of resistance to motion κ in (7) is to make the solution the product of $e^{-\kappa t}$ with a sinusoidal term of frequency $n'/2\pi$. The motion is called *damped harmonic motion.* $n'/2\pi$ will be called the *damped natural frequency*, and κ the *damping coefficient*. The motion given by (7) is not strictly periodic since each oscillation is different from the preceding one, but it does possess certain periodic properties: for example $x = 0$ when

$$t = \frac{r\pi}{n'} - \frac{B}{n'} \quad (r = 1, 2,...), \tag{9}$$

that is, at a series of instants separated by the half-period π/n'. Also the velocity has the same property, for

$$\dot{x} = Ae^{-\kappa t}\{n' \cos(n't+B) - \kappa \sin(n't+B)\}, \tag{10}$$

and so $\dot{x} = 0$, when

$$t = \frac{r\pi}{n'} + \frac{1}{n'}\tan^{-1}\frac{n'}{\kappa} - \frac{B}{n'} \quad (r = 1, 2,...). \tag{11}$$

Comparing (9) and (11) it appears that the particle always takes time

$$\frac{1}{n'}\tan^{-1}\frac{n'}{\kappa}$$

to swing from its equilibrium position $x = 0$ to a point at which it is at rest, and time

$$\frac{1}{n'}\left\{\pi - \tan^{-1}\frac{n'}{\kappa}\right\}$$

to swing back from rest to $x = 0$. It thus moves back more slowly than it moves out. This may be seen from Fig. 5, in which the curves $e^{-\kappa t}$, $\sin n't$, and $e^{-\kappa t}\sin n't$ are shown.

Substituting the times (11) in (7) gives the displacements of the particle at successive instants when it is at rest. These are seen to be in geometric progression with common ratio

$$-e^{-\pi\kappa/n'}. \tag{12}$$

The amplitudes of successive swings on the same side of the origin diminish by the factor

$$e^{-2\pi\kappa/n'},$$

and the logarithm of this, namely,

$$\delta = 2\pi\kappa/n', \tag{13}$$

is called the logarithmic decrement.† This quantity, which is the ratio of the damping coefficient κ to the damped natural frequency $n'/2\pi$, appears also in connexion with resonance. The quantity in electric circuit theory analogous to $n'/2\kappa$ is called the 'Q' of a circuit.

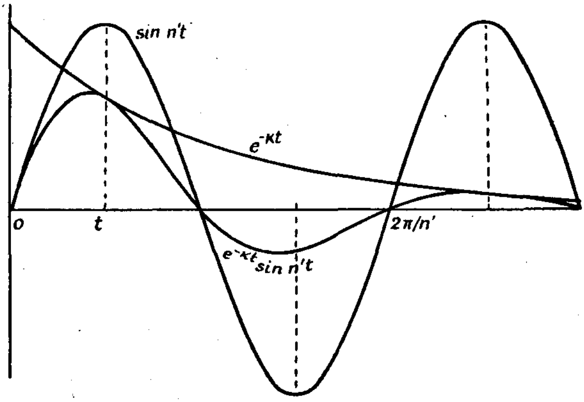


FIG. 5.

Finally it should be remarked that if κ/n is small, the damped natural frequency

$$\frac{1}{2\pi}n' = \frac{1}{2\pi}\sqrt{(n^2-\kappa^2)} = \frac{n}{2\pi}\left(1-\frac{\kappa^2}{2n^2}+\ldots\right) \tag{14}$$

differs very little from the undamped natural frequency $n/2\pi$. Thus a small amount of resistance to motion has a much more important effect on the amplitude than on the frequency.

(ii) *The case* $\kappa > n$

Here the general solution of (4) is

$$x = Ae^{-\{\kappa-\sqrt{(\kappa^2-n^2)}\}t}+Be^{-\{\kappa+\sqrt{(\kappa^2-n^2)}\}t}. \tag{15}$$

The velocity $\dot{x}$ can vanish once at most. The motion is not oscillatory and consists either of a single swing or of a creep back

† Several different definitions are in use. Some authors use half this quantity which corresponds to comparing successive swings on *opposite* sides of the origin. Sometimes, also, the quantity δ in (13) is called the decrement.

to the origin, according to the circumstances of projection. This case $\kappa > n$ is referred to as 'dead beat'.

(iii) *The case of 'critical damping'*, $\kappa = n$

Here the auxiliary equation to (4) has equal roots n and the general solution is
$$(A+Bt)e^{-nt}. \tag{16}$$

As in case (ii) the motion consists of at most a single swing and is not oscillatory.

Finally we consider the effect of increasing the damping coefficient of a system with undamped natural frequency n. If $\kappa < n$ we have from (7)
$$|x| < |A|e^{-\kappa t};$$
thus as κ is increased towards n, the motion, while remaining oscillatory, dies away like
$$e^{-\kappa t}. \tag{17}$$
When $\kappa = n$, the motion ceases to be oscillatory and dies away like
$$te^{-nt}, \tag{18}$$
by (16). Finally, when $\kappa > n$ the motion given by (15) dies away like
$$e^{-\{\kappa-\sqrt{(\kappa^2-n^2)}\}t}. \tag{19}$$

Since both (17) for any value of $\kappa < n$, and (19) for any value of $\kappa > n$, are larger than (18) for sufficiently large values of t, it follows that, in order to make the motion die away as rapidly as possible we must give κ the critical value n. If this value is exceeded the motion does not die away so rapidly. Thus in recording instruments which have to take up a reading as quickly as possible, critical damping is often aimed at.

Ex. *The particle is set in motion with velocity V at $t = 0$ from its equilibrium position $x = 0$: to find the motion.*

The conditions $x = 0$, $\dot{x} = V$, when $t = 0$ require that in (7)
$$A \sin B = 0,$$
$$A\{n' \cos B - \kappa \sin B\} = V.$$
Thus $B = 0$, $A = V/n'$ and the solution is
$$x = \frac{V}{n'} e^{-\kappa t} \sin n't, \quad \text{if } \kappa < n.$$
Similarly from (16) and (15) we find
$$x = Vte^{-nt}, \quad \text{if } \kappa = n,$$
$$x = \frac{V}{\sqrt{(\kappa^2-n^2)}} e^{-\kappa t} \sinh t\sqrt{(\kappa^2-n^2)}, \quad \text{if } \kappa > n.$$

30. The harmonic oscillator with external force applied to it

In this section we consider the harmonic oscillator with applied forces of various types; the most important case, namely harmonic applied force, will be treated at length in § 31. Resistance to motion proportional to velocity is not considered, but it could be added in all cases.

(i) *Constant applied force*

The effect of such a force, as remarked in §§ 28, 29, is to give oscillation about a displaced position of equilibrium.

Ex. 1. *A particle of mass m is hung vertically by a spring of stiffness mn^2. At $t = 0$, when the spring is unstrained, the particle is released.*

Taking the x-axis vertically downwards, with the origin at the position in which the spring is unstrained, the equation of motion of the particle, § 28 (1), is

$$m\ddot{x} = -mn^2x+mg,$$

or

$$\ddot{x}+n^2x = g. \tag{1}$$

The general solution of this is

$$x = A\sin nt+B\cos nt+\frac{g}{n^2}. \tag{2}$$

The constants A and B in (2) have to be determined from the conditions $x = \dot{x} = 0$, when $t = 0$. These give

$$B+\frac{g}{n^2} = 0,$$

$$A = 0.$$

Therefore the solution is

$$x = \frac{g}{n^2}(1-\cos nt),$$

an oscillation about $x = g/n^2$, which is the equilibrium position of the mass when hanging from the spring.

(ii) *The applied force any function $F(t)$ of the time*

In this case the equation of motion, § 29 (2), is

$$(D^2+n^2)x = \frac{1}{m}F(t). \tag{3}$$

By § 14 (54) the general solution of (3) is

$$x = A\sin nt+B\cos nt+\frac{1}{mn}\int\limits_0^t \sin n(t-\xi)F(\xi)\,d\xi. \tag{4}$$

Ex. 2. *The particle is set in motion at $t = 0$ from rest in its equilibrium position by a constant force F_0 which acts for time T and then ceases.*

In this case $F(\xi)$ in (4) is F_0 for $0 < t < T$, and zero for $t > T$. We have to determine A and B in (4) to give $x = 0$ and $\dot{x} = 0$ when $t = 0$. These require

$$B = 0,$$

$$nA + \left[\frac{F_0}{m}\int\limits_0^t \cos n(t-\xi)\,d\xi\right]_{t=0} = 0.$$

Thus $A = B = 0$, and the solution is

$$x = \frac{F_0}{mn}\int\limits_0^t \sin n(t-\xi)\,d\xi = \frac{F_0}{mn^2}(1-\cos nt) \quad (0 < t < T),$$

$$x = \frac{F_0}{mn}\int\limits_0^T \sin n(t-\xi)\,d\xi$$

$$= \frac{F_0}{mn^2}\{\cos n(t-T)-\cos nt\} = \frac{2F_0}{mn^2}\sin n(t-\tfrac{1}{2}T)\sin \tfrac{1}{2}nT \quad (t > T).$$

Thus the residual effect after the force has ceased is an oscillation of amplitude $(2F_0/mn^2)\sin \frac{1}{2}nT$. This, of course, could have been found by studying the motions for $0 < t < T$ and for $t > T$ separately.

(iii) *Solid or coulomb friction*

In this case, as remarked in § 28, the direction of motion must be known before the differential equation is written down, and the frictional force, μ times the normal reaction, is in the direction opposite to the direction of motion. This differential equation is valid only for motion in this direction.

It should be remarked that the assumption that kinetic friction is equal to a constant times the normal reaction is a very crude first approximation which is used because it gives linear equations of motion.

Actually, for most rubbing substances the frictional force varies with the velocity of sliding according to a complicated law such as that of Fig. 6 (*a*) in which for low velocities of sliding the frictional force is less than the static value. A second approximation which leads to interesting results is to assume a constant coefficient of kinetic friction μ which is less than the static value μ'; cf. Ex. 4 below.

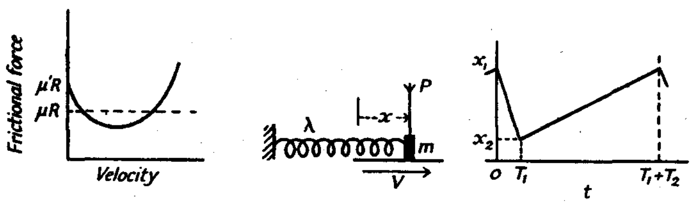
Ex. 3. *A particle of mass m rests on a horizontal plane, the coefficient of friction being μ. It is attached to a fixed point by an elastic spring of stiffness mn^2. The particle is displaced a distance a from its equilibrium position and then released.*

Initially we have $x = a$, $\dot{x} = 0$, when $t = 0$. Also, when released, the particle will start to move backwards and so the frictional force will act forwards. The equation of motion for this part of the motion is then

$$m\ddot{x} = -mn^2x+\mu mg,$$

or

$$\ddot{x}+n^2x = \mu g. \tag{5}$$



(a) (b) (c)

FIG. 6.

The general solution of this is

$$x = A\sin nt+B\cos nt+\frac{\mu g}{n^2}.$$

The conditions $x = a$, $\dot{x} = 0$, when $t = 0$ give $A = 0$, $B = a-\mu g/n^2$, and so the solution is

$$x = \frac{\mu g}{n^2}+\left(a-\frac{\mu g}{n^2}\right)\cos nt, \tag{6}$$

$$\dot{x} = -n\left(a-\frac{\mu g}{n^2}\right)\sin nt. \tag{7}$$

The equation of motion (5) and the solutions (6) and (7) hold so long as $\dot{x}$ is negative. This is the case until $t = \pi/n$, at which time the particle comes to rest, its displacement then being

$$x = -a+\frac{2\mu g}{n^2},$$

that is, a distance $(a-2\mu g/n^2)$ on the opposite side of its equilibrium position.

The same argument will show that it will next come to rest at time $2\pi/n$ at a distance $(a-4\mu g/n^2)$ from the equilibrium position, and so on.

Thus the particle oscillates with the period $2\pi/n$ which it would have in the absence of friction, but the amplitude diminishes by a constant amount $2\mu g/n^2$ in each half-swing. This continues until the particle comes to rest so near the equilibrium position that the restoring force is less than the frictional force. Motion then ceases.

If this happens after r half-swings we must have

$$2r-1 < \frac{n^2a}{\mu g} < 2r+1,$$

that is, r must be the first integer greater than $\frac{1}{2}(n^2a/\mu g)-\frac{1}{2}$.

Ex. 4. *Chattering in a system in which kinetic friction is less than static friction.*

This type of problem has many important technical applications. The motion is periodic but is quite different from the harmonic oscillations treated hitherto and is related to the 'relaxation oscillations' discussed in § 59.

As a definite problem suppose that a mass m is pressed by force P against a plane which moves with velocity V, motion of the mass with the plane being resisted by a spring of stiffness mn^2 [Fig. 6 (*b*)]. The coefficients of static and dynamic friction between the mass and the plane are μ' and μ, with $\mu' > \mu$.

We suppose the mass to be moving with the plane in the direction of the x-axis; this will continue until its displacement from the position in which the spring is unstrained is

$$x_1 = \frac{\mu' P}{mn^2}. \tag{8}$$

At this instant, which we shall take as the origin of time, $t = 0$, it will commence to slip and the frictional force on it will be dynamic friction acting forwards. That is, its equation of motion is

$$m\ddot{x} = -mn^2x+\mu P, \tag{9}$$

to be solved with $\dot{x} = V$, $x = \mu' P/mn^2$, when $t = 0$. The solution of (9) with these initial conditions is

$$x = \frac{\mu P}{mn^2}+\frac{V}{n}\sin nt+\frac{(\mu'-\mu)P}{mn^2}\cos nt, \tag{10}$$

$$\dot{x} = V\cos nt-\frac{(\mu'-\mu)P}{mn}\sin nt. \tag{11}$$

This motion continues until the velocity of the mass relative to the plane vanishes, and then sliding ceases. This happens at time T_1 given by the smallest (non-zero) root of

$$V\cos nt-\frac{(\mu'-\mu)P}{mn}\sin nt = V,$$

i.e. of

$$\tan\tfrac{1}{2}nt = -\frac{(\mu'-\mu)P}{mnV}. \tag{12}$$

This gives

$$T_1 = \frac{2\pi}{n}-\frac{2}{n}\tan^{-1}\frac{(\mu'-\mu)P}{mnV}. \tag{13}$$

At this time its displacement x_2 is, by (10),

$$x_2 = \frac{(2\mu-\mu')P}{mn^2}. \tag{14}$$

When the particle has come to rest relative to the plane it stays moving with it until its displacement is x_1 given by (8), when the process repeats itself. The time taken in moving from x_2 to x_1 is

$$T_2 = \frac{x_1 - x_2}{V} = \frac{2(\mu' - \mu)P}{mn^2 V}, \tag{15}$$

and the period of the whole process is $T_1 + T_2$ given by (13) and (15). The curve of displacement against time for the motion is sketched in Fig. 6 (c).

31. The damped harmonic oscillator: forced oscillations

We now consider the system of § 29, namely a mass m with restoring force mn^2 times its displacement x, and resistance to motion $2m\kappa$ times its velocity $\dot{x}$, acted on by an external force $F_0 \sin(\omega t + \beta)$.

Putting $F(t) = F_0 \sin(\omega t + \beta)$ in § 29 (2), the equation of motion is

$$(D^2 + 2\kappa D + n^2)x = \frac{F_0}{m} \sin(\omega t + \beta). \tag{1}$$

The complementary function of (1) has been discussed in § 29; we now have to find a particular integral. Proceeding as in § 14 (iv), we find a particular integral of

$$(D^2 + 2\kappa D + n^2)x = \frac{F_0}{m} e^{i(\omega t + \beta)} \tag{2}$$

and take its imaginary part. We seek a particular integral of (2) of the form $x = Xe^{i\omega t}$: substituting this in (2) gives

$$(n^2 - \omega^2 + 2\kappa i\omega)X = \frac{F_0}{m} e^{i\beta},$$

and the required particular integral of (2) becomes

$$x = \frac{F_0}{m(n^2 - \omega^2 + 2\kappa i\omega)} e^{i(\omega t + \beta)}$$

$$= \frac{F_0}{m\{(n^2 - \omega^2)^2 + 4\kappa^2\omega^2\}^{\frac{1}{2}}} e^{i(\omega t + \beta - \phi)}, \tag{3}$$

where
$$\phi = \arg\{(n^2 - \omega^2) + 2\kappa i\omega\}. \tag{4}$$

Taking the imaginary part of (3), the particular integral of (1) is

$$x = \frac{F_0}{m\{(n^2 - \omega^2)^2 + 4\kappa^2\omega^2\}^{\frac{1}{2}}} \sin(\omega t + \beta - \phi), \tag{5}$$

where ϕ is defined in (4).

Adding the complementary function [§ 29 (7) if $\kappa < n$, or the corresponding expression of § 29 if $\kappa \geqslant n$] gives the general solution of (1)

$$x = Ae^{-\kappa t}\sin(n't+B)+$$
$$+\frac{F_0}{m\{(n^2-\omega^2)^2+4\kappa^2\omega^2\}^{\frac{1}{2}}}\sin(\omega t+\beta-\phi), \quad (6)$$

where $n' = \sqrt{(n^2-\kappa^2)}$, and A and B are constants to be determined from the initial conditions. $n'/2\pi$ is the damped natural frequency. The first term of (6) dies away exponentially as the time increases and is termed the *transient* part of the solution; for large values of the time, only the second term of (6), which is the particular integral (5) of (1), remains. This is called the *steady state solution* or *forced oscillation*; it has the same frequency $\omega/2\pi$ as the applied force and a lag in phase of ϕ behind it. In the time during which the transient is not negligible the oscillation builds up from its initial value to the final steady state value: the smaller the damping coefficient, the longer the time taken in this building-up process.

We proceed to discuss in detail the way in which the amplitude and phase of the forced oscillation (5) vary as ω varies between 0 and ∞. The velocity of the mass in the forced oscillation (5) is

$$\dot{x} = \frac{F_0\,\omega}{m\{(n^2-\omega^2)^2+4\kappa^2\omega^2\}^{\frac{1}{2}}}\sin\{\omega t+\beta-(\phi-\tfrac{1}{2}\pi)\}, \quad (7)$$

and we discuss also the variation of the amplitude and phase lag $(\phi-\frac{1}{2}\pi)$ of this.

Treating the phase lag first, we need the value of ϕ from (4). If $\omega < n$, $n^2-\omega^2$ is positive, and

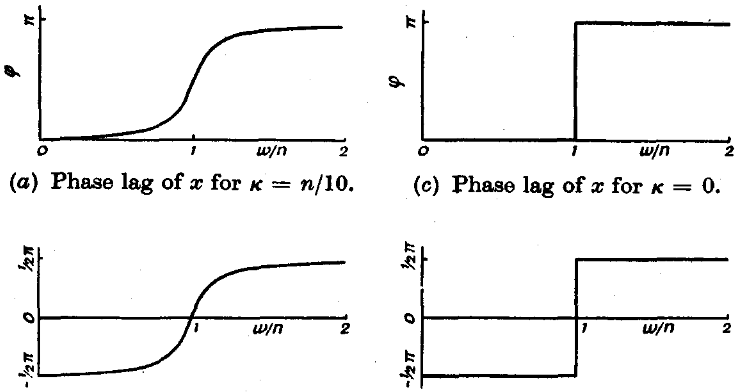
$$\phi = \tan^{-1}\frac{2\kappa\omega}{n^2-\omega^2} \quad (\omega < n). \quad (8)$$

If $\omega > n$, $n^2-\omega^2$ is negative, and [cf. § 14 (25)]

$$\phi = \pi-\tan^{-1}\frac{2\kappa\omega}{\omega^2-n^2} \quad (\omega > n). \quad (9)$$

It appears from (8) and (9) that as $\omega \to 0$, $\phi \to 0$, that is the phase lag ϕ of the displacement is small for small ω; as $\omega \to n$,

$\phi \to \frac{1}{2}\pi$, a phase lag of $\frac{1}{2}\pi$ when $\omega/2\pi$ is equal to the undamped natural frequency; as $\omega \to \infty$, $\phi \to \pi$, a phase lag of π for high frequencies. Since by (7) the phase lag of the velocity is $\phi - \frac{1}{2}\pi$ it follows that this is $-\frac{1}{2}\pi$ for very low frequencies, $\frac{1}{2}\pi$ for very high frequencies, and zero at the undamped natural frequency. Graphs of ϕ and $\phi - \frac{1}{2}\pi$ are shown in Fig. 7 (*a*) and (*b*) respectively, for the case $\kappa = n/10$.



(*a*) Phase lag of x for $\kappa = n/10$.

(*c*) Phase lag of x for $\kappa = 0$.

(*b*) Phase lag of $\dot{x}$ for $\kappa = n/10$.

(*d*) Phase lag of $\dot{x}$ for $\kappa = 0$.

FIG. 7

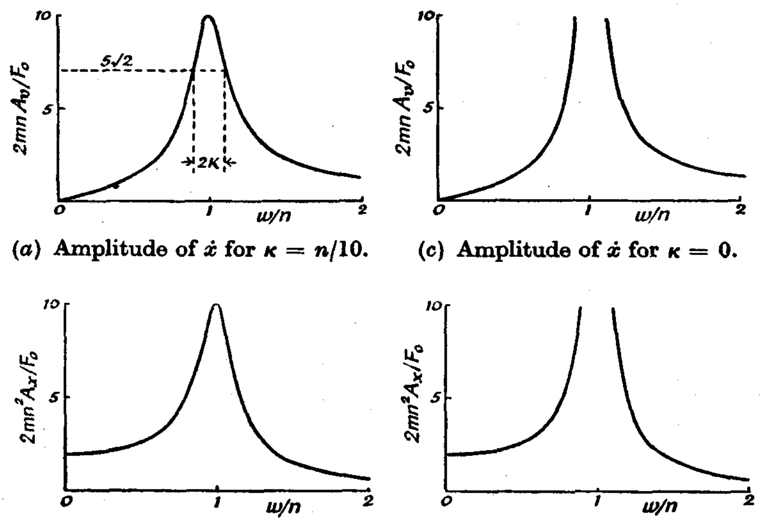
Next we have to consider the variation with ω of the amplitude of the forced oscillation, and we study first the amplitude

$$A_v = \frac{F_0}{2\kappa m\{1+(n^2-\omega^2)^2/4\kappa^2\omega^2\}^{\frac{1}{2}}} \tag{10}$$

of the velocity $\dot{x}$ given by (7). Clearly $A_v = 0$ when $\omega = 0$, and $A_v \to 0$ as $\omega \to \infty$. Also, when $\omega = n$, the denominator of (10) has its least value, so that A_v has a maximum of $F_0/2\kappa m$ when $\omega = n$, that is when the frequency is equal to the undamped natural frequency. The curve of A_v against ω is shown in Fig. 8 (*a*) for $\kappa = n/10$. Clearly, the smaller the value of the damping coefficient κ, the larger the value of the maximum. Thus when the frequency of the applied force is equal to the undamped natural frequency of the system, vibrations of large amplitude can be set up, particularly if the damping coefficient is small. This phenomenon is called *resonance*.

It is of importance to have some information about the sharpness of the peak of the curve of A_v against ω. To specify this we determine the values of ω at which A_v has a value $(1/s)$th of its maximum. By (10) this is the case when

$$\frac{1}{\{1+(n^2-\omega^2)^2/4\kappa^2\omega^2\}^{\frac{1}{2}}} = \frac{1}{s}, \tag{11}$$



(a) Amplitude of $\dot{x}$ for $\kappa = n/10$. (c) Amplitude of $\dot{x}$ for $\kappa = 0$.

(b) Amplitude of x for $\kappa = n/10$. (d) Amplitude of x for $\kappa = 0$.

FIG. 8.

or, squaring, when

$$(n^2-\omega^2)^2 = 4\kappa^2\omega^2(s^2-1),$$

that is, when
$$\omega^2 \pm 2\kappa\omega(s^2-1)^{\frac{1}{2}}-n^2 = 0. \tag{12}$$

The roots of this quadratic are

$$\omega = \mp\kappa(s^2-1)^{\frac{1}{2}}+\{\kappa^2(s^2-1)+n^2\}^{\frac{1}{2}}, \tag{13}$$

where the positive sign has been chosen for the second square root since we are only interested in positive values of ω.

These two values of ω differ by $2\kappa\sqrt{(s^2-1)}$, so this is the width of the peak at the point where the amplitude is $(1/s)$th of its maximum value. In particular, the width of the curve is 2κ at $1/\sqrt{2} = 0{\cdot}7071$ of the maximum amplitude, and the values (13) of ω for this value of s are $(\kappa^2+n^2)^{\frac{1}{2}}\pm\kappa$. In Fig. 9 curves of A_v against ω for various values of κ/n are shown in order to illustrate the effect of decreasing κ on the sharpness of the curve.

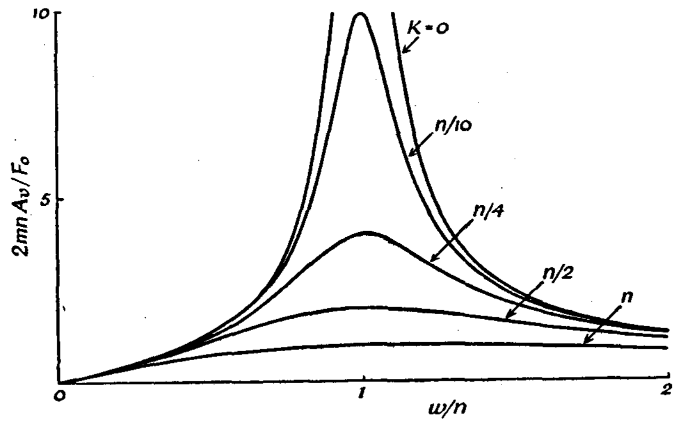


FIG. 9.

The amplitude of the velocity was considered in detail above because it is the more important in the analogous problem in electric circuit theory (cf. § 43) and also because it leads to the simple exact result (13). We now consider the amplitude A_x of the displacement x, which by (3) is

$$A_x = \frac{F_0}{m\{(n^2-\omega^2)^2+4\kappa^2\omega^2\}^{\frac{1}{2}}}. \tag{14}$$

When $\omega = 0$, $A_x = F_0/mn^2$, and as $\omega \to \infty$, $A_x \to 0$. Also

$$\frac{dA_x}{d\omega} = \frac{2\omega F_0\{n^2-2\kappa^2-\omega^2\}}{m\{(n^2-\omega^2)^2+4\kappa^2\omega^2\}^{\frac{3}{2}}}. \tag{15}$$

It follows that A_x has a minimum at $\omega = 0$ and a maximum at $\omega = \sqrt{(n^2-2\kappa^2)}$, provided $n^2 > 2\kappa^2$. If $n^2 < 2\kappa^2$ the curve decreases steadily and there is no maximum. The maximum value of A_x is

$$\frac{F_0}{2\kappa m(n^2-\kappa^2)^{\frac{1}{2}}}, \tag{16}$$

and it should be noticed that the frequency $(n^2-2\kappa^2)^{\frac{1}{2}}/2\pi$ at which it is attained does not coincide with either the undamped or the damped natural frequency. The curve of A_x against ω is shown in Fig. 8 (*b*).

As before we specify the sharpness of the peak by seeking the values of ω at which A_x has $(1/s)$th of its maximum value (16). These are given by

$$\omega = \{(n^2-2\kappa^2)\pm 2\kappa(s^2-1)^{\frac{1}{2}}(n^2-\kappa^2)^{\frac{1}{2}}\}^{\frac{1}{2}}. \tag{17}$$

If we assume that κ/n is small and use the binomial theorem in (17), we get approximately, neglecting terms in κ^2/n^2,

$$\omega = n \pm \kappa\sqrt{(s^2-1)}, \tag{18}$$

which gives the same value as (13) for the width of the peak, but (18) is approximate whereas (13) was exact.

Finally, we treat the case $\kappa = 0$ *ab initio* in order to show clearly the connexion between the results for this case and those obtained earlier. This connexion is of considerable importance, since the study of more complicated systems involving several masses is not particularly difficult if damping is neglected, but becomes extremely complicated if it is included. It is therefore desirable to be able to obtain a qualitative idea of the behaviour of such a system from the solution for the case of no damping.

If $\kappa = 0$, the differential equation (1) becomes

$$(D^2+n^2)x = \frac{F_0}{m}\sin(\omega t+\beta), \tag{19}$$

and its particular integral, found as before, is

$$x = \frac{F_0}{m(n^2-\omega^2)}\sin(\omega t+\beta), \tag{20}$$

provided $\omega \neq n$.

If $\omega < n$, x is thus exactly in phase with the applied force. As $\omega \to n$ the amplitude of x tends to infinity. When $\omega > n$, x becomes negative and may be written

$$x = \frac{F_0}{m(\omega^2-n^2)}\sin(\omega t+\beta-\pi),$$

which corresponds to an oscillation of amplitude $F_0/m(\omega^2-n^2)$ with a phase lag of π behind the applied force. Thus (20) may be written

$$x = \frac{F_0}{m|n^2-\omega^2|}\sin(\omega t+\beta-\phi), \tag{21}$$

where the phase lag

$$\phi = 0 \quad (\omega < n)$$

$$\phi = \pi \quad (\omega > n)$$

changes discontinuously by π as ω passes through n. Also the amplitude tends to infinity as ω tends to n. These curves are shown in Figs. 7 (c) and 8 (d), respectively, and are obvious limiting cases of those for finite damping as $\kappa \to 0$.

In the same way the velocity is

$$\dot{x} = \frac{\omega F_0}{m|n^2-\omega^2|}\sin(\omega t+\beta+\tfrac{1}{2}\pi-\phi).$$

Its phase lag and amplitude are shown in Figs. 7 (d) and 8 (c).

The general solution of (19) is

$$A\sin(nt+B)+\frac{F_0}{m(n^2-\omega^2)}\sin(\omega t+\beta), \tag{22}$$

where A and B are arbitrary constants to be found from the initial conditions. It should be remarked that in this ideal case the complementary function $A\sin(nt+B)$ does not die away for large values of the time.

It remains to consider what happens in the case of no damping when ω has the resonance frequency n, so that (19) becomes

$$(D^2+n^2)x = \frac{F_0}{m}\sin(nt+\beta). \tag{23}$$

The particular integral of this has been found in § 14, Ex. 9, to be

$$-\frac{F_0}{2mn}t\cos(nt+\beta). \tag{24}$$

It thus corresponds to an oscillation of steadily increasing amplitude. If a system with negligible damping is set in motion by a harmonic force of the resonance frequency, the amplitude of its oscillations will increase steadily until they are so great that the assumed equations of motion cease to hold.

32. The harmonic oscillator: forced oscillations caused by motion of the support

The differential equations discussed in detail in § 31 arise again here but in a slightly different manner.

Suppose a particle of mass m is attached to a point S by a spring of stiffness mn^2, and that there is resistance to motion $2m\kappa$ times the velocity of the particle. Let the point S be given a prescribed motion $\xi(t)$ in the direction of the spring. If the origin of x is taken so that the spring is unstrained when $x = 0$

and $\xi = 0$, the restoring force on the particle for any values of x and ξ will be
$$mn^2\{x-\xi(t)\},$$
and the equation of motion of the particle will be
$$m\ddot{x} = -mn^2\{x-\xi(t)\}-2m\kappa\dot{x},$$
or
$$\ddot{x}+2\kappa\dot{x}+n^2x = n^2\xi(t), \tag{1}$$
which is the same as § 29 (2) except for the change of notation.

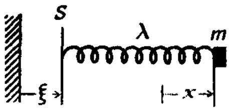


FIG. 10.

If the point of support is given a harmonic motion
$$\xi = a\sin(\omega t+\beta),$$
(1) becomes
$$\ddot{x}+2\kappa\dot{x}+n^2x = n^2a\sin(\omega t+\beta), \tag{2}$$
which is the same as § 31 (1) with n^2a in place of F_0/m.

Ex. *A machine of mass M is supported on springs of total stiffness $(M+m)n^2$, and its vibration is damped by resistance $2(M+m)\kappa$ times its velocity. It carries a mass m which executes a vertical simple harmonic motion $\xi = a\sin(\omega t+\beta)$ relative to the mass M. Find the steady periodic motion of the bed.*

Let x be the displacement of the mass M measured upwards from the position in which the springs are unstrained, so that $(x+\xi)$ is the position of the mass m relative to the same origin. Let $f(\xi)$ be the force exerted by the machine on the mass m, then the equation of motion of the mass m is
$$m(\ddot{x}+\ddot{\xi}) = -mg+f(\xi), \tag{3}$$
and the equation of motion of the mass M is
$$M\ddot{x} = -(M+m)n^2x-2(M+m)\kappa\dot{x}-f(\xi)-Mg. \tag{4}$$

Adding (3) and (4) gives
$$\ddot{x}+2\kappa\dot{x}+n^2x = -g-\frac{m}{M+m}\ddot{\xi}$$
$$= -g+\frac{ma\omega^2}{M+m}\sin(\omega t+\beta). \tag{5}$$

Using § 31 (4) and (5), the particular integral of (5) is
$$-\frac{g}{n^2}+\frac{ma\omega^2}{(M+m)\{(n^2-\omega^2)^2+4\kappa^2\omega^2\}^{\frac{1}{2}}}\sin(\omega t+\beta-\phi),$$
corresponding to a forced oscillation about the equilibrium position $-g/n^2$ of the machine.

33. Systems of several masses

When several masses are involved there is a great increase in algebraical complexity. We discuss in detail a case involving two masses to show the new ideas involved. In this section we shall assume that there is no resistance to motion; the same problem, with resistance included, will be studied in § 34.

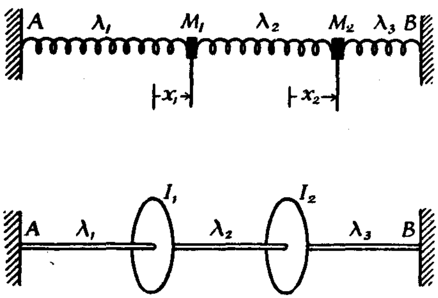


FIG. 11.

Suppose that two masses M_1 and M_2 are connected as shown in Fig. 11 by springs of stiffnesses λ_1, λ_2, and λ_3 to fixed points A and B, and that they oscillate in the straight line AB. Let x_1 and x_2 be the displacements of M_1 and M_2 from their equilibrium positions. Then, if there are no external forces, the equations of motion of M_1 and M_2 are

$$M_1\ddot{x}_1 = \lambda_2(x_2-x_1)-\lambda_1 x_1, \tag{1}$$

$$M_2\ddot{x}_2 = -\lambda_2(x_2-x_1)-\lambda_3 x_2. \tag{2}$$

Writing

$$n_1^2 = \frac{\lambda_1}{M_1}, \qquad n_2^2 = \frac{\lambda_2}{M_2}, \qquad n_{12}^2 = \frac{\lambda_2}{M_1}, \qquad n_{23}^2 = \frac{\lambda_3}{M_2}, \tag{3}$$

these become

$$(D^2+n_1^2+n_{12}^2)x_1-n_{12}^2 x_2 = 0, \tag{4}$$

$$-n_2^2 x_1+(D^2+n_2^2+n_{23}^2)x_2 = 0. \tag{5}$$

(4) and (5) are a pair of simultaneous linear differential equations for x_1 and x_2. We could solve them by the methods of § 15, but it is more usual in problems of this type to use the

equivalent method indicated at the end of § 13 and seek a solution in which x_1 and x_2 are proportional to $e^{\alpha t}$. Thus we assume†

$$x_1 = X_1 e^{\alpha t}, \qquad x_2 = X_2 e^{\alpha t}, \tag{6}$$

where X_1 and X_2 are independent of t. Substituting (6) in (4) and (5) gives

$$(\alpha^2+n_1^2+n_{12}^2)X_1-n_{12}^2 X_2 = 0, \tag{7}$$

$$-n_2^2 X_1+(\alpha^2+n_2^2+n_{23}^2)X_2 = 0. \tag{8}$$

These may be written

$$\frac{X_1}{X_2} = \frac{n_{12}^2}{\alpha^2+n_1^2+n_{12}^2} = \frac{\alpha^2+n_2^2+n_{23}^2}{n_2^2}. \tag{9}$$

The second of equations (9) gives the equation

$$(\alpha^2+n_2^2+n_{23}^2)(\alpha^2+n_1^2+n_{12}^2)-n_2^2 n_{12}^2 = 0 \tag{10}$$

for α, and corresponding to each root of this, the first of equations (9) gives the associated value of the ratio $X_1:X_2$. (9) is the auxiliary equation of the differential equations (4) and (5) for x_1 and x_2. It will be called the *frequency equation*, since it will appear that its roots determine the natural frequencies of the system.

For simplicity we now restrict ourselves to the case of equal masses and springs, so that

$$M_1 = M_2, \qquad \lambda_1 = \lambda_2 = \lambda_3, \qquad n_1 = n_2 = n_{12} = n_{23},$$

and (9) and (10) become

$$\frac{X_1}{X_2} = \frac{n_1^2}{\alpha^2+2n_1^2} = \frac{\alpha^2+2n_1^2}{n_1^2}, \tag{11}$$

and

$$\alpha^4+4\alpha^2 n_1^2+3n_1^4 = 0. \tag{12}$$

The roots of (12) are $\alpha^2 = -n_1^2$ and $\alpha^2 = -3n_1^2$.

The root $\alpha^2 = -n_1^2$, $\alpha = \pm i n_1$, gives by (11)

$$\frac{X_1}{X_2} = 1. \tag{13}$$

† It will appear in (13) that α is pure imaginary so that we might have started by assuming solutions proportional to e^{int}, and this is often done. The form (6) is used here, partly because of the correspondence with the work of Chapter II, and partly because it is a little better when there is resistance to motion as in § 34.

Thus the solution of type (6) is

$$x_1 = Ce^{in_1t}+De^{-in_1t},$$
$$x_2 = Ce^{in_1t}+De^{-in_1t},$$

where C and D are any constants. These may be written

$$\left.\begin{aligned} x_1 &= A_1\sin(n_1t+\beta_1) \\ x_2 &= A_1\sin(n_1t+\beta_1) \end{aligned}\right\}, \tag{14}$$

where A_1 and β_1 are any constants.

Thus in this solution, x_1 and x_2 execute harmonic oscillations of the same amplitude and phase, and with period $2\pi/n_1$.

The other root $\alpha^2 = -3n_1^2$, $\alpha = \pm in_1\surd 3$, of (12) gives from (11)

$$\frac{X_1}{X_2} = -1. \tag{15}$$

For this root the solutions (6) take the form

$$\left.\begin{aligned} x_1 &= A_2\sin(n_1t\surd 3+\beta_2) \\ x_2 &= A_2\sin(n_1t\surd 3+\beta_2-\pi) \end{aligned}\right\}, \tag{16}$$

where A_2 and β_2 are arbitrary constants. Thus x_1 and x_2 execute harmonic vibrations of the same amplitude, of period $2\pi/n_1\surd 3$, and 180° out of phase.

These two solutions (14) and (16) are called the two *normal modes of oscillation* of the system, and their frequencies $n_1/2\pi$ and $n_1\surd 3/2\pi$ are called the *natural frequencies* of the system. In the 'lower mode', that is the mode of lower natural frequency, Fig. 12 (*a*), the displacements of the particles are equal and in phase; in the higher mode, Fig. 12 (*b*), they are equal and out of phase by 180°.

The general solution of (4) and (5) consists of a combination of the two normal modes (14) and (16) with arbitrary amplitudes and phases, that is,

$$\left.\begin{aligned} x_1 &= A_1\sin(n_1t+\beta_1)+A_2\sin(n_1t\surd 3+\beta_2) \\ x_2 &= A_1\sin(n_1t+\beta_1)+A_2\sin(n_1t\surd 3+\beta_2-\pi) \end{aligned}\right\}. \tag{17}$$

This contains four arbitrary constants as it should (cf. § 15), and these can be determined from the initial displacements and velocities of the two masses. For initial value problems of this sort it is better to use the Laplace transformation method as in § 40; for most purposes a knowledge of the normal modes and natural frequencies is sufficient.

The same type of result holds in general: if there had been r masses, the equation for α^2 would have had r roots, corresponding to r natural frequencies, each of these gives a normal

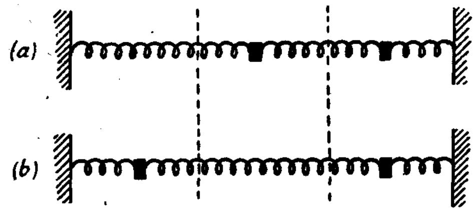


FIG. 12.

mode in which the relative amplitudes and phases of the masses are known.

Next we consider forced oscillations of the system of Fig. 11 due to a force $F_0 \sin \omega t$ applied to the mass M_1.

The differential equations (4) and (5) are replaced by

$$(D^2+n_1^2+n_{12}^2)x_1-n_{12}^2 x_2 = (F_0/M_1)\sin \omega t, \tag{18}$$

$$-n_2^2 x_1+(D^2+n_2^2+n_{23}^2)x_2 = 0. \tag{19}$$

We now require a particular integral of these, and as usual replace $\sin \omega t$ in (18) by $e^{i\omega t}$, giving

$$(D^2+n_1^2+n_{12}^2)x_1-n_{12}^2 x_2 = (F_0/M_1)e^{i\omega t}, \tag{20}$$

find a particular integral of (19) and (20), and take its imaginary part. To find this particular integral we seek solutions of (19) and (20) of the form

$$x_1 = X_1 e^{i\omega t}, \qquad x_2 = X_2 e^{i\omega t}. \tag{21}$$

Substituting (21) in (19) and (20) gives

$$-n_2^2 X_1+(n_2^2+n_{23}^2-\omega^2)X_2 = 0,$$

$$(n_1^2+n_{12}^2-\omega^2)X_1-n_{12}^2 X_2 = F_0/M_1.$$

Solving we find

$$X_1 = \frac{F_0(n_2^2+n_{23}^2-\omega^2)}{M_1\{(n_1^2+n_{12}^2-\omega^2)(n_2^2+n_{23}^2-\omega^2)-n_2^2 n_{12}^2\}}, \tag{22}$$

$$X_2 = \frac{n_2^2 F_0}{M_1\{(n_1^2+n_{12}^2-\omega^2)(n_2^2+n_{23}^2-\omega^2)-n_2^2 n_{12}^2\}}. \tag{23}$$

X_1 and X_2 are both real, so, using these values in (21) and taking the imaginary part, the required particular integrals of (18) and (19) are found to be

$$x_1 = X_1 \sin \omega t, \tag{24}$$

$$x_2 = X_2 \sin \omega t. \tag{25}$$

To study the behaviour of these in greater detail we consider again the special case $M_1 = M_2$, $\lambda_1 = \lambda_2 = \lambda_3$ discussed earlier. In this case X_1 and X_2 become

$$X_1 = \frac{F_0(2n_1^2-\omega^2)}{M_1(3n_1^2-\omega^2)(n_1^2-\omega^2)}, \tag{26}$$

$$X_2 = \frac{n_1^2 F_0}{M_1(3n_1^2-\omega^2)(n_1^2-\omega^2)}. \tag{27}$$

The behaviour of X_1 and X_2, given by (26) and (27), as functions of ω is shown in Fig. 13 (*a*) and (*b*). They tend to infinity as ω tends to either of the values n_1 or $n_1\sqrt{3}$ corresponding to natural frequencies of the system, and they change sign on passing through these points. As in § 31 (21) we regard x_1 and x_2 as oscillations of amplitudes $|X_1|$ and $|X_2|$ respectively, and when X_1 or X_2 is negative we express this as a phase lag of π of x_1 or x_2 behind the applied force. The amplitudes of x_1 and x_2 are shown in Fig. 13 (*c*) and (*d*) and their phases in Fig. 13 (*e*) and (*f*). These may be compared with the curves of Fig. 15 for the same system when resistance to motion is taken into account.

The case of harmonic force applied to one of the masses only has been considered above. Clearly results of the same general type will be obtained if harmonic force is applied to both the masses, or if, as in § 32, one of the supports is given a harmonic oscillation.

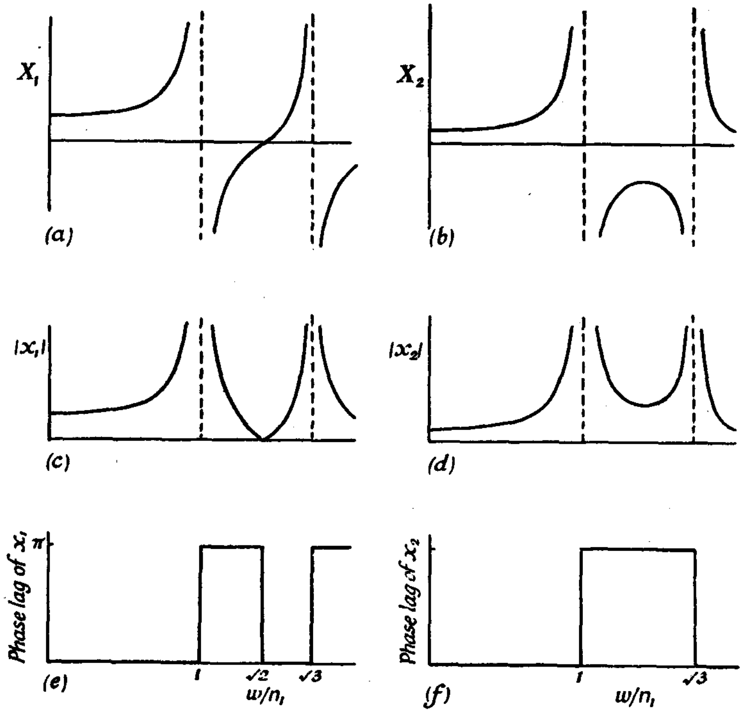


FIG. 13.

34. Systems of several masses with resistance proportional to velocity

To illustrate the effect of resistance to motion we consider again in detail the system of § 33, but now include resistance terms. These resistance terms may arise in two ways: there may be resistance proportional to the absolute velocities $\dot{x}_1$ and $\dot{x}_2$ of the masses, and there may in addition be resistance proportional to the relative velocity $(\dot{x}_2-\dot{x}_1)$ of the masses. These are indicated by the appropriate symbols in Fig. 14.

The equations of motion of this system when subject to no external forces are

$$M_1\ddot{x}_1 = \lambda_2(x_2-x_1)-\lambda_1 x_1-k_1\dot{x}_1+k_{12}(\dot{x}_2-\dot{x}_1), \tag{1}$$

$$M_2\ddot{x}_2 = -\lambda_2(x_2-x_1)-\lambda_3 x_2-k_2\dot{x}_2-k_{12}(\dot{x}_2-\dot{x}_1). \tag{2}$$

Using the abbreviations § 33 (3) and in addition

$$2\kappa_1 = \frac{k_1}{M_1}, \qquad 2\kappa_2 = \frac{k_2}{M_2}, \qquad 2\kappa_{12} = \frac{k_{12}}{M_1}, \qquad 2\kappa_{21} = \frac{k_{12}}{M_2}, \tag{3}$$

these become

$$\{D^2+2(\kappa_1+\kappa_{12})D+n_1^2+n_{12}^2\}x_1-(2\kappa_{12}D+n_{12}^2)x_2 = 0, \tag{4}$$

$$-(2\kappa_{21}D+n_2^2)x_1+\{D^2+2(\kappa_2+\kappa_{21})D+n_2^2+n_{23}^2\}x_2 = 0. \tag{5}$$

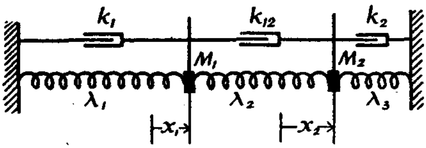


FIG. 14.

As before, we seek solutions of (4) and (5) of the form

$$x_1 = X_1 e^{\alpha t}, \qquad x_2 = X_2 e^{\alpha t}, \tag{6}$$

where X_1 and X_2 are independent of t. Substituting from (6) in (4) and (5) gives the equations for X_1 and X_2, namely,

$$\{\alpha^2+2(\kappa_1+\kappa_{12})\alpha+n_1^2+n_{12}^2\}X_1-(2\kappa_{12}\alpha+n_{12}^2)X_2 = 0, \tag{7}$$

$$-(2\kappa_{21}\alpha+n_2^2)X_1+\{\alpha^2+2(\kappa_2+\kappa_{21})\alpha+n_2^2+n_{23}^2\}X_2 = 0. \tag{8}$$

To simplify the algebra we now restrict ourselves to the special case considered in detail in § 33, namely, $M_1 = M_2$, $\lambda_1 = \lambda_2 = \lambda_3$, so that $n_2^2 = n_{12}^2 = n_{23}^2 = n_1^2$, and in addition we take $k_{12} = 0$ so that $\kappa_{12} = \kappa_{21} = 0$, and $k_1 = k_2$ so that $\kappa_2 = \kappa_1$. The equations (7) and (8) then become

$$(\alpha^2+2\kappa_1\alpha+2n_1^2)X_1-n_1^2X_2 = 0, \tag{9}$$

$$-n_1^2X_1+(\alpha^2+2\kappa_1\alpha+2n_1^2)X_2 = 0. \tag{10}$$

These require

$$\frac{X_1}{X_2} = \frac{\alpha^2+2\kappa_1\alpha+2n_1^2}{n_1^2} = \frac{n_1^2}{\alpha^2+2\kappa_1\alpha+2n_1^2}. \tag{11}$$

The second of equations (11) gives the frequency equation

$$(\alpha^2+2\kappa_1\alpha+2n_1^2)^2-n_1^4 = 0,$$

or

$$(\alpha^2+2\kappa_1\alpha+n_1^2)(\alpha^2+2\kappa_1\alpha+3n_1^2) = 0. \tag{12}$$

The roots of this are

$$\alpha = -\kappa_1 \pm i\sqrt{(n_1^2-\kappa_1^2)}, \tag{13}$$

and

$$\alpha = -\kappa_1 \pm i\sqrt{(3n_1^2-\kappa_1^2)}, \tag{14}$$

provided that $n_1 > \kappa_1$.

The roots (13) correspond to an oscillation with time factor

$$e^{-\kappa_1 t \pm it\sqrt{(n_1^2-\kappa_1^2)}}$$

and in which, using (13) in the first of equations (11),

$$\frac{X_1}{X_2} = 1.$$

This gives the normal mode

$$\left.\begin{aligned} x_1 &= A_1 e^{-\kappa_1 t}\sin\{t\sqrt{(n_1^2-\kappa_1^2)}+\theta_1\} \\ x_2 &= A_1 e^{-\kappa_1 t}\sin\{t\sqrt{(n_1^2-\kappa_1^2)}+\theta_1\} \end{aligned}\right\}, \tag{15}$$

where A_1 and θ_1 are arbitrary constants. This differs only from the corresponding result § 33 (14) for the undamped case by the occurrence of the damping factor $e^{-\kappa_1 t}$ and the change in the natural frequency from $n_1/2\pi$ to $(n_1^2-\kappa_1^2)^{\frac{1}{2}}/2\pi$.

In the same way, using (14) in the first of equations (11) gives $X_1/X_2 = -1$, and (14) leads to the normal mode

$$\left.\begin{aligned} x_1 &= A_2 e^{-\kappa_1 t}\sin\{t\sqrt{(3n_1^2-\kappa_1^2)}+\theta_2\} \\ x_2 &= A_2 e^{-\kappa_1 t}\sin\{t\sqrt{(3n_1^2-\kappa_1^2)}+\theta_2-\pi\} \end{aligned}\right\}; \tag{16}$$

cf. § 33 (16). It should be remarked that the fact that the ratios X_1/X_2 are the same as those of the undamped case of § 33 is accidental: usually they become complex.

Considering now forced oscillations, suppose that, as in § 33, a force $F_0 \sin \omega t$ is applied to the first of the masses. We require a particular integral of the equations

$$(D^2+2\kappa_1 D+2n_1^2)x_1-n_1^2 x_2 = (F_0/M_1)e^{i\omega t}, \tag{17}$$

$$-n_1^2 x_1+(D^2+2\kappa_1 D+2n_1^2)x_2 = 0, \tag{18}$$

and we assume this to be of the form

$$x_1 = X_1 e^{i\omega t}, \qquad x_2 = X_2 e^{i\omega t}. \tag{19}$$

Substituting (19) in (17) and (18) gives

$$(2n_1^2-\omega^2+2\kappa_1 i\omega)X_1-n_1^2 X_2 = F_0/M_1,$$

$$-n_1^2 X_1+(2n_1^2-\omega^2+2\kappa_1 i\omega)X_2 = 0.$$

Solving for X_1 gives

$$X_1 = \frac{F_0(2n_1^2-\omega^2+2\kappa_1 i\omega)}{M_1(3n_1^2-\omega^2+2\kappa_1 i\omega)(n_1^2-\omega^2+2\kappa_1 i\omega)} = A_1 e^{-i\phi}, \quad (20)$$

where

$$A_1 = \frac{F_0}{M_1}\left\{\frac{(2n_1^2-\omega^2)^2+4\kappa_1^2\omega^2}{[(3n_1^2-\omega^2)^2+4\kappa_1^2\omega^2][(n_1^2-\omega^2)^2+4\kappa_1^2\omega^2]}\right\}^{\frac{1}{2}}, \quad (21)$$

$$\phi = \arg(3n_1^2-\omega^2+2\kappa_1 i\omega)+\arg(n_1^2-\omega^2+2\kappa_1 i\omega)-$$
$$-\arg(2n_1^2-\omega^2+2\kappa_1 i\omega). \quad (22)$$

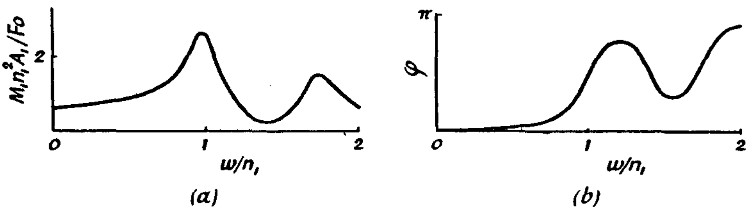


FIG. 15.

Using (20) in (19) and taking the imaginary part gives for the forced oscillation of x_1

$$x_1 = A_1 \sin(\omega t-\phi), \quad (23)$$

a vibration of frequency $\omega/2\pi$, amplitude A_1, and phase lag ϕ behind the applied force.

We now have to discuss the variation of A_1 and ϕ with ω as was done in § 31, and to compare the results with those of § 33.

The discussion of ϕ is easy: it consists of three terms of type § 31 (4), Fig. 7 (*a*), and is shown in Fig. 15 (*b*) for the case $\kappa_1 = n_1/10$.

The amplitude A_1 contains the product of two terms of type § 31 (14) and thus may be expected to have two maxima near the values n_1 and $n_1\sqrt{3}$ of ω. The curve is too complicated for general discussion: its graph for the case $\kappa_1 = n_1/10$ is shown in Fig. 15 (*a*).

The correspondence between the case $\kappa = 0$ of Fig. 13 (*c*) and (*e*) with the case of small damping, Fig. 15 (*a*) and (*b*), shows clearly on comparing these figures.

35. Systems of several masses: variation of the natural frequencies with the number of masses

In § 33 the natural frequencies of two equal masses constrained by three equal springs as in Fig. 11 have been studied: in this section we consider the effect of adding more masses connected

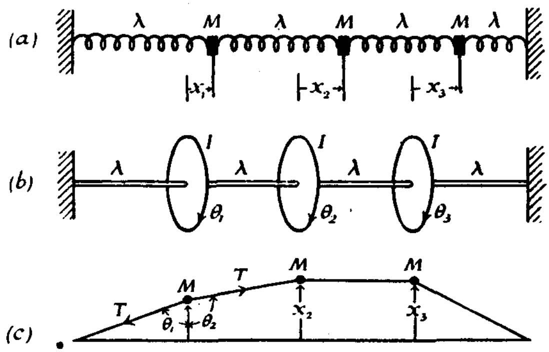


FIG. 16.

in the same way. Resistance to motion will not be included—if it is added, a change corresponding to that from the results of § 33 to those of § 34 appears.

The system to be considered here is that of Fig. 16 (a), longitudinal vibrations of three equal masses M constrained by equal springs of stiffness λ. The corresponding system for torsional oscillation of a shaft is shown in Fig. 16 (b). In order to introduce another interesting system, and because the normal oscillations can be represented more picturesquely for it, we shall state the problem in terms of the system of Fig. 16 (c), *small transverse oscillations of three equal masses attached at distances l, $2l$, $3l$ along a light elastic string of length $4l$ stretched to tension T.*

Let x_1, x_2, x_3 be the displacements of the masses; these are assumed to be so small that they do not affect the tension of the string. The restoring force on the first mass whose displacement is x_1 is

$$T\cos\theta_1 + T\cos\theta_2 = T\frac{x_1}{l} - T\frac{x_2 - x_1}{l}, \tag{1}$$

approximately. Those on the second and third masses can be written down in the same way.

Thus the equations of motion of the three masses are

$$M\ddot{x}_1 = -\frac{Tx_1}{l}+\frac{T(x_2-x_1)}{l},$$

$$M\ddot{x}_2 = -\frac{T(x_2-x_1)}{l}+\frac{T(x_3-x_2)}{l},$$

$$M\ddot{x}_3 = -\frac{T(x_3-x_2)}{l}-\frac{Tx_3}{l}.$$

Writing $n^2 = T/Ml$ these become

$$(D^2+2n^2)x_1-n^2x_2 = 0, \tag{2}$$

$$-n^2x_1+(D^2+2n^2)x_2-n^2x_3 = 0, \tag{3}$$

$$-n^2x_2+(D^2+2n^2)x_3 = 0. \tag{4}$$

These are identical with the equations which would be written down as in § 33 for the system of Fig. 16 (*a*) with $n^2 = \lambda/M$, or for the system of Fig. 16 (*b*) with $n^2 = \lambda/I$. In the problem of Fig. 16 (*c*) the linear equations (2) to (4) are only an approximation for small oscillations—the restoring forces in fact contain in addition terms involving the squares and higher powers of the displacements which we neglect. The equations of the problems of Fig. 16 (*a*) and (*b*) are accurately linear if we assume the springs and shafts to be perfectly elastic.

To solve (2) to (4) we assume a solution

$$x_1 = X_1 e^{\alpha t}, \qquad x_2 = X_2 e^{\alpha t}, \qquad x_3 = X_3 e^{\alpha t}, \tag{5}$$

and substitute in (2) to (4) which give

$$(\alpha^2+2n^2)X_1-n^2X_2 = 0, \tag{6}$$

$$-n^2X_1+(\alpha^2+2n^2)X_2-n^2X_3 = 0, \tag{7}$$

$$-n^2X_2+(\alpha^2+2n^2)X_3 = 0. \tag{8}$$

(6)–(8) are three homogeneous linear equations for X_1, X_2, X_3, and, in order that they may have a solution, α must be a root of

$$\begin{vmatrix} \alpha^2+2n^2 & -n^2 & 0 \\ -n^2 & \alpha^2+2n^2 & -n^2 \\ 0 & -n^2 & \alpha^2+2n^2 \end{vmatrix} = 0, \tag{9}$$

which is the frequency equation. Expanding the determinant this becomes

$$(\alpha^2+2n^2)\{(\alpha^2+2n^2)^2-2n^4\} = 0. \tag{10}$$

The roots of (10) are

$$\alpha^2 = -2n^2, \quad \alpha^2 = -(2+\sqrt{2})n^2, \quad \alpha^2 = -(2-\sqrt{2})n^2, \tag{11}$$

corresponding to the natural frequencies of

$$[n\sqrt{(2-\sqrt{2})}]/2\pi, \quad [n\sqrt{2}]/2\pi, \quad [n\sqrt{(2+\sqrt{2})}]/2\pi. \tag{12}$$

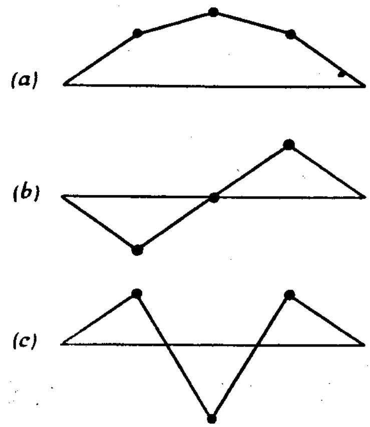


FIG. 17.

To find the normal mode of oscillation corresponding to each natural frequency we insert the value of α in any two of (6)–(8) and solve for the ratio $X_1:X_2:X_3$.

Thus if

$$\alpha^2 = -(2-\sqrt{2})n^2, \qquad X_1:X_2:X_3 = 1:\sqrt{2}:1, \tag{13}$$

if
$$\alpha^2 = -2n^2, \qquad X_1:X_2:X_3 = 1:0:-1, \tag{14}$$

if
$$\alpha^2 = -(2+\sqrt{2})n^2, \qquad X_1:X_2:X_3 = 1:-\sqrt{2}:1. \tag{15}$$

The displacements of the particles in the three normal modes are shown in Fig. 17: (*a*), corresponding to (13), is the lowest mode; while (*c*), corresponding to (15), is the highest.

The method of solving (6)–(8) used above is the general one applicable to any number of equations; the simpler method used in §§ 33, 34 for the case of two equations is equivalent to it.

If a fourth mass is added in the chain of Fig. 16, the frequency equation corresponding to (9) becomes

$$\begin{vmatrix} \alpha^2+2n^2 & -n^2 & 0 & 0 \\ -n^2 & \alpha^2+2n^2 & -n^2 & 0 \\ 0 & -n^2 & \alpha^2+2n^2 & -n^2 \\ 0 & 0 & -n^2 & \alpha^2+2n^2 \end{vmatrix} = 0, \tag{16}$$

which has roots $\alpha^2 = -0{\cdot}382n^2, \alpha^2 = -1{\cdot}382n^2, \alpha^2 = -2{\cdot}618n^2$, $\alpha^2 = -3{\cdot}618n^2$, corresponding to natural frequencies

$$0{\cdot}618n/2\pi, \quad 1{\cdot}176n/2\pi, \quad 1{\cdot}618n/2\pi, \quad 1{\cdot}902n/2\pi. \tag{17}$$

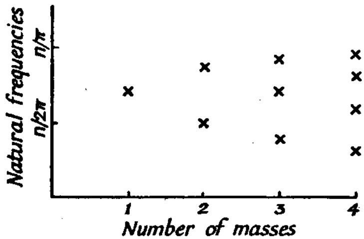


FIG. 18.

The natural frequencies for the same system with two masses have been found in § 33 to be $n/2\pi$ and $(n\sqrt{3})/2\pi$, while the natural frequency for the system with one mass only is the same as that for the mass attached to a spring of twice the stiffness, namely, $(n\sqrt{2})/2\pi$.

These results are shown by crosses in Fig. 18: it appears that as more masses are added to the system the highest natural frequency increases and tends to n/π, and the lowest decreases and tends to zero. The individual natural frequencies steadily approach closer together. The general result for any number of masses is given in Ex. 15 at the end of the chapter.

36. Systems of several masses, variation of the natural frequencies with the masses: vibration dampers

Although the general frequency equation for any values of the masses and stiffnesses of the springs was given in § 33, only the simple case of equal masses and stiffnesses was studied in order to have simple numerical results for discussion. The question of how the natural frequencies vary when the masses are varied naturally arises, and in this section we discuss another system from this point of view.

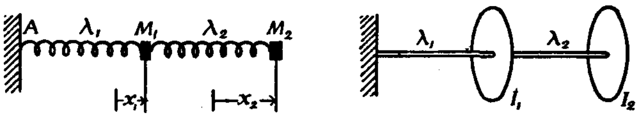


FIG. 19.

The problem considered here is that of longitudinal vibrations of two masses M_1 and M_2, connected by a spring of stiffness λ_2; M_1 is attached to a fixed point by a spring of stiffness λ_1. As before, resistance to motion is neglected: it may be included as in § 34.

Let x_1 and x_2 be the displacements of the masses from their equilibrium positions. Then, if there are no external forces, the equations of motion are

$$M_1 \ddot{x}_1 = -\lambda_1 x_1 + \lambda_2(x_2 - x_1), \tag{1}$$

$$M_2 \ddot{x}_2 = -\lambda_2(x_2 - x_1). \tag{2}$$

Writing

$$n_1^2 = \frac{\lambda_1}{M_1}, \qquad n_2^2 = \frac{\lambda_2}{M_2}, \qquad n_{12}^2 = \frac{\lambda_2}{M_1}, \tag{3}$$

these become

$$(D^2 + n_1^2 + n_{12}^2)x_1 - n_{12}^2 x_2 = 0, \tag{4}$$

$$-n_2^2 x_1 + (D^2 + n_2^2)x_2 = 0. \tag{5}$$

As usual, we seek solutions

$$x_1 = X_1 e^{\alpha t}, \qquad x_2 = X_2 e^{\alpha t}. \tag{6}$$

Substituting in (4) and (5) gives

$$(\alpha^2 + n_1^2 + n_{12}^2)X_1 - n_{12}^2 X_2 = 0,$$

$$-n_2^2 X_1 + (\alpha^2 + n_2^2)X_2 = 0.$$

That is

$$\frac{X_1}{X_2} = \frac{n_{12}^2}{\alpha^2 + n_1^2 + n_{12}^2} = \frac{\alpha^2 + n_2^2}{n_2^2}. \tag{7}$$

The second of equations (7) is the frequency equation

$$\alpha^4 + \alpha^2(n_1^2 + n_2^2 + n_{12}^2) + n_1^2 n_2^2 = 0. \tag{8}$$

We propose to discuss the way in which the roots of this vary with M_2, and, in order to have only one variable parameter we suppose that the stiffnesses of the springs are equal. Writing $M_2 = M_1/k$ we have

$$n_2^2 = kn_1^2, \qquad n_{12}^2 = n_1^2, \tag{9}$$

and (8) becomes

$$\left(\frac{\alpha}{n_1}\right)^4 + \left(\frac{\alpha}{n_1}\right)^2(2+k)+k = 0. \tag{10}$$

The roots of this are

$$\left(\frac{\alpha}{n_1}\right)^2 = -(1+\tfrac{1}{2}k)\pm(1+\tfrac{1}{4}k^2)^{\frac{1}{2}}. \tag{11}$$

Corresponding to each of these roots the first of equations (7) gives the ratio X_1/X_2.

If $k = 1$, *both masses equal*, the roots are

$$\alpha = \pm 0{\cdot}618in_1, \quad \text{with } X_1/X_2 = 0{\cdot}618,$$

and $$\alpha = \pm 1{\cdot}618in_1, \quad \text{with } X_1/X_2 = -1{\cdot}618.$$

If $k = 2$, *the second mass half the first*, the roots are

$$\alpha = \pm 0{\cdot}766in_1, \quad \text{with } X_1/X_2 = 0{\cdot}707,$$

and $$\alpha = \pm 1{\cdot}848in_1, \quad \text{with } X_1/X_2 = -0{\cdot}707.$$

If $k = 0{\cdot}5$, *the second mass double the first*, the roots are

$$\alpha = \pm 0{\cdot}468in_1, \quad \text{with } X_1/X_2 = 0{\cdot}562,$$

and $$\alpha = \pm 1{\cdot}510in_1, \quad \text{with } X_1/X_2 = -3{\cdot}562.$$

As $k \to 0$, *i.e.* $M_2 \to \infty$, the roots are, neglecting k^2,

$$\alpha = \pm in_1(1+\tfrac{1}{8}k)\surd 2, \quad \text{with } X_1/X_2 = -2/k, \tag{12}$$

and $$\alpha = \pm in_1\surd(\tfrac{1}{2}k), \quad \text{with } X_1/X_2 = \tfrac{1}{2}(1+\tfrac{1}{4}k). \tag{13}$$

In the limiting case $M_2 \to \infty$ its position becomes fixed and the system becomes that of a single mass attached to two fixed points by equal springs. The motion in this case has a single natural frequency given by (12). (13) shows that the second natural frequency of the system of two masses tends to zero as $M_2 \to \infty$.

As $k \to \infty$, *i.e.* $M_2 \to 0$, writing (11) in the form

$$\left(\frac{\alpha}{n_1}\right)^2 = -\frac{k}{2}\left\{1+\frac{2}{k}\mp\left(1+\frac{2}{k^2}+\ldots\right)\right\}, \tag{14}$$

it appears that, neglecting terms in $1/k^2$, (14) has roots

$$\alpha = \pm in_1\left(1-\frac{1}{2k}\right), \quad \text{with } X_1/X_2 = 1-1/k, \tag{15}$$

and
$$\alpha = \pm in_1\left(1+\frac{1}{2k}\right)\sqrt{k}, \quad \text{with } X_1/X_2 = -1/k. \tag{16}$$

In the limiting case $M_2 = 0$ the second mass is not present and the system has the single natural frequency (15): the second natural frequency, given by (16), tends to infinity as $M_2 \to 0$.

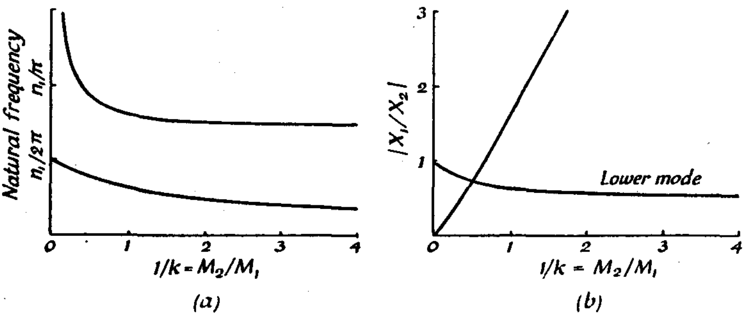


FIG. 20.

These results are plotted in Fig. 20. The variation of the natural frequencies with the values of the second mass is shown in Fig. 20 (*a*): the effect of a small second mass is to introduce a higher second natural frequency; as the second mass is increased both natural frequencies decrease. The ratio of the amplitudes of the first and second masses is shown in Fig. 20 (*b*). For the vibration of lower frequency this decreases steadily from 1 to $\frac{1}{2}$ as the second mass is increased: for the higher mode it increases steadily.

Results of this type have important practical applications: most practical mechanical systems inevitably have natural frequencies; also they are usually designed to operate over a fixed range of frequency. If any of the natural frequencies fall in the range of operating frequencies resonance will occur and vibrations of large amplitude will be set up. It appears from Fig. 20 that the natural frequencies of a system may be altered by the addition of extra components: for example the addition of a suitable second mass M_2 to a single mass will remove the natural frequencies of the system from the range between $n_1/2\pi$ and $n_1/4\pi$. The main object of so-called 'vibration dampers' is to remove all natural frequencies of a system from a specified range in this way; a secondary object is to damp some of the vibrations. A simple example, related to the system of Fig. 19, is that of Fig. 21.

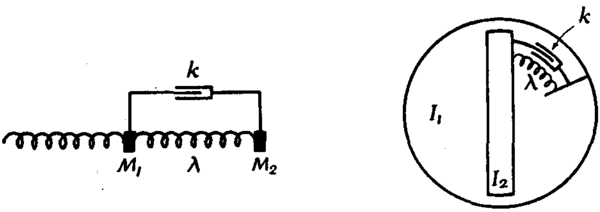


FIG. 21.

The mass M_1 is a portion of a mechanical system and it is required to damp its vibration; a second, suitably chosen, mass M_2 shifts the natural frequencies to a safe region, and a dash-pot between M_1 and M_2 giving resistance to motion proportional to relative velocity damps the vibration. The corresponding rotation system for two wheels I_1 and I_2 on the same shaft is widely used.

37. Geared systems

If a number of shafts carrying wheels are connected by gears, or a number of masses by levers, a small additional complication occurs. We discuss it here for the rather more interesting case of gearing.

Suppose a gear of moment of inertia I_1 and radius a is in mesh with a gear of moment of inertia I_2 and radius b, [Fig. 22 (a)]. I_1 is connected to a wheel of moment of inertia I by a shaft of stiffness λ_1, and I_2 to a wheel of moment of inertia I_3 by a shaft of stiffness λ_2. Let θ and θ_1 be the angular displacements of the wheels I and I_1 from fixed reference positions, and θ_2 and θ_3 those of the wheels I_2 and I_3, as shown in Fig. 22 (a). It is convenient to measure θ and θ_1 in the same direction, opposite to that of θ_2 and θ_3, since the wheels I_1 and I_2 rotate in opposite directions.

The relationship imposed by the gears can be seen from Fig. 22 (b). Firstly, since the circumferences of the two wheels must travel the same distance

$$a\theta_1 = b\theta_2. \tag{1}$$

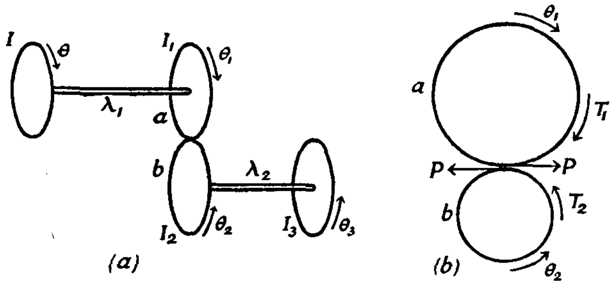


FIG. 22.

Secondly, if T_1 and T_2 are the torques on the two gear-wheels, measured in the directions of θ_1 and θ_2 increasing, respectively, and if P is the reaction at the point of contact of the wheels,

$$T_2 = Pb, \qquad T_1 = -Pa,$$

and therefore
$$aT_2+bT_1 = 0. \tag{2}$$

The equations of motion of the wheels are

$$I\ddot{\theta} = \lambda_1(\theta_1-\theta), \tag{3}$$
$$I_1\ddot{\theta}_1 = -\lambda_1(\theta_1-\theta)+T_1, \tag{4}$$
$$I_2\ddot{\theta}_2 = T_2+\lambda_2(\theta_3-\theta_2), \tag{5}$$
$$I_3\ddot{\theta}_3 = -\lambda_2(\theta_3-\theta_2). \tag{6}$$

Eliminating T_1 and T_2 from (4) and (5) by (2), and expressing θ_2 in terms of θ_1 by (1), we obtain the following three equations for θ, θ_1, and θ_3:

$$(D^2+n^2)\theta-n^2\theta_1 = 0, \tag{7}$$
$$-n_{12}^2\theta+\{(r^2+\rho)D^2+n_{12}^2+n_2^2r^2\}\theta_1-rn_2^2\theta_3 = 0, \tag{8}$$
$$-rn_3^2\theta_1+(D^2+n_3^2)\theta_3 = 0, \tag{9}$$

where

$$r = \frac{a}{b}, \quad \rho = \frac{I_1}{I_2}, \quad n^2 = \frac{\lambda_1}{I}, \quad n_{12}^2 = \frac{\lambda_1}{I_2}, \quad n_2^2 = \frac{\lambda_2}{I_2}, \quad n_3^2 = \frac{\lambda_2}{I_3}. \tag{10}$$

If as usual we seek a solution of (7)–(9) of the form

$$\theta_1 = \Theta_1 e^{\alpha t}, \qquad \theta_2 = \Theta_2 e^{\alpha t}, \qquad \theta_3 = \Theta_3 e^{\alpha t}, \tag{11}$$

the frequency equation for α is

$$\begin{vmatrix} \alpha^2+n^2 & -n^2 & 0 \\ -n_{12}^2 & (r^2+\rho)\alpha^2+n_{12}^2+n_2^2 r^2 & -rn_2^2 \\ 0 & -rn_3^2 & \alpha^2+n_3^2 \end{vmatrix} = 0. \tag{12}$$

On expanding, (12) becomes

$$\alpha^2\{(r^2+\rho)\alpha^4+[(r^2+\rho)(n^2+n_3^2)+n_{12}^2+r^2n_2^2]\alpha^2+$$
$$+[(r^2+\rho)n^2n_3^2+r^2n^2n_2^2+n_3^2n_{12}^2]\} = 0. \tag{13}$$

There is thus a zero root in α^2 as well as a pair of roots giving rise to a pair of natural frequencies with their associated normal modes. A factor α^2 in the auxiliary equation gives rise to a term of type

$$A+Bt$$

in the solution, corresponding to steady rotation of the system.

38. Mechanical models illustrating the rheological behaviour of common substances

In § 28 we described three simple elements, namely perfectly elastic solid, perfectly viscous liquid, and coulomb friction, from which idealized mechanical systems such as those of §§ 29–37 could be built up.

The same elements may be combined to give a useful approximation to the behaviour of many common substances ranging from metals and rubber to flour dough. All substances exhibit to a greater or less extent phenomena such as creep, elastic hysteresis, plastic flow, etc. We give below a number of typical examples.

(i) *Rubber-like substances*

These may be represented by the combination of a spring giving restoring force λ times the displacement and a dash-pot giving resistance k times the velocity. λ and k are regarded as constants of the rubber itself. The extension of a piece of rubber when constant stress S_0 is applied to it is thus given by the equation

$$k\dot{x}+\lambda x = S_0, \tag{1}$$

or

$$\left(D+\frac{\lambda}{k}\right)x = \frac{S_0}{k}. \tag{2}$$

If the stress S_0 is applied at $t = 0$ when $x = 0$, the solution of (2) is

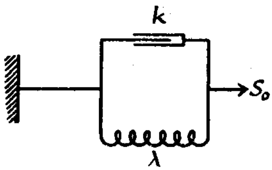


FIG. 23.

$$x = \frac{S_0}{\lambda}(1-e^{-\lambda t/k}). \tag{3}$$

That is, the substance does not take up the deflexion S_0/λ instantaneously as a perfectly elastic solid does, but moves out to it according to the law (3).

If a mass M is attached to a piece of rubber its equation of motion will be that of § 29: by measurements of two quantities such as the natural frequency and damping coefficient of the motion the constants λ and k of the rubber may be determined.

Next we illustrate the phenomenon of elastic after effect which occurs for such substances. Instead of applying a constant stress suddenly we apply a cyclically varying stress: the simplest case, which we consider here, is that of a stress S which increases linearly for time T and then decreases again to zero in time T: this is given by

$$\left.\begin{aligned} S &= S_0 t, && 0 < t < T \\ S &= S_0(2T-t), && T < t < 2T \\ S &= 0, && t > 2T \end{aligned}\right\}, \tag{4}$$

where S_0 is a constant.

For $0 < t < T$, the differential equation is

$$\left(D+\frac{\lambda}{k}\right)x = \frac{S_0}{k}t, \tag{5}$$

with $x = 0$, when $t = 0$.

The solution of this is

$$x = \frac{S_0 t}{\lambda}-\frac{S_0 k}{\lambda^2}(1-e^{-\lambda t/k}). \tag{6}$$

When $t = T$, this gives

$$x = \frac{S_0 T}{\lambda} - \frac{S_0 k}{\lambda^2}(1-e^{-\lambda T/k}). \tag{7}$$

When $T < t < 2T$, the differential equation is

$$\left(D+\frac{\lambda}{k}\right)x = \frac{S_0}{k}(2T-t), \tag{8}$$

to be solved with the value (7) of x when $t = T$.

The solution is

$$x = \frac{S_0}{\lambda}\left(2T-t+\frac{k}{\lambda}\right)+\frac{S_0 k}{\lambda^2}(1-2e^{\lambda T/k})e^{-\lambda t/k}. \tag{9}$$

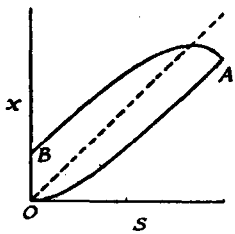


FIG. 24.

When $t = 2T$, so that the stress has returned to zero, the strain x is

$$x = \frac{kS_0}{\lambda^2}(1-e^{\lambda T/k})^2 e^{-2\lambda T/k}. \tag{10}$$

Finally, when $t > 2T$, the differential equation is

$$\left(D+\frac{\lambda}{k}\right)x = 0, \tag{11}$$

with initial value (10) of x. This has solution

$$x = \frac{kS_0}{\lambda^2}(1-e^{\lambda T/k})^2 e^{-\lambda t/k}. \tag{12}$$

If we plot the displacement x against the stress S as in Fig. 24 we get the curve OAB, OA being the portion $0 < t < T$, and AB that for $T < t < 2T$. OB is the residual deflexion when the stress is zero.

(ii) *Substances exhibiting plastic flow*

To include the phenomena of plastic flow an element providing constant friction F has to be introduced.

The simplest system is that of Fig. 25 (a) which corresponds roughly to the behaviour of metals. This is elastic until the stress S is equal to F (the yield point) which is the greatest stress the substance can resist.

The system of Fig. 25 (b) (a Bingham solid) is again elastic until S reaches F: for values of $S > F$ it flows at a rate which increases with S. If x is the displacement in this system

$$x = \frac{S}{\lambda} \qquad (S < F), \tag{13}$$

$$x = \frac{F}{\lambda}+\frac{1}{k}(S-F)t \quad (S > F). \tag{14}$$

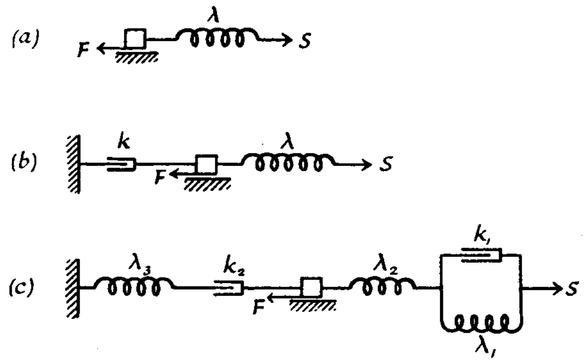


FIG. 25.

Finally Fig. 25 (c) is a model which has been proposed for flour dough. The displacement x is given by

$$x = \frac{S}{\lambda_1}(1-e^{-\lambda_1 t/k_1})+\frac{S}{\lambda_2} \qquad (S < F), \tag{15}$$

$$x = \frac{S}{\lambda_1}(1-e^{-\lambda_1 t/k_1})+\frac{S}{\lambda_2}+\frac{(S-F)t}{k_2}+\frac{S-F}{\lambda_3} \quad (S > F). \tag{16}$$

39. Impulsive motion

We shall speak of an 'impulsive force' or 'a blow of impulse P' meaning a very great force applied for a time τ so short that the motion of its point of application during this time is negligible. while the force is so large that its integral over this time is finite —this will be called the impulse P of the blow.

Thus if $F(t)$ is an impulsive force applied at $t = 0$, we have

$$\int\limits_0^\tau F(t)\,dt = P. \tag{1}$$

Suppose now that a particle of mass m is struck by a blow of impulse P at $t = 0$. The equation of motion is

$$m\ddot{x} = F(t). \tag{2}$$

Integrating (2) over the small time τ during which the blow operates, we find

$$m[\dot{x}]_0^\tau = \int\limits_0^\tau F(t)\,dt = P,$$

or
$$mu_f - mu_i = P, \tag{3}$$

where u_i and u_f are the velocities of the particle before and after the blow. Thus the change in momentum of the particle in the direction of the blow is equal to the impulse of the blow.

If the particle is attached to a spring with viscous damping [§ 29 (1)] the same result holds: (2) is replaced by

$$m\ddot{x} + k\dot{x} + \lambda x = F(t).$$

Integrating as before gives

$$m[\dot{x}]_0^\tau + k[x]_0^\tau + \lambda\int\limits_0^\tau x\,dt = P. \tag{4}$$

Now we have postulated above that τ shall be negligibly small, and that the change of x in this time shall be negligible, so the second and third terms of the left-hand side of (4) are negligible and we regain (3).

Thus setting a particle of mass m in motion by a blow of impulse P is equivalent to giving it an initial velocity P/m.

The same result applies to systems of masses such as those considered in this chapter; since the motion of any particle is negligible during the blow, it can exert no influence on its neighbours. If the particles are in contact or rigidly connected this is not the case.

Using the δ-function notation, a blow of impulse P at $t = 0$ may be regarded as a force $P\,\delta(t)$, and the above results obtained.

Ex. *Motion of a damped harmonic oscillator maintained by impulses.*

We consider the oscillator of § 29, Fig. 4, with no applied forces. Choosing the origin of time so that $x = 0$ when $t = 0$, the solution § 29 (7)

$$x = Ae^{-\kappa t}\sin n't \tag{5}$$

represents an oscillation which dies away as discussed in § 29.

Now suppose that each time the mass passes through the position $x = 0$, that is, when

$$t = r\pi/n' \quad (r = 1, 2,...), \tag{6}$$

the particle is given a blow P just sufficient to raise the magnitude of its velocity to the value it had when $t = 0$. This is roughly the action of the escapement of a clock. If this is done, all half-swings of the mass will be the same, and the motion is maintained indefinitely. To calculate P we have from (5)

$$\dot{x} = Ae^{-\kappa t}\{n'\cos n't - \kappa \sin n't\}, \tag{7}$$

and so

$$\dot{x} = n'A, \quad \text{when } t = 0,$$

$$\dot{x} = -n'Ae^{-\kappa\pi/n'}, \quad \text{when } t = \pi/n'.$$

Thus the magnitude of the velocity when $t = \pi/n'$ is less than that when $t = 0$ by an amount

$$n'A(1-e^{-\kappa\pi/n'}), \tag{8}$$

and a blow of impulse

$$P = mn'A(1-e^{-\kappa\pi/n'}) \tag{9}$$

is required.

If we regard P as a given quantity, (9) gives A, and the motion is

$$x = \frac{P}{mn'(1-e^{-\kappa\pi/n'})} e^{-\kappa t} \sin n't, \tag{10}$$

for $0 < t < \pi/n'$, all subsequent half-swings being the same.

40. Initial value problems: use of the Laplace transformation

In the preceding sections the general solutions of a number of problems have been found: in initial value problems sufficient information is always given in the initial conditions to find the arbitrary constants in these solutions. In this section we shall solve some typical initial value problems by the Laplace transformation method—it is for such problems that the method was designed and its advantage over the older methods increases as the complexity of the problem increases.

Ex. 1. *Force $F_0 \sin \omega t$ is applied at $t = 0$ to the damped harmonic oscillator of §§ 29, 31, the initial displacement and velocity of the mass being zero.*

We have to solve

$$m(D^2+2\kappa D+n^2)x = F_0 \sin \omega t \tag{1}$$

with $x = Dx = 0$, when $t = 0$.

The subsidiary equation, § 18 (26), is

$$m(p^2+2\kappa p+n^2)\bar{x} = \frac{F_0 \omega}{p^2+\omega^2}.$$

Thus, considering the case $n^2 > \kappa^2$, and writing $n'^2 = n^2-\kappa^2$,

$$m\bar{x} = \frac{F_0\omega}{(p+\kappa-in')(p+\kappa+in')(p-i\omega)(p+i\omega)}$$

$$= \left\{\frac{F_0\omega}{2in'(p+\kappa-in')\{\omega^2+(\kappa-in')^2\}}+\text{conjugate}\right\}+$$

$$+\left\{\frac{F_0\omega}{2i\omega(p-i\omega)(n^2-\omega^2+2\kappa i\omega)}+\text{conjugate}\right\}, \quad (2)$$

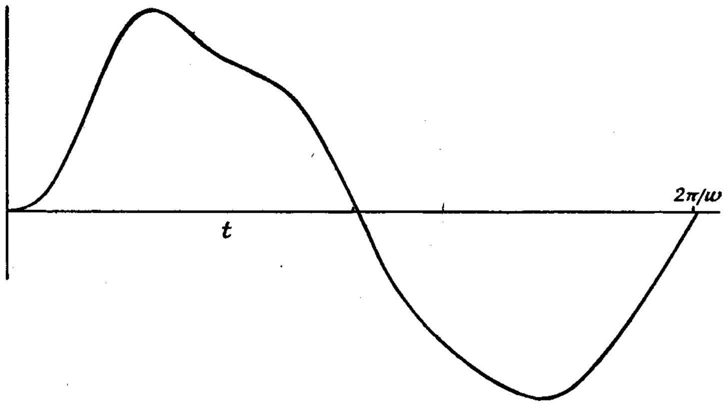


FIG. 26.

on putting $\bar{x}$ in partial fractions by § 18 (29). Writing 'conjugate' implies a similar term except that that the sign of i is changed. From (2) and § 18 (11)

$$mx = \left\{\frac{F_0\omega}{2in'(\omega^2+\kappa^2-n'^2-2in'\kappa)}e^{-\kappa t+in't}+\text{conjugate}\right\}+$$

$$+\left\{\frac{F_0\omega}{2i\omega(n^2-\omega^2+2\kappa i\omega)}e^{i\omega t}+\text{conjugate}\right\}$$

$$= \frac{F_0\omega}{n'Z_1}e^{-\kappa t}\sin(n't-\theta)+\frac{F_0}{\{(n^2-\omega^2)^2+4\kappa^2\omega^2\}^{\frac{1}{2}}}\sin(\omega t-\phi), \quad (3)$$

where $$\phi = \arg(n^2-\omega^2+2\kappa i\omega), \quad (4)$$

and Z_1 and θ are the modulus and argument of

$$\omega^2+\kappa^2-n'^2-2n'\kappa i. \quad (5)$$

The second term of (3) is the forced oscillation § 31 (5), and the first the transient part of § 31 (6) with the arbitrary constants determined to make $x = \dot{x} = 0$ when $t = 0$.

The function (3) is graphed in Fig. 26 for the case $n = 4\omega$, $\kappa = n/10$ and shows how the starting transient dies out, leaving the forced oscillation.

Ex. 2. *The system of § 33, Fig. 11, with equal masses and springs. The first mass is set in motion with velocity V at $t = 0$ when the masses are at rest in their equilibrium positions.*

We have to solve

$$(D^2+2n^2)x_1-n^2x_2 = 0,$$
$$-n^2x_1+(D^2+2n^2)x_2 = 0,$$

where $n^2 = \lambda/M$, with $x_1 = x_2 = \dot{x}_2 = 0$, $\dot{x}_1 = V$, when $t = 0$.

The subsidiary equations are

$$(p^2+2n^2)\bar{x}_1-n^2\bar{x}_2 = V,$$
$$-n^2\bar{x}_1+(p^2+2n^2)\bar{x}_2 = 0.$$

Solving for $\bar{x}_1$ we get

$$\bar{x}_1 = \frac{(p^2+2n^2)V}{(p^2+3n^2)(p^2+n^2)} = \frac{V}{2(p^2+n^2)}+\frac{V}{2(p^2+3n^2)}.$$

Therefore $$x_1 = \frac{V}{2n}\sin nt+\frac{V}{2n\sqrt{3}}\sin nt\sqrt{3}.$$

EXAMPLES ON CHAPTER IV

1. The displacement x in an undamped harmonic oscillation $x = a\cos nt$ may be written in the form $x = \mathbf{R}(ae^{int})$ and so interpreted as the real part of the complex number $z = ae^{int}$ which describes a circle in the z-plane with constant speed. Show that the velocity $\dot{x}$ may be represented in the same way as $\mathbf{R}(nae^{i(nt+\frac{1}{2}\pi)})$. Show that damped harmonic motion $x = ae^{-\kappa t}\cos n't$ may be treated in the same way, but the complex number z now describes an equiangular spiral.

2. For forced oscillations of the damped harmonic oscillator of § 31 show that the phase lag of the velocity is $\pm\tan^{-1}(s^2-1)^{\frac{1}{2}}$ at the points at which the amplitude of the velocity has $(1/s)$th of its maximum value. In particular it is $\pm\frac{1}{4}\pi$ if $s = \sqrt{2}$.

Show also that the slope of the curve of the phase-lag ϕ against ω is $1/\kappa$ when $\omega = n$.

3. Force $F_0\sin\omega t$ is applied to the damped harmonic oscillator of § 29 with critical damping, $\kappa = n$. Show that the displacement in the forced oscillation is

$$\frac{F_0}{m(n^2+\omega^2)}\sin(\omega t-2\phi),$$

where $\phi = \tan^{-1}(\omega/n)$.

4. In the damped harmonic motion $x = Ae^{-\kappa t}\sin n't$ of § 29 the potential energy V of the system is $\frac{1}{2}\lambda x^2$ and the kinetic energy T is $\frac{1}{2}m\dot{x}^2$. Show that the total energy $T+V$ is given by

$$T+V = \tfrac{1}{2}mA^2e^{-2\kappa t}\{n^2-\kappa^2\cos 2n't-\kappa n'\sin 2n't\},$$

and that the average value of this quantity over the cycle $2r\pi/n'$ to $2(r+1)\pi/n'$ is

$$(mA^2n'^3/8\pi\kappa)\{e^{-4\kappa r\pi/n'}-e^{-4\kappa(r+1)\pi/n'}\}.$$

Find the loss in total energy in the same cycle, and show that if Q is defined as 2π times the ratio of the average total energy over a cycle to the energy loss in the cycle,

$$Q = n'/2\kappa = \pi/\delta,$$

where δ is defined in § 29 (13). This provides a physical interpretation for the important quantity Q mentioned in § 29.

5. A mass m attached to a spring of stiffness mn^2 is set in motion from rest at $t = 0$ in its equilibrium position by a force which is $F_0 \sin \omega t$ for $0 < t < \pi/\omega$, and zero for $t > \pi/\omega$. Show that its displacement x is

$$x = F_0\{\omega \sin nt - n \sin \omega t\}/\{mn(\omega^2 - n^2)\} \qquad (0 < t < \pi/\omega).$$

$$x = \{2\omega F_0/mn(\omega^2 - n^2)\}\sin n(t - \pi/2\omega)\cos(n\pi/2\omega) \quad (t > \pi/\omega).$$

6. A mass m is constrained to move in a straight line and is attached to a fixed point in the line by a spring of stiffness mn^2. Its motion is resisted by coulomb friction μR. The mass can be set in any position by an adjusting screw whose end touches it. At $t = 0$, when the mass is at rest and the spring compressed by an amount x_1, the screw is turned so that its end moves away from the mass with constant velocity v. Show that the end of the screw and the mass will lose contact until a time t given by the smallest root of

$$(x_1 - \mu R/mn^2)(1 - \cos nt) = vt.$$

This implies that if the screw is released with constant velocity a jerky motion always results. Show that if the inertia of the screw is taken into account it is possible to have a smooth motion.

7. A mass hangs at rest in its equilibrium position at the end of a spring whose unstretched length is a and whose stretched length is b. At $t = 0$ the point of support is given a downwards motion $c \sin \omega t$. Show that the length of the spring at time t is

$$b - \frac{cn\omega}{n^2 - \omega^2}\sin nt + \frac{c\omega^2}{n^2 - \omega^2}\sin \omega t,$$

where $n/2\pi$ is the natural frequency of oscillation of the mass, and $n \neq \omega$.

8. In the system of Fig. 19, $M_2 = 2M_1/3$, $\lambda_1 = \lambda_2$; show that the natural frequencies are $(n_1\sqrt{3})/2\pi$ and $n_1/(2\pi\sqrt{2})$, where $n_1^2 = \lambda_1/M_1$, and find the normal modes of oscillation.

9. In the system of Fig. 19 the masses M and the stiffnesses λ of the springs are equal, and in addition a third mass M is connected by an equal spring λ. Writing $n^2 = \lambda/M$, show that the natural frequencies are

$$0{\cdot}445n/2\pi, \quad 1{\cdot}247n/2\pi, \quad 1{\cdot}802n/2\pi.$$

10. Show that if in the system of Fig. 19 the stiffnesses of the springs are equal, the squares of the natural frequencies are rational if

$$M_1 : M_2 = r : s,$$

where r and s are integers such that $r^2 + 4s^2$ is a perfect square. Show that such integers can be found by splitting up any odd perfect square into a sum of integers differing by unity, e.g. $49 = 24+25$ gives $24^2+7^2 = 25^2$, corresponding to $M_1 : M_2 = 7:12$.

11. Three wheels A, B, and C are of moment of inertia I. A and B, and B and C, are connected by shafts of stiffness λ. Show that the natural frequencies of the system are $n/2\pi$ and $(n\surd 3)/2\pi$, where $n^2 = \lambda/I$.

12. If in the system of Fig. 14, $M_1 = M_2$, $\lambda_1 = \lambda_2 = \lambda_3$, $k_1 = k_2 = 0$, show that the normal modes of oscillation are

$$x_1 = x_2 = A_1 \sin(nt+\beta_1),$$
$$x_1 = -x_2 = A_2 e^{-2\kappa t}\sin\{t\surd(3n^2-4\kappa^2)+\beta_2\},$$

where $n^2 = \lambda_1/M_1$, and $2\kappa = k_{12}/M_1$.

13. If in the system of Fig. 14, $M_1 = M_2$, $\lambda_1 = \lambda_2 = \lambda_3$, $k_2 = k_{12} = 0$, show that the frequency equation is

$$\alpha^4+2\kappa\alpha^3+4n^2\alpha^2+4\kappa\alpha n^2+3n^4 = 0,$$

where $n^2 = \lambda_1/M_1$ and $2\kappa = k_1/M_1$. Show that if κ is so small that κ^2 is negligible, the natural frequencies are still $n/2\pi$ and $(n\surd 3)/2\pi$, but the oscillations both have a damping factor $\exp(-\frac{1}{2}\kappa t)$.

14. n particles, each of mass m, are attached at equal distances along a string of length $(n+1)l$ which is stretched to tension T and whose ends are fixed. If the particles execute small transverse oscillations and x_r is the displacement of the rth particle, show that

$$\ddot{x}_r = c^2(x_{r+1}-2x_r+x_{r-1}) \quad (r = 1,\dots, n),$$

with $x_0 = x_{n+1} = 0$, and $c^2 = T/ml$.

15. Show that the system of equations of Ex. 14 is satisfied by

$$x_r = (Ae^{r\beta}+Be^{-r\beta})e^{\alpha t},$$

where β is a root of $\qquad \cosh\beta = 1+(\alpha^2/2c^2)$.

Show that the conditions $x_0 = x_{n+1} = 0$ require $\beta = s\pi i/(n+1)$, and thus that the natural frequencies of the system are

$$c\{2-2\cos s\pi/(n+1)\}^{\frac{1}{2}}/2\pi \quad (s = 1, 2,\dots, n).$$

16. A wheel of moment of inertia I is connected to a gear-box by a shaft of stiffness λ. The gear-box gives a step up of r, and is connected to a wheel of moment of inertia I_1 by a shaft of stiffness λ_1. Show that if the moments of inertia of the gears in the gear-box are negligible, the natural frequency of the system is $\omega/2\pi$, where

$$\omega^2 = \frac{\lambda\lambda_1(I_1 r^2+I)}{II_1(\lambda+r^2\lambda_1)}.$$

17. If the system of Ex. 16 is at rest with the shafts unstrained, and the wheel I is set in motion at $t = 0$ with angular velocity Ω, show that its subsequent angular velocity is

$$\Omega(I+I_1 r^2\cos\omega t)/(I_1 r^2+I),$$

where ω is defined in Ex. 16.

18. The motion of a mass M acted on by a force $F \sin \omega t$ and constrained by supports of rubber in shear is given by

$$(MD^2+KD+S)x = F \sin \omega t,$$

where K and S depend on the nature of the rubber. Find the amplitude of the forced oscillation and show that, neglecting terms in K^2, its maximum value is approximately $F/2\pi K\Omega$, where Ω, the resonance frequency, is approximately $(S/M)^{\frac{1}{2}}/2\pi$.

19. Show that if a varying stress $S(t)$ is applied to a rubber-like substance [cf. § 38 (i)] for $t > 0$ with zero initial strain, the strain x at time t is given by

$$kx = e^{-\lambda t/k} \int\limits_0^t e^{-\lambda\xi/k} S(\xi)\, d\xi.$$

If $S(\xi) = \sin \omega\xi$ show that, writing $\alpha = \lambda/k$, $\phi = \tan^{-1}(k\omega/\lambda)$,

$$x = \frac{\omega}{k(\alpha^2+\omega^2)} e^{-\alpha t} + \frac{1}{k(\alpha^2+\omega^2)^{\frac{1}{2}}} \sin(\omega t - \phi).$$

20. In the system of Fig. 11 the masses and springs are equal. At $t = 0$, when the masses are at rest and the springs unstrained, a point of support is given the motion $a \sin \omega t$, starting in the direction towards the masses. Show that, if ω is not equal to either natural frequency, the displacement of the nearer mass is, writing $n^2 = \lambda_1/M_1$,

$$\frac{na\omega}{2(\omega^2-n^2)} \sin nt + \frac{na\omega}{2(\omega^2-3n^2)\sqrt{3}} \sin nt\sqrt{3} + \frac{an^2(2n^2-\omega^2)}{(n^2-\omega^2)(3n^2-\omega^2)} \sin \omega t.$$

21. A mass $M/3$ is connected by a light string of length l to a mass M which is connected to a fixed point by an equal string. At $t = 0$, when the two masses are hanging vertically and at rest, the mass M is given a small horizontal blow of impulse P; show that its subsequent displacement is

$$P\{3 \sin nt\sqrt{2} + \sqrt{3} \sin nt\sqrt{(2/3)}\}/(4nM\sqrt{2}),$$

where $n^2 = g/l$.

22. In the geared system of Fig. 22 (a), $I = I_1 = I_2 = I_3$, $\lambda_1 = \lambda_2$, and $a = b$. For $t < 0$ the wheels I and I_1 are at rest, the gears are not in mesh, and the wheels I_1 and I_2 are rotating with constant angular velocity ω. At $t = 0$ the gears are forced into mesh. Show that the angular velocity of I_3 at time t is $\frac{1}{2}\omega(1+\cos nt)$, where $n^2 = \lambda_1/I$.

V

ELECTRIC CIRCUIT THEORY

41. Introductory

In this chapter we shall consider the elementary theory of electric circuits with 'lumped' or concentrated properties. This theory is so closely related to that of the mechanical systems of Chapter IV that they can best be studied simultaneously.

We regard electric circuits as being built up of 'elements' of three types: namely, inductance L, resistance R, and capacitance C. The current I at a point of a circuit in a certain direction is the rate at which positive charge passes that point in that direction. In the problems with which we shall be concerned, the current is usually caused by a 'voltage' V applied to two terminals, one of which we select, by convention, as the positive one, so that V is positive when this terminal is at the higher voltage and the current I is positive when flowing away from it.

These circuit elements are illustrated in Fig. 27. The information provided by the theory of electricity which we shall assume is as follows.

(i) The voltage drop across a resistance R is R times the current in it [Fig. 27 (*a*)],

$$RI = V. \tag{1}$$

(ii) The voltage drop across an inductance L is L times the rate of change of current in it [Fig. 27 (*b*)],

$$L\frac{dI}{dt} = V. \tag{2}$$

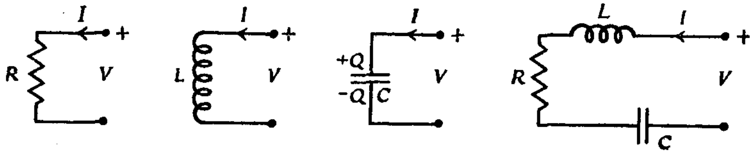
(iii) The voltage drop across a capacitance C is $(1/C)$ times the charge Q on it [Fig. 27 (*c*)],

$$V = \frac{Q}{C}. \tag{3}$$

Also, since the current I is the rate of flow of positive charge, we must have

$$I = \frac{dQ}{dt}, \tag{4}$$

and, with the conventions of sign introduced above, the current is positive when flowing towards the high-voltage side of the capacitance.



(a) $RI = V$ (b) $L\frac{dI}{dt} = V$ (c) $\frac{Q}{C} = V$ (d) $L\frac{dI}{dt}+RI+\frac{Q}{C} = V$

FIG. 27.

The equations (1)–(4) are practically all that is needed for the work of this chapter; they are linear, and it is because of this that it is possible to go so far in electric circuit theory. They are, of course, approximations, but they are very good approximations (except for iron-cored inductances), so results calculated by using them will be very near the truth. The more accurate equations are non-linear and will be discussed in § 57.

More complicated circuits can be regarded as being built up of the simple elements described above, but it is a little shorter to take as the fundamental unit the 'L, R, C circuit' consisting of inductance L, resistance R, and capacitance C in series [Fig. 27 (*d*)]. If voltage V is applied to such a circuit, the sum of the voltage drops over the inductance, resistance, and capacitance, given by (1), (2), and (3), respectively, must be equal to V, that is

$$L\frac{dI}{dt}+RI+\frac{Q}{C} = V. \tag{5}$$

(4) and (5) are the fundamental equations for this circuit; they are a pair of simultaneous ordinary linear differential equations for the two unknowns I and Q in terms of V, which is supposed to be a given function of the time. If we substitute from (4) in (5) we get the equation

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C} = V \tag{6}$$

for Q. Writing
$$\kappa = \frac{R}{2L}, \qquad n^2 = \frac{1}{LC}, \tag{7}$$
this becomes
$$\frac{d^2Q}{dt^2}+2\kappa\frac{dQ}{dt}+n^2Q = \frac{V}{L}, \tag{8}$$
which is the same as § 29 (2), except that it has Q in place of x, and V/L in place of F/m. This is a special case of a far-reaching analogy between mechanical and electrical systems which will be discussed in §§ 42, 43. For the present we simply note that the similarity in form of (8) and § 29 (2) allows us to quote many of the results and much of the general discussion of §§ 29–32; some examples of this are given below.

Ex. 1. *Constant voltage E applied at $t = 0$ to the circuit of Fig. 27 (d) with the initial values $Q = 0$ and $I = 0$ when $t = 0$.*

We have to solve
$$(D^2+2\kappa D+n^2)Q = \frac{E}{L}. \tag{9}$$
A particular integral of this is
$$\frac{E}{n^2L} = CE.$$
Adding the complementary function given by § 29 (7), the general solution is found to be
$$Q = ae^{-\kappa t}\sin n't+be^{-\kappa t}\cos n't+CE, \tag{10}$$
where a and b are arbitrary constants, $n' = \sqrt{(n^2-\kappa^2)}$, and we assume that this is real (the other cases follow precisely as in § 29). The initial conditions
$$Q = 0, \qquad I = \frac{dQ}{dt} = 0, \quad \text{when } t = 0$$
give the equations for a and b
$$b+CE = 0, \qquad n'a-\kappa b = 0.$$
Thus the final solution is
$$Q = CE\{1-e^{-\kappa t}\cos n't-(\kappa/n')e^{-\kappa t}\sin n't\},$$
and the current I is by (4)
$$I = (E/n'L)e^{-\kappa t}\sin n't.$$

Ex. 2. *Alternating voltage $E\sin(\omega t+\beta)$ applied to the circuit of Fig. 27 (d).*

The differential equation is now
$$(D^2+2\kappa D+n^2)Q = \frac{E}{L}\sin(\omega t+\beta),$$
and its general solution, obtained by quoting § 31 (6) with x replaced by Q and F_0/m by E/L, is
$$Q = Ae^{-\kappa t}\sin(n't+B)+\frac{E}{L\{(n^2-\omega^2)^2+4\kappa^2\omega^2\}}\sin(\omega t+\beta-\phi), \tag{11}$$
where $n' = \sqrt{(n^2-\kappa^2)}$ and ϕ is defined in § 31 (4).

Differentiating, we have by (4)

$$I = Ae^{-\kappa t}\{n'\cos(n't+B)-\kappa\sin(n't+B)\}+$$
$$+\frac{E\omega}{L\{(n^2-\omega^2)^2+4\kappa^2\omega^2\}}\sin\{\omega t+\beta-(\phi-\tfrac{1}{2}\pi)\}. \quad (12)$$

The first term of (12) is the 'transient' current which dies away with time; the second is the steady state alternating current. The variation of its amplitude and phase with frequency are shown in Figs. 8 (*a*) and (*c*) and Figs. 7 (*b*) and (*d*).

If the voltage is supposed to be switched on at time $t = 0$ the constants A and B can be determined from the given values of the charge Q and current I at this instant, either by direct substitution, or as in § 40, Ex. 1, where the complete solution of this problem is given.

42. Electrical networks

The principles of § 41 may be extended immediately to any network, however complicated. A network can be divided up into a number of simple 'branches' AB, BC,..., Fig. 28 (*a*), which are connected at 'junctions' or 'nodes' A, B,.... For definiteness, we suppose each branch to contain L, R, C in series, together, possibly, with a source of applied voltage.

First we assign by convention a positive direction for current in each branch. Suppose that in the branch AB this direction is from A to B, that I_1 is the current in this branch, and that V_1 is the external voltage applied at terminals in the branch AB, V_1 being reckoned positive if it would cause a current to flow in the direction from A to B. These conventions are shown in Fig. 28 (*b*).

Then, as in § 41, the voltage drop v_1 over AB is the sum of the voltage drops over the elements L_1, R_1, C_1, and the applied voltage V_1 in AB, that is

$$L_1\frac{dI_1}{dt}+R_1 I_1+\frac{Q_1}{C_1}-V_1 = v_1. \quad (1)$$

V_1 appears in the left-hand side of (1) with a negative sign, since, with the convention chosen above, it is a *rise* in voltage.

Now consider any closed circuit round several branches of Fig. 28 (*a*), e.g. the closed circuit AB, BC, CA. If v_1, v_2, v_3 are the voltage drops from A to B, B to C, and C to A, respectively, the algebraic sum of these must be zero. Adding the three corresponding equations of type (1), we get the result that *the algebraic* *sum of the voltage drops across the elements and the sources of applied voltage in a closed circuit is zero.* Clearly this is true for any closed circuit, and the statement is known as *Kirchhoff's first law.* Alternatively it may be stated in the form that *the algebraic sum of the voltage drops across the elements of a closed circuit is equal to the algebraic sum of the applied voltages in the closed circuit.*

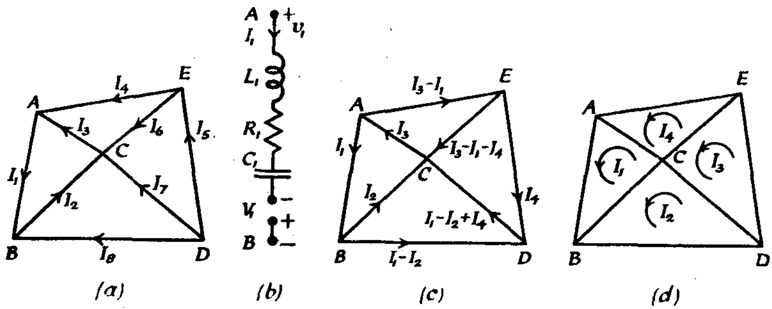


FIG. 28.

We also require *Kirchhoff's second law*, which states that *the algebraic sum of the currents flowing towards any junction is zero.* This is simply the expression of the physical fact that charge does not accumulate at the junctions.

By using Kirchhoff's two laws we can always write down sufficient equations to determine all the currents and charges in any network. There are various ways of shortening and systematizing this work which are commonly used by engineers in solving such problems. These will be indicated briefly later: for the present we solve a number of problems from first principles.

Ex. 1. *The circuit of Fig. 29 with applied voltage V.*

Let I_1, I_2, and I_3 be the currents in AB, BC, and BE in the directions of the arrows. Let Q_1, Q_2, and Q_3 be the charges on C_1, C_3, and C_2; then by § 41 (4)

$$I_1 = DQ_1, \qquad I_2 = DQ_2, \qquad I_3 = DQ_3. \tag{2}$$

Next, Kirchhoff's second law at the junction B gives

$$I_1 - I_2 - I_3 = 0. \tag{3}$$

Kirchhoff's first law for the closed circuit $ABEF$ gives

$$(L_1 D+R_1)I_1+\frac{Q_1}{C_1}+\frac{Q_3}{C_2} = V, \tag{4}$$

and for the closed circuit $BCDE$ it gives

$$(L_2 D+R_2)I_2+\frac{Q_2}{C_3}-\frac{Q_3}{C_2} = 0. \tag{5}$$

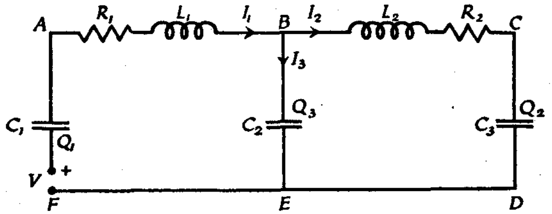


FIG. 29.

(2), (3), (4), and (5) are six equations for the six unknowns $I_1,\ldots, Q_3$. If we substitute for the I in terms of the Q by (2), equations (4), (5), and (3), respectively, become

$$\left(L_1 D^2+R_1 D+\frac{1}{C_1}\right)Q_1+\frac{Q_3}{C_2} = V, \tag{6}$$

$$\left(L_2 D^2+R_2 D+\frac{1}{C_3}\right)Q_2-\frac{Q_3}{C_2} = 0, \tag{7}$$

$$D(Q_1-Q_2-Q_3) = 0. \tag{8}$$

Integrating (8) gives

$$Q_1-Q_2-Q_3 = Q, \tag{9}$$

where Q is a constant to be determined from the known conditions at the instant when the voltage V was switched on. For simplicity we assume that $Q = 0$; this would be the case, for example, if the condensers were initially uncharged. We then have

$$Q_3 = Q_1-Q_2, \tag{10}$$

and substituting this in (6) and (7) gives

$$\left(L_1 D^2+R_1 D+\frac{1}{C_1}+\frac{1}{C_2}\right)Q_1-\frac{Q_2}{C_2} = V, \tag{11}$$

$$-\frac{Q_1}{C_2}+\left(L_2 D^2+R_2 D+\frac{1}{C_2}+\frac{1}{C_3}\right)Q_2 = 0. \tag{12}$$

Writing

$$\begin{aligned} \kappa_1 &= \frac{R_1}{2L_1}, \quad n_1^2 = \frac{1}{L_1 C_1}, \quad n_{12}^2 = \frac{1}{L_1 C_2}, \\ \kappa_2 &= \frac{R_2}{2L_2}, \quad n_2^2 = \frac{1}{L_2 C_2}, \quad n_{23}^2 = \frac{1}{L_2 C_3}, \end{aligned} \tag{13}$$

(11) and (12) become

$$(D^2+2\kappa_1 D+n_1^2+n_{12}^2)Q_1-n_{12}^2 Q_2 = V/L_1, \tag{14}$$

$$-n_2^2 Q_1+(D^2+2\kappa_2 D+n_2^2+n_{23}^2)Q_2 = 0. \tag{15}$$

These equations are exactly those for forced oscillations of the mechanical system of Fig. 14 with $k_{12} = 0$ and Q_1 and Q_2 replacing x_1 and x_2 (cf. § 34 (4) and (5)) and it follows that the whole discussion of free and forced oscillations of this system in § 34 can be taken over bodily: there will be two natural frequencies, each with its own damping factor and normal mode of oscillation of the currents and charges.

For the special case $L_1 = L_2$, $R_1 = R_2$, $C_1 = C_2 = C_3$, so that

$$n_1^2 = n_{12}^2 = n_2^2 = n_{23}^2, \qquad \kappa_1 = \kappa_2, \tag{16}$$

(14) and (15) become

$$(D^2+2\kappa_1 D+2n_1^2)Q_1-n_1^2 Q_2 = V/L_1, \tag{17}$$

$$-n_1^2 Q_1+(D^2+2\kappa_1 D+2n_1^2)Q_2 = 0. \tag{18}$$

Free oscillations for this case have been discussed in detail in § 34 (9) and (10), and forced oscillations in § 34 (17) and (18).

Ex. 2. *A system containing mutual inductance.*

In addition to inductance, resistance, and capacitance in the branches of a circuit there may be mutual inductance between some of the branches. If there is current I_r in the rth branch and current I_s in the sth branch, Fig. 30 (a), and mutual inductance M between them, there will be voltage drops

$$M\frac{dI_r}{dt} \quad \text{and} \quad M\frac{dI_s}{dt}, \tag{19}$$

respectively in the sth and rth branches. The mutual inductance M may be positive or negative according to the way in which the coils are wound.

As an example we write down the equations for the circuit of Fig. 30 (b) which has mutual inductance M between the inductances L_1 and L_2. Choosing currents I_1, I_2, and I_3 in the directions of the arrows, Kirchhoff's second law at the junction C gives

$$I_1-I_2-I_3 = 0. \tag{20}$$

The first law for the circuits $ABCDEA$ and $CGFDC$, respectively, gives

$$(L_1 D+R_1)I_1 - MDI_2 + L_3 DI_3 = V,$$

$$(L_2 D+R_2)I_2 - L_3 DI_3 - MDI_1 = 0.$$

Eliminating I_3 from these by (20) gives

$$\{(L_1+L_3)D+R_1\}I_1-(M+L_3)DI_2 = V, \tag{21}$$

$$-(M+L_3)DI_1+\{(L_2+L_3)D+R_2\}I_2 = 0. \tag{22}$$

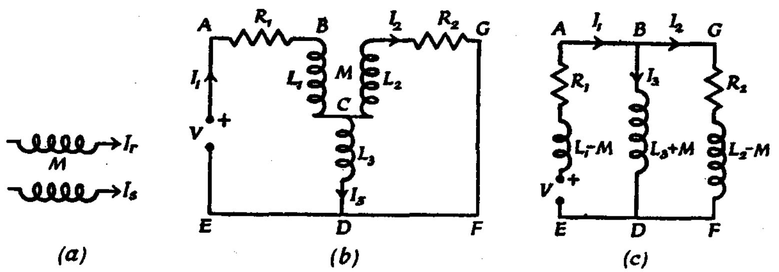


FIG. 30.

It should be noticed that the equations (21) and (22) are the same as those which would be found for the network of Fig. 30 (c) which contains self-inductances and resistances only. This device is much used in circuit theory.

Ex. 3. *The transformer.*

Two circuits L_1, R_1 and L_2, R_2 are coupled solely by mutual inductance M. Alternating voltage $E\sin\omega t$ is applied to the primary.

If I_1 and I_2 are the primary and secondary currents, Fig. 31, the circuit equations are

$$(L_1 D+R_1)I_1+MDI_2 = E\sin\omega t, \tag{23}$$

$$MDI_1+(L_2 D+R_2)I_2 = 0. \tag{24}$$

Suppose we wish to find the forced oscillations of the system. As usual we replace $\sin\omega t$ by $e^{i\omega t}$ and find a particular integral of the resulting system by assuming

$$I_1 = I_1' e^{i\omega t}, \qquad I_2 = I_2' e^{i\omega t}, \tag{25}$$

where I_1' and I_2' are constants. This gives

$$(R_1+L_1 i\omega)I_1'+Mi\omega I_2' = E,$$

$$Mi\omega I_1'+(R_2+L_2 i\omega)I_2' = 0.$$

Solving for I_2' we get

$$I_2' = -\frac{ME}{(L_1 R_2+R_1 L_2)+i\{(L_1 L_2-M^2)\omega-R_1 R_2/\omega\}}.$$

Thus the steady value of I_2 is

$$-\frac{ME}{Z}\sin(\omega t-\phi),$$

where $\tan\phi = [(L_1 L_2-M^2)\omega^2-R_1 R_2]/(L_1 R_2+R_1 L_2)\omega,$

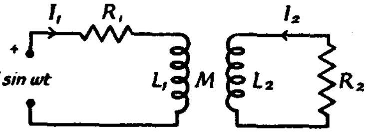


FIG. 31.

and $Z = \{(L_1 R_2+R_1 L_2)^2-2R_1 R_2 \sigma+\sigma^2\omega^2+R_1^2 R_2^2/\omega^2\}^{\frac{1}{2}},$

where $\sigma = L_1 L_2-M^2$.

$Z \to \infty$ as $\omega \to 0$ or $\omega \to \infty$. It has a minimum when $\omega = (R_1 R_2/\sigma)^{\frac{1}{2}}$.

In the examples above we have written down the equations directly from Kirchhoff's two laws. There are two ways in which this procedure can be shortened.

(i) Instead of assuming an unknown current in each branch of the circuit and writing down algebraic equations connecting them by Kirchhoff's second law at the junctions, we may choose the unknown currents to satisfy the second law automatically. For example in Fig. 28 (c), if we assume unknown currents I_1 and I_2 in AB and BC, the current in BD must be I_1-I_2. Proceeding in this way, it appears that only four unknown currents appear in Fig. 28 (c) instead of the eight in Fig. 28 (a). We shall often use this procedure in future.

(ii) The same result is achieved if we assume a 'mesh-current' to be circulating round each of the four meshes ABC, BDC,... of Fig. 28 (d). In this case the current in BC, for example, is I_1-I_2, and Kirchhoff's second law is satisfied automatically at B and similarly at all the other junctions.

43. Mechanical analogies

In §§ 41, 42 it was seen that certain electrical circuits led to precisely the same differential equations as corresponding mechanical systems, so that the algebra of the solutions is the same for both. The obvious correspondence appears on comparing (5) and (4) of § 41, namely,

$$L\frac{dI}{dt}+RI+\frac{Q}{C} = V, \tag{1}$$

$$\frac{dQ}{dt} = I, \tag{2}$$

with the equations of motion § 29 (1) for a mass m attached to a spring of stiffness λ, with resistance to motion k times the velocity v, and acted on by force F. These are

$$m\frac{dv}{dt}+kv+\lambda x = F, \tag{3}$$

$$\frac{dx}{dt} = v. \tag{4}$$

These two sets of equations correspond precisely if we replace Q by x, I by v, L by m, $1/C$ by λ, V by F and R by k. This is the most common and useful form of analogy.

These considerations can be carried a good deal farther. First we notice that by integrating (2) with respect to the time from $t = 0$ to $t = t$ we get

$$Q = \overset{0}{Q}+\int_0^t I\,dt, \tag{5}$$

where $\overset{0}{Q}$ is the value of Q when $t = 0$. For shortness we shall write (5) in the form

$$Q = \int I\,dt, \tag{6}$$

and other integrals of the same type below are to be understood in the same sense.

Using (6) in (1) gives

$$L\frac{dI}{dt}+RI+\frac{1}{C}\int I\,dt = V, \tag{7}$$

which is a form in which the fundamental equation for the L, R, C circuit is often written. (7) is called an integrodifferential equation since it contains both the integral and the differential coefficient of the unknown I.

In § 41 the equations

$$RI = V, \qquad L\frac{dI}{dt} = V, \qquad \frac{1}{C}Q = \frac{1}{C}\int I\,dt = V, \tag{8}$$

were given for determining the current in a resistance, inductance, or capacitance in terms of the voltage V applied to it.

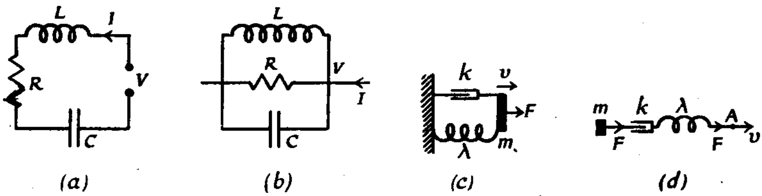


Fig. 32.

Now suppose we regard the *current* I in each of these elements as known and wish to find the voltage drop V in terms of it. The equations (8) may be rewritten from this point of view as†

$$\frac{1}{R}V = I, \qquad \frac{1}{L}\int V\,dt = I, \qquad C\frac{dV}{dt} = I, \tag{9}$$

where the second and third of (9) are obtained by integrating the second and differentiating the third of (8), respectively.

Next consider the 'L, R, C parallel circuit', Fig. 32 (b), in which the total current I into the combination of L, R, and C in parallel is regarded as given, and it is required to find the voltage drop across the elements. Adding the currents in the separate elements given by (9), we get

$$C\frac{dV}{dt} + \frac{1}{R}V + \frac{1}{L}\int V\,dt = I. \tag{10}$$

This is of the same form as (7) with V and I interchanged, L and C interchanged, and R replaced‡ by $1/R$. The circuit Fig. 32 (b) is called the 'dual' of Fig. 32 (a), and this duality can be extended to more complicated circuits.

† These equations may be used to develop the theory of electrical networks in the same way that we have used (8).

‡ It will be noticed that in (7) and (10) all of L, R, C and their reciprocals appear. It is usual to give these reciprocals names and to use them when they occur: thus $G = 1/R$ is the conductance, $S = 1/C$ the elastance, etc.

Similar relations apply to mechanical systems: writing x for displacement, $v = \dot{x}$ for velocity, we have

$$kv = F, \qquad m\frac{dv}{dt} = F, \qquad \lambda x = \lambda \int v\,dt = F, \tag{11}$$

for a force F applied respectively to a dash-pot which gives resistance to motion k times the velocity, to a mass m, and to a spring of stiffness λ. Combining these we have for the mechanical system, Fig. 32 (c),

$$m\frac{dv}{dt}+kv+\lambda \int v\,dt = F \tag{12}$$

as in (3) and (4). If, on the other hand, we regard the velocity as given and wish to determine the force across the elements, we write (11) in the form

$$\frac{1}{k}F = v, \qquad \frac{1}{m}\int F\,dt = v, \qquad \frac{1}{\lambda}\frac{dF}{dt} = v. \tag{13}$$

The equation of motion of the system of Fig. 32 (d), in which the point A is moved in a straight line with velocity v which is a known function of t, is

$$\frac{1}{\lambda}\frac{dF}{dt}+\frac{1}{k}F+\frac{1}{m}\int F\,dt = v, \tag{14}$$

which, again, is of the same form as (7), (10), and (12). The system of Fig. 32 (d) is the dual of the system of Fig. 32 (c), and all four systems of Fig. 32 lead to equations of the same type in symbols which correspond as follows:

Electrical		*Mechanical*	
L, R, C series *Fig. 32 (a)*	*Dual* *Fig. 32 (b)*	*Damped harmonic oscillator* *Fig. 32 (c)*	*Dual* *Fig. 32 (d)*
L	C	m	$1/\lambda$
R	$1/R$	k	$1/k$
C	L	$1/\lambda$	m
I	V	v	F
V	I	F	v

It is possible, by using the relations given above, to set up electrical analogues of dynamical systems. The correspondence between the circuit of Fig. 29 and a special case of the mechanical system of Fig. 14 has been noted in § 42.

In the same way the circuits of Fig. 33 (*a*), (*b*), and (*c*) will be found to be the analogues of the mechanical systems of Figs. 14, 19, and 16 (*a*), respectively, of the type in which inductance corresponds to mass, etc. For example, if the differential equations for Q_1 and Q_2 in Fig. 33 (*a*) are written down they will be

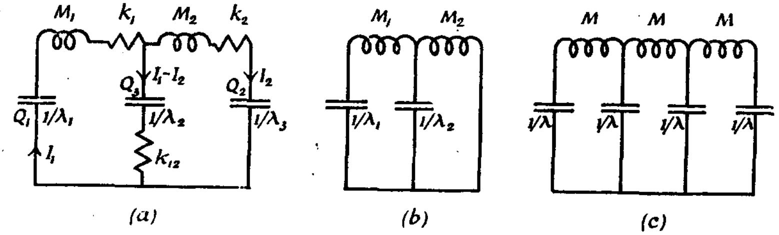


FIG. 33.

found to be identical with § 34 (4) and (5) with Q_1 and Q_2 replacing x_1 and x_2. In doing this the charge in any branch is taken to be the total amount of charge transported by the current in this branch, so that Q_3, the charge associated with I_1-I_2, is equal to Q_1-Q_2; cf. § 42 (10).

The procedure for setting up such analogous systems has been extensively studied and extended to systems containing gearing and also to acoustical systems.†

44. Steady state theory. Impedance

It has been shown in the last three sections that there is no fundamental difference between the equations to be solved in problems of electric circuit theory and of mechanical vibrations. But owing to the enormous complexity of many practical circuits, special techniques have been developed for their study which concentrate attention on the quantities of interest to the electrical engineer. This leads to two important changes in point of view.

Firstly, the electrical engineer is most often interested in the steady response of a circuit to an alternating voltage rather than in its transient behaviour. Thus he does not usually look for

† Cf. Olson, *Dynamical Analogies* (van Nostrand, 1943).

natural frequencies as such, but regards them as frequencies at which resonance occurs: this corresponds to the second of the two methods developed in Chapter IV. Similarly, the normal modes of oscillation corresponding to the natural frequencies are rarely calculated—for a complete solution of a transient problem

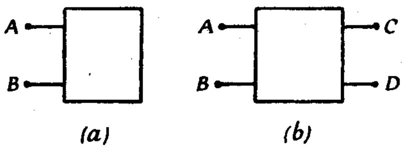


FIG. 34.

by the classical methods it is necessary to do this, but the labour can be avoided by the use of the Laplace transformation or operational methods which have, in fact, largely been developed for problems of this type.

Secondly, the engineer is very little interested in what goes on in many parts of the complicated circuits with which he deals. Instead, he regards them as boxes, with, for example, two terminals A, B, Fig. 34 (a), which is a *two-terminal network*, or with four terminals, Fig. 34 (b), which is a *four-terminal network*.

The commonest types of problem are then: (i) to find the steady state current flowing into the network of Fig. 34 (a) due to sinusoidal voltage applied to the terminals AB, or (ii) to find the steady state current in a known load connected to the terminals CD of Fig. 34 (b) when a sinusoidal voltage is applied to the terminals AB.

In a complicated circuit it would be a waste of labour to write down a complete set of equations for all the currents in all the branches in the boxes of Fig. 34 and this labour can often be avoided by the methods now to be discussed.

Consider first the L, R, C circuit, Fig. 35 (a), with sinusoidal voltage $E\cos(\omega t+\beta)$ applied to it, and suppose we require the steady state current in the circuit. The equations for the current I in the circuit are, by § 41 (5),

$$(LD+R)I+\frac{Q}{C} = E\cos(\omega t+\beta), \tag{1}$$

$$DQ = I. \tag{2}$$

To find the steady state current we replace $E\cos(\omega t+\beta)$ in (1) by $E'e^{i\omega t}$, where

$$E' = Ee^{i\beta}. \tag{3}$$

Then we seek a solution of the form

$$I = I'e^{i\omega t}, \qquad Q = Q'e^{i\omega t}, \tag{4}$$

and take its real part. Substituting (4) in (1) and (2) gives

$$(R+Li\omega)I'+\frac{Q'}{C} = E', \qquad i\omega Q' = I'.$$

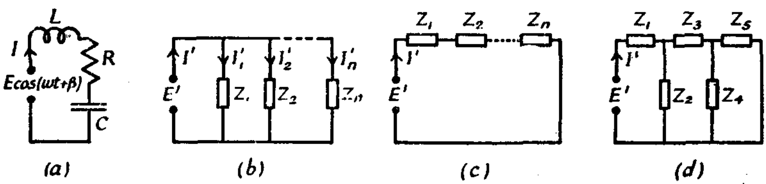


FIG. 35.

Therefore, eliminating Q',

$$\left\{R+i\left(L\omega-\frac{1}{C\omega}\right)\right\}I' = E'. \tag{5}$$

We call
$$z = R+i\left(L\omega-\frac{1}{C\omega}\right) \tag{6}$$

the *complex impedance* of the circuit. Its imaginary part

$$X = L\omega-\frac{1}{C\omega} \tag{7}$$

is called the *reactance* of the circuit, and its modulus

$$|z| = \left\{R^2+\left(L\omega-\frac{1}{C\omega}\right)^2\right\}^{\frac{1}{2}} = \{R^2+X^2\}^{\frac{1}{2}} \tag{8}$$

is called the *impedance* of the circuit. The angle

$$\theta = \tan^{-1}\frac{X}{R} \tag{9}$$

is called the phase angle.

With this notation, (5) gives

$$I' = \frac{E'}{z}. \tag{10}$$

Using (10) and (3) in (4), the steady state current is thus the real part of

$$\frac{E'}{z}e^{i\omega t} = \frac{E}{|z|}e^{i(\omega t+\beta-\theta)},$$

that is,

$$\frac{E}{|z|}\cos(\omega t+\beta-\theta), \tag{11}$$

where $|z|$ and θ are defined in (8) and (9). This is the result† which was quoted in § 41 (12), and the argument above is merely a repetition of that of § 31 in which this particular integral of the equations (1) and (2) was found. One minor change has been made, namely including the phase angle β of $E\cos(\omega t+\beta)$ in the complex quantity E' of (3), but the main point has been the expressing of the solution in terms of the complex impedance z which is now regarded as a quantity which can be written down immediately for this circuit.

For shortness, we shall call E' the *complex voltage* applied to the circuit and I' the *complex current* in it, it being understood that these terms refer to the steady state only, and that to get actual real voltage or current we multiply E' or I' by $e^{i\omega t}$ and take the real part. Thus (10) states that the complex current in an L, R, C circuit is obtained by dividing the complex voltage across it by the complex impedance z. This relation has the same form as Ohm's law for direct current with complex impedance z replacing resistance; thus we can write down formulae for the impedance of complicated networks by the same rules which allow us to write down direct current resistance.

First we define the complex impedance of any two-terminal network, Fig. 34 (*a*), as the complex voltage across its terminals divided by the complex current into them.

Next consider the circuit of Fig. 35 (*b*) with n impedances $z_1,\ldots, z_n$ in parallel and complex voltage E' applied to it. Let $I'_1,\ldots, I'_n$ be the complex currents in the impedances, then

$$I'_1 = \frac{E'}{z_1}, \quad \ldots, \quad I'_n = \frac{E'}{z_n}.$$

† Except that the voltage has been taken here to be $E\cos(\omega t+\beta)$ to conform with the usual engineering practice. It should be added, also, that engineers use the small letters e and i for the quantities denoted here by E' and I'.

Thus if I' is the total complex current into the network

$$I' = I'_1 + \ldots + I'_n = E'\left(\frac{1}{z_1} + \ldots + \frac{1}{z_n}\right). \tag{12}$$

Thus the complex impedance z of the n impedances in parallel is given by

$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}. \tag{13}$$

Similarly the complex impedance z of a number of impedances $z_1, \ldots, z_n$ in series, Fig. 35 (c), is

$$z = z_1 + z_2 + \ldots + z_n. \tag{14}$$

For more complicated circuits the complex impedance may be written down by a combination of these rules. For example, for the 'ladder' network of Fig. 35 (d), it is

$$z_1 + \cfrac{1}{\cfrac{1}{z_2} + \cfrac{1}{z_3 + \cfrac{1}{(1/z_4) + (1/z_5)}}}. \tag{15}$$

The reciprocal $1/z$ of the complex impedance of a circuit is its complex *admittance*. Both quantities are equally useful in the theory: for example (13) may be better stated in the form that the admittance of n circuits in parallel is the sum of their admittances.

For four-terminal networks the position is a little more complicated. Consider the network of Fig. 34 (b) and suppose a load of complex impedance z_L is connected to the terminals CD. If I'_L is the complex current in this load caused by complex voltage E' applied at the terminals AB, the quantity

$$E'/I'_L$$

is called the *transfer impedance* between the pairs of terminals AB and CD: it can be calculated from a knowledge of the network and the load impedance. To do this it is usually necessary to use Kirchhoff's laws discussed in § 42. In the steady state with which we are now concerned the voltage drop over any element of complex impedance z carrying complex current I' will be $zI'e^{i\omega t}$ and the applied voltage will be $E'e^{i\omega t}$:

the time-factors $e^{i\omega t}$ cancel and we are left with equations connecting the complex currents and voltages. Kirchhoff's first law then becomes *the algebraic sum of the complex voltage drops over the elements of a closed circuit is equal to the algebraic sum of the complex applied voltages in the circuit.* The second law becomes *the algebraic sum of the complex currents at a junction is zero.*

45. Variation of impedance with frequency. Filter circuits

It was remarked in § 44 that most of the information required by engineers could be obtained by writing down the complex impedance of a circuit and studying its variation with frequency. In this section we discuss some simple networks from this point of view.

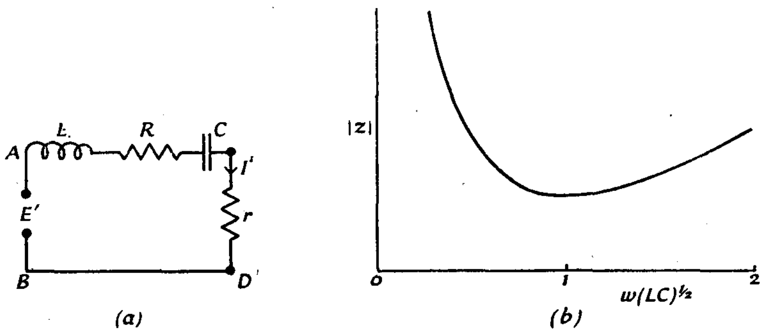


FIG. 36.

(i) *The L, R, C circuit in series with a load*

We suppose that complex voltage E' is applied to the terminals AB of Fig. 36 (*a*), and that the complex current I' in a terminating resistance r is to be calculated. By § 44 (10)

$$I' = \frac{E'}{z},$$

where
$$z = R+r+\left(L\omega-\frac{1}{C\omega}\right)i$$

is the complex impedance.

The impedance is

$$|z| = \left\{(R+r)^2+\left(L\omega-\frac{1}{C\omega}\right)^2\right\}^{\frac{1}{2}}, \tag{1}$$

and this has its least value $(R+r)$ when $\omega^2 = 1/LC$, that is, at the undamped natural frequency of the circuit. The variation of $|z|$ with ω is shown in Fig. 36 (*b*). When $\omega = (LC)^{-\frac{1}{2}}$, the impedance is least and the current greatest: thus this circuit tends to favour the passage of this frequency relative to others. The problem is essentially that of § 31 and Fig. 8 (*a*) except that here we have studied the impedance instead of the amplitude which is proportional to its reciprocal.

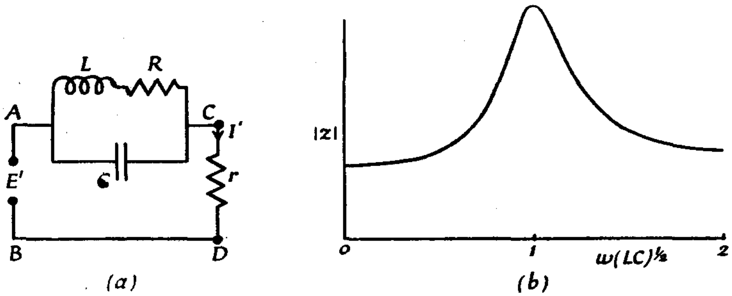


FIG. 37.

(ii) *The 'choke' circuit, Fig. 37 (a)*

Here we require the complex current I' in a load consisting of resistance r when complex voltage E' is applied at the terminals AB.

The complex impedance z of the system $ACDB$ is, by § 44 (13) and (14),

$$z = r + \frac{1}{C\omega i + 1/(R+L\omega i)}$$

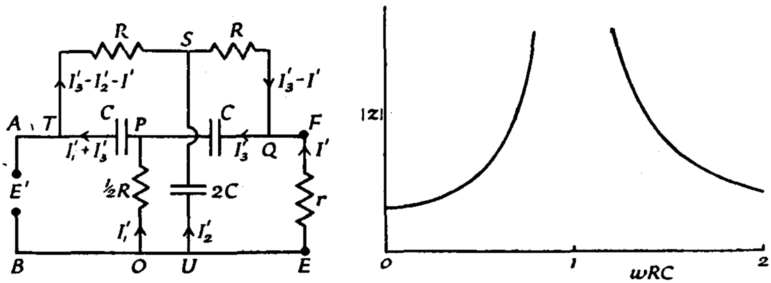
$$= r + \frac{R+L\omega i}{(1-LC\omega^2)+RC\omega i}.$$

$$|z| = \left\{\frac{\{r(1-LC\omega^2)+R\}^2+(RCr\omega+L\omega)^2}{(1-LC\omega^2)^2+R^2C^2\omega^2}\right\}^{\frac{1}{2}}. \tag{2}$$

Here, if R is small which is usually the case, the denominator of (2) is least for a value of ω near $(LC)^{-\frac{1}{2}}$. Thus the impedance has a maximum value near the undamped natural frequency of the L, C circuit—its variation is shown in Fig. 37 (*b*). Thus this circuit tends to discourage the passage of the frequency $(LC)^{-\frac{1}{2}}/2\pi$ and to favour higher or lower frequencies.

(iii) *The 'parallel-T' circuit*

The circuit is shown in Fig. 38 (a). Complex voltage E' is applied at the terminals AB, and we require the complex current I' in a load (which is taken to be a resistance r) connected across the terminals EF.



(a) (b)

FIG. 38.

This is a more complicated problem than those discussed above because we require the current in a different branch to that in which the voltage is applied. We use Kirchhoff's laws for complex currents and voltages as in § 44. Let I_1', I_2', and I_3' be the complex currents in the branches OP, US, and QP of Fig. 38 (a), those in the other branches being chosen to satisfy Kirchhoff's second law automatically. Then Kirchhoff's first law for the complex voltage drops round the closed circuits $PABOP$, $OEFPO$, $USQFEU$, and $PTSQP$, respectively, give

$$-\tfrac{1}{2}RI_1'+\frac{i}{C\omega}(I_1'+I_3') = E', \tag{3}$$

$$rI'-\frac{i}{C\omega}I_3'-\tfrac{1}{2}RI_1' = 0, \tag{4}$$

$$-\frac{i}{2C\omega}I_2'+R(I_3'-I')-rI' = 0, \tag{5}$$

$$-\frac{i}{C\omega}(I_1'+I_3')+R(I_3'-I_2'-I')+R(I_3'-I')-\frac{i}{C\omega}I_3' = 0. \tag{6}$$

If we write $$x = \omega RC, \qquad k = r/R, \tag{7}$$

and solve for I' we find

$$-\frac{RI'}{E'} = \begin{vmatrix} -\frac{1}{2} & 0 & -i/x \\ 0 & -i/2x & 1 \\ -i/x & -1 & 2(1-i/x) \end{vmatrix} \div \begin{vmatrix} 0 & (\frac{1}{2}-i/x) & 0 & -i/x \\ k & -\frac{1}{2} & 0 & -i/x \\ -(k+1) & 0 & -i/2x & .1 \\ -2 & -i/x & -1 & 2(1-i/x) \end{vmatrix}$$

$$= \frac{x^2-1}{(kx^2-2-k)-2ix(1+2k)}. \tag{8}$$

If we write $$I' = E'/z,$$
where, in the notation of § 44, z is the transfer impedance between the branches AB and EF, we find from (8)
$$|z| = \frac{R\{(kx^2-2-k)^2+4x^2(1+2k)^2\}^{\frac{1}{2}}}{|x^2-1|}. \tag{9}$$
The variation of $|z|$ with $x = \omega RC$ is shown in Fig. 38 (*b*). The impedance is infinite at the frequency $1/2\pi RC$.

(iv) *Filter circuits*

In the above examples it has been seen that simple combinations of circuit elements have rather crude filtering properties. Thus the circuit of Fig. 36 (*a*) tends to 'stop' low and high frequencies and to 'pass' those near $(LC)^{-\frac{1}{2}}/2\pi$. The circuits of Figs. 37 (*a*) and 38 (*a*), on the other hand,

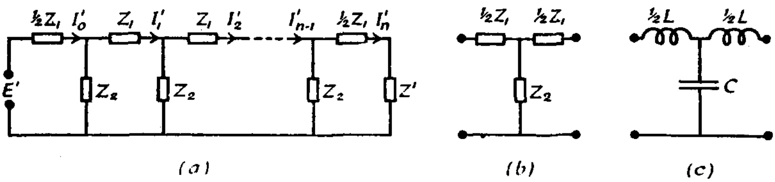


FIG. 39.

pass the low and high frequencies and stop, in part or wholly, intermediate frequencies. This behaviour can be sharpened by connecting together a number of these combinations. The simplest example of this is the 'ladder' network of Fig. 39 (*a*) which may be regarded as composed of n of the sections of Fig. 39 (*b*) connected in tandem and terminated by a complex impedance z'.

We suppose complex voltage E' to be applied to the circuit and we wish to find the complex current I'_n in the terminating impedance z'. Let I'_0, I'_1,..., I'_n be the mesh currents as shown in Fig. 39 (*a*). Then Kirchhoff's laws for the successive meshes give
$$\tfrac{1}{2}z_1 I'_0+z_2(I'_0-I'_1) = E', \tag{10}$$
$$z_1 I'_r+z_2(I'_r-I'_{r+1})-z_2(I'_{r-1}-I'_r) = 0 \quad (r = 1,..., n-1), \tag{11}$$
$$(\tfrac{1}{2}z_1+z')I'_n-z_2(I'_{n-1}-I'_n) = 0. \tag{12}$$
We seek a solution† of the set of equations (11) which connect the complex currents in three successive impedances z_1 of the form
$$I'_r = Ae^{r\theta}, \tag{13}$$

† Equations such as this are called difference equations; for other examples see Chapter IV, Ex. 14 and § 85 (12). They have a general theory similar to that of differential equations and the present procedure may be regarded as being suggested by that at the end of § 13.

where A and θ are independent of r. Substituting in (11) gives

$$(z_1+2z_2)e^{r\theta}-z_2\,e^{(r+1)\theta}-z_2\,e^{(r-1)\theta} = 0.$$

Or
$$\cosh\theta = 1+\frac{z_1}{2z_2}. \tag{14}$$

Since (14) gives two values of θ with opposite signs, we get the final result that all the equations (11) are satisfied by

$$I'_r = Ae^{r\theta}+Be^{-r\theta}, \tag{15}$$

where A and B are independent of r, and θ is given by (14). The constants A and B are determined by substituting (15) in (10) and (12) which give, using (14),

$$(A+B)\cosh\theta-(Ae^{\theta}+Be^{-\theta}) = E'/z_2$$

$$(\cosh\theta+z'/z_2)(Ae^{n\theta}+Be^{-n\theta})-(Ae^{(n-1)\theta}+Be^{-(n-1)\theta}) = 0.$$

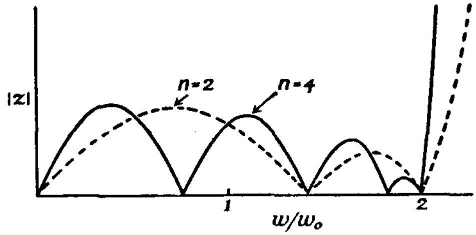


FIG. 40.

Solving for A and B and putting these values in (15) we get, after some reduction,

$$I'_r = \frac{E'[\sinh\theta\cosh(n-r)\theta+(z'/z_2)\sinh(n-r)\theta]}{z_2\sinh\theta[\sinh n\theta\sinh\theta+(z'/z_2)\cosh n\theta]}. \tag{16}$$

For a simple special case to study in detail, we consider I'_n in the case in which $z' = 0$ (the output terminals short-circuited) for the circuit of Fig. 39 (c) in which

$$z_1 = L\omega i, \qquad z_2 = -\frac{i}{C\omega}, \tag{17}$$

so that
$$\cosh\theta = 1-\tfrac{1}{2}LC\omega^2. \tag{18}$$

In this case (16) gives
$$I'_n = E'/z, \tag{19}$$

where z, the transfer impedance in which we are interested, is given by

$$\begin{aligned} z &= z_2\sinh\theta\sinh n\theta \\ &= -(i/C\omega)2^{n-1}(\cosh\theta-1)(\cosh\theta-\cos\pi/n)\ldots\times \\ &\qquad\times(\cosh\theta-\cos(n-1)\pi/n)(\cosh\theta+1), \end{aligned} \tag{20}$$

where in (20) we have used the finite product for $\sinh\theta\sinh n\theta$. Using the value of (18) of $\cosh\theta$ in (20) we get finally

$$z = 2^{n-2}iL\omega(1-\cos\pi/n-\tfrac{1}{2}LC\omega^2)(1-\cos 2\pi/n-\tfrac{1}{2}LC\omega^2)\ldots(2-\tfrac{1}{2}LC\omega^2). \tag{21}$$

In Fig. 40, $|z|$ is plotted against ω/ω_0, where $\omega_0 = (LC)^{-\frac{1}{2}}$, for $n = 2$ and $n = 4$.

It appears that $|z|$ is small if $\omega < 2\omega_0$, but $|z|$ increases rapidly as ω is increased beyond $2\omega_0$ and that this tendency increases as n, the number of sections, is increased. The filter is a 'low-pass' filter with 'cut-off' frequency ω_0/π.

46. Circuits containing vacuum tubes

We shall consider only triodes which illustrate most of the important properties of such circuits.

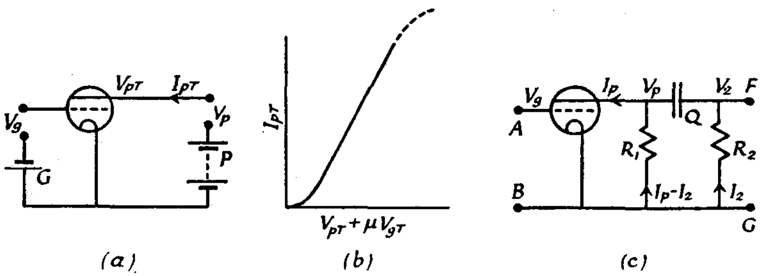


FIG. 41.

When a triode forms part of a circuit [Fig. 41 (a)] there are batteries of voltages P and G in the plate and grid circuits which by themselves would cause steady direct current S to flow to the plate and would apply steady voltages V_P and V_G to the plate and grid respectively. Superposed on these steady voltages and currents are varying voltages V_p and V_g applied to the plate and grid from the remainder of the circuit, and it is mainly in these that we are interested. We shall write V_{pT} and V_{gT} for the total voltages applied to the plate and grid from these two causes, and I_{pT} for the total plate current, reckoned positive when flowing towards the plate. Then we have

$$V_{pT} = V_p + V_P, \tag{1}$$

$$V_{gT} = V_g + V_G, \tag{2}$$

$$I_{pT} = I_p + S, \tag{3}$$

where I_p is the varying part of the total plate current.

The plate current is determined in terms of the plate and grid voltages by the family of characteristics of the triode. Thus

$$I_{pT} = \phi(V_{pT}, V_{gT}), \tag{4}$$

and also

$$S = \phi(V_P, V_G), \tag{5}$$

where, in general, ϕ is a non-linear function of two variables. However, it is found experimentally that the effect of changing V_{gT} is, to a very close approximation over a very wide range of values, a constant, μ, times the effect of changing V_{pT}, so that the function of two variables in (4) can be replaced by a function of the single variable $V_{pT}+\mu V_{gT}$ and we have

$$I_{pT} = f(V_{pT}+\mu V_{gT}). \tag{6}$$

This relation we shall call the *characteristic* of the triode. μ is called the *amplification factor*. The form of the function (6) is sketched in Fig. 41 (*b*). In general it has a long straight portion which flattens off to zero for small, and to a horizontal asymptote for large, values of $V_{pT}+\mu V_{gT}$. Thus, although the complete graph of the function is far from linear, a linear theory will be a good approximation if operation is confined to the straight portion of the characteristic. This is the case in many practical applications, but in many others the non-linearity is of great importance.

On the linear portion of the characteristic (6) becomes

$$I_{pT} = gV_{pT}+\mu g V_{gT}+K, \tag{7}$$

where g is a constant called the plate conductance, and K is a constant. The same relation

$$S = gV_P+\mu g V_G+K \tag{8}$$

holds for the steady current and voltages, and subtracting (7) and (8) and using (1) to (3) we get

$$I_p = gV_p+\mu g V_g, \tag{9}$$

which is a relation connecting the varying parts of the plate current and the grid and plate voltages.

When the tube is connected in a circuit, contributions to V_p, I_p, etc., will come only from varying currents and voltages in the circuit elements, so, when we are dealing with the linear theory, we may omit entirely the batteries and their constant effects from the circuit diagram [cf. Fig. 41 (*c*)] and deal only with the varying currents and voltages superposed on them, using (9) as the characteristic of the tube. But when dealing with non-linear problems on the accurate characteristic, Fig. 41 (*b*), it is often better to work with the total voltages and currents.

As an example of the linear theory we study the resistance coupled amplifier, Fig. 41 (*c*). We wish to find the varying voltage V_2 across the terminals FG caused by varying voltage V_g applied to the grid. Neglecting the small capacitances between grid and plate, etc., the circuit equations are

$$R_1(I_p-I_2) = -V_p, \tag{10}$$

$$R_2 I_2+\frac{Q}{C} = -V_p, \tag{11}$$

$$DQ = I_2, \tag{12}$$

together with (9).

These are the differential equations which have to be solved if we wish to study transients. For the steady state alternating current we suppose that complex voltage E' is applied to the grid, and that I'_p and I'_2 are the complex currents in the plate and in R_2. Then the equations become

$$R_1(I'_p-I'_2) = -V'_p,$$

$$R_2 I'_2-\frac{i}{C\omega}I'_2 = -V'_p,$$

$$I'_p = gV'_p+g\mu E'.$$

Solving these, we get for the complex voltage drop V'_2 over FG

$$V'_2 = -R_2 I'_2 = -\frac{R_2 g\mu E'}{(1+gR_2+R_2/R_1)-i(gR_1+1)/R_1 C\omega}. \tag{13}$$

The ratio of the magnitudes of V'_2 and E' is

$$\frac{R_2 g\mu}{\{(1+gR_2+R_2/R_1)^2+(gR_1+1)^2/R_1^2 C^2\omega^2\}^{\frac{1}{2}}}, \tag{14}$$

which increases from zero when $\omega = 0$ to

$$\frac{R_2 g\mu}{(1+gR_2+R_2/R_1)} \tag{15}$$

as $\omega \to \infty$. The phase of V_2 lags behind that of E' by an amount

$$\pi-\tan^{-1}\frac{(gR_1+1)}{C\omega(R_1+gR_1R_2+R_2)}. \tag{16}$$

47. Stability. Oscillator circuits

In dealing with the solutions of linear differential equations in Chapter II it was found that the solution of

$$(a_0 D^n+...+a_n)y = 0 \tag{1}$$

was of type

$$A_1 e^{\alpha_1 t}+...+A_n e^{\alpha_n t}, \tag{2}$$

where $A_1,..., A_n$ were arbitrary constants, and $\alpha_1,..., \alpha_n$ were the roots of the auxiliary equation

$$a_0 \alpha^n+...+a_n = 0. \tag{3}$$

For the moment we suppose $\alpha_1,..., \alpha_n$ to be all different. In the solutions found up to the present, all the α have been either real and negative, or complex with negative real parts, corresponding to exponential decay or to damped oscillations. If, however, any of the roots has a positive real part, the solution would contain a term of exponentially increasing amplitude, and the system is said to be unstable. Such effects arise most commonly in systems in which energy is supplied from an external source, such as valve circuits, servomechanisms, etc. It must always be understood that the linear equations of motion with which we deal are only approximate, and only hold for a restricted range of the dependent variable; thus if we say a system is unstable it implies that its displacement will become so large that it will move into a region in which our linear equations no longer hold.

The criterion for stability is that *all* the roots of (3) should have negative real parts. For the quadratic

$$\alpha^2+a_1\alpha+a_2 = 0, \tag{4}$$

this requires simply that

$$a_1 > 0, \qquad a_2 > 0; \tag{5}$$

the roots are complex if, in addition, $4a_2 > a_1^2$.

Suppose, now, that in (3) $a_0 > 0$ and that the left-hand side can be resolved into linear and quadratic factors in the form

$$a_0(\alpha+b_1)(\alpha+b_2)...(\alpha^2+c_1\alpha+d_1)(\alpha^2+c_2\alpha+d_2)... = 0. \tag{6}$$

For stability, all the quantities $b_1, b_2,...$ in (6) must be positive, and, by (5), all the quantities $c_1, d_1, c_2, d_2,...$ must also be positive. Multiplying out the left-hand side of (6), it follows that *if* $a_0 > 0$ *it is a necessary condition for stability that the coefficients of all the powers of* α *in* (3) *be positive.*

This simple criterion is often useful but it is not sufficient, and the complete conditions for equation (3) with $a_0 > 0$ are that all the determinants

$$a_1, \quad \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \dots, \quad \begin{vmatrix} a_1 & a_0 & 0 & 0 & . & . & . & 0 \\ a_3 & a_2 & a_1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & 0 \\ a_{2n-1} & a_{2n-2} & . & . & . & . & . & a_n \end{vmatrix} \tag{7}$$

should be positive, where, in writing them down, a_r is replaced by zero if $r > n$. If any of the determinants in (7) is negative the system is unstable. The result (7) is known as Hurwitz's criterion;† it is related to Routh's rule and provides a compact statement though rather a clumsy one to use.

For the cubic
$$\alpha^3+a_1\alpha^2+a_2\alpha+a_3 = 0, \tag{8}$$
the conditions (7) give
$$a_1 > 0, \quad a_1 a_2-a_3 > 0, \quad a_3 > 0. \tag{9}$$

Criteria for stability are important in two ways:

(i) *Systems required to be stable.* In systems such as servomechanisms, amplifiers, etc., in which energy is supplied to the system, oscillations of large amplitude may be set up by a chance disturbance unless the system is stable. The criteria for stability set limits to some of the circuit parameters; cf. § 49.

(ii) *Systems required to be unstable.* In studying oscillation generators the first step is usually to write down approximate

† For a proof see Frank–von Mises, *Differentialgleichungen der Physik*, vol. 1, p. 162. For Routh's rule see his *Advanced Rigid Dynamics*, chap. vi.

linear equations of motion. If this system is unstable it will execute large oscillations whose amplitude is determined by the accurate non-linear equations. We proceed to discuss a simple triode oscillator circuit from this point of view.

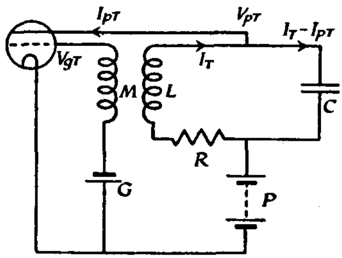


FIG. 42.

The circuit is shown in Fig. 42. I_{pT}, V_{pT}, V_{gT} are the total plate current, and plate and grid potentials, as in § 46; I_T is the total current in the inductance L. The characteristic § 46 (6) is

$$I_{pT} = f(V_{pT}+\mu V_{gT}). \tag{10}$$

Assuming† that the current to the grid is zero, the circuit equations are

$$I_T - I_{pT} = C\frac{dV_{pT}}{dt}, \tag{11}$$

$$-L\frac{dI_T}{dt} - RI_T = V_{pT} - P, \tag{12}$$

$$-M\frac{dI_T}{dt} = V_{gT} - G. \tag{13}$$

Eliminating V_{pT}, V_{gT}, and I_{pT} from (10)–(13) gives the differential equation for I_T

$$LC\frac{d^2I_T}{dt^2} + RC\frac{dI_T}{dt} - f\left\{P+\mu G - RI_T - (L+\mu M)\frac{dI_T}{dt}\right\} + I_T = 0. \tag{14}$$

This is the accurate equation; it will be discussed further in

† This is very nearly true for most valve circuits under most operating conditions.

§ 48. Here we make the assumption § 46 (7) that the characteristic is linear, in which case (14) becomes

$$LC\frac{d^2I_T}{dt^2}+[RC+g(L+\mu M)]\frac{dI_T}{dt}+(1+Rg)I_T = g(P+\mu G)+K. \tag{15}$$

This has a particular integral $[g(P+\mu G)+K]/(1+Rg)$ which is the direct current to the plate. The complementary function corresponds to an oscillation about this value. By (5) this oscillation is unstable if

$$RC+g(L+\mu M) < 0.$$

That is, if
$$M < -\frac{L}{\mu}-\frac{RC}{g\mu}. \tag{16}$$

Since M can be negative this condition can be satisfied; it is the condition for the circuit to act as an oscillation generator. As remarked above, the non-linearity of the characteristic becomes important as the oscillations become larger and ultimately it determines their amplitude. This point will be discussed in § 48.

48. Further discussion of triode oscillations

In order to study the effect of non-linearity we now consider the equations of § 47 in greater detail. First, it is convenient, as in § 46, to work with the variable parts V_p, V_g, I_p, I of the voltages and currents. Using § 46 (1) to (3) and the corresponding equation for the current I in L,

$$I_T = I+S, \tag{1}$$

in § 47 (11)–(13) we get the following equations connecting the variable parts of V_p, V_g, I_p, and I:

$$I-I_p = C\frac{dV_p}{dt}, \tag{2}$$

$$-L\frac{dI}{dt}-RI = V_p, \tag{3}$$

$$-M\frac{dI}{dt} = V_g, \tag{4}$$

where in (3) and (4) we have used $V_P = P-RS$ and $V_G = G$.

Also, treating the characteristic of the triode § 46 (6) in the same way, we get

$$I_p = f(V_p+\mu V_g+V_P+\mu V_G)-f(V_P+\mu V_G). \quad (5)$$

Expanding the first term of (5) by Taylor's theorem we get

$$I_p = g(V_p+\mu V_g)+g_1(V_p+\mu V_g)^2+g_2(V_p+\mu V_g)^3+..., \quad (6)$$

where g, g_1, g_2,... are constants determined by the triode and the values of P and G. Retaining only the first term corresponds to the linear assumption § 46 (9). We may simplify (6) further since in practice R is usually small, so that neglecting the term RI in (3)

$$\frac{V_p}{V_g} = \frac{L}{M}. \quad (7)$$

Using (7) in (6) gives

$$I_p = kgV_g+g_1 k^2V_g^2+g_2 k^3 V_g^3+..., \quad (8)$$

where, for shortness, we write

$$k = (L+\mu M)/M. \quad (9)$$

From (2) and (3)

$$LC\frac{d^2I}{dt^2}+RC\frac{dI}{dt}+I = I_p = kgV_g+k^2g_1 V_g^2+k^3g_2 V_g^3+..., \quad (10)$$

using (8). Using (4) in this gives

$$LC\frac{d^2I}{dt^2}+$$

$$+\frac{dI}{dt}\left\{RC+kgM-k^2g_1 M^2\frac{dI}{dt}+k^3g_2 M^3\left(\frac{dI}{dt}\right)^2-...\right\}+I = 0. \quad (11)$$

Alternatively, differentiating (10) with respect to the time and using (4) gives an equation for V_g

$$LC\frac{d^2V_g}{dt^2}+\frac{dV_g}{dt}\{(RC+kgM)+2k^2g_1 MV_g+3k^3g_2 MV_g^2+...\}+V_g = 0. \quad (12)$$

Either (11) or (12) may be chosen as the fundamental type of non-linear differential equation requiring further study. Taking (12) we notice first that it is non-linear because of the terms in V_g, V_g^2,... multiplying dV_g/dt which are caused by the terms in $g_1, g_2,...$ in (8). If we neglect these, (12) becomes linear, and by § 46 (5) its solution is unstable if

$$RC+kgM < 0, \tag{13}$$

that is, using the definition (9) of k, if

$$M < -\frac{RC}{\mu g}-\frac{L}{\mu}, \tag{14}$$

as in § 47 (16). If the circuit is to act as an oscillator M must be negative, and also, by (13), k must be positive. The signs of g_1, g_2, etc., have to be determined from the form of the characteristic (8), and it appears that as this curve is concave upwards for small values of V_g and concave downwards for large ones g_2 must be negative and g_1 positive. It will appear later that the term involving g_1 is not important.

To summarize this information we write

$$\begin{aligned} n^2 &= \frac{1}{LC}, & \epsilon &= -\frac{RC+kgM}{LC}, \\ \alpha &= -\frac{2k^2g_1 M}{RC+kgM}, & \beta &= -\frac{3k^3g_2 M}{RC+kgM}, \end{aligned} \tag{15}$$

and (12) becomes

$$\frac{d^2V_g}{dt^2}-\epsilon(1-\alpha V_g-\beta V_g^2-...)\frac{dV_g}{dt}+n^2V_g = 0, \tag{16}$$

where ϵ and β are certainly positive. Usually ϵ is small, and if the circuit commences to oscillate with a small value of V_g the oscillations will at first increase in amplitude like $\exp(\frac{1}{2}\epsilon t)$. But when V_g becomes so large that $\beta V_g^2 > 1-\alpha V_g$, the sign of the coefficient of dV_g/dt in (16) changes from negative to positive, corresponding to a stable oscillation. This behaviour will be studied further in § 58.

It should be mentioned that the power series (6) is obviously not a particularly good method of representing a curve such as Fig. 41 (*b*); it has been used here because of the importance of equations of types (11) and (12) in non-linear mechanics (cf. § 58). Other methods of approximating to the effect of a non-linear characteristic have also been used.†

† e.g. Pidduck, *Math. Gazette*, **29** (1945), 206, who uses a characteristic of the form $f(x) = 0$, $x < a$; $f(x) = k(x-a)$, $x > a$.

49. Servomechanisms

The basic ideas involved may be understood from Fig. 43. Suppose a heavy rotor of moment of inertia I is to be turned about an axis so that it always follows the motion of a light pointer. We call θ_i, the angular displacement of the pointer from a fixed direction, the *input displacement*: this is a prescribed

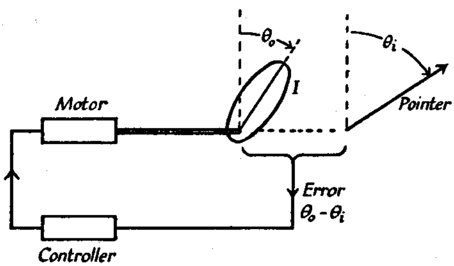


FIG. 43.

function of the time. The angular displacement of the rotor from the same direction, θ_0, is the *output displacement*, and $\theta = \theta_0 - \theta_i$ is the *error*.

In order to make θ_0 follow θ_i, the error θ is fed into a controller which determines the torque T applied to the rotor I by the driving motor. The simplest case is that of 'proportional control' in which T is proportional to the error, so that

$$T = -\lambda(\theta_0 - \theta_i), \tag{1}$$

where λ is a constant. Then, if there is frictional resistance $k\dot{\theta}_0$ to the motion of the rotor, its equation of motion is

$$I\ddot{\theta}_0 = -k\dot{\theta}_0 - \lambda(\theta_0 - \theta_i),$$

or

$$(ID^2 + kD + \lambda)\theta_0 = \lambda\theta_i. \tag{2}$$

The way in which θ_0 follows θ_i may be studied by giving θ_i a simple prescribed motion, calculating θ_0, and comparing the two. Suppose, for example, that θ_i is given a steady rotation

$$\theta_i = \omega t, \tag{3}$$

starting at $t = 0$ when $\theta_0 = D\theta_0 = 0$. The solution of (2) with these initial values is

$$\theta_0 = \omega t - \frac{k\omega}{\lambda} + \frac{k\omega}{\lambda} e^{-\kappa t} \cos n't + \left(\frac{\kappa k\omega}{\lambda n'} - \frac{\omega}{n'}\right) e^{-\kappa t} \sin n't, \tag{4}$$

where $\kappa = k/2I, \qquad n'^2 = \lambda/I - \kappa^2.$

Comparing (3) and (4) it appears that the error $\theta_0 - \theta_i$ contains an oscillation which dies away like $\exp(-\kappa t)$, and that for large values of the time θ_0 lags behind θ_i by $k\omega/\lambda$. If k is increased in order to make the oscillation die away more rapidly, the time lag is also increased. Thus this simple system cannot be made really efficient.

In the next stage of complexity terms depending on the successive derivatives and integrals of the error are included in T. Thus

$$T = -\lambda(\theta_0 - \theta_i) - \lambda_1 \int_0^t (\theta_0 - \theta_i)\, dt \tag{5}$$

has a term proportional to the error, together with a term proportional to the integral of the error. This gives the equation of motion

$$(ID^2 + kD + \lambda)\theta_0 + \lambda_1 \int_0^t \theta_0\, dt = \lambda\theta_i + \lambda_1 \int_0^t \theta_i\, dt, \tag{6}$$

or, differentiating,

$$(ID^3 + kD^2 + \lambda D + \lambda_1)\theta_0 = \lambda D\theta_i + \lambda_1 \theta_i. \tag{7}$$

If, as before, we take $\theta_i = \omega t$ as the input displacement, the particular integral of (7) is just ωt, and so there is now no constant lag as there was in (4).

The question of stability of the system must be studied. By § 47 (9) this requires

$$k\lambda > I\lambda_1, \tag{8}$$

which sets a limit to the coefficients in (5).

In practical systems the controller is in effect a complicated amplifier circuit, and, instead of simple relationships such as (1) or (5), expressions involving the properties of this circuit appear.†

† See, for example, MacColl, *Servomechanisms* (van Nostrand, 1945).

50. Impulsive motion

In § 39 the effect of an impulsive force or blow on a particle was studied: analogous problems arise in electric circuit theory, as might be expected from § 43. We define an impulsive voltage E_0 at $t = 0$ as a voltage $V(t)$ which is very large over the vanishingly small time-interval $0 < t < \tau$, zero outside this interval, and such that its time integral has the finite value E_0, i.e.

$$\int_{-\infty}^{\infty} V(t)\,dt = E_0. \tag{1}$$

In the notation of the δ-function, § 17, this voltage is $E_0\,\delta(t)$.

Ex. 1. *Impulsive voltage E_0 applied at $t = 0$ to an L, R, C circuit with initial current I_i and initial charge Q_i.*

The differential equations, § 41 (4) and (5) are

$$(LD+R)I+\frac{Q}{C} = V(t), \tag{2}$$

$$DQ = I. \tag{3}$$

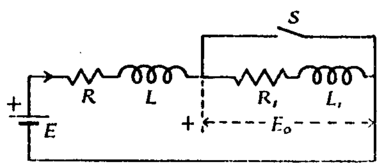


FIG. 44.

As in the analogous problem of § 39 we integrate these over the small time τ. I and Q must be finite, so that their integrals over this time will be of the order of τ. Omitting these small quantities (2) and (3) give

$$L[I]_{t=0}^{t=\tau} = E_0, \qquad [Q]_{t=0}^{t=\tau} = 0. \tag{4}$$

Writing I_f and Q_f for the values of I and Q when $t = \tau$, we get from (4)

$$I_f = I_i+E_0/L, \qquad Q_f = Q_i. \tag{5}$$

The current is thus changed instantaneously by an amount E_0/L [cf. § 39 (3)] by the impulsive voltage, and the new initial current (5) must be used in calculations on the subsequent behaviour of the circuit.

Ex. 2. *Steady current E/R is flowing in the circuit of Fig. 44 with the switch S closed, when at $t = 0$ the switch S is opened.*

In switching problems of this type there is an impulsive redistribution of currents between the inductances analogous to the redistribution of velocity when two masses collide. Suppose there is an impulsive voltage

E_0 over L_1, R_1 caused by the closing of the switch. Then by (5) the currents in L_1, R_1, and L, R, respectively, change to

$$E_0/L_1 \quad \text{and} \quad E/R - E_0/L. \tag{6}$$

Now for $t > 0$ these must be equal, so

$$E_0 = \frac{ELL_1}{R(L+L_1)}, \tag{7}$$

and the initial value of the current in the circuit with the switch open is

$$\frac{EL}{R(L+L_1)}. \tag{8}$$

51. Transient problems. Use of the Laplace transformation

In transient problems differential equations such as those written down earlier have to be solved with given values of the initial currents and charges in the system. The only difficulty is the amount of algebra involved, much of which can be avoided by the use of the Laplace transformation discussed in § 18.

We begin with the fundamental equations § 41 (5), (4) for an L, R, C circuit, namely,

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = V, \tag{1}$$

$$\frac{dQ}{dt} = I. \tag{2}$$

Let $\bar{I}$, $\bar{Q}$, and $\bar{V}$ be the Laplace transforms of I, Q, and V, and let $\overset{0}{I}$ and $\overset{0}{Q}$ be the values of I and Q when $t = 0$. Then the subsidiary equations corresponding to (1) and (2), formed as in § 18, are

$$(Lp+R)\bar{I} + \frac{\bar{Q}}{C} = \bar{V} + L\overset{0}{I}, \tag{3}$$

$$p\bar{Q} = \bar{I} + \overset{0}{Q}. \tag{4}$$

(4) gives
$$\bar{Q} = \frac{\bar{I}}{p} + \frac{\overset{0}{Q}}{p}, \tag{5}$$

and substituting this in (3) gives

$$\left(Lp + R + \frac{1}{Cp}\right)\bar{I} = \bar{V} + L\overset{0}{I} - \frac{\overset{0}{Q}}{Cp}. \tag{6}$$

(6) will be called the *subsidiary equation for an L, R, C circuit*: it gives the Laplace transform of I, and is regarded as fundamental. If Q is needed, it is found from (5).

Ex. 1. As a simple example on the L, R, C circuit, suppose that the condenser has initial charge $\overset{0}{Q}$, and is discharged at $t = 0$ through L and R. Then in (6), $\bar{V} = 0$, $\overset{0}{I} = 0$, and (6) becomes

$$\bar{I} = -\frac{\overset{0}{Q}}{(LCp^2+RCp+1)} = -\frac{\overset{0}{Q}}{LC\{(p+\kappa)^2+n'^2\}}, \tag{7}$$

where $$\kappa = R/2L \quad \text{and} \quad n'^2 = 1/LC-\kappa^2. \tag{8}$$

It follows from § 18, Theorem 1, and § 18 (13), (8), and (15), respectively, that

$$I = -\frac{\overset{0}{Q}}{n'LC}e^{-\kappa t}\sin n't, \qquad \text{if } n'^2 > 0, \tag{9}$$

$$I = -\frac{\overset{0}{Q}t}{LC}e^{-\kappa t}, \qquad \text{if } n'^2 = 0, \tag{10}$$

$$I = -\frac{\overset{0}{Q}}{kLC}e^{-\kappa t}\sinh kt, \qquad \text{if } n'^2 = -k^2 < 0. \tag{11}$$

The negative sign in these indicates that the current is flowing away from the high-voltage side of the condenser.

More complicated circuits are regarded as being built up of L, R, C, or simpler circuits, and sufficient equations can be written down to determine the Laplace transforms of all the currents in a circuit by adding equations of type (6) round closed circuits. When this is done, as in Kirchhoff's first law, only the transforms of applied voltages appear on the right-hand sides together with terms involving the initial conditions in which each inductance is given its own initial current and each condenser its own initial charge. The statement corresponding to Kirchhoff's second law is that the algebraic sum of the Laplace transforms of the currents flowing towards any junction is zero.

Ex. 2. *Voltage V applied at $t = 0$ to the circuit of Fig. 38 (a), the condensers being initially uncharged.*

Choosing currents as shown in Fig. 38 (a) (omitting the dashes, i.e. I_1 is the current in OP, etc.), Kirchhoff's second law is satisfied automatically. Then the subsidiary equations corresponding to Kirchhoff's first law for the circuits *PABOP*, *OEFPO*, *USQFEU*, *PTSQP*, respectively, are

$$\tfrac{1}{2}R\bar{I}_1+\frac{1}{Cp}(\bar{I}_1+\bar{I}_3) = -\bar{V}, \tag{12}$$

$$r\bar{I}+\frac{1}{Cp}\bar{I}_3-\tfrac{1}{2}R\bar{I}_1 = 0, \tag{13}$$

$$\frac{1}{2Cp}\bar{I}_2+R(\bar{I}_3-\bar{I})-r\bar{I} = 0, \tag{14}$$

$$\frac{1}{Cp}(\bar{I}_1+\bar{I}_3)+R(\bar{I}_3-\bar{I}_2-\bar{I})+R(\bar{I}_3-\bar{I})+\frac{1}{Cp}\bar{I}_3 = 0. \tag{15}$$

Writing $$\omega_0 = 1/RC, \quad k = r/R, \tag{16}$$

and solving for $\bar{I}$ we get

$$\bar{I} = -\frac{\bar{V}(p^2+\omega_0^2)}{R\{kp^2+2p\omega_0(1+2k)+(2+k)\omega_0^2\}}. \tag{17}$$

Suppose, for example, that

$$V = \sin\omega_0 t, \qquad \bar{V} = \frac{\omega_0}{p^2+\omega_0^2}; \tag{18}$$

then from (17)

$$\bar{I} = -\frac{\omega_0}{R\{kp^2+2p\omega_0(1+2k)+\omega_0^2(2+k)\}}. \tag{19}$$

Therefore $$I = \frac{\omega_0}{kR(\lambda_2-\lambda_1)}\{e^{\lambda_1 t}-e^{\lambda_2 t}\}, \tag{20}$$

where λ_1 and λ_2 are the roots (both real and negative) of

$$kp^2+2p\omega_0(1+2k)+\omega_0^2(2+k) = 0.$$

The current I thus dies away exponentially and has no component of frequency $\omega_0/2\pi$. In the steady-state treatment of § 45 it was found that this frequency was 'stopped' by the circuit: here we have verified this and found the transient caused by switching on the voltage.

It is instructive to compare the treatment of this section with that of § 45. If the bars in (12) to (17) are replaced by dashes, $\bar{V}$ is replaced by E' and p by $i\omega$, we get the equations § 45 (3) to (8). By reversing the process we may pass from § 45 (8) to equation (17) above which gives the complete transient solution. If the condensers are initially charged, there will be other terms on the right-hand sides of (12)–(15) and this simple correspondence no longer holds. There is a general correspondence between the Laplace transformation and the steady-state theory which is developed in works on the subject. As another example we notice that if $\overset{0}{I} = \overset{0}{Q} = 0$, (6) may be written

$$z(p)\bar{I} = \bar{V}, \tag{21}$$

where
$$z(p) = Lp+R+\frac{1}{Cp} \tag{22}$$

is called the *generalized impedance* of the L, R, C circuit. The complex impedance, defined in § 44 (6), is just $z(i\omega)$. The rules given in § 44 (13), (14), (15) for writing down complex impedances also apply to generalized impedances.

Finally we remark that it can be shown that the Laplace transformation method always gives the correct result when applied to switching problems such as those of § 50 in which the currents or charges in a circuit redistribute themselves instantaneously at the instant when a switch is opened or closed.

Ex. 3. *The circuit of Fig. 31. At $t = 0$, when steady current E/R_1 is flowing in the primary from a battery of voltage E, and the secondary current is zero, the primary circuit is opened.*

We have $\overset{0}{I_1} = E/R_1$, $\overset{0}{I_2} = 0$, and the subsidiary equation for the secondary § 42 (24) gives

$$Mp\bar{I}_1+(L_2p+R_2)\bar{I}_2 = ME/R_1. \tag{23}$$

Now $I_1 = 0$ for $t > 0$ and so

$$\bar{I}_1 = \int\limits_0^\infty e^{-pt}I_1\,dt = 0.$$

Using this in (23) gives

$$\bar{I}_2 = \frac{ME}{R_1(L_2p+R_2)},$$

$$I_2 = \frac{ME}{R_1L_2}e^{-R_2t/L_2}. \tag{24}$$

The solution (24) shows that the secondary current changes discontinuously from zero to ME/R_1L_2 at the instant the primary circuit is broken.

Ex. 4. *Impulsive voltage E_0 applied at $t = 0$ to an L, R, C circuit with zero initial current and charge.*

The voltage V in (1) is $E_0\,\delta(t)$, and using § 18 (10) the subsidiary equation is

$$\left(Lp+R+\frac{1}{Cp}\right)\bar{I} = E_0. \tag{25}$$

Thus, by (8),

$$\bar{I} = \frac{E_0\{(p+\kappa)-\kappa\}}{L\{(p+\kappa)^2+n'^2\}},$$

$$I = \frac{E_0}{L}e^{-\kappa t}\left\{\cos n't-\frac{\kappa}{n'}\sin n't\right\}. \tag{26}$$

EXAMPLES ON CHAPTER V

1. A condenser of capacitance C is charged to voltage E_0 and discharged at $t = 0$ into a non-inductive resistance R. Show that the charge on the condenser at time t is

$$CE_0\,e^{-t/RC}.$$

2. A battery of voltage E_0 is applied at $t = 0$ to a circuit consisting of inductance L and capacitance C in series. Show that, if the initial charge and current are zero, the current in the circuit at time t is

$$(E_0/nL)\sin nt,$$

where $n = (LC)^{-\frac{1}{2}}$. Show also that if the circuit is opened at the instant π/n when the current changes sign, the condenser will then be charged to voltage $2E_0$.

3. A circuit consists of an inductive resistance L, R, a capacitance C, and a resistance $1/G$ in parallel (this corresponds to the effect of a leaky condenser). Show that its natural frequency of oscillation is

$$\frac{1}{2\pi}\left[\frac{1}{LC}-\frac{1}{4}\left(\frac{R}{L}-\frac{G}{C}\right)^2\right]^{\frac{1}{2}},$$

and that the oscillations have the damping factor $\exp[-(R/2L+G/2C)t]$.

4. A circuit consists of three branches in parallel. Two of them contain equal inductive resistances L, R, and the third a capacitance C. Show that the general solution contains an exponentially decaying term, together with a damped oscillation of frequency $(2n^2-\kappa^2)^{\frac{1}{2}}/2\pi$ and damping factor $\exp(-\kappa t)$, where $n^2 = 1/LC$, $\kappa = R/2L$. What is the mechanical analogue of the circuit?

5. A circuit consists of inductance L and capacitance C in series with a battery of voltage E_0. At $t = 0$, $I = I_0$, and the condenser is uncharged. Show that if I_1 is the current at the time when the voltage drop over the condenser attains the value $V < E_0$, then

$$E_0^2+(I_0/nC)^2 = (E_0-V)^2+(I_1/nC)^2,$$

where $n^2 = 1/LC$. Discuss the behaviour of such a circuit in which the condenser is automatically discharged when the voltage drop over it attains the value V.

6. The primary circuit of a transformer consists of inductance L_1 and capacitance C_1 in series, and the secondary consists of inductance L_2 and capacitance C_2 in series. There is mutual inductance M between

L_1 and L_2. Find an equation for the natural frequencies, and show that in the special case $L_1 = L_2$, $C_1 = C_2$, these are

$$n(1+k)^{-\frac{1}{2}}/2\pi \quad \text{and} \quad n(1-k)^{-\frac{1}{2}}/2\pi,$$

where $n^2 = 1/L_1 C_1$ and $k = M/L_1$.

7. The primary of a transformer contains inductance L_1, resistance R_1, and capacitance C_1 in series, and the secondary L_2, R_2, C_2 in series. They are coupled by mutual inductance M. Alternating voltage $E'e^{i\omega t}$ is applied to the primary; show that the steady-state secondary current is

$$(E'/Z_0)e^{i(\omega t+\delta-\pi)},$$

where

$$Z_0 e^{i\delta} = \{R_1 X_2 + R_2 X_1 + i(M^2\omega^2 + R_1 R_2 - X_1 X_2)\}/M\omega,$$

and $$X_1 = L_1\omega - 1/C_1\omega, \qquad X_2 = L_2\omega - 1/C_2\omega.$$

8. Show that the complex impedance of the circuit of Fig. 29 with $C_1 = C_2 = C_3$, $L_1 = L_2 = 0$, $R_1 = R_2$ is

$$\frac{4R_1 C_1\omega + i(R_1^2 C_1^2\omega^2 - 3)}{2C_1\omega + R_1 C_1^2\omega^2 i}.$$

9. The branches AB and CD of a circuit $ABCDA$ contain resistance R, and the branches BC and DA contain capacitance C. Show that if complex voltage E' is applied to the terminals A and C, the complex voltage drop between D and B is $E'\exp(2\gamma i)$, where $\gamma = \tan^{-1}(1/RC\omega)$. The magnitude of the voltage across D and B is equal to that of the applied voltage, but its phase may be varied at will by changing R or C.

10. Show that if complex voltage E' is applied to a three-stage ($n = 3$) filter circuit, Fig. 39 (a), with z_1 a capacitance C, z_2 an inductance L, and $z' = 0$, the complex output current I_3' is

$$I_3' = 4i\omega^7 E'\{L\omega_0^2(\omega^2-\omega_0^2)(3\omega^2-\omega_0^2)(4\omega^2-\omega_0^2)\}^{-1},$$

where $\omega_0^2 = 1/LC$. Show that the filter cuts off frequencies less than $\frac{1}{2}\omega_0$, i.e. it is a 'high-pass' filter.

11. A telephone line has resistance R_1 between posts and a leakage resistance R_2 at each post. If a battery of voltage E is connected to it half-way between two posts and the line is broken between the nth and $(n+1)$th posts [$z' \to \infty$ in § 45 (16)], show that the battery current is

$$\frac{E\sinh n\theta}{R_2\sinh\theta\cosh n\theta},$$

where $\cosh\theta = 1+(R_1/2R_2)$.

12. Show that $|I_n'|$, the current in the terminal impedance of the filter circuit of Fig. 39 (a), decreases exponentially as n is increased unless ω is such that θ defined in § 45 (14) is pure imaginary, and that the condition for this is that $\cosh\theta$ lie between ± 1.

For the circuit of Fig. 39 (a) in which z_1 consists of inductance L_1 and capacitance C_1 in series, while z_2 consists of inductance L_2, show that this criterion is satisfied if $(L_1 C_1 + 4L_2 C_1)^{-\frac{1}{2}} < \omega < (L_1 C_1)^{-\frac{1}{2}}$. The filter is a 'band-pass' filter and this is its pass band.

13. If the resistance-coupled amplifier of Fig. 41 is operating on a linear portion of its characteristic and the grid voltage is suddenly changed by E_0, show that the resulting change of voltage over FG is

$$-\frac{R_1 R_2 \mu g E_0}{(R_1+R_2+R_1 R_2 g)} \exp\left\{-\frac{(R_1 g+1)t}{C(R_1+R_2+R_1 R_2 g)}\right\}.$$

14. A valve circuit is that of Fig. 42 with the following modifications: (*i*) the condenser C is absent; (*ii*) there is no mutual inductance between the inductance L and the inductance L_1 in the grid circuit, and the resistances of these inductances are negligible; (iii) there is a capacitance C_1 between the grid and the plate, it being assumed, as usual, that no current flows to the grid. Show that the circuit will act as an oscillation generator if $L < \mu L_1$, and find the corresponding result if the resistances of L_1 and L_2 are not neglected.

15. In the servomechanism of Fig. 43 the torque applied to the rotor is arranged to be $-\lambda\theta - k\dot{\theta}$, where $\theta = \theta_0 - \theta_i$ is the error (cf. § 49). Show that if the system is initially at rest and the pointer is given a displacement $\theta_i = \Omega t$, the displacement of the rotor is

$$\Omega t - (\Omega/n')e^{-\kappa t}\sin n't,$$

where $\kappa = 2k/I$, $n'^2 = (\lambda/I) - \kappa^2$. Friction is neglected.

16. In the circuit of Fig. 29 with $C_1 = C_2 = C_3$, $R_1 = R_2$, $L_1 = L_2 = 0$, the condenser C_1 is charged to voltage E_0 and discharged into the system at $t = 0$, the other two condensers being then uncharged. Show that the charge on the condenser C_1 at any time is

$$C_1 E_0\{2+3e^{-\alpha t}+e^{-3\alpha t}\}/6,$$

where $\alpha = 1/R_1 C_1$.

17. Voltage $E\sin(\omega t+\beta)$ is applied at $t = 0$ to an inductance L and a resistance R in series, the initial current in the inductance being zero. Show that the current at time t is

$$E\{\sin(\gamma-\beta)e^{-Rt/L}+\sin(\omega t+\beta-\gamma)\}(R^2+L^2\omega^2)^{-\frac{1}{2}},$$

where $\gamma = \tan^{-1}(L\omega/R)$.

18. An inductive resistance L, R is carrying steady current E_0/R from a battery of voltage E_0, when at $t = 0$ an inductive resistance L_1, R_1 is switched into series with it. Show that the subsequent battery current is

$$\frac{E_0}{R+R_1} - \frac{E_0(RL_1-LR_1)}{R(L+L_1)(R+R_1)} e^{-t(R+R_1)/(L+L_1)}.$$

VI

VECTORS

52. Vector algebra

In applied mathematics we are concerned with two different types of quantity: firstly, quantities such as temperature, mass, electric charge, etc., which are specified by their magnitude only and are called scalars, and secondly, quantities such as force, velocity, electric and magnetic field strength, etc., with which are associated both magnitude and a direction in space. It is for the handling of these latter that vector analysis has been developed. Its place in mathematical teaching is still a matter of argument; on the one hand it is possible to work exclusively in rectangular Cartesian coordinates or others of similar type—most of the classical English text-books do this and the student must be prepared to follow arguments set out in this way; on the other hand, vector methods do provide a conspicuous simplification in many problems of three-dimensional statics, dynamics, and electromagnetism, though, again, it should be said that they are not sufficiently powerful to handle many of the more complicated problems and that for these still other methods have been developed.

The most common practice, and that which will be adopted here, is to use vector analysis as a tool in subjects for which it is well adapted. Its main virtues from the present point of view are: (i) that it provides a useful physical picture so that it is better to *think* of quantities such as force, angular velocity, electric field, etc., as vectors even if manipulation is done with their components, and (ii) that the vector product arises naturally in fundamental formulae such as § 53 (11) of statics, § 53 (6) of dynamics, and § 60 (23) of electromagnetism, and that by the use of its elementary properties their discussion is greatly simplified.

In this chapter a brief account† of the fundamentals of vector

† For a complete treatment see Milne, *Vectorial Mechanics* (Methuen); Weatherburn, *Elementary Vector Analysis* and *Advanced Vector Analysis* (Bell).

algebra is given, and subsequently vector treatments will be used for problems particularly well adapted to them, while in other cases the vector equivalent of Cartesian formulae will be noted.

A vector $\mathbf{r}$ is defined as a number r associated with a definite direction in space.† The number r is called the magnitude of the

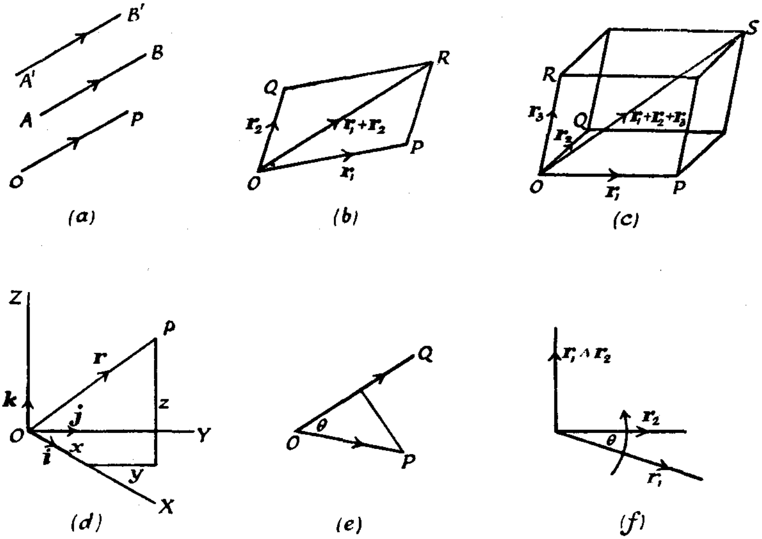


FIG. 45.

vector and is to be positive or zero. A vector may thus be represented by *any* line such as AB in Fig. 45(a) which has length r and is such that the direction from A to B is parallel to the given direction and is in the same sense. Any other line equal and parallel to AB represents the vector $\mathbf{r}$ equally well, for example, $A'B'$, and, in particular, just one such line OP can be drawn from a chosen origin O.

The sum of two vectors. If OP and OQ, drawn from an origin O, represent the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, then the sum $\mathbf{r}_1+\mathbf{r}_2$ is defined

† Clarendon type is usually used in print for vectors. In manuscript a bar or a wavy line above or below the letter is convenient; if written below it cannot be confused with the bar used to denote an average or a Laplace transform. $|\mathbf{r}|$ is sometimes written for the magnitude of $\mathbf{r}$. Normally the space with which we are dealing is three-dimensional; the results, except those involving vector products, still hold in two dimensions.

as the vector represented by the diagonal OR of the parallelogram $OPRQ$; cf. Fig. 45 (b). The sum of three non-coplanar vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, represented by OP, OQ, OR in Fig. 45 (c), will be represented by the diagonal OS of the parallelepiped with sides OP, OQ, OR.

Algebraic laws such as

$$\mathbf{r}_1+\mathbf{r}_2 = \mathbf{r}_2+\mathbf{r}_1, \tag{1}$$

$$\mathbf{r}_1+(\mathbf{r}_2+\mathbf{r}_3) = (\mathbf{r}_1+\mathbf{r}_2)+\mathbf{r}_3 = \mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3 \tag{2}$$

follow immediately from the geometry of Figs. 45 (b) and (c).

$(-\mathbf{r})$ is defined as a vector of the same magnitude r as $\mathbf{r}$, but oppositely directed. The difference $\mathbf{r}_1-\mathbf{r}_2$ is defined as $\mathbf{r}_1+(-\mathbf{r}_2)$.

The product $k\mathbf{r}$ of a number k with a vector $\mathbf{r}$ is defined as a vector whose magnitude is $|k|r$, and which is in the same direction as $\mathbf{r}$ if $k > 0$, and in the opposite direction to $\mathbf{r}$ if $k < 0$.

An immediate consequence of the parallelogram law of addition is that a vector represented by OS, Fig. 45 (c), may be expressed as a sum of vectors represented by OP, OQ, OR in three chosen (non-coplanar) directions. These vectors are called the components of the vector in the chosen directions. The most important case is that in which the directions are mutually perpendicular. Suppose that OX, OY, OZ are a right-handed system of rectangular axes, and that $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are vectors of unit magnitude ('unit vectors') in these directions, Fig. 45 (d). Then if OP represents a vector $\mathbf{r}$ of magnitude r, and (x,y,z) are the Cartesian coordinates of P relative to OX, OY, OZ, we have

$$\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}. \tag{3}$$

In future we shall always use right-handed rectangular axes, and for shortness will describe x, y, z as the 'components' of the vector $\mathbf{r}$; strictly, of course, the components, as defined above, are $x\mathbf{i}$, $y\mathbf{j}$, $z\mathbf{k}$. The magnitude r of the vector $\mathbf{r}$ is

$$(x^2+y^2+z^2)^{\frac{1}{2}}. \tag{4}$$

If x_1, y_1, z_1 are the components of another vector $\mathbf{r}_1$ relative to the same axes, the components of the sum $\mathbf{r}+\mathbf{r}_1$, are $x+x_1$, $y+y_1$, $z+z_1$.

The product of two vectors. The choice of a definition is rather arbitrary, since any expression which involves the product of the magnitudes of the vectors might be regarded as a product. Two of these expressions are chosen because of their usefulness: they are called the 'scalar' (or 'dot') product, and the 'vector' (or 'cross') product.

If two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are represented by OP and OQ, Fig. 45 (e), the angle θ such that $0 \leqslant \theta \leqslant \pi$ between OP and OQ is called the angle between the vectors. The *scalar product* $\mathbf{r}_1 . \mathbf{r}_2$ of the two vectors is then defined by

$$\mathbf{r}_1 . \mathbf{r}_2 = r_1 r_2 \cos \theta. \tag{5}$$

As its name implies, it is a scalar or pure number. It follows immediately that if the two vectors are perpendicular their scalar product is zero. Also that

$$\mathbf{r}_1 . \mathbf{r}_2 = \mathbf{r}_2 . \mathbf{r}_1, \tag{6}$$

and that

$$\mathbf{r}_1 . \mathbf{r}_1 = r_1^2. \tag{7}$$

Finally, it is easy to show geometrically that (cf. Ex. 1)

$$\mathbf{r}_1 . (\mathbf{r}_2 + \mathbf{r}_3) = \mathbf{r}_1 . \mathbf{r}_2 + \mathbf{r}_1 . \mathbf{r}_3. \tag{8}$$

For the system of mutually perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ introduced above the scalar products are

$$\mathbf{i} . \mathbf{i} = \mathbf{j} . \mathbf{j} = \mathbf{k} . \mathbf{k} = 1, \tag{9}$$

$$\mathbf{i} . \mathbf{j} = \mathbf{j} . \mathbf{k} = \mathbf{k} . \mathbf{i} = 0. \tag{10}$$

If x_1, y_1, z_1 and x_2, y_2, z_2 are the components of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to the axes of Fig. 45 (d), we have by (3)

$$\begin{aligned} \mathbf{r}_1 . \mathbf{r}_2 &= (x_1 \mathbf{i} + y_1 \mathbf{j} + z_1 \mathbf{k})(x_2 \mathbf{i} + y_2 \mathbf{j} + z_2 \mathbf{k}) \\ &= x_1 x_2 \mathbf{i} . \mathbf{i} + y_1 y_2 \mathbf{j} . \mathbf{j} + z_1 z_2 \mathbf{k} . \mathbf{k} + y_1 z_2 \mathbf{j} . \mathbf{k} + z_1 y_2 \mathbf{k} . \mathbf{j} + \dots \\ &= x_1 x_2 + y_1 y_2 + z_1 z_2, \end{aligned} \tag{11}$$

using (9) and (10).

The vector product $\mathbf{r}_1 \wedge \mathbf{r}_2$ of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is defined as the vector whose direction is perpendicular to the plane of $\mathbf{r}_1$ and $\mathbf{r}_2$ and in the sense of the progression of a right-handed screw rotating from $\mathbf{r}_1$ towards $\mathbf{r}_2$, and whose magnitude is

$$r_1 r_2 \sin \theta, \tag{12}$$

where θ is the angle between the directions of $\mathbf{r}_1$ and $\mathbf{r}_2$ [Fig. 45 (f)].

It follows that† $\quad \mathbf{r}_2 \wedge \mathbf{r}_1 = -\mathbf{r}_1 \wedge \mathbf{r}_2. \qquad (13)$

For the unit vectors of Fig. 45 (d) the vector products are

$$\mathbf{i} \wedge \mathbf{i} = \mathbf{j} \wedge \mathbf{j} = \mathbf{k} \wedge \mathbf{k} = 0, \tag{14}$$

$$\mathbf{i} \wedge \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \wedge \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \wedge \mathbf{i} = \mathbf{j}. \tag{15}$$

It is easy to show that (cf. Ex. 2)

$$\mathbf{r}_1 \wedge (\mathbf{r}_2 + \mathbf{r}_3) = \mathbf{r}_1 \wedge \mathbf{r}_2 + \mathbf{r}_1 \wedge \mathbf{r}_3. \tag{16}$$

If $\mathbf{r}_1$ and $\mathbf{r}_2$ are vectors of components x_1, y_1, z_1 and x_2, y_2, z_2, we have by (3)

$$\begin{aligned} \mathbf{r}_1 \wedge \mathbf{r}_2 &= (x_1 \mathbf{i} + y_1 \mathbf{j} + z_1 \mathbf{k}) \wedge (x_2 \mathbf{i} + y_2 \mathbf{j} + z_2 \mathbf{k}) \\ &= (y_1 z_2 - z_1 y_2)\mathbf{i} + (z_1 x_2 - x_1 z_2)\mathbf{j} + (x_1 y_2 - y_1 x_2)\mathbf{k}, \end{aligned} \tag{17}$$

on multiplying out and using (14) and (15). This may be written in the form

$$\mathbf{r}_1 \wedge \mathbf{r}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \tag{18}$$

which is the easiest way of remembering the result.

The vector product of two vectors being a vector, the scalar and vector products of it with a third vector will be of importance. These are called scalar and vector triple products.

Using (11) and (18) it follows that the scalar triple product

$$\mathbf{r}_1 . (\mathbf{r}_2 \wedge \mathbf{r}_3) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \tag{19}$$

and thus from the properties of the determinant in (19) that

$$\mathbf{r}_1 . (\mathbf{r}_2 \wedge \mathbf{r}_3) = \mathbf{r}_3 . (\mathbf{r}_1 \wedge \mathbf{r}_2) = \mathbf{r}_2 . (\mathbf{r}_3 \wedge \mathbf{r}_1). \tag{20}$$

† That is, the commutative law of multiplication does not hold. In developing any calculus of this sort, it is necessary to prove the laws of elementary algebra as in (1), (2), (6), (8), (16), etc.; this makes the early stages tedious, but it is essential since exceptions such as (13) do occur. The same point arises in connexion with the operator D; cf. § 12.

The vector triple product is by (18)

$$\mathbf{r}_1 \wedge (\mathbf{r}_2 \wedge \mathbf{r}_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ y_2 z_3 - y_3 z_2 & z_2 x_3 - z_3 x_2 & x_2 y_3 - y_2 x_3 \end{vmatrix}$$

$$= \mathbf{i}\{x_2(x_3 x_1 + y_3 y_1 + z_3 z_1) - x_3(x_1 x_2 + y_1 y_2 + z_1 z_2)\} +$$
$$+ \mathbf{j}\{y_2(x_3 x_1 + y_3 y_1 + z_3 z_1) - y_3(x_1 x_2 + y_1 y_2 + z_1 z_2)\} +$$
$$+ \mathbf{k}\{z_2(x_3 x_1 + y_3 y_1 + z_3 z_1) - z_3(x_1 x_2 + y_1 y_2 + z_1 z_2)\}$$
$$= \mathbf{r}_2(\mathbf{r}_3 . \mathbf{r}_1) - \mathbf{r}_3(\mathbf{r}_1 . \mathbf{r}_2). \qquad (21)$$

This result is of the greatest importance. The reason for its form may be seen from the following considerations: $\mathbf{r}_2 \wedge \mathbf{r}_3$ is perpendicular to the plane of $\mathbf{r}_2$ and $\mathbf{r}_3$, and since the vector product $\mathbf{r}_1 \wedge (\mathbf{r}_2 \wedge \mathbf{r}_3)$ is perpendicular to $\mathbf{r}_2 \wedge \mathbf{r}_3$ it must lie in the plane of $\mathbf{r}_2$ and $\mathbf{r}_3$, that is, there must be a relation of type

$$\mathbf{r}_1 \wedge (\mathbf{r}_2 \wedge \mathbf{r}_3) = \alpha \mathbf{r}_2 + \beta \mathbf{r}_3,$$

where α and β are scalars. The result† (21) gives the values of α and β.

53. The vector quantities of applied mathematics

The vector algebra developed in § 52 is really the pure mathematics of a type of generalized number. We now have to consider its relation to the mathematics of the physical quantities in which we are interested. The first stage in the development of any branch of applied mathematics consists essentially of establishing the vectorial character of many of the fundamental quantities which occur in it. This process needs considerable care, not only since this character is different for different quantities, but also because it does not follow without proof (and in fact is not always true) that if a quantity can be specified by a vector the combined effect of two such quantities is the same as the effect of the quantity specified by the sum of the vectors. In this section we sketch the foundations of statics and dynamics, very briefly, from this point of view.

Vectors as defined in § 52 only specify a magnitude and a

† Many direct proofs of (21) are available, cf. *Math. Gazette*, **23** (1939), 35; **33** (1949), 125, 212; Milne, loc. cit., § 31.

direction:† some physical quantities can be completely described in this way, for example a translation of a rigid body without rotation (i.e. in such a way that the displacements of all points of the body are equal and parallel); others are specified by a magnitude and a direction *at a point*, and for these a vector is required to specify the magnitude and direction, and another to specify the point.

(i) *Position.* The position of a point P relative to an origin O can be specified by a vector $\mathbf{r}$ of length and direction given by OP. As in Fig. 45 (*a*), OP is the one line which can be drawn from O which represents the vector $\mathbf{r}$: it will be called a *position vector*.

If the position vector of O relative to another origin O' is $\mathbf{r}'$, then the position vector of P relative to O' is $\mathbf{r}+\mathbf{r}'$.

(ii) *Velocity and acceleration.* To introduce velocity we have to define differentiation of a vector with respect to a scalar quantity, which in our case will be the time t. If the position vector of a moving point P relative to an origin O is $\mathbf{r}$ at time t and $\mathbf{r}+\delta\mathbf{r}$ at time $t+\delta t$, we define the velocity $\mathbf{v}$ or $\dot{\mathbf{r}}$ as

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t}. \tag{1}$$

From its definition, cf. Fig. 46 (*a*), the velocity of the point P can be represented by a vector directed along the tangent to its path. The acceleration $\ddot{\mathbf{r}}$ is defined similarly as $d\mathbf{v}/dt$.

Rules for differentiating sums and products of vectors are proved in the same way as for scalars. Thus,

$$\frac{d}{dt}(\mathbf{r}_1+\mathbf{r}_2) = \dot{\mathbf{r}}_1+\dot{\mathbf{r}}_2, \tag{2}$$

$$\frac{d}{dt}(\mathbf{r}_1\,.\,\mathbf{r}_2) = \mathbf{r}_1\,.\,\dot{\mathbf{r}}_2+\dot{\mathbf{r}}_1\,.\,\mathbf{r}_2, \tag{3}$$

$$\frac{d}{dt}(\mathbf{r}_1\wedge\mathbf{r}_2) = \mathbf{r}_1\wedge\dot{\mathbf{r}}_2+\dot{\mathbf{r}}_1\wedge\mathbf{r}_2. \tag{4}$$

† They are often called *free vectors* to emphasize the fact that they are not restricted to any particular line and to distinguish them from *localized vectors* which have a definite line of action. Thus a couple in statics is a free vector, and a force a localized vector.

It follows from (2) that velocities combine according to the law of vector addition, that is, if $\dot{\mathbf{r}}_1$ is the velocity of a point P relative to an origin O, and $\dot{\mathbf{r}}_2$ is the velocity of O relative to another origin O', then $\dot{\mathbf{r}}_1+\dot{\mathbf{r}}_2$ is the velocity of P relative to O'.

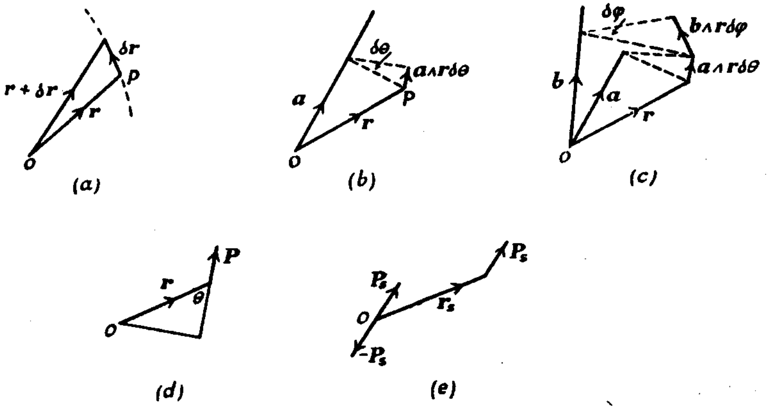


FIG. 46.

(iii) *Rotation.* Suppose a rigid body is rotated through a small angle $\delta\theta$ about an axis through the origin O whose direction is specified by a unit vector $\mathbf{a}$. Then the displacement of the point P whose position vector relative to O is $\mathbf{r}$ is given in magnitude and direction by

$$\mathbf{a}\wedge\mathbf{r}\,\delta\theta \tag{5}$$

[cf. Fig. 46 (*b*)] provided $\delta\theta$ is so small that $\delta\theta^2$ is negligible.

If the body is rotating about the axis, so that $\delta\theta$ is the angle turned through in time δt and

$$\dot{\theta} = \lim_{\delta t\to 0}\frac{\delta\theta}{\delta t},$$

then, taking the limit as $\delta t \to 0$, (5) gives for the velocity $\mathbf{v}$ of the point P

$$\mathbf{v} = \boldsymbol{\omega}\wedge\mathbf{r}, \tag{6}$$

where $\boldsymbol{\omega} = \mathbf{a}\dot{\theta}$. $\boldsymbol{\omega}$ is called the angular velocity of the body. Returning to (5), suppose that following the rotation $\delta\theta$ about $\mathbf{a}$, the whole system, including the axis $\mathbf{a}$, is given another small rotation $\delta\phi$ about an axis through O specified by the unit vector **b**. The displacement of P due to the combined effect of these two small rotations is given by

$$\mathbf{a}\wedge\mathbf{r}\,\delta\theta+\mathbf{b}\wedge\mathbf{r}\,\delta\phi = (\mathbf{a}\,\delta\theta+\mathbf{b}\,\delta\phi)\wedge\mathbf{r}, \tag{7}$$

cf. Fig. 46 (c), and thus is the same as the displacement caused by a single rotation specified by the vector sum† $\mathbf{a}\,\delta\theta+\mathbf{b}\,\delta\phi$. Thus small rotations combine as vectors, and in the same way angular velocities combine as vectors.

The most general displacement of a rigid body can be specified by the displacement of a marked point on it from a fixed origin, together with a rotation of the body about an axis through the marked point (cf. Ex. 10). Thus its motion will be specified by the velocity **v** of the marked point and the angular velocity $\boldsymbol{\omega}$ of rotation about it.

(iv) *Force*. A force is specified by its magnitude, direction, and line of action. So far as magnitude and direction are concerned, it may be specified by a vector **P**.

It is taken as an experimental fact (or, if preferred, as an axiom of statics) that a force may be regarded as applied at any point of its line of action (or that forces **P** and $-\mathbf{P}$ applied at any two points of this line annul each other). Thus a force is completely specified by **P** and the position vector **r** of any point of its line of action. We shall speak of 'a force **P** applied at a point **r**'.

It is a second experimental fact (or an axiom of statics) that concurrent forces combine as vectors, that is, that forces $\mathbf{P}_1$ and $\mathbf{P}_2$ applied at a point O are equivalent to a force $\mathbf{P}_1+\mathbf{P}_2$ applied at O.

The work done by a force **P** in a small displacement $\delta\mathbf{r}$ of its point of application is

$$\mathbf{P}.\delta\mathbf{r}. \tag{8}$$

(v) *The equation of motion of a particle*. If the resultant of the

† The need for care is illustrated by the fact that this is not true of *finite* rotations. Such rotations can be *specified* by vectors, but the combined effect of two finite rotations is not equal to that of a single rotation corresponding to the sum of the vectors. In some treatments of vectors the parallelogram law of addition is included in the definition, and before any quantity is regarded as a vector quantity it has to be verified that such quantities combine according to the parallelogram law.

forces on a particle of constant mass m is $\mathbf{P}$, Newton's second law gives the equation of motion

$$\frac{d}{dt}(m\dot{\mathbf{r}}) = m\ddot{\mathbf{r}} = \mathbf{P}. \tag{9}$$

If the mass of the particle is not constant, Newton's law becomes

$$\frac{d}{dt}(m\dot{\mathbf{r}}) = \mathbf{P}, \tag{10}$$

but this is not applicable to all cases; cf. § 63.

(vi) *The moment of a force about a point.* The moment of a force $\mathbf{P}$ about a point O is defined as the vector whose magnitude is P times the perpendicular distance of O from the line of action of $\mathbf{P}$, and whose direction is normal to the plane of O and $\mathbf{P}$ and is related by the right-hand screw law to the direction of turning of $\mathbf{P}$ about O. If $\mathbf{r}$ is the position vector, relative to O, of any point on the line of action of $\mathbf{P}$, the moment is exactly [cf. Fig. 46 (d)]

$$\mathbf{r} \wedge \mathbf{P}. \tag{11}$$

(vii) *The general conditions of statical equilibrium.* Suppose a rigid body is acted on by forces $\mathbf{P}_1$, $\mathbf{P}_2,\ldots,$ $\mathbf{P}_n$ applied at the points $\mathbf{r}_1$, $\mathbf{r}_2,\ldots,$ $\mathbf{r}_n$, respectively. The general conditions for the body to be in equilibrium under this system of forces are

$$\sum_{s=1}^{n} \mathbf{P}_s = 0, \tag{12}$$

$$\sum_{s=1}^{n} \mathbf{r}_s \wedge \mathbf{P}_s = 0. \tag{13}$$

These may be established in either of two ways. Here the equations of motion of a rigid body, § 66 (8) and (9), will be established without reference to them, and (12) and (13) may be deduced from the condition that the body is to remain at rest under the application of these forces.

Alternatively, the system of forces is reduced in the following way. Corresponding to the force $\mathbf{P}_s$ at $\mathbf{r}_s$ we add two forces $\mathbf{P}_s$ and $-\mathbf{P}_s$ at the origin O: these have no effect on the system [cf. Fig. 46 (e)]. The force $\mathbf{P}_s$ at $\mathbf{r}_s$ and the force $-\mathbf{P}_s$ at O are defined to form a couple of moment $\mathbf{r}_s \wedge \mathbf{P}_s$ and the properties of this studied (cf. Ex. 11, 12). When this has been done for all forces, we are left with concurrent forces $\mathbf{P}_1,\ldots, \mathbf{P}_n$ at the origin, and (12) is the condition that the resultant of these vanish, while (13) is the condition that the resultant of the moments of the couples vanish.

(viii) *Linear momentum.* Let m be the mass of a typical particle of a rigid body (or any assemblage of particles), let $\mathbf{r}$ be its position vector relative to an origin and $\dot{\mathbf{r}}$ its velocity. Its linear momentum is defined as $m\dot{\mathbf{r}}$, and, using $\sum$ to denote a summation over all particles of the system, the linear momentum of the system is

$$\sum m\dot{\mathbf{r}}. \tag{14}$$

(ix) *Angular momentum.* If m is a typical particle of any assemblage of particles as in (viii), its linear momentum is $m\dot{\mathbf{r}}$, and the moment of this momentum about the origin O is as in (11)

$$m\mathbf{r}\wedge\dot{\mathbf{r}}.$$

This quantity, summed over all particles of the system, viz.

$$\sum m\mathbf{r}\wedge\dot{\mathbf{r}}, \tag{15}$$

is called the angular momentum of the system about the origin.

(x) *Vector fields.* In a vector field, at each point of space a vector is defined whose magnitude and direction are functions of the position of the point. Electric and magnetic fields are of this type.

(xi) *Vectors in electric circuit theory.* A complex number $z = x+iy = |z|e^{i\theta}$ may be represented by a vector in the (x,y)-plane of magnitude $|z|$ and in a direction inclined at θ to the x-axis. The sum of several complex numbers may be represented by the sum of the corresponding vectors. The product and quotient of two complex numbers z_1 and z_2 may be represented by vectors of magnitudes $|z_1|\times|z_2|$ and $|z_1|/|z_2|$, respectively, in the directions $\theta_1\pm\theta_2$, and so on; in particular, multiplying a complex number by i corresponds to rotating the vector which represents it through 90°.

In this way a representation of the currents and voltages in an electric circuit can be given which has been much used by engineers. Its most important application is to the steady state theory, cf. § 44, where it gives a representation of the complex currents and voltages, I' and E', and the complex impedances z in the various parts of the circuit, which shows the connexions between them very clearly: this may be used either as a diagram for illustrating and proving circuit properties or as a drawing-board method for calculating them numerically. The actual currents and voltages, I and E, are obtained by multiplying I' and E' by $e^{i\omega t}$ and taking the real part: since $e^{i\omega t}$ is represented by a vector of unit length which rotates steadily with angular velocity ω, the connexion between I and E is found by regarding the diagram connecting I' and E' as rotating steadily with angular velocity ω.

EXAMPLES ON CHAPTER VI

1. If OP and OQ represent the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, show that $\mathbf{r}_1.\mathbf{r}_2$ is equal to the product of the length of OP and the projection of OQ on it.

OPR is any triangle. Show that the sum of the projections of OP and PR on any line through O is equal to the projection of OR on it. Deduce § 52 (8).

2. Let OP, OQ, OR, OS represent the vectors $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_2+\mathbf{r}_3$, and $\mathbf{r}_1$, respectively. Let OP', OQ', OR' be the projections of OP, OQ, and OR on the plane through O perpendicular to OS. Show that the vector products of $\mathbf{r}_1$ with $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_2+\mathbf{r}_3$ are represented by r_1 times the lines obtained by rotating OP', OQ', and OR' through 90°. Deduce § 52 (16).

3. If OP, OQ, OR represent $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, show that $\mathbf{r}_1.(\mathbf{r}_2\wedge\mathbf{r}_3)$ is equal in magnitude to the volume of a parallelepiped of sides OP, OQ, OR.

4. Derive § 52 (20) by expressing $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ in terms of their components in the right-handed system of rectangular axes parallel to $\mathbf{r}_2$, $\mathbf{r}_2\wedge\mathbf{r}_3$, and $\mathbf{r}_2\wedge(\mathbf{r}_2\wedge\mathbf{r}_3)$.

5. Show that

(i) $$(\mathbf{r}_1\wedge\mathbf{r}_2).(\mathbf{r}_3\wedge\mathbf{r}_4) = \begin{vmatrix} \mathbf{r}_1.\mathbf{r}_3 & \mathbf{r}_1.\mathbf{r}_4 \\ \mathbf{r}_2.\mathbf{r}_3 & \mathbf{r}_2.\mathbf{r}_4 \end{vmatrix}.$$

(ii) $$(\mathbf{r}_1\wedge\mathbf{r}_2).(\mathbf{r}_1\wedge\mathbf{r}_2) = r_1^2 r_2^2-(\mathbf{r}_1.\mathbf{r}_2)^2.$$

(iii) $$\begin{aligned}(y_1 z_2-z_1 y_2)^2+(z_1 x_2-x_1 z_2)^2+(x_1 y_2-y_1 x_2)^2 \\ = (x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)-(x_1 x_2+y_1 y_2+z_1 z_2)^2.\end{aligned}$$

6. Show that

$$\begin{aligned}(\mathbf{a}\wedge\mathbf{b})\wedge(\mathbf{c}\wedge\mathbf{d}) &= [\mathbf{a}.(\mathbf{c}\wedge\mathbf{d})]\mathbf{b}-[\mathbf{b}.(\mathbf{c}\wedge\mathbf{d})]\mathbf{a} \\ &= [\mathbf{a}.(\mathbf{b}\wedge\mathbf{d})]\mathbf{c}-[\mathbf{a}.(\mathbf{b}\wedge\mathbf{c})]\mathbf{d}.\end{aligned}$$

Deduce that any vector $\mathbf{r}$ can be represented in terms of three non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ by the formula

$$\mathbf{r} = \frac{[\mathbf{r}.(\mathbf{b}\wedge\mathbf{c})]\mathbf{a}+[\mathbf{r}.(\mathbf{c}\wedge\mathbf{a})]\mathbf{b}+[\mathbf{r}.(\mathbf{a}\wedge\mathbf{b})]\mathbf{c}}{\mathbf{a}.(\mathbf{b}\wedge\mathbf{c})}.$$

7. Show that the solution of the vector equation

$$\mathbf{a}x+\mathbf{b}y+\mathbf{c}z = \mathbf{d}$$

is

$$x = \frac{\mathbf{d}.(\mathbf{b}\wedge\mathbf{c})}{\mathbf{a}.(\mathbf{b}\wedge\mathbf{c})},$$

etc., unless the denominator vanishes. Deduce the usual rule for solving linear equations using determinants.

8. Show that the equation of a straight line may be written

$$\mathbf{r} = \mathbf{a}+t\mathbf{b},$$

where $\mathbf{r}$ is the position vector of any point on it. Show that the shortest distance between the above line and the line $\mathbf{r} = \mathbf{a}'+t\mathbf{b}'$ is

$$\frac{\mathbf{b}.(\mathbf{b}'\wedge(\mathbf{a}-\mathbf{a}'))}{|\mathbf{b}\wedge\mathbf{b}'|}.$$

9. Show that

(i) The equation of the plane through $\mathbf{r}_1$ with its normal in the direction of a unit vector $\mathbf{n}$ is

$$(\mathbf{r}-\mathbf{r}_1).\mathbf{n} = 0,$$

where $\mathbf{r}$ is the position vector of any point on the plane.

(ii) If the length of the perpendicular from the origin to the plane is p, the equation of the plane is $\mathbf{n}.\mathbf{r} = p$.

(iii) The equation of the plane through three points whose position vectors are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, is

$$\mathbf{r}.(\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a}+\mathbf{a}\wedge\mathbf{b}) = \mathbf{a}.(\mathbf{b}\wedge\mathbf{c}).$$

10. The point O of a rigid body is fixed. OA and OB are two marked lines in the body, and OA' and OB' are their positions after the body has been moved in any way about O. Show that the body can be brought from its first position to its second by a rotation about the line of intersection of the plane which is perpendicular to the plane AOA' and bisects the angle AOA' with the plane related in the same way to BOB'.

11. A force $\mathbf{P}$ applied at $\mathbf{r}_1$ and a force $-\mathbf{P}$ at $\mathbf{r}_2$ constitute a couple. The sum of the moments of the forces about O is $(\mathbf{r}_1-\mathbf{r}_2)\wedge\mathbf{P}$, which is independent of the position of O and is called the moment of the couple.

(i) Show that the above couple and a couple consisting of $\mathbf{Q}$ at $\mathbf{r}_1$ and $-\mathbf{Q}$ at $\mathbf{r}_2$, where $\mathbf{Q}$ is such that $(\mathbf{r}_1-\mathbf{r}_2)\wedge\mathbf{Q} = -(\mathbf{r}_1-\mathbf{r}_2)\wedge\mathbf{P}$, are in static equilibrium. [Add forces $-\mathbf{P}-\mathbf{Q}$ at $\mathbf{r}_1$, and $\mathbf{P}+\mathbf{Q}$ at $\mathbf{r}_2$.]

(ii) Show that the above couple and a couple consisting of $-\mathbf{P}$ at $\mathbf{r}_1+\mathbf{r}$ and $\mathbf{P}$ at $\mathbf{r}_2+\mathbf{r}$ are in static equilibrium.

(iii) Deduce that a couple is statically equivalent to any couple of the same moment.

12. Show that any two couples are equivalent to a couple whose moment is the sum of their moments.

13. Show that forces $\mathbf{P}_1$ at $\mathbf{r}_1$,..., $\mathbf{P}_n$ at $\mathbf{r}_n$ are equivalent to a force

$$\mathbf{R} = \sum \mathbf{P}$$

at the origin O, together with a couple of moment

$$\mathbf{G} = \sum \mathbf{r} \wedge \mathbf{P}.$$

Deduce the conditions of statical equilibrium, § 53 (12) and (13).

If the origin is moved from O to the point O' whose position vector relative to O is $\mathbf{s}$, show that the resultant force and couple are $\mathbf{R}$ and $\mathbf{G}-\mathbf{s}\wedge\mathbf{R}$.

14. Show that the system of forces in Ex. 13 is equivalent to a single force if and only if $\mathbf{R}.\mathbf{G} = 0$, and that its line of action relative to the origin O is

$$\frac{\mathbf{R}\wedge\mathbf{G}}{R^2}+t\mathbf{R}.$$

15. Show that if the point O' in Ex. 13 is chosen on the line

$$\frac{\mathbf{R}\wedge\mathbf{G}}{R^2}+t\mathbf{R},$$

the couple and force are parallel and the ratio of their magnitudes is

$$(\mathbf{R}.\mathbf{G})/R^2.$$

16. If forces $\mathbf{P}_1,\ldots, \mathbf{P}_n$ act on a rigid body at the points $\mathbf{r}_1,\ldots, \mathbf{r}_n$, respectively, and the body is given a small displacement (which must be consistent with any constraints on the body) consisting of a translation $\delta\mathbf{a}$ and a rotation $\delta\theta$ about an axis specified by the unit vector $\mathbf{n}$, show that the work done by the forces is

$$\sum_{s=1}^{n} \mathbf{P}_s.\delta\mathbf{a}+\sum_{s=1}^{n} \mathbf{n}.(\mathbf{r}_s\wedge\mathbf{P}_s)\,\delta\theta.$$

Deduce that this work is zero if the forces are in equilibrium, and conversely that if the work is zero for all possible small displacements the forces are in equilibrium.

17. Complex voltage E' of frequency $\omega/2\pi$ is applied to an L, R, C circuit. Draw vector diagrams showing the voltage drops over the inductance, resistance, and capacitance, and their relation to E'.

18. Draw vector diagrams representing the combinations of impedances in Fig. 35 (*b*), (*c*), (*d*).

19. The voltage drop V over portion of a circuit carrying steady state alternating current is the real part of $V'e^{i\omega t}$, and the current I in it is the real part of $I'e^{i\omega t}$. The average power P_{av} in this portion of the circuit is defined as the average of VI over a period. Show that if $V' = V_1+iV_2$, $I' = I_1+iI_2$, then

$$P_{av} = \tfrac{1}{2}(V_1I_1+V_2I_2).$$

This provides a meaning for the scalar product of the vectors V' and I'.

VII
PARTICLE DYNAMICS

54. Introductory

In §§ 4, 28 the motion of a particle whose position is specified by a single coordinate, say x, with any laws of force and resistance, was discussed and found to lead to the equation

$$m\ddot{x} = f(x, \dot{x}, t), \tag{1}$$

and frequently to the simpler equation

$$m\ddot{x} = f(x)+g(\dot{x})+h(t). \tag{2}$$

The important special case in which $f(x)$ and $g(\dot{x})$ are proportional to x and $\dot{x}$, respectively, so that (2) becomes linear, has been discussed in the preceding chapters.

We now return to the non-linear equations (1) and (2). There are no general methods for solving these, but two important special cases can be treated in detail, namely those in which the right-hand sides consist of functions of x or $\dot{x}$ only. These cases are considered in §§ 55, 56. Occasionally, exact solutions of more complicated equations can be obtained in the same way; for example, the equation

$$\ddot{x}+\dot{x}^2 f(x)+g(x) = 0 \tag{3}$$

is reduced to a first-order linear equation in v^2 by § 55 (3), but the discussion of such equations has usually been restricted to the study of cases in which the non-linear terms are small. These are treated by the method of successive approximations described in § 58. When the non-linear terms are not small, numerical or graphical integration is often resorted to.

The remainder of the chapter is devoted to a study of the motion of particles in two or more dimensions.

In all cases it is assumed without further statement that the mass of the particles involved is constant. The motion of a particle whose mass varies is considered in § 63.

55. The force a function of position only

In this case the equation of motion § 54 (2) becomes

$$m\ddot{x} = f(x). \tag{1}$$

The methods of this section and the next consist essentially of transforming the second-order equation § 54 (2) into a separable first-order equation in either the velocity $v = \dot{x}$ or in v^2. We have

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} \tag{2}$$

$$= \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx}. \tag{3}$$

Using the form (3) of $\ddot{x}$ we may write (1) in the form

$$\tfrac{1}{2}m\frac{d(v^2)}{dx} = f(x), \tag{4}$$

and thus v^2 is determined in terms of x by a simple integration

$$\tfrac{1}{2}mv^2 = \int f(x)\,dx + C. \tag{5}$$

The arbitrary constant C in (5) is to be determined from the initial conditions. Alternatively, using these, a definite integral for v^2 may be written down immediately; suppose that, when $t = 0$, $x = a$ and $v = V$, then, integrating (4) with respect to x between the limits a and x, we get

$$\tfrac{1}{2}m[v^2]_a^x = \int_a^x f(x)\,dx,$$

or

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mV^2 = \int_a^x f(x)\,dx. \tag{6}$$

A slightly different method of integrating (1) which is often used is as follows: multiply both sides of (1) by $\dot{x}$ which gives

$$m\dot{x}\ddot{x} = \dot{x}f(x); \tag{7}$$

and integrate with respect to the *time* which gives

$$\tfrac{1}{2}m\dot{x}^2 = \int f(x)\frac{dx}{dt}\,dt + C = \int f(x)\,dx + C, \tag{8}$$

as before, since

$$\frac{d}{dt}(\dot{x}^2) = 2\dot{x}\ddot{x}. \tag{9}$$

Equations (5) or (6) give the velocity as a function of position, and are referred to as the *first integral* of the equation of motion. Each of them is the energy equation for the motion and will be discussed from this point of view in § 73. For the present we merely remark that, for motion to be possible, v must be real and thus v^2 given by (6) must be positive.

To complete the solution we have from (6)

$$\frac{dx}{dt} = \pm\left\{V^2+\frac{2}{m}\int\limits_a^x f(x)\,dx\right\}^{\frac{1}{2}}. \tag{10}$$

The sign before the square root is determined by the circumstances of projection, that is, by the sign of dx/dt at the instant $t = 0$ when the particle was set in motion. Finally, integrating (10) between the corresponding limits of 0 to t in t, and a to x in x, gives

$$t = \pm\int\limits_a^x \left\{V^2+\frac{2}{m}\int\limits_a^x f(x)\,dx\right\}^{-\frac{1}{2}} dx. \tag{11}$$

Needless to say, these formulae should not be quoted; the whole process should be gone through in each special case, remembering only the initial step (4) or (7).

Ex. 1. *The simple pendulum.*

A particle of mass m is attached to a light rigid rod of length l, freely hinged at a fixed point O, and moves in a vertical plane. If θ is the inclination of the rod to the vertical, the equation of motion of m is [cf. § 61 (11)]

$$ml\ddot{\theta} = -mg\sin\theta, \tag{12}$$

or

$$\ddot{\theta}+n^2\sin\theta = 0, \tag{13}$$

where

$$n^2 = g/l. \tag{14}$$

For small oscillations $\sin\theta$ in (13) may be replaced by θ, and (13) becomes the linear equation

$$\ddot{\theta}+n^2\theta = 0. \tag{15}$$

Here we do not make this assumption, but solve (13) by the methods described earlier. As in (4) we write (13) in the form

$$\frac{1}{2}\frac{d}{d\theta}(\dot{\theta}^2) = -n^2\sin\theta,$$

so that, integrating,

$$\dot{\theta}^2 = 2n^2\cos\theta+C. \tag{16}$$

Suppose that the pendulum is released from rest at $\theta = \alpha$, that is, when $t = 0$, $\theta = \alpha$, $\dot{\theta} = 0$. Substituting these values in (16) gives $C = -2n^2\cos\alpha$, and thus (16) becomes

$$\dot{\theta}^2 = 2n^2(\cos\theta - \cos\alpha).$$

Therefore
$$\frac{d\theta}{dt} = -n(2\cos\theta - 2\cos\alpha)^{\frac{1}{2}}, \tag{17}$$

where the negative sign has been chosen for the square root in (17) since the particle begins to move backwards. Integrating (17) between the limits 0 and t in t, and α and θ in θ, gives

$$nt = -\int_{\alpha}^{\theta} \frac{d\theta}{(2\cos\theta - 2\cos\alpha)^{\frac{1}{2}}}. \tag{18}$$

This integral cannot be expressed in terms of elementary functions, but, like many that arise in the present context, it is an elliptic integral.

The elliptic integrals $F(k,\phi)$ and $E(k,\phi)$ of the first and second kinds, respectively, are defined by

$$F(k,\phi) = \int_0^{\phi} \frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{1}{2}}} = \int_0^{\sin\phi} \frac{dt}{(1-t^2)^{\frac{1}{2}}(1-k^2t^2)^{\frac{1}{2}}}, \tag{19}$$

$$E(k,\phi) = \int_0^{\phi} (1-k^2\sin^2\psi)^{\frac{1}{2}}\,d\psi = \int_0^{\sin\phi} \left\{\frac{1-k^2t^2}{1-t^2}\right\}^{\frac{1}{2}} dt, \tag{20}$$

and are tabulated functions.† Many integrals can be expressed in terms of them, for example integrals whose integrands are the reciprocals of the square roots of cubics or quartics.

To reduce (18) to one of these forms we write it as

$$2nt = \int_{\theta}^{\alpha} \frac{d\theta}{(\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta)^{\frac{1}{2}}}. \tag{21}$$

In this put
$$\sin\tfrac{1}{2}\theta = \sin\tfrac{1}{2}\alpha\sin\phi,$$

$$\frac{d\theta}{d\phi} = \frac{2\sin\frac{1}{2}\alpha\cos\phi}{\cos\frac{1}{2}\theta} = \frac{2\sin\frac{1}{2}\alpha\cos\phi}{(1-\sin^2\frac{1}{2}\alpha\sin^2\phi)^{\frac{1}{2}}},$$

† Cf. Jahnke-Emde, *Tables of Functions* (Teubner). They give also many formulae for expressing integrals in terms of F and E.

and (21) becomes

$$nt = \int\limits_{\phi}^{\frac{1}{2}\pi} \frac{d\phi}{(1-\sin^2\frac{1}{2}\alpha\sin^2\phi)^{\frac{1}{2}}} \tag{22}$$

$$= F(\sin\tfrac{1}{2}\alpha, \tfrac{1}{2}\pi) - F(\sin\tfrac{1}{2}\alpha, \phi). \tag{23}$$

The period T of the oscillation is

$$T = \frac{4}{n} F(\sin\tfrac{1}{2}\alpha, \tfrac{1}{2}\pi). \tag{24}$$

While (23) and (24) are accurate solutions of the problem in terms of tabulated functions, they are not particularly informative in the sense that they do not show how solutions based on the approximate linear equation (15) go wrong for larger values of the amplitude. This may be seen by expanding the integrand of (22) by the binomial theorem. Suppose we wish to find the effect of the amplitude of the oscillation on the period: by (22) the period T is

$$T = \frac{4}{n}\int\limits_{0}^{\frac{1}{2}\pi} (1-\sin^2\tfrac{1}{2}\alpha\sin^2\phi)^{-\frac{1}{2}}\, d\phi$$

$$= \frac{4}{n}\int\limits_{0}^{\frac{1}{2}\pi} (1+\tfrac{1}{2}\sin^2\tfrac{1}{2}\alpha\sin^2\phi+\tfrac{3}{8}\sin^4\tfrac{1}{2}\alpha\sin^4\phi+\ldots)\, d\phi$$

$$= \frac{4}{n}\left(\frac{\pi}{2}+\frac{1}{2}\frac{\pi}{4}\sin^2\tfrac{1}{2}\alpha+\frac{3}{8}\frac{3\pi}{16}\sin^4\tfrac{1}{2}\alpha+\ldots\right). \tag{25}$$

Neglecting fourth and higher powers of α this gives

$$T = \frac{2\pi}{n}\left(1+\frac{\alpha^2}{16}\right), \tag{26}$$

thus the period increases with increasing amplitude.

Ex. 2. *The anharmonic oscillator.*

The linear or harmonic oscillator has equation of motion

$$\ddot{x} = -n^2 x. \tag{27}$$

In many important cases the law of force is not that of (27), but there is a change of restoring force with displacement which may be expressed by adding higher powers of x to the right-hand side of (27). The next power for which the restoring force is independent of the sign of x is the third, giving

$$\ddot{x} = -n^2 x - bx^3. \tag{28}$$

If b is positive in (28) the restoring force increases steadily above the linear value as the displacement increases; this is the case with a non-linear spring whose stiffness increases with increase of displacement. If b is negative the restoring force falls below the linear value: the case $b = -\frac{1}{6}n^2$, corresponding to retaining the first two terms in the expansion of $\sin\theta$, is often used as a second approximation to the equation of motion (13) of a pendulum.

If a quadratic term in x is included as in

$$\ddot{x} = -n^2x - ax^2, \tag{29}$$

or

$$\ddot{x} = -n^2x - ax^2 - bx^3, \tag{30}$$

the restoring force is not symmetrical about $x = 0$.

Exact solutions can easily be written down. Taking the law (30), suppose that when $t = 0$, $\dot{x} = V$, and $x = 0$. Then, as in (6),

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}V^2 = -\tfrac{1}{2}n^2x^2 - \tfrac{1}{3}ax^3 - \tfrac{1}{4}bx^4, \tag{31}$$

$$v = \{V^2 - n^2x^2 - \tfrac{2}{3}ax^3 - \tfrac{1}{2}bx^4\}^{\frac{1}{2}},$$

and

$$t = \int\limits_0^x \{V^2 - n^2x^2 - \tfrac{2}{3}ax^3 - \tfrac{1}{2}bx^4\}^{-\frac{1}{2}}\,dx. \tag{32}$$

As remarked above, the integral (32) can be expressed in terms of the elliptic integral $F(k, \phi)$ defined in (19), but if higher powers of x than the third appear in the equation of motion (30), this cannot be done.

Ex. 3. *A rigid cone of semi-vertical angle α and mass m, moving with velocity V in the direction of its axis, impinges normally on a plastic substance which provides resistance to motion of the cone which may be represented by a uniform pressure F (the 'flow pressure') over the region of contact.*

Let x be the depth of penetration of the apex of the cone at time t after the instant of contact $t = 0$. Then at $t = 0$ we have $x = 0$, $\dot{x} = V$.

The force on an element of area δA of the surface of the cone is $-F\,\delta A$; the resolved part of this in the direction of the axis is $-F\,\delta A \sin\alpha$, which is just $-F$ times the projection of the area δA on the plane $x = 0$. Thus the total force on the cone in the direction of its axis is $-F$ times the area of the impression made by the cone in the plane $x = 0$, that is

$$-F\pi x^2 \tan^2\alpha.$$

Thus the equation of motion of the cone is

$$m\ddot{x} = -\pi F x^2 \tan^2\alpha. \tag{33}$$

Integrating as in (6) and using the initial conditions, the velocity v of the cone is found to be

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mV^2 - \tfrac{1}{3}\pi F x^3 \tan^2\alpha. \tag{34}$$

The cone comes to rest when $v = 0$, that is, when the depth of penetration is

$$\left(\frac{3mV^2}{2\pi F \tan^2\alpha}\right)^{\frac{1}{3}}.$$

F can be determined by measuring the size of the impression. The time of penetration to any depth can be expressed as an elliptic integral.

Ex. 4. *Collision of equal spheres.*

Suppose a sphere A of mass m and radius a, moving with velocity V along the x-axis, collides with an equal sphere B at rest with its centre on the x-axis. We choose the origins of time and distance so that the spheres touch at $t = 0$, and the centres of the spheres A and B are at $x = 0$ and $x = 2a$ at that time.

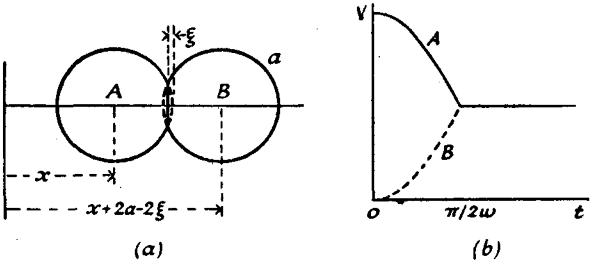


FIG. 47.

In the process of collision a small flat is squashed on either sphere, and, since we have assumed that the spheres are of the same size and material, this squashing will be the same for both and the surface of separation will be plane. Let ξ be the depth of the impression on either sphere at time t, then if x is the position of the centre of the sphere A at time t, that of the sphere B will be $x+2a-2\xi$.

Also, when $t = 0$, $x = 0$, $\xi = 0$, $\dot{x} = V$, and the velocity of the centre of the sphere B is zero, that is

$$\frac{d}{dt}(x+2a-2\xi) = 0,$$

i.e.

$$\dot{\xi} = \tfrac{1}{2}V, \quad \text{when } t = 0. \tag{35}$$

The spheres exert forces on each other across the area of contact: suppose the force is $f(\xi)$, a function of the depth of the impression which will be discussed later. The equations of motion of the spheres A and B are

$$m\ddot{x} = -f(\xi), \tag{36}$$

$$m(\ddot{x}-2\ddot{\xi}) = f(\xi). \tag{37}$$

Adding (36) and (37) gives

$$\ddot{x}-\ddot{\xi} = 0.$$

Integrating, and using $\dot{x} = V$, $\dot{\xi} = \frac{1}{2}V$, when $t = 0$, gives

$$\dot{x}-\dot{\xi} = \tfrac{1}{2}V. \tag{38}$$

This result could have been written down by the principle of conservation of momentum.

Subtracting (36) and (37) gives

$$m\ddot{\xi} = -f(\xi), \tag{39}$$

an equation for ξ of the type being studied in this section.

If the spheres are perfectly elastic, $f(\xi)$ can be calculated by the methods of the theory of elasticity, being the force necessary to squash a flat of depth ξ on a sphere, and it is found that $f(\xi) = k\xi^3$, where k depends on the radius of the sphere and its elastic properties.

Here we shall consider the case in which the spheres are perfectly plastic, that is, that they exert a constant pressure P (the flow pressure, characteristic of the material) over the area of contact while the spheres are approaching one another. If r is the radius of contact, so that, for small ξ, $r^2 = 2a\xi$ nearly, we have

$$f(\xi) = \pi r^2 P = 2\pi a\xi P,$$

and (39) becomes

$$\ddot{\xi}+\omega^2\xi = 0, \tag{40}$$

where

$$\omega^2 = 2\pi aP/m. \tag{41}$$

(40) is linear and its solution with $\xi = 0$, $\dot{\xi} = \frac{1}{2}V$, when $t = 0$, is

$$\xi = \frac{V}{2\omega}\sin\omega t. \tag{42}$$

The spheres stop approaching when $\dot{\xi} = 0$, that is, when $t = \pi/2\omega$ and the value of ξ then is $V/2\omega$.

The velocity of the sphere A is by (38) and (42)

$$\dot{x} = \tfrac{1}{2}V+\dot{\xi} = \tfrac{1}{2}V(1+\cos\omega t). \tag{43}$$

The velocity of the sphere B is

$$\frac{d}{dt}(x+2a-2\xi) = \dot{x}-2\dot{\xi} = \tfrac{1}{2}V-\dot{\xi} = \tfrac{1}{2}V(1-\cos\omega t). \tag{44}$$

The velocities of the spheres A and B, given by (43) and (44), are shown in Fig. 47 (b). For $t > \pi/2\omega$ the spheres both move with velocity $\frac{1}{2}V$.

56. Motion with resistance a function of the velocity

In this section we consider motion with differential equation

$$m\ddot{x} = f(\dot{x}). \tag{1}$$

The process of solution is very simple provided the integrals can be evaluated. As in § 55 (2) and (3), we write

$$v = \dot{x}$$

for the velocity. Then

$$\ddot{x} = \frac{dv}{dt}, \tag{2}$$

and

$$\ddot{x} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}. \tag{3}$$

The method of solution is now as follows:

(i) *Velocity in terms of distance*

Writing $\ddot{x}$ in (1) in the form (3) gives

$$mv\frac{dv}{dx} = f(v), \tag{4}$$

a separable first-order equation for v in terms of x.

(ii) *Velocity in terms of time*

Writing $\ddot{x}$ in (1) in the form (2) gives

$$m\frac{dv}{dt} = f(v), \tag{5}$$

a separable first-order equation for v in terms of t.

(iii) *Position in terms of time*

The simplest way of finding this is usually to eliminate v between the solutions of (4) and (5). Alternatively, the solution of (5) gives dx/dt as a function of t, and another integration gives x as a function of t.

The function $f(\dot{x})$ in (1) may contain a term independent of the velocity, that is, a constant force F. In this case we write $f(\dot{x}) = F-\phi(\dot{x})$, where $\phi(\dot{x})$ is the resistance to the motion. Then (1) becomes

$$m\ddot{x} = F-\phi(\dot{x}). \tag{6}$$

Usually $\phi(\dot{x})$ is an increasing function of $\dot{x}$, so there will be a velocity V at which the resistance to motion is equal to the applied force F. That is

$$\phi(V) = F. \tag{7}$$

When $\dot{x}$ has the value V, $\ddot{x}$ is zero by (6), and thus the particle continues to move with constant velocity V. For this reason V is called the *terminal velocity*; clearly it will appear as a natural parameter in many solutions.

Before proceeding to solve problems it is useful to consider the commonly occurring laws of resistance to motion. Resistance proportional to velocity arises from shearing of ideal viscous fluid and has been studied in Chapter IV. It also occurs for the slow fall of small spheres through viscous fluid. If a is the radius of the sphere and ρ and ν the density and kinematic viscosity of the fluid, Stokes's law states that the resistance to motion of the sphere at velocity v is

$$6\pi a\rho\nu v. \tag{8}$$

If ρ_0 is the density of the sphere, its terminal velocity when falling in the fluid under gravity is by (7) and (8)

$$\frac{2(\rho_0-\rho)ga^2}{9\rho\nu}. \tag{9}$$

The law (8) is of considerable importance since it determines the motion of small raindrops and small particles settling in liquid (sedimentation).

(8) is valid for motion in which the fluid is not turbulent. The degree of turbulence of the fluid is determined by the Reynolds number R, a dimensionless quantity which in the present case is

$$R = \frac{2va}{\nu}. \tag{10}$$

The resistance to motion of the sphere is found experimentally to be given by

$$\tfrac{1}{2}\pi\rho a^2 C(R)v^2, \tag{11}$$

where $C(R)$ is a function of R only. If $R < 10^{-1}$, $C(R) = 24/R$, and (11) reduces to (8). For $10^{-1} < R < 10^3$ there is a transition region; while for $10^3 < R < 10^5$, C is very nearly constant so that the resistance to motion is proportional to the square of the velocity.

The variation of $C(R)$ with R is shown in Fig. 48.

Enough has been said to show that even for the simple case of a falling sphere the variation of resistance with velocity is extremely complicated. For other bodies, projectiles, etc., the resistance is usually specified graphically or as a power series in the velocity.

Since resistance proportional to the square of the velocity has been seen above to have some physical significance and leads to fairly simple results, we consider it in the examples below.

Ex. 1. *A particle of mass m falls from rest under gravity and resistance to motion km times the square of the velocity.*

Taking the origin so that $x = 0$ when $t = 0$, and the x-axis vertically downwards, the equation of motion is

$$m\ddot{x} = mg - mk\dot{x}^2. \tag{12}$$

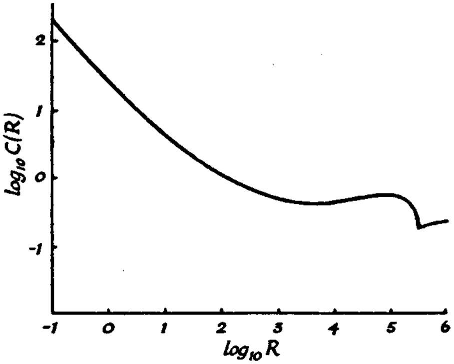


FIG. 48.

The terminal velocity V is given by

$$V^2 = g/k. \tag{13}$$

To find velocity in terms of distance, as in (4) we write (12) in the form

$$v\frac{dv}{dx} = k(V^2 - v^2). \tag{14}$$

Integrating, and using $v = 0$ when $x = 0$, we get

$$\int_0^v \frac{v\,dv}{V^2 - v^2} = kx.$$

That is,
$$-\tfrac{1}{2}\ln\left(\frac{V^2 - v^2}{V^2}\right) = kx.$$

Therefore
$$v^2 = V^2(1 - e^{-2kx}). \tag{15}$$

This shows the way in which $v \to V$ as $x \to \infty$.

Next, to find velocity in terms of time, as in (5) we write (12) in the form

$$\frac{dv}{dt} = k(V^2 - v^2). \tag{16}$$

Since $v = 0$ when $t = 0$ this gives

$$\int_0^v \frac{dv}{V^2 - v^2} = kt.$$

Therefore $$v = V \tanh kVt. \tag{17}$$

To find the position at any time, we eliminate v between (15) and (17). This gives

$$\tanh^2 kVt = 1 - e^{-2kx},$$

or $$x = \frac{1}{k} \ln \cosh kVt. \tag{18}$$

Ex. 2. *A particle is projected vertically upwards with velocity U under gravity and resistance to motion mk times the square of the velocity.*

Choosing the x-axis vertically upwards with the origin at the point of projection, the equation of motion is

$$m\ddot{x} = -mg - mk\dot{x}^2, \tag{19}$$

to be solved with $\dot{x} = U$, $x = 0$, when $t = 0$.

To find velocity as a function of distance we write (19) as

$$v\frac{dv}{dx} = -k(V^2 + v^2), \tag{20}$$

where V^2 is defined in (13). The solution of (20) is

$$\int \frac{v\,dv}{V^2+v^2} = -kx + C, \qquad \tfrac{1}{2}\ln(V^2+v^2) = -kx + C.$$

Since $v = U$ when $x = 0$, this gives

$$2kx = \ln \frac{V^2+U^2}{V^2+v^2}. \tag{21}$$

The greatest height attained, which is the value of x when $v = 0$, is

$$\frac{1}{2k} \ln\left(1 + \frac{U^2}{V^2}\right). \tag{22}$$

To find velocity as a function of the time we write (19) as

$$\frac{dv}{dt} = -k(V^2 + v^2).$$

The solution of this with $v = U$ when $t = 0$ is

$$kVt = \tan^{-1}\frac{U}{V} - \tan^{-1}\frac{v}{V}. \tag{23}$$

After the particle has come to rest at height given by (22), and at time

$$\frac{1}{kV} \tan^{-1}\frac{U}{V}, \tag{24}$$

it commences to fall, and the equation (19) ceases to hold since the resistance to motion now acts upwards.

Ex. 3. *A particle of mass m is blown along the x-axis by a wind of velocity U, the force on the particle being $mk(U-v)^2$, where v is the velocity of the particle. If it starts from rest at $x = 0$ at $t = 0$, find the motion.*

The equation of motion is

$$\frac{dv}{dt} = k(U-v)^2, \tag{25}$$

and its solution with $v = 0$ when $t = 0$ is

$$\frac{1}{U-v} - \frac{1}{U} = kt,$$

or

$$v = \frac{ktU^2}{1+ktU}. \tag{26}$$

To find the position at time t we have from (26)

$$x = \int\limits_0^t \frac{ktU^2\,dt}{1+ktU} = Ut - \frac{1}{k}\ln(1+ktU). \tag{27}$$

57. Non-linear problems in electric circuit theory

In Chapter V the equations of electric circuit theory were discussed on the linear assumptions § 41 (1)–(3). For most purposes these are adequate, and also the general non-linear equations are quite intractable. There is a number of cases in which non-linearity is of great importance, and in which exact solutions of simple special problems can be obtained by methods similar to those of §§ 55, 56. Here we discuss briefly non-linear resistances and iron-cored inductances.

(i) *Non-linear resistance*

Many semi-conductors and electron tubes may be treated as resistances in which the voltage drop V is connected with the current I by the relation

$$V = \phi(I) \tag{1}$$

in place of § 41 (1).

Suppose, for example, that constant voltage E is applied at $t = 0$ to such a resistance in series with an inductance L obeying § 41 (2). Then, as in § 41, the circuit equation is

$$L\frac{dI}{dt} + \phi(I) = E. \tag{2}$$

If the current in the inductance is zero when $t = 0$, the solution of (2) is

$$t = L\int_{0}^{I} \frac{dI}{E-\phi(I)}. \tag{3}$$

This can be evaluated for simple forms of $\phi(I)$, the one most used in practice being the power law $I = KV^n$.

(ii) *Iron-cored inductance*

In this case the linear relation § 41 (2) does not hold. It is now most convenient to work with the flux Φ linked with the inductance. The voltage drop V across the inductance is then

$$\frac{d\Phi}{dt} = V. \tag{4}$$

The flux Φ, instead of being LI as in the linear case, is now connected with I by a non-linear relation

$$I = f(\Phi). \tag{5}$$

For example, suppose that constant voltage E is applied at $t = 0$ to an inductance satisfying (4) and (5), in series with a linear resistance R satisfying § 41 (1). Then the circuit relations are

$$\frac{d\Phi}{dt} + Rf(\Phi) = E, \tag{6}$$

and if $I = 0$ when $t = 0$, the solution is

$$t = \int_{0}^{\Phi} \frac{d\Phi}{E-Rf(\Phi)}. \tag{7}$$

Clearly other simple problems of types (i) and (ii) can be solved explicitly in the same way. When we come to slightly more general problems, such as the L, R, C circuit, we usually reach second-order non-linear equations which cannot be solved explicitly. For example, the circuit equation for oscillations in a closed L, R, C circuit with linear resistance and iron-cored inductance is

$$\frac{d\Phi}{dt} + RI + \frac{Q}{C} = 0,$$

or

$$\frac{d^2\Phi}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0, \tag{8}$$

where I and Φ are connected by (5). Taking the simple form

$$I = A\Phi + B\Phi^3 \tag{9}$$

as the approximation next in order of simplicity to the linear one, (8) becomes

$$\frac{d^2\Phi}{dt^2} + R(A+3B\Phi^2)\frac{d\Phi}{dt} + \frac{1}{C}\Phi(A+B\Phi^2) = 0. \tag{10}$$

58. Oscillations of non-linear systems

It was remarked in § 54 that the types of problem discussed in §§ 55–7 are the only important non-linear ones for which exact solutions are obtainable by elementary methods. Because of the difficulty of studying more general non-linear problems, attention has largely been concentrated on what is perhaps the most important problem of this type, namely the effect of small non-linear terms on the oscillations of a system.

Thus we study equations of motion of the type

$$\ddot{x} + n^2x + \epsilon f(x, \dot{x}) = 0, \tag{1}$$

where ϵ is small. If we neglect ϵ altogether, the equation becomes

$$\ddot{x} + n^2x = 0, \tag{2}$$

whose solution,

$$a\sin(nt+\phi), \tag{3}$$

is a harmonic oscillation whose period $2\pi/n$ is independent of the amplitude of the oscillation. We wish to see what effect the small non-linear terms in (1) have on the simple solution (3). It may be remarked that, even if an exact solution could be found, it probably would not be very useful for this purpose (e.g. § 55 (24) is not) so that in any case new methods have to be devised.

First we list some problems of the type envisaged, most of which will be used as examples later:

(i) The motion of a mass m with linear restoring force and resistance to motion proportional to the square of the velocity†

$$\ddot{x} + k\dot{x}|\dot{x}| + n^2x = 0. \tag{4}$$

(ii) The anharmonic oscillator

$$\ddot{x} + n^2x + bx^3 = 0. \tag{5}$$

† Notice that $\dot{x}\,|\dot{x}|$ changes sign with the velocity, so (4) is applicable to both directions of motion whereas § 54 (3) is not.

(iii) The equation § 57 (10) for electrical oscillations in a circuit containing an iron-cored inductance, viz.

$$\ddot{\Phi}+R(A+3B\Phi^2)\dot{\Phi}+\frac{1}{C}\Phi(A+B\Phi^2) = 0. \tag{6}$$

(iv) The equations § 48 (16) and (11) which arose in connexion with triode oscillations, namely

$$\ddot{v}-\epsilon(1-\alpha v-\beta v^2-...)\dot{v}+n^2v = 0, \tag{7}$$

$$\ddot{I}-\epsilon(1-a\dot{I}-b\dot{I}^2-...)\dot{I}+n^2I = 0. \tag{8}$$

The most important equation of this type is van der Pol's

$$\ddot{v}-\epsilon(1-v^2)\dot{v}+n^2v = 0, \tag{9}$$

which is equivalent to (7) if only the terms $(1-\beta v^2)$ in the coefficient of $\dot{v}$ are retained.

There are various methods of treating (1); we shall first give a sketch of that used by Kryloff and Bogoliuboff† since it leads to a single formula applicable to a wide variety of cases. The other methods of approach will be treated more briefly later.

If we neglect ϵ in (1) we get the solution (3), that is,

$$x = a\sin(nt+\phi), \tag{10}$$

$$\dot{x} = na\cos(nt+\phi), \tag{11}$$

in which a and ϕ are constants. We now attempt to find a solution of (1) in which x and $\dot{x}$ are still given by (10) and (11), but a and ϕ in these are now functions of the time. If x is given by (10) with a and ϕ functions of the time, we have

$$\dot{x} = an\cos(nt+\phi)+\dot{a}\sin(nt+\phi)+a\dot{\phi}\cos(nt+\phi), \tag{12}$$

but since we have assumed that $\dot{x}$ is given by (11), the sum of the last two terms of (12) must be zero, that is

$$\dot{a}\sin(nt+\phi)+a\dot{\phi}\cos(nt+\phi) = 0. \tag{13}$$

Differentiating (11) gives

$$\ddot{x} = -n^2a\sin(nt+\phi)+n\dot{a}\cos(nt+\phi)-na\dot{\phi}\sin(nt+\phi). \tag{14}$$

† *Introduction to Non-Linear Mechanics* (Princeton University Press, 1947).

Substituting (10), (11), and (14) in (1), and writing for shortness

$$\psi = nt+\phi, \tag{15}$$

we get

$$n\dot{a}\cos\psi - na\dot{\phi}\sin\psi = -\epsilon f(a\sin\psi, na\cos\psi). \tag{16}$$

Solving (13) and (16) we get

$$\dot{a} = -\frac{\epsilon}{n} f(a\sin\psi, an\cos\psi)\cos\psi, \tag{17}$$

$$\dot{\phi} = \frac{\epsilon}{na} f(a\sin\psi, an\cos\psi)\sin\psi. \tag{18}$$

These equations give the way in which the amplitude a and the phase ϕ of the solution (10) of (1) vary with time. Since both $\dot{a}$ and $\dot{\phi}$ are proportional to the small quantity ϵ, it follows that they are small, that is, that a and ϕ are slowly varying functions of the time. Thus in time $2\pi/n$, $\psi = nt+\phi$ will increase by nearly 2π while a and ϕ will have changed very little. Thus in calculating $\dot{a}$ and $\dot{\phi}$ from (17) and (18) we may, as a first approximation, replace the right-hand sides by their average values over a range 2π in ψ, regarding a as constant when taking the average; that is we take

$$\frac{da}{dt} = -\frac{\epsilon}{2\pi n}\int\limits_0^{2\pi} f(a\sin\psi, an\cos\psi)\cos\psi\, d\psi, \tag{19}$$

$$\frac{d\phi}{dt} = \frac{\epsilon}{2\pi na}\int\limits_0^{2\pi} f(a\sin\psi, an\cos\psi)\sin\psi\, d\psi. \tag{20}$$

It should be emphasized that (17) and (18) are exact; (19) and (20) are simply obtained from them here as approximations which are physically reasonable. For a complete justification of (19) and (20) and higher approximations (the method is to expand the right-hand sides of (17) and (18) in Fourier series (cf. Chap. XI) of which (19) and (20) are the first terms) the reader is referred to Kryloff and Bogoliuboff (loc. cit.).

Ex. 1. *The anharmonic oscillator* (5).

In this case (19) and (20) give

$$\frac{da}{dt} = -\frac{b}{2\pi n}\int\limits_0^{2\pi} a^3\sin^3\psi\cos\psi\,d\psi = 0, \tag{21}$$

$$\frac{d\phi}{dt} = \frac{b}{2\pi na}\int\limits_0^{2\pi} a^3\sin^4\psi\,d\psi = \frac{3ba^2}{8n}. \tag{22}$$

By (21) the amplitude is independent of time, and by (22) the phase increases linearly with time. Thus the solution is

$$x = a\sin nt\left(1+\frac{3ba^2}{8n^2}\right). \tag{23}$$

Thus the period, which is

$$\frac{2\pi}{n}\left(1+\frac{3ba^2}{8n^2}\right)^{-1}, \tag{24}$$

decreases with increasing amplitude.

If we take $b = -\frac{1}{6}n^2$ in (5), we get the approximation to the equation § 55 (13) for the simple pendulum obtained by replacing $\sin\theta$ by $(\theta-\frac{1}{6}\theta^3)$. With this value of b, (24) gives for the period in this case

$$\frac{2\pi}{n}\left(1-\frac{a^2}{16}\right)^{-1} = \frac{2\pi}{n}\left(1+\frac{a^2}{16}\right), \tag{25}$$

neglecting terms in a^4. This agrees with § 55 (26).

Ex. 2. *Equation* (4).

For this equation (19) and (20) give

$$\frac{da}{dt} = -\frac{k}{2\pi n}\int\limits_0^{2\pi} a^2n^2\cos^2\psi|\cos\psi|\,d\psi = -\frac{4ka^2n}{3\pi}, \tag{26}$$

$$\frac{d\phi}{dt} = \frac{k}{2\pi na}\int\limits_0^{2\pi} a^2n^2\cos\psi|\cos\psi|\sin\psi\,d\psi = 0. \tag{27}$$

By (27), ϕ is constant, that is, the period is unaffected by the damping to this approximation. (26) is a differential equation for a, and its solution is

$$\frac{1}{a}-\frac{1}{a_0} = \frac{4knt}{3\pi}, \tag{28}$$

where a_0 is the value of a when $t = 0$. In a half-swing, t increases by π/n, $1/a$ increases by $4k/3$, that is, the amplitude a decreases by $4ka^2/3$, approximately.

Ex. 3. *The triode equation* (7).

We consider the equation

$$\ddot{x}-\epsilon(1-\alpha x-\beta x^2)\dot{x}+n^2x = 0, \tag{29}$$

for which (19) and (20) give

$$\frac{da}{dt} = \frac{\epsilon}{2\pi n}\int\limits_0^{2\pi}(1-\alpha a\sin\psi-\beta a^2\sin^2\psi)an\cos^2\psi\,d\psi$$

$$= \tfrac{1}{2}\epsilon a(1-\tfrac{1}{4}\beta a^2), \tag{30}$$

$$\frac{d\phi}{dt} = -\frac{\epsilon}{2\pi na}\int\limits_0^{2\pi}(1-\alpha a\sin\psi-\beta a^2\sin^2\psi)an\cos\psi\sin\psi\,d\psi$$

$$= 0. \tag{31}$$

From (31), ϕ is constant, that is the period is not affected to this approximation. Also, since α does not appear in (30) it does not affect the amplitude to this approximation.

The differential equation (30) for a has solution

$$\int\frac{2da}{a(1-\frac{1}{4}\beta a^2)} = \epsilon t+C,$$

$$\ln\frac{a^2}{1-\frac{1}{4}\beta a^2} = \epsilon t+C,$$

$$\frac{a^2}{1-\frac{1}{4}\beta a^2} = \frac{a_0^2}{1-\frac{1}{4}\beta a_0^2}e^{\epsilon t},$$

where a_0 is the value of a when $t = 0$. Thus, finally,

$$a^2 = \frac{a_0^2 e^{\epsilon t}}{1+\frac{1}{4}\beta a_0^2(e^{\epsilon t}-1)}. \tag{32}$$

(32) shows how the amplitude varies with time. As $t \to \infty$, $a \to 2\beta^{-\frac{1}{2}}$ whatever the initial value of the amplitude, and thus the final oscillation of the system is

$$x = \frac{2}{\beta^{\frac{1}{2}}}\sin(nt+\phi), \tag{33}$$

where ϕ is a constant. It should be observed that the solution tends to (33) *whether the initial amplitude is larger or smaller than* $2\beta^{-\frac{1}{2}}$. The way in which the oscillations build up if a_0 is small is shown in Fig. 49 (*b*).

In the next two examples we illustrate other methods of attack which are often used. They are a little *ad hoc* and have to be used with care as it frequently happens that while they work well in some cases they need modifications in others.

Ex. 4. *Solution in a trigonometric series.*

As an example we consider the anharmonic oscillator with applied force $F\sin\omega t$. The differential equation is

$$\ddot{x}+n^2x+bx^3 = F\sin\omega t. \tag{34}$$

We seek a solution
$$x = A\sin\omega t, \tag{35}$$
where A is a constant; substituting this in (34) requires

$$(-\omega^2+n^2)A\sin\omega t+bA^3\sin^3\omega t = F\sin\omega t.$$

Using the result

$$\sin^3\omega t = \tfrac{3}{4}\sin\omega t-\tfrac{1}{4}\sin 3\omega t,$$

this becomes

$$\{(n^2-\omega^2)A+\tfrac{3}{4}bA^3-F\}\sin\omega t-\tfrac{1}{4}bA^3\sin 3\omega t = 0. \tag{36}$$

Clearly this cannot be satisfied exactly for all values of t. But if

$$(n^2-\omega^2)A+\tfrac{3}{4}bA^3 = F, \tag{37}$$

the coefficient of $\sin\omega t$ is zero, so that if we ignore the term in $\sin 3\omega t$ in (36) the amplitude of the solution (35) is given by the solution of the cubic (37). In particular if $\omega = n$, corresponding to resonance if the non-linear term is neglected,

$$A = (4F/3b)^{\frac{1}{3}}, \tag{38}$$

so that the amplitude depends on the cube root of the applied force.

The crude treatment above in which the term $\sin 3\omega t$ in (36) has been neglected may be improved as follows. Instead of (35) we assume a series

$$x = A\sin\omega t+A_2\sin 2\omega t+A_3\sin 3\omega t+\ldots, \tag{39}$$

substitute in (34), and equate the coefficients of the successive trigonometric functions to zero. The first equation so obtained will be (37) which determines A, subsequent equations determine A_2, etc.

The free oscillation corresponding to $F = 0$ in (34) may be studied in the same way, but ω in the substitution (35) must be regarded as an unknown to be determined.

Ex. 5. *The method of iteration.*

We again consider the anharmonic oscillator

$$\ddot{x}+n^2x+bx^3 = 0, \tag{40}$$

where b is small. The first approximation, neglecting the small term, is

$$x = A\sin nt. \tag{41}$$

The general method of iteration consists of substituting the first approximation in the small terms of the original equation and solving again. This process, theoretically, has to be repeated indefinitely, and it has to be proved that the set of results so obtained converges to a definite solution. The difficulty which appears at the outset is that if the effect of the non-linear terms in (40) is to change the period, the assumed first approximation (41) will after a short time cease to be a valid approximation. Thus instead of (41) we can only assume as first approximation

$$x = A\sin mt, \tag{42}$$

where m is unknown but does not differ greatly from n.

Substituting (42) in the small term of (40) gives

$$\begin{aligned}\ddot{x}+n^2x &= -bA^3\sin^3 mt \\ &= -\tfrac{1}{4}bA^3(3\sin mt-\sin 3mt).\end{aligned} \tag{43}$$

We now seek a solution of (43) of the form

$$x = A\sin mt+B\sin 3mt. \tag{44}$$

Substituting (44) in (43) requires

$$\{A(n^2-m^2)+\tfrac{3}{4}bA^3\}\sin mt+\{B(n^2-9m^2)-\tfrac{1}{4}bA^3\}\sin 3mt = 0.$$

That is

$$B = \frac{bA^3}{4(n^2-9m^2)}, \tag{45}$$

$$m^2 = n^2+\tfrac{3}{4}bA^2. \tag{46}$$

(46) shows the way in which the period varies with the amplitude: it agrees with the result (24) found previously.

59. Relaxation oscillations

This is the name given to periodic oscillations of a system in which energy is supplied from outside during part of a period and dissipated within the system in another part of the period, so that the total energy in the system oscillates periodically. Most practical systems in which oscillations are maintained are of this type.

One simple example which has been analysed in detail is the system of § 30, Ex. 4. Here the potential energy stored in the spring increases steadily in the static phase, and this energy is dissipated by friction while slipping occurs. The system has a well-defined period determined largely by the velocity of the moving plane.

An analogous electrical system is shown in Fig. 49 (a). This is idealized, but represents in principle the working of many practical circuits. A battery of voltage E is connected to resistance R and capacitance C in series: when the voltage drop across C reaches the value $E_1 < E$, a diode D across it suddenly discharges it completely.

The charge Q on the condenser satisfies

$$R\frac{dQ}{dt}+\frac{Q}{C} = E, \tag{1}$$

and if we assume $Q = 0$ when $t = 0$, the solution is

$$Q = CE(1-e^{-t/RC}).$$

The voltage drop across the condenser is E_1 when

$$E-Ee^{-t/RC} = E_1,$$

that is, when
$$t = RC\ln\frac{E}{E-E_1}. \tag{2}$$

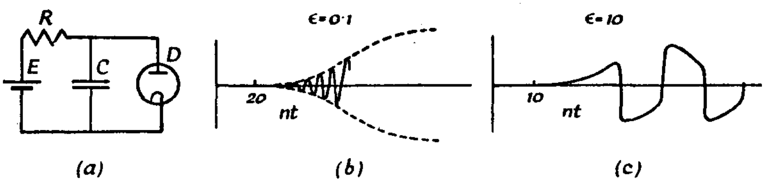


FIG. 49.

The condenser is then discharged, and the process repeats itself with period given by (2).

The most important example of relaxation oscillations is the behaviour of the triode studied in §§ 48, 58. This is typical of the behaviour of systems specified by differential equations of type
$$\ddot{x}-\epsilon(1-x^2)\dot{x}+n^2x = 0 \tag{3}$$
in which the damping coefficient is negative and the system unstable for small displacements, while for large displacements the damping coefficient becomes positive. Thus small oscillations of the system tend to grow, and large oscillations tend to diminish, and a stable oscillation results. The nature of the solutions of (3) for various values of the parameter ϵ has been exhaustively studied by van der Pol. Small values of ϵ have been studied in § 58 and a first approximation to the method of growth and the final steady state has been found: the results are shown in Fig. 49 (*b*).

The case of large ϵ has been examined by van der Pol; the way in which the oscillations build up is shown in Fig. 49 (*c*). It appears that the final wave form is very far from sinusoidal—this is a characteristic of such oscillations.

60. Motion in two or more dimensions

Problems of this type usually are very difficult to handle unless they separate into a number of equations of the types previously discussed. We discuss a number of examples in which this is the case.

Ex. 1. *The simple projectile.*

A particle of mass m is projected at $t = 0$ with velocity u at an angle α to the horizontal. Taking the origin O at the point of projection, and the axes of x and y horizontal and vertical, the equations of motion are

$$m\ddot{x} = 0, \tag{1}$$

$$m\ddot{y} = -mg. \tag{2}$$

Integrating twice and using the initial values $x = y = 0$, $\dot{x} = u\cos\alpha$, $\dot{y} = u\sin\alpha$, we get

$$x = ut\cos\alpha, \tag{3}$$

$$y = ut\sin\alpha - \tfrac{1}{2}gt^2. \tag{4}$$

Eliminating t between (3) and (4) gives the equation of the path

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha. \tag{5}$$

Ex. 2. *The projectile of Ex. 1 but with resistance to motion a function $f(v)$ of the velocity.*

Let ψ be the slope of the path at the point (x, y). The resistance $f(v)$ is directed backwards along the tangent to the path, so its components in the x- and y-directions are $-f(v)\cos\psi$ and $-f(v)\sin\psi$. Thus the equations of motion are now

$$m\ddot{x} = -f(v)\cos\psi \tag{6}$$

$$m\ddot{y} = -f(v)\sin\psi - mg, \tag{7}$$

where, if s is the arc measured along the path,

$$v = \dot{s}; \qquad \cos\psi = dx/ds; \qquad \sin\psi = dy/ds. \tag{8}$$

The equations now do not separate into two for x and y except in the case of resistance proportional to velocity in which

$$f(v) = mkv = mk\frac{ds}{dt}$$

which we now consider. In this case (6) and (7) become

$$m\ddot{x} = -mk\frac{ds}{dt}\frac{dx}{ds} = -mk\dot{x} \tag{9}$$

$$m\ddot{y} = -mk\frac{ds}{dt}\frac{dy}{ds}-mg = -mk\dot{y}-mg. \tag{10}$$

Equations (9) and (10) are linear second-order equations. The general solution of (9) is

$$x = A+Be^{-kt};$$

choosing A and B to make $x = 0$, $\dot{x} = u\cos\alpha$, when $t = 0$, we get

$$x = \frac{u\cos\alpha}{k}(1-e^{-kt}). \tag{11}$$

Similarly, from (10) with $y = 0$, $\dot{y} = u\sin\alpha$, when $t = 0$,

$$y = -\frac{gt}{k}+\frac{1}{k}\left(u\sin\alpha+\frac{g}{k}\right)(1-e^{-kt}). \tag{12}$$

Eliminating the time between (11) and (12) gives the equation of the path. From (11)

$$t = -\frac{1}{k}\ln\left(1-\frac{kx}{u\cos\alpha}\right), \tag{13}$$

and substituting (13) and (11) in (12), we get

$$y = \frac{g}{k^2}\ln\left(1-\frac{kx}{u\cos\alpha}\right)+\left(u\sin\alpha+\frac{g}{k}\right)\frac{x}{u\cos\alpha}. \tag{14}$$

Although the linear law of resistance is rather artificial the simple results (11), (12), (14) may be used to illustrate the general effects of resistance to motion. From (11) and (12) it follows that

$$x \to \frac{u\cos\alpha}{k}, \quad y \to -\infty, \quad \text{as } t \to \infty, \tag{15}$$

that is, the curve has a vertical asymptote at this value of x. The range on the horizontal plane, of course, is less than this; it is obtained by putting $y = 0$ in (14) which gives a transcendental equation for x.

Finally, if k is small, we may expand the logarithm in (14) by the logarithmic series. This gives

$$y = \frac{g}{k^2}\left\{-\frac{kx}{u\cos\alpha}-\frac{k^2x^2}{2u^2\cos^2\alpha}-\frac{k^3x^3}{3u^3\cos^3\alpha}-\ldots\right\}+$$
$$+\left(u\sin\alpha+\frac{g}{k}\right)\frac{x}{u\cos\alpha}$$
$$= x\tan\alpha-\frac{gx^2}{2u^2}\sec^2\alpha-\frac{gkx^3}{3u^3}\sec^3\alpha-\ldots. \tag{16}$$

The first two terms of (16) are the path (5) in the absence of resistance. The terms of the series give the change in the path caused by resistance.

Ex. 3. *The motion of a particle of mass m in the xy-plane with restoring force $m\nu^2x$ parallel to the x-axis and mn^2y parallel to the y-axis.*

The equations of motion are

$$\ddot{x}+\nu^2x = 0, \tag{17}$$
$$\ddot{y}+n^2y = 0, \tag{18}$$

corresponding to two simple harmonic motions. With an appropriate choice of the origin of time their general solutions may be written

$$x = a\sin(\nu t+\theta), \tag{19}$$
$$y = b\sin nt. \tag{20}$$

The path of the particle is obtained by eliminating t between (19) and (20). If ν is a simple multiple of n the paths are known as Lissajous figures.

If $\nu = n$ the path is

$$\left(\frac{x}{a}-\frac{y}{b}\cos\theta\right)^2 = \left(1-\frac{y^2}{b^2}\right)\sin^2\theta,$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\theta = \sin^2\theta, \tag{21}$$

an ellipse whose shape and orientation depend on the relative phase θ of the two vibrations.

If $\nu = 2n$ the path is

$$\frac{x}{a} = 2\frac{y}{b}\left(1-\frac{y^2}{b^2}\right)^{\frac{1}{2}}\cos\theta+\left(1-\frac{2y^2}{b^2}\right)\sin\theta. \tag{22}$$

The motion of an electron in electric and magnetic fields.

The force $\mathbf{F}$ on an electron of charge $-e$ in a static electric field $\mathbf{E}$ and a static magnetic field $\mathbf{H}$ is

$$\mathbf{F} = -e\mathbf{E}-\frac{e}{c}\mathbf{v}\wedge\mathbf{H}, \tag{23}$$

where $\mathbf{v}$ is the vector velocity of the electron and c is the velocity of light. Here e and $\mathbf{E}$ are measured in e.s.u. and $\mathbf{H}$ in e.m.u. The force of gravity is usually negligible but may be included if desired.

If the magnetic field is uniform and H is its magnitude, and we choose the z-axis in the direction of this field, the components of $\mathbf{F}$ in the x, y, and z directions become

$$F_x = -eE_x - \frac{eH}{c}\dot{y}, \tag{24}$$

$$F_y = -eE_y + \frac{eH}{c}\dot{x}, \tag{25}$$

$$F_z = -eE_z. \tag{26}$$

Ex. 4. *No electric field. The particle projected from the origin at $t = 0$ with velocity V in a direction which makes an angle θ with the magnetic field.*

Taking the x-axis to be in the plane containing the z-axis and the direction of projection, the initial conditions are

$$x = y = z = 0; \quad \dot{x} = V\sin\theta, \quad \dot{y} = 0, \quad \dot{z} = V\cos\theta, \tag{27}$$

when $t = 0$.

The equations of motion are

$$m\ddot{x} = -\frac{eH}{c}\dot{y}, \tag{28}$$

$$m\ddot{y} = \frac{eH}{c}\dot{x}, \tag{29}$$

$$m\ddot{z} = 0. \tag{30}$$

(30) gives immediately

$$\dot{z} = V\cos\theta, \qquad z = Vt\cos\theta. \tag{31}$$

Writing $$\omega = \frac{eH}{mc}, \qquad u = \dot{x}, \qquad v = \dot{y}, \tag{32}$$

(28) and (29) become

$$Du + \omega v = 0, \tag{33}$$

$$Dv - \omega u = 0. \tag{34}$$

These give $$(D^2+\omega^2)u = 0, \tag{35}$$

the general solution of which is

$$u = A\sin\omega t+B\cos\omega t. \tag{36}$$

Then, by (33)

$$v = -\frac{1}{\omega}Du = -A\cos\omega t+B\sin\omega t. \tag{37}$$

The initial conditions, $u = V\sin\theta$, $v = 0$, when $t = 0$, give $A = 0$, $B = V\sin\theta$, and we get

$$\dot{x} = u = V\sin\theta\cos\omega t, \tag{38}$$

$$\dot{y} = v = V\sin\theta\sin\omega t. \tag{39}$$

It follows that the speed of the electron is

$$(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}} = V. \tag{40}$$

Integrating (38) and (39) and using the initial conditions (27) gives

$$x = \frac{V}{\omega}\sin\theta\sin\omega t, \tag{41}$$

$$y = \frac{V}{\omega}\sin\theta(1-\cos\omega t), \tag{42}$$

and therefore $$x^2+\left(y-\frac{V}{\omega}\sin\theta\right)^2 = \frac{V^2\sin^2\theta}{\omega^2}. \tag{43}$$

The projection of the path on the plane $z = 0$ is thus a circle of radius $(V\sin\theta)/\omega$ and with centre at $(0, (V/\omega)\sin\theta)$. The path of the particle is, by (31), a helix with axis along the direction of the magnetic field described with constant speed V. If $\theta = \frac{1}{2}\pi$, so that the motion is wholly in the plane $z = 0$, the path is a circle of radius (V/ω).

Ex. 5. *No electric field. The particle projected in the xy-plane from the point (a,b) with speed V in a direction inclined at ϕ to the x-axis.*

This is a problem similar to that of Ex. 4, but we shall now solve it by a useful alternative method. The equations of motion are

$$\ddot{x}+\omega\dot{y} = 0, \tag{44}$$

$$\ddot{y}-\omega\dot{x} = 0, \tag{45}$$

where ω is defined in (32). Writing $\zeta = x+iy$ and adding i times (45) to (44) we get

$$\ddot{\zeta}-i\omega\dot{\zeta} = 0. \tag{46}$$

This is a single differential equation for the complex quantity ζ. It has to be solved with the initial values

$$\zeta = a+ib, \qquad \dot{\zeta} = Ve^{i\phi}, \qquad \text{when } t = 0. \tag{47}$$

The general solution of (46) is

$$\zeta = Ae^{i\omega t}+B, \tag{48}$$

where now the arbitrary constants A and B are complex. Using (48) in (47) gives A and B, and we get finally

$$\zeta = a+ib+(V/\omega)[e^{i\omega t}-1]e^{i(\phi-\frac{1}{2}\pi)}. \tag{49}$$

The path of the particle may either be found by writing down the real and imaginary parts of (49) and discussing them as in Ex. 4, or, better, by noticing that for all values of t the points (49) lie on a circle of radius V/ω whose centre is at

$$a+ib-(V/\omega)e^{i(\phi-\frac{1}{2}\pi)}. \tag{50}$$

The speed of the particle is

$$(\dot{x}^2+\dot{y}^2)^{\frac{1}{2}} = |\dot{\zeta}| = V. \tag{51}$$

Another example of the method is given in Ex. 8.

Ex. 6. *The problem of Ex. 4 with, in addition, a constant electric field E parallel to the magnetic field.*

The only change is that (30) is replaced by

$$m\ddot{z} = -eE \tag{52}$$

and thus the electron has constant acceleration in the z direction.

Ex. 7. *Constant electric field E along the x-axis and constant magnetic field H along the z-axis. Zero initial velocity and displacement.*

The equations of motion are now

$$m\ddot{x} = -eE-\frac{eH}{c}\dot{y}, \tag{53}$$

$$m\ddot{y} = \frac{eH}{c}\dot{x}, \tag{54}$$

$$m\ddot{z} = 0. \tag{55}$$

Making the substitution (32) as before, (53) and (54) become

$$Du+\omega v = -\frac{e}{m}E, \tag{56}$$

$$Dv-\omega u = 0. \tag{57}$$

As before we find $$u = A\sin\omega t + B\cos\omega t, \tag{58}$$

and from (56) $$v = -\frac{e}{m\omega}E - A\cos\omega t + B\sin\omega t. \tag{59}$$

The initial conditions $u = v = 0$ when $t = 0$ give
$$B = 0, \qquad A = -eE/m\omega,$$
and we get
$$\dot{x} = u = -\frac{eE}{m\omega}\sin\omega t, \tag{60}$$
$$\dot{y} = v = \frac{eE}{m\omega}(\cos\omega t - 1). \tag{61}$$

Integrating (60) and (61) with $x = y = 0$ when $t = 0$ gives
$$x = \frac{eE}{m\omega^2}(\cos\omega t - 1), \tag{62}$$
$$y = \frac{eE}{m\omega^2}(\sin\omega t - \omega t). \tag{63}$$

The path is therefore a cycloid. The maximum value of x is $2eE/m\omega^2$.

Ex. 8. *An electron of mass m is attracted to a centre of force at the origin by a force λ times its displacement. There is a magnetic field H along the z-axis. Motion in the xy-plane only will be considered.*

The equations of motion are now
$$m\ddot{x} = -\lambda x - \frac{eH}{c}\dot{y}, \tag{64}$$
$$m\ddot{y} = -\lambda y + \frac{eH}{c}\dot{x}. \tag{65}$$

Writing
$$n^2 = \frac{\lambda}{m}, \qquad \omega = \frac{eH}{mc}, \tag{66}$$
$$\zeta = x + iy, \tag{67}$$
and adding i times (65) to (64) gives
$$\ddot{\zeta} - i\omega\dot{\zeta} + n^2\zeta = 0. \tag{68}$$

The general solution of this is
$$\zeta = A\exp\{\tfrac{1}{2}i\omega + i(n^2 + \tfrac{1}{4}\omega^2)^{\frac{1}{2}}\}t + B\exp\{\tfrac{1}{2}i\omega - i(n^2 + \tfrac{1}{4}\omega^2)^{\frac{1}{2}}\}t,$$
where A and B are constants which may be complex. Taking the real part, x has the form
$$x = C\cos[\{(n^2 + \tfrac{1}{4}\omega^2)^{\frac{1}{2}} - \tfrac{1}{2}\omega\}t + D] + E\cos[\{(n^2 + \tfrac{1}{4}\omega^2)^{\frac{1}{2}} + \tfrac{1}{2}\omega\}t + F], \tag{69}$$
where C, D, E, F are real constants.

In the absence of a magnetic field the frequency of the oscillations is $n/2\pi$; the effect of a magnetic field is to introduce in place of this a pair of frequencies which are $(n \pm \tfrac{1}{2}\omega)/2\pi$, provided ω^2 can be neglected. Problems of this type occur in the classical theory of the Zeeman effect.

Ex. 9. *The motion of a charged particle in the field of a magnetic dipole.*† In this case the field is a function of position, and equation (23) gives

$$m\ddot{x} = -(\dot{y}H_z - \dot{z}H_y)e/c, \tag{70}$$

$$m\ddot{y} = -(\dot{z}H_x - \dot{x}H_z)e/c, \tag{71}$$

$$m\ddot{z} = -(\dot{x}H_y - \dot{y}H_x)e/c. \tag{72}$$

It follows that $\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z} = 0$, and therefore

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = V^2, \tag{73}$$

where V^2 is a constant. That is, the speed of a charged particle in any pure magnetic field is constant. If we take the dipole to be of moment μ and to be situated at the origin with its axis along the z-axis, its magnetic field‡ at (x, y, z) has components

$$H_x = 3\mu xzr^{-5}, \qquad H_y = 3\mu yzr^{-5}, \qquad H_z = \mu(3z^2-r^2)r^{-5}, \tag{74}$$

where

$$r^2 = x^2+y^2+z^2. \tag{75}$$

Putting (74) in (70) and (71) we get

$$\ddot{x} = k\{\dot{y}(3z^2-r^2)-3\dot{z}yz\}r^{-5}, \tag{76}$$

$$\ddot{y} = k\{3\dot{z}xz-\dot{x}(3z^2-r^2)\}r^{-5}, \tag{77}$$

where $k = -\mu e/mc$.

Multiplying (77) by x and (76) by y and subtracting gives

$$x\ddot{y}-y\ddot{x} = k\{3z\dot{z}(x^2+y^2)-(3z^2-r^2)(x\dot{x}+y\dot{y})\}r^{-5}. \tag{78}$$

Using the value (75) of r^2 this may be written

$$\frac{d}{dt}(x\dot{y}-y\dot{x})+k\frac{d}{dt}\left\{\frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right\} = 0. \tag{79}$$

Integrating, using the notation (75) and writing $R^2 = x^2+y^2$, we get

$$x\dot{y}-y\dot{x} = C-\frac{kR^2}{r^3}, \tag{80}$$

where C is a constant of integration. The quantity on the left is proportional to the angular momentum of the particle about the z-axis. If V is the constant speed of the particle and θ is the angle its direction at any point makes with the plane through the particle and the z-axis, the left-hand side of (80) is $VR\sin\theta$, and (80) becomes finally

$$VR\sin\theta = C-\frac{kR^2}{r^3}. \tag{81}$$

This first integral has been extensively studied by Störmer in connexion with the theory of aurorae. Using the fact that $|\sin\theta| < 1$, he found that charged particles incident from space can only reach the earth's surface in certain regions.

† This problem is given as an example of rather complicated manipulation in Cartesians. It is an interesting exercise to study it vectorially.

‡ The field due to a magnetic dipole is calculated in exactly the same way as that for an electric dipole; cf. § 72 (23).

61. Motion on a fixed plane curve

When a particle moves on a fixed curve the forces on the particle in the directions of the tangent and normal to the curve are usually known, and to write down equations of motion we need expressions for the accelerations in these directions.

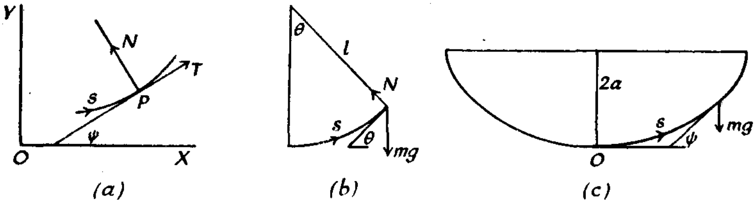


FIG. 50.

Suppose the intrinsic equation of the curve is $s = f(\psi)$, where s is the arc measured along the curve from a fixed point to the point P, and ψ is the angle the tangent at P makes with a fixed direction OX, which we shall take as one of the axes OX, OY of a rectangular coordinate system. The radius of curvature ρ of the curve at P is given by

$$\rho = \frac{ds}{d\psi}. \tag{1}$$

The particle moves along the curve with speed $\dot{s} = ds/dt$. Its components of velocity parallel to OX and OY are

$$\dot{x} = \dot{s}\cos\psi, \tag{2}$$

$$\dot{y} = \dot{s}\sin\psi. \tag{3}$$

Differentiating these we get for its components of acceleration in the directions of OX and OY

$$\ddot{x} = \ddot{s}\cos\psi - \dot{s}\dot{\psi}\sin\psi = \ddot{s}\cos\psi - (\dot{s}^2/\rho)\sin\psi, \tag{4}$$

$$\ddot{y} = \ddot{s}\sin\psi + \dot{s}\dot{\psi}\cos\psi = \ddot{s}\sin\psi + (\dot{s}^2/\rho)\cos\psi, \tag{5}$$

where in (4) and (5) we have used the result

$$\dot{\psi} = \frac{d\psi}{ds}\frac{ds}{dt} = \frac{\dot{s}}{\rho}. \tag{6}$$

The *tangential acceleration* of the particle is by (4) and (5)

$$\ddot{x}\cos\psi + \ddot{y}\sin\psi = \ddot{s}. \tag{7}$$

The *normal acceleration* (in the direction of the inward normal) is

$$-\ddot{x}\sin\psi+\ddot{y}\cos\psi = \dot{s}^2/\rho. \tag{8}$$

If the forces on the particle in the directions of the tangent and inward normal to the curve are T and N, respectively, its equations of motion are

$$m\ddot{s} = T, \tag{9}$$

$$\frac{m\dot{s}^2}{\rho} = N. \tag{10}$$

Ex. 1. *Motion on a smooth vertical circle of radius l: the simple pendulum.*

Measuring s from the lowest point of the circle (Fig. 50 (b)) and θ from the downward vertical we have $s = l\theta$, $\psi = \theta$.

The forces on the particle are N, the normal reaction of the circle, and mg, the force of gravity. Thus (9) and (10) give

$$ml\ddot{\theta} = -mg\sin\theta, \tag{11}$$

$$ml\dot{\theta}^2 = N-mg\cos\theta. \tag{12}$$

The integration of (11) has been discussed in § 55, and when $\dot{\theta}^2$ is known, (12) gives N. If the particle is constrained to move in the circle by being attached to O by a light rigid rod freely hinged at O, or if the particle is a bead sliding on a smooth circular wire, N may have either sign. On the other hand, if the particle is connected to O by a flexible string or slides on the inside of a smooth circular cylinder, N must be positive: if at any stage N becomes zero, the particle will leave the circle and the equations (11) and (12) no longer hold.

Ex. 2. *Motion on a rough vertical circle of radius a.*

Suppose the particle to be moving in the direction of increasing θ. Then if the coefficient of friction is μ, frictional force μN acts tangentially in the direction of decreasing θ. The equations of motion (9) and (10) now become

$$ma\ddot{\theta} = -\mu N-mg\sin\theta, \tag{13}$$

$$ma\dot{\theta}^2 = N-mg\cos\theta. \tag{14}$$

Eliminating N these give

$$a\ddot{\theta}+\mu a\dot{\theta}^2 = -g\sin\theta-\mu g\cos\theta. \tag{15}$$

Putting $\qquad \dot{\theta}^2 = u, \qquad \ddot{\theta} = \frac{1}{2}\frac{du}{d\theta},$

(15) becomes a linear first-order equation for u, namely

$$\frac{du}{d\theta}+2\mu u = -\frac{2g}{a}(\sin\theta+\mu\cos\theta). \tag{16}$$

Ex. 3. *Motion on a smooth vertical cycloid with its vertex downwards.*

Measuring s from the vertex O (Fig. 50(c)), the intrinsic equation of the cycloid is

$$s = 4a\sin\psi, \tag{17}$$

where $2a$ is the distance from the vertex to the line of cusps.

The equation of motion (9) gives

$$m\ddot{s} = -mg\sin\psi, \tag{18}$$

or, using (17),

$$\ddot{s}+\frac{g}{4a}s = 0. \tag{19}$$

The solution of this is

$$s = A\sin\{t(g/4a)^{\frac{1}{2}}+B\},$$

an oscillation whose period $2\pi(4a/g)^{\frac{1}{2}}$ is independent of its amplitude.

62. Central forces

If the force on a particle consists of attraction or repulsion from a fixed point O, it is convenient to work in plane polar coordinates with this point as origin.

Let (x,y) be the rectangular Cartesian coordinates of the particle P, and let (r,θ) be its polar coordinates with O as origin and θ measured from OX. Then

$$x = r\cos\theta, \tag{1}$$

$$y = r\sin\theta, \tag{2}$$

$$\dot{x} = \dot{r}\cos\theta-r\dot{\theta}\sin\theta, \tag{3}$$

$$\dot{y} = \dot{r}\sin\theta+r\dot{\theta}\cos\theta, \tag{4}$$

$$\ddot{x} = \ddot{r}\cos\theta-2\dot{r}\dot{\theta}\sin\theta-r\dot{\theta}^2\cos\theta-r\ddot{\theta}\sin\theta, \tag{5}$$

$$\ddot{y} = \ddot{r}\sin\theta+2\dot{r}\dot{\theta}\cos\theta-r\dot{\theta}^2\sin\theta+r\ddot{\theta}\cos\theta. \tag{6}$$

The *radial velocity* of P, that is, its component of velocity along OP, is

$$\dot{x}\cos\theta+\dot{y}\sin\theta = \dot{r}, \tag{7}$$

by (3) and (4).

The *transverse velocity* of P, that is, its component of velocity perpendicular to OP in the direction of θ increasing, is

$$\dot{y}\cos\theta-\dot{x}\sin\theta = r\dot{\theta}. \tag{8}$$

The *radial acceleration* of P is

$$\ddot{x}\cos\theta+\ddot{y}\sin\theta = \ddot{r}-r\dot{\theta}^2. \tag{9}$$

The *transverse acceleration* of P is

$$\ddot{y}\cos\theta-\ddot{x}\sin\theta = 2\dot{r}\dot{\theta}+r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}). \tag{10}$$

The problem to be considered is that of the motion of a particle of mass m attracted to a centre of force O by force $mf(r)$ and under no other forces. Let (r, θ) be polar coordinates of the particle in the plane containing O and the direction of projection of the particle: since there are no forces perpendicular to this plane the particle will always remain in it.

The equations of motion are, by (9) and (10),

$$\ddot{r}-r\dot{\theta}^2 = -f(r), \tag{11}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0. \tag{12}$$

(12) gives immediately $$r^2\dot{\theta} = h, \tag{13}$$

where h is a constant. Since the transverse velocity of the particle is $r\dot{\theta}$, $mr^2\dot{\theta} = mh$ is the constant angular momentum of the particle about the centre of force.

By using (13) we can eliminate the time from (11) and get the differential equation of the orbit. Instead of working in terms of the radius r it is more convenient to use its reciprocal

$$u = \frac{1}{r}. \tag{14}$$

Then

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}, \tag{15}$$

using (13). Also

$$\ddot{r} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}, \tag{16}$$

again using (13). Substituting (16) and (13) in (11) gives

$$-h^2u^2\frac{d^2u}{d\theta^2}-h^2u^3 = -f(1/u),$$

or $$\frac{d^2u}{d\theta^2}+u = \frac{1}{h^2u^2}f\left(\frac{1}{u}\right). \tag{17}$$

This is the differential equation of the orbit: it may be solved by the methods of § 55.

The general problem is: given the circumstances of projection, find the orbit. Normally, at the instant of projection, $t = 0$, we are given the distance a, the speed V, and the angle β that the direction of motion makes with the outward radius vector. Thus the constant h of (13) is

$$h = Va\sin\beta. \tag{18}$$

The angle ϕ between the tangent and radius vector of the curve at any point is given by

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = u\frac{d}{d\theta}\left(\frac{1}{u}\right) = -\frac{1}{u}\frac{du}{d\theta}. \tag{19}$$

When $t = 0$ we have

$$u = 1/a,\ \phi = \beta, \text{ and so } \frac{du}{d\theta} = -\frac{1}{a}\cot\beta. \tag{20}$$

Thus the initial conditions required for the solution of (17) are known in terms of V, a, and β.

It follows from (19) that when $\phi = 90°$, that is, the particle is moving perpendicular to the radius vector, $du/d\theta = 0$. Points at which this occurs are called apses.

Finally the speed v at any point of the orbit is, by (7) and (8),

$$v^2 = \dot{r}^2+r^2\dot{\theta}^2 = h^2\left(\frac{du}{d\theta}\right)^2+h^2u^2, \tag{21}$$

using (15) and (13).

We now consider the solution of (17). First we remark that if the law of force is the inverse square, or the inverse cube, or a combination of the two, namely

$$f(r) = \frac{\mu}{r^2}+\frac{\lambda}{r^3},$$

so that

$$f\left(\frac{1}{u}\right) = \mu u^2+\lambda u^3,$$

(17) is linear and its general solution can be written down immediately.

For other laws of force, as in § 55 (7), we multiply (17) by $h^2\,du/d\theta$. This gives

$$h^2\frac{d^2u}{d\theta^2}\frac{du}{d\theta}+h^2u\frac{du}{d\theta} = \frac{1}{u^2}f\left(\frac{1}{u}\right)\frac{du}{d\theta}.$$

Integrating gives

$$\tfrac{1}{2}h^2\left(\frac{du}{d\theta}\right)^2+\tfrac{1}{2}h^2u^2 = \int\frac{du}{u^2}f\left(\frac{1}{u}\right)+C$$

$$= -\int f(r)\,dr+C. \tag{22}$$

By (21) this is $$\tfrac{1}{2}v^2+\int f(r)\,dr = C, \tag{23}$$

which is the energy equation, § 73.

In the above we have studied the motion of a particle attracted to a fixed point. The case of practical importance, which is that of two particles of masses m_1 and m_2 attracted to one another by force $f(r)$, can be reduced to this (cf. § 76, Ex. 2). The centre of mass of the two particles moves with constant velocity, and the motion of m_2 relative to m_1 is the same as if m_1 were fixed and the attractive force were $\{(m_1+m_2)/m_1\}f(r)$.

Ex. 1. *The particle is projected from distance a with velocity V at an angle β to the radius vector. The law of force is the attractive inverse square $f(r) = \mu/r^2$.*

The differential equation (17) is

$$\frac{d^2u}{d\theta^2}+u = \frac{\mu}{h^2}, \tag{24}$$

where, by (18), $$h = Va\sin\beta. \tag{25}$$

If we measure θ from the radius vector to the point of projection, it follows from (20) that (24) has to be solved with

$$u = \frac{1}{a}, \qquad \frac{du}{d\theta} = -\frac{1}{a}\cot\beta, \quad \text{when } \theta = 0. \tag{26}$$

The general solution of (24) is

$$u = \frac{\mu}{h^2}\{1+e\cos(\theta+\alpha)\}, \tag{27}$$

where e and α are unknown constants to be determined from (26).

(27) may be written

$$\frac{l}{r} = 1+e\cos(\theta+\alpha), \tag{28}$$

where $$l = h^2/\mu = (Va\sin\beta)^2/\mu. \tag{29}$$

Now the polar equation of a conic of semi-latus rectum l and eccentricity e, referred to a focus as origin and with θ measured from its axis, is

$$\frac{l}{r} = 1+e\cos\theta. \tag{30}$$

Thus the orbit (28) is a conic of semi-latus rectum (29), eccentricity e, and with its axis inclined at α to the radius vector to the point of projection. To find e and α we have from (26) and (28)

$$\frac{l}{a} = 1+e\cos\alpha, \tag{31}$$

$$-\frac{l}{a}\cot\beta = -e\sin\alpha. \tag{32}$$

From (31) and (32)
$$\tan\alpha = \frac{l\cot\beta}{l-a}, \tag{33}$$

$$\begin{aligned} e^2 &= \left(\frac{l}{a}-1\right)^2+\frac{l^2}{a^2}\cot^2\beta \\ &= \frac{l^2}{a^2}\operatorname{cosec}^2\beta-\frac{2l}{a}+1 \\ &= 1-\frac{2V^2a\sin^2\beta}{\mu}+\frac{V^4a^2\sin^2\beta}{\mu^2}. \end{aligned} \tag{34}$$

Now the conic (30) is an ellipse, parabola, or hyperbola according as $e < = > 1$. Thus from (34) the orbit is an ellipse, parabola, or hyperbola according as

$$V^2 < = > \frac{2\mu}{a}. \tag{35}$$

Ex. 2. *A particle is projected with velocity V at a very great distance from an inverse square centre of repulsive force, $f(r) = -\mu/r^2$, in a direction whose perpendicular distance from the centre of force is p. It is required to find the angle ψ between the initial and final directions of its path.*

The equation (17) is

$$\frac{d^2u}{d\theta^2}+u = -\frac{\mu}{h^2}, \tag{36}$$

and its general solution is

$$u = A\cos\theta+B\sin\theta-\mu/h^2. \tag{37}$$

From the circumstances of projection we know that $h = pV$. Also at the point of projection r is very large, u and θ are very small, and

$$p = r\sin\theta = r\theta, \tag{38}$$

very nearly. Since θ is small at the point of projection, we may replace $\sin\theta$ by θ, and $\cos\theta$ by 1, in (37), so that, using (38), (37) becomes

$$\frac{\theta}{p} = A+B\theta-\frac{\mu}{h^2}. \tag{39}$$

It follows that $A = \mu/h^2$, $B = 1/p$, and the equation of the path is

$$u = \frac{\mu}{h^2}(\cos\theta-1)+\frac{1}{p}\sin\theta. \tag{40}$$

From (40)
$$\frac{du}{d\theta} = -\frac{\mu}{h^2}\sin\theta+\frac{1}{p}\cos\theta. \tag{41}$$

Therefore $du/d\theta = 0$ when

$$\tan\theta = \frac{h^2}{\mu p} = \frac{pV^2}{\mu}.$$

By symmetry the required angle ψ is twice this angle, that is

$$\psi = 2\tan^{-1}\frac{pV^2}{\mu}. \tag{42}$$

This result leads to Rutherford's scattering formula. Suppose that there are N scattering centres per unit area of a plane normal to the initial direction of the particle. Then the chance of the initial direction lying between distances p and $p+\delta p$ from one of these is

$$2\pi Np\,\delta p. \tag{43}$$

If $\phi = \pi-\psi$ is the angle of deflexion of the particle, (43) is the chance of the particle being deflected through an angle between ϕ and $\phi+\delta\phi$, and using (42) in the form

$$p = \frac{\mu}{V^2}\cot\tfrac{1}{2}\phi,$$

it becomes

$$\frac{\pi\mu^2 N}{V^4}\cot\tfrac{1}{2}\phi\,\mathrm{cosec}^2\,\tfrac{1}{2}\phi\,\delta\phi. \tag{44}$$

Ex. 3. *Stability of a circular orbit.*

Writing, for convenience, $f(1/u) = \phi(u)$ in (17), this becomes

$$\frac{d^2u}{d\theta^2}+u = \frac{1}{h^2u^2}\phi(u). \tag{45}$$

A circular orbit with $u = b$, $d^2u/d\theta^2 = 0$, is possible under any law of force provided the velocity in the orbit is chosen so that

$$h^2 = \frac{1}{b^3}\phi(b). \tag{46}$$

We have to consider whether such an orbit is stable, that is, if the motion is disturbed slightly, whether the particle will oscillate about the circular orbit or will diverge from it. Suppose the particle is disturbed by a small radial impulse so that h will remain unchanged. In the equation (45) for the orbit put

$$u = b+x, \tag{47}$$

where x is supposed small, and (45) becomes

$$\frac{d^2x}{d\theta^2}+b+x = \frac{1}{h^2(b+x)^2}\phi(b+x) \tag{48}$$

$$= \frac{1}{b^2h^2}\left(1-\frac{2x}{b}+\ldots\right)\{\phi(b)+x\phi'(b)+\ldots\}$$

$$= \frac{\phi(b)}{b^2h^2}+\frac{x}{b^2h^2}\left(\phi'(b)-\frac{2}{b}\phi(b)\right), \tag{49}$$

where to get (49) we have expanded the right-hand side of (48), using Taylor's theorem and retaining only the terms in x. Using (46) this becomes

$$\frac{d^2x}{d\theta^2}+x\left\{3-\frac{b\phi'(b)}{\phi(b)}\right\} = 0. \tag{50}$$

If the coefficient of x in (50) is positive the solution consists of trigonometric terms and the particle oscillates about the circular orbit, that is, the motion is stable. If this coefficient is negative, x contains an exponentially increasing term and the motion is unstable. Thus the condition for stability is

$$3 > \frac{b\phi'(b)}{\phi(b)}. \tag{51}$$

For example, for the inverse nth power attraction $f(r) = \mu r^{-n}$, $\phi(b) = \mu b^n$, (51) requires $n < 3$. Thus circular motion with the inverse nth power law is unstable if $n > 3$.

63. Motion of a particle whose mass varies

If the mass m of a particle is not constant, Newton's law of motion must be used in the form § 53 (10), namely,

$$\frac{d}{dt}(mv) = X, \tag{1}$$

where v is the component of the velocity of the particle in the direction of the x-axis and X is the component of the force on it in this direction.

Integrating (1) with respect to the time from t to $t+\delta t$ gives

$$\big[mv\big]_t^{t+\delta t} = X\,\delta t, \tag{2}$$

that is, $X\,\delta t$ is the increase in time δt of the momentum of the particle in the direction of the x-axis.

In many problems of varying mass, for example the motion of a raindrop which grows by absorbing smaller drops with which it collides, or the motion of a rocket, the mass gained or lost has momentum itself, and (1) must be generalized to account for this. Only the case of motion in one dimension will be considered here.

Suppose that at time t the mass is m and its velocity v, and suppose that in time t to $t+\delta t$ it gains mass δm which moves with velocity V. The gain in momentum in the time t to $t+\delta t$ is thus

$$(m+\delta m)(v+\delta v)-mv-V\,\delta m = (v-V)\,\delta m+m\,\delta v.$$

By (2), if force X acts on the particle the gain of momentum must be $X\,\delta t$, and so

$$(v-V)\,\delta m+m\,\delta v = X\,\delta t,$$

or, in the limit as $\delta t \to 0$

$$m\frac{dv}{dt}+(v-V)\frac{dm}{dt} = X, \tag{3}$$

or

$$\frac{d}{dt}(mv)-V\frac{dm}{dt} = X. \tag{4}$$

(4) reduces to (1) only if $V = 0$.

In the case of a rocket the velocity of efflux U of the burnt gases relative to the rocket is known, and in (4) we have $V = v-U$. Thus the equations of motion of a rocket under no forces are

$$\frac{d}{dt}(mv)-(v-U)\frac{dm}{dt} = 0,$$

or

$$m\frac{dv}{dt} = -U\frac{dm}{dt}. \tag{5}$$

If U is constant the solution of this is

$$\frac{v}{U} = -\ln m+C,$$

so that, if the rocket starts from rest with initial mass M,

$$v = U\ln\frac{M}{m}. \tag{6}$$

If m_0 is the mass of fuel carried, so that $m = M-m_0$ when all the fuel has been used, the maximum velocity attained is

$$U\ln\frac{M}{M-m_0}. \tag{7}$$

If the rocket is projected vertically upwards under gravity, (4) becomes

$$m\frac{dv}{dt}+U\frac{dm}{dt} = -mg. \tag{8}$$

This may be written

$$U\frac{dm}{dt} = -m\frac{d(v+gt)}{dt}.$$

The solution of this with $m = M$, $v = 0$, when $t = 0$, is

$$v = U \ln \frac{M}{m} - gt, \tag{9}$$

assuming, as before, that U is constant.

64. Moving axes

Hitherto the position of the particle being studied has always been referred to axes fixed in space. But it is often desirable to

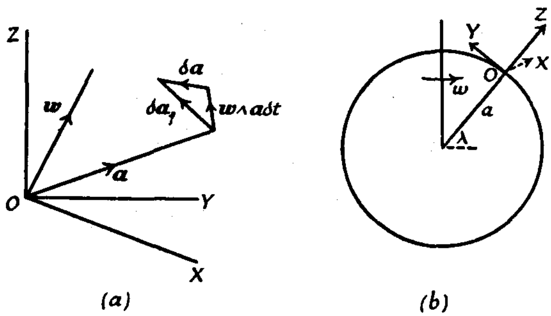


FIG. 51.

use a set of axes which are moving. For example, in problems on motion relative to the earth it is natural to use axes fixed at the point of observation, such as the north and east directions, and these axes, being fixed to the earth, move and rotate with it.

When we come to write down equations of motion, this has to be done relative to fixed axes, and we choose fixed axes along the instantaneous positions of the moving axes at the instant under consideration.

Consider first the case of rotating axes with a fixed origin O. For definiteness we shall take the rotating axes to be right-handed rectangular axes OX, OY, OZ, and we suppose that this system is rotating with angular velocity $\boldsymbol{\omega}$ relative to a set of fixed axes along the instantaneous directions of OX, OY, OZ.

Suppose that $\mathbf{a}$ is a vector specified relative to the moving system OX, OY, OZ, and that we require the value of its time-rate of change relative to the fixed system.

The change $(\delta\mathbf{a})_f$ in the vector $\mathbf{a}$ relative to the fixed axes in an element of time δt will be made up of two parts; (i) the change $\delta\mathbf{a}$ relative to the moving system OX, OY, OZ, and, (ii) the change $\boldsymbol{\omega}\wedge\mathbf{a}\,\delta t$ due to the motion of the system OX, OY, OZ carrying the vector $\mathbf{a}$ with it; cf. Fig. 51 (a). That is,

$$(\delta\mathbf{a})_f = \delta\mathbf{a}+\boldsymbol{\omega}\wedge\mathbf{a}\,\delta t. \tag{1}$$

Writing
$$\dot{\mathbf{a}} = \lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t} \tag{2}$$

for the rate of change of $\mathbf{a}$ relative to the moving system, and taking the limit of (1) as $\delta t\to 0$ we get for the rate of change of $\mathbf{a}$ relative to the fixed system

$$\left(\frac{d\mathbf{a}}{dt}\right)_f = \dot{\mathbf{a}}+\boldsymbol{\omega}\wedge\mathbf{a}. \tag{3}$$

If (a_1, a_2, a_3) and $(\omega_1, \omega_2, \omega_3)$ are the components of $\mathbf{a}$ and $\boldsymbol{\omega}$, the components of (3) are

$$(\dot a_1+\omega_2 a_3-\omega_3 a_2,\ \dot a_2+\omega_3 a_1-\omega_1 a_3,\ \dot a_3+\omega_1 a_2-\omega_2 a_1). \tag{4}$$

We shall use these results to calculate velocities and accelerations: they take into account the effect of rotation of the axes about the origin O. If, in addition, the origin O is in motion, its velocity and acceleration must be added to the values calculated from (3) to get results relative to an origin and axes at rest.

Ex. 1. *Velocities and accelerations in plane polar coordinates.*

Let the polar coordinates of a point be (r,θ): choose OX along the direction to the point from the origin, OY perpendicular to this in the coordinate plane, and OZ perpendicular to the plane. Then the components of the angular velocity of the moving axes are $(0, 0, \dot\theta)$. The point we are interested in is $(r, 0, 0)$.

By (4) the rate of change of its position is

$$(\dot r, r\dot\theta, 0), \tag{5}$$

and this is its velocity.

To find its acceleration we need the rate of change of the vector (5), and by (4) this is

$$(\ddot r-r\dot\theta^2,\ r\ddot\theta+2\dot r\dot\theta,\ 0), \tag{6}$$

as in § 62 (9) and (10). Velocities and accelerations in other coordinate systems may be found in the same way.

Ex. 2. *Motion relative to the earth.*

Consider a point on the earth's surface in latitude λ. Choose axes OZ inclined at $\frac{1}{2}\pi-\lambda$ to the earth's axis; OX easterly, perpendicular to the plane of OZ and the axis; and OY to make a right-handed system (Fig. 51 (b)). The earth rotates with angular velocity ω about its axis, and so the components of the angular velocity of the system OX, OY, OZ about their instantaneous positions are

$$(0,\ \omega\cos\lambda,\ \omega\sin\lambda). \tag{7}$$

If the position of a particle is (x, y, z), its components of velocity relative to O along the instantaneous directions of OX, OY, OZ are, by (4),

$$(\dot{x}+\omega z\cos\lambda-\omega y\sin\lambda,\ \dot{y}+\omega x\sin\lambda,\ \dot{z}-\omega x\cos\lambda). \tag{8}$$

To find the acceleration of the particle relative to the origin O we need the components of the rate of change of the vector (8) along the instantaneous directions of the axes. By (4) these are

$$\ddot{x}-2\omega\dot{y}\sin\lambda+2\omega\dot{z}\cos\lambda-\omega^2x, \tag{9}$$

$$\ddot{y}+2\omega\dot{x}\sin\lambda+\omega^2z\sin\lambda\cos\lambda-\omega^2y\sin^2\lambda, \tag{10}$$

$$\ddot{z}-2\omega\dot{x}\cos\lambda-\omega^2z\cos^2\lambda+\omega^2y\sin\lambda\cos\lambda. \tag{11}$$

In (9) to (11) the terms in ω^2x, ω^2y, and ω^2z are negligible.

If the perpendicular from the origin O to the earth's axis is of length p, the origin has acceleration ω^2p towards the axis. Therefore components of acceleration

$$(0,\ \omega^2p\sin\lambda,\ -\omega^2p\cos\lambda) \tag{12}$$

have to be added to (9)–(11) to get accelerations relative to fixed axes.

Since the earth is spheroidal, gravity will act in the plane YOZ: suppose its direction is inclined at a (small) angle α to OZ. Then the equations of motion of a particle of mass m under gravity are (neglecting the small terms in ω^2x, etc.)

$$\ddot{x}-2\omega\dot{y}\sin\lambda+2\omega\dot{z}\cos\lambda = 0, \tag{13}$$

$$\ddot{y}+2\omega\dot{x}\sin\lambda = -g\sin\alpha-\omega^2p\sin\lambda, \tag{14}$$

$$\ddot{z}-2\omega\dot{x}\cos\lambda = -g\cos\alpha+\omega^2p\cos\lambda. \tag{15}$$

The right-hand side of these equations is a force in the direction obtained by compounding the earth's attraction with centrifugal force. If this direction (the direction of apparent gravity) is inclined at $\frac{1}{2}\pi-\theta$ to the axis (θ is the geographical latitude) and we take OZ in this direction and OY correspondingly, the equations (13)–(15) become

$$\ddot{x}-2\omega\dot{y}\sin\theta+2\omega\dot{z}\cos\theta = 0, \tag{16}$$

$$\ddot{y}+2\omega\dot{x}\sin\theta = 0, \tag{17}$$

$$\ddot{z}-2\omega\dot{x}\cos\theta = -g. \tag{18}$$

Suppose the particle falls from rest at the origin when $t = 0$. Integrating (16)–(18) gives

$$\dot{x}-2\omega y\sin\theta+2\omega z\cos\theta = 0, \tag{19}$$

$$\dot{y}+2\omega x\sin\theta = 0, \tag{20}$$

$$\dot{z}-2\omega x\cos\theta = -gt. \tag{21}$$

Using (20) and (21) in (16),

$$\ddot{x}+4\omega^2x = 2\omega gt\cos\theta.$$

Therefore, using $x = \dot{x} = 0$, when $t = 0$

$$x = -\frac{g\cos\theta}{4\omega^2}\sin 2\omega t+\frac{gt}{2\omega}\cos\theta. \tag{22}$$

Since ωt is small, expanding $\sin 2\omega t$ in (22) gives, very nearly,

$$x = \tfrac{1}{3}g\omega t^3\cos\theta,$$

a small deviation in an easterly direction. A much smaller deviation in a southerly direction can be found from (20).

EXAMPLES ON CHAPTER VII

1. A simple pendulum is set in motion from the downward vertical position with angular velocity $2n$, where $n^2 = g/l$. Show that it has just sufficient energy to reach the upward vertical position, and that the time it takes to reach the angle θ is

$$\frac{1}{n}\ln\tan\left(\frac{\pi}{4}+\frac{\theta}{4}\right).$$

2. The earth's attraction on a particle of mass m at height h above its surface is $mga^2(a+h)^{-2}$, where a is the earth's radius. If a particle is projected vertically upwards with velocity V, show that (neglecting air resistance and the rotation of the earth) its velocity at height h is

$$[aV^2+h(V^2-2ga)]^{\frac{1}{2}}(a+h)^{-\frac{1}{2}}.$$

Show that if $V^2 = 2ga$, the time it takes to reach the height h is

$$2[(a+h)^{\frac{3}{2}}-a^{\frac{3}{2}}]/3Va^{\frac{1}{2}}.$$

3. An elastic sphere of mass m and radius a, moving with velocity V, collides with an equal sphere which is directly in its path. If the force between the spheres is kr^3, where r is the radius of the impression on either sphere, show that the maximum depth of the impression is

$$\left[\frac{5mV^2}{16k(2a)^{\frac{1}{2}}}\right]^{\frac{2}{5}},$$

and discuss the motion. [Cf. § 55, Ex. 4.]

4. The force on a particle of mass m is $max^{-2}-mbx^{-3}$. It is released from rest at $x = X$; show that it oscillates between $x = X$ and $x = Xb/(2Xa-b)$ with period $2\pi aX^3(2Xa-b)^{-\frac{3}{2}}$.

5. The motion of a mass m in the direction of x increasing is resisted by a compressed air spring which provides restoring force $P(1-kx)^{-\gamma}$, where P, k, and γ are constants. If a constant force $P' > P$ is applied to the mass when it is at rest and $x = 0$, show that it comes to rest when

$$kP'x(1-\gamma)+P(1-kx)^{1-\gamma} = P.$$

6. The region $0 < x < a$ contains a space charge of electrons (as in a vacuum tube). The electric potential V in the region satisfies

$$\frac{d^2V}{dx^2} = KJV^{-\frac{1}{2}},$$

where K is a constant and J is the density of electric current (also constant). If $V = 0$ when $x = 0$, $V = V_a$ when $x = a$, and $dV/dx = 0$ when $x = 0$, show that $J = 4V_a^{\frac{3}{2}}/(9Ka^2)$.

7. Show that the solution of

$$\frac{d^2v}{dx^2}+\beta e^v = 0,$$

with $dv/dx = 0$ and $v = v_0$ when $x = 0$, is

$$v = v_0-2\ln\cosh x(\tfrac{1}{2}\beta e^{v_0})^{\frac{1}{2}}.$$

This corresponds to motion with a restoring force which increases exponentially with the displacement. In the theory of the thermal breakdown of a dielectric, the temperature v satisfies the above equation with $dv/dx = 0$ when $x = 0$, and $v = 0$ when $x = 1$. Find an equation for $\sqrt{(\frac{1}{2}\beta e^{v_0})}$, where v_0 is the temperature at $x = 0$, and discuss its solution graphically.

8. A particle moves in a straight line under resistance to motion mkv^3, where v is its velocity. If the initial velocity of the particle is V, show that the distance described in time t is

$$\{(2ktV^2+1)^{\frac{1}{2}}-1\}/kV.$$

9. A particle is projected vertically upwards with velocity U in a medium whose resistance varies as the square of the velocity. Show that the particle returns to its starting-point with velocity $UV(U^2+V^2)^{-\frac{1}{2}}$, where V is the terminal velocity.

10. A particle of mass m is attached to a spring of stiffness mn^2, and its motion is resisted by a force mkv^2, where v is its velocity. Derive a linear first-order equation for v^2. Show that if the particle is released from rest at $x = a$, it next comes to rest at $x = b$ given by

$$(1+2ak)e^{-2ak} = (1+2bk)e^{-2bk}.$$

11. A battery of voltage E is applied at $t = 0$ to an inductance L in series with a non-linear resistance for which $I = KV^n$; cf. § 57 (i). If the initial current in the inductance is zero, show that if $n = 3$ the voltage drop across the resistance at time t is given by

$$t = -3KE^2L\left\{\ln\left(1-\frac{V}{E}\right)+\frac{V}{E}\left(1+\frac{V}{2E}\right)\right\},$$

while the corresponding result for the case $n = 3/2$ is

$$t = 3KLE^{\frac{1}{2}}\{\tanh^{-1}(V/E)^{\frac{1}{2}}-(V/E)^{\frac{1}{2}}\}.$$

12. A battery of voltage E is applied at $t = 0$ to a capacitance C in series with a non-linear resistance for which $I = KV^n$. If the condenser is initially uncharged, show that the voltage drop V over the resistance at time t is given by

$$(n-1)KE^{n-1}V^{n-1}t = CE^{n-1}-CV^{n-1}.$$

13. A non-linear inductance in which the flux ϕ is related to the current I by $I = A\phi+B\phi^3$ is in series with a linear resistance R. If the flux is ϕ_0 at time $t = 0$, show that its value at time t is given by

$$\frac{\phi}{(A+B\phi^2)^{\frac12}} = \frac{\phi_0}{(A+B\phi_0^2)^{\frac12}}e^{-RAt}.$$

14. For the equation of type § 58 (8),

$$\ddot{I}-\epsilon(1-\alpha\dot{I}-\beta\dot{I}^2)\dot{I}+n^2I = 0,$$

show that if the amplitude of an oscillation at $t = 0$ is a_0, its value at time t is

$$a_0 e^{\frac12\epsilon t}\{1+\tfrac34\beta a_0^2 n^2(e^{\epsilon t}-1)\}^{-\frac12}.$$

15. Show that the period of oscillation in a circuit containing a non-linear inductance specified by § 57 (9) and (10) is approximately

$$\frac{2\pi}{n}\left(1+\frac{3Ba^2}{8A}\right)^{-1},$$

where $n^2 = A/C$, and a is the amplitude of the oscillation. Discuss the variation of a with time.

16. A particle of mass m is pressed against a plane by a force P, the coefficients of static and dynamic friction between the particle and the plane being μ' and μ respectively, and $\mu' > \mu$. The particle is attached to a spring of stiffness λ, the other end of which is moved with constant velocity V. Discuss the motion of the particle, and show that it will perform stick-slip relaxation oscillations and find their period.

17. A particle is projected vertically upwards from ground level with velocity U into air in which a horizontal wind of velocity u is blowing. Assuming that the wind causes a horizontal force $mk(u-v)^2$ on the particle, where v is the horizontal component of its velocity, and that there is resistance to vertical motion mk times the square of the vertical component of its velocity, show that it reaches the ground again after time T given by

$$T = \frac{1}{kV}\left\{\tan^{-1}\frac{U}{V}+\tanh^{-1}\frac{U}{(U^2+V^2)^{\frac12}}\right\},$$

and that its direction of motion makes then an angle

$$\tan^{-1}\{UV(1+kTu)/[kTu^2(U^2+V^2)^{\frac12}]\}$$

with the ground, where $V = \sqrt{(g/k)}$.

18. A particle is projected with velocity U at an angle α to the horizontal in a medium in which the resistance is proportional to the

velocity. Show that for the range on a horizontal plane to be a maximum, α must be given by

$$U(V+U\sin\alpha) = V(U+V\sin\alpha)\ln[1+(U/V)\operatorname{cosec}\alpha],$$

where V is the terminal velocity.

19. An electron gun emits a slightly divergent beam of electrons from a point on the z-axis; all of these have the same speed V, but their directions are inclined at angles up to θ with the z-axis. Show that, if a magnetic field H is applied along the z-axis, all the electrons will cross this axis again at points whose distances from the gun lie in the range from $(2\pi Vmc/eH)$ to $(2\pi Vmc/eH)\cos\theta$. Thus if θ is small the electrons can very nearly be focused at a point.

20. An electron is acted on by an electric field $E_0 \sin nt$ along the x-axis, and a magnetic field H along the z-axis. Show that if $\omega \neq n$, where $\omega = eH/mc$, and the electron is emitted at the origin $t = 0$ with zero velocity, its path is

$$x = eE_0\{n\sin\omega t-\omega\sin nt\}/m\omega(\omega^2-n^2),$$

$$y = eE_0\{(\omega^2-n^2)+n^2\cos\omega t-\omega^2\cos nt\}/m\omega n(n^2-\omega^2).$$

Discuss also the case $\omega = n$.

21. An electron moves in the region between two concentric cylinders parallel to the z-axis of radii a and b, $b > a$. It is acted on by a magnetic field H parallel to the z-axis, and by an electrostatic field between the cylinders such that the force on the electron is radial, and its magnitude is k/r, where r is the distance from the z-axis and k is a constant (the components of this force along the x- and y-axes will be kx/r^2 and ky/r^2). Considering only motion in the xy-plane, show that if v is the speed of the electron at any point, two first integrals of the motion are

$$\tfrac{1}{2}mv^2 = k\ln r+C, \quad \text{and} \quad x\dot{y}-y\dot{x} = \tfrac{1}{2}\omega r^2+C',$$

where C and C' are arbitrary constants, and $\omega = eH/mc$.

If $v = 0$ when $r = a$, and when $r = b$ the electron is moving in a direction perpendicular to the radius vector (that is, it reaches the cylinder $r = b$ at grazing incidence), show that

$$H = \frac{2cb}{b^2-a^2}\left(\frac{2mk}{e^2}\ln(b/a)\right)^{\frac{1}{2}}.$$

This is the theory of the cylindrical magnetron, H being the field which is just sufficient to stop electrons from reaching the anode. If the anode potential is V, the constant k above has the value $eV/\ln(b/a)$.

22. Deduce the result of Ex. 21 by working (i) in cylindrical polar coordinates, and (ii) vectorially.

23. A particle is projected with velocity u at an angle α to the horizontal under gravity in a medium in which the resistance to motion is mk times the square of the velocity. If s is the length of the arc of its path

from the point of projection, and ψ is the slope of the tangent at s, show that

$$\dot{x} = u\cos\alpha e^{-ks},$$

$$\sec^3\psi\frac{d\psi}{ds}+(g/u^2)e^{2ks}\sec^2\alpha = 0.$$

Find the intrinsic equation of the path by integrating the last equation.

24. A particle is moving under a central force to the origin. Show that the rate at which area is swept over by the radius vector to the particle (the areal velocity) is $\frac{1}{2}h$, where h is defined in § 62 (13).

If the path of the particle is known to be the ellipse $l = r(1+e\cos\theta)$, show that the attractive force must be mkr^{-2}, where $k = h^2/l$, and that the periodic time is $2\pi a^{\frac{3}{2}}k^{-\frac{1}{2}}$, where a is the semi-major axis of the ellipse.

25. A particle is projected at distance c from a centre of force at the origin with velocity v in a direction inclined at β to the radius vector. If the law of force is an inverse square repulsion $m\mu/r^2$, show that the orbit is a hyperbola of eccentricity

$$\left\{1+\frac{2v^2c\sin^2\beta}{\mu}+\frac{v^4c^2\sin^2\beta}{\mu^2}\right\}^{\frac{1}{2}},$$

and that
$$v^2 = \frac{\mu}{a}-\frac{2\mu}{r},$$

where a is the semi-major axis of the hyperbola.

26. A particle is attracted to the origin O by the force $m\mu r^{-2}+m\lambda r^{-3}$. Show that it is moving perpendicular to the radius vector when

$$\theta = n\pi h(h^2-\lambda)^{-\frac{1}{2}}+C,$$

where C is a constant depending on the circumstances of projection, n is any integer, and h is defined in § 62 (13).

27. The equation of motion of a particle of mass m attracted by a central force $mf(r)$ to an origin O may be written

$$\ddot{\mathbf{r}} = -\mathbf{r}\{f(r)/r\}, \tag{1}$$

where $\mathbf{r}$ is the position vector of the particle relative to the origin O.

(i) Deduce the constancy of the angular momentum $\mathbf{H} = m\mathbf{r}\wedge\dot{\mathbf{r}}$ of the particle about the origin by taking the vector product of (1) with $\mathbf{r}$.

(ii) Deduce the energy equation, § 62 (23), by taking the scalar product of (1) with $\dot{\mathbf{r}}$.

(iii) Show that
$$\mathbf{H}\wedge\ddot{\mathbf{r}} = -mr^2f(r)\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right).$$

28. Show that the equation of motion of an electron of charge $-e$ and mass m in a magnetic field $\mathbf{H}$ is

$$\ddot{\mathbf{r}} = -(e/mc)\dot{\mathbf{r}}\wedge\mathbf{H}.$$

Deduce that, whatever the nature of $\mathbf{H}$ as a function of position, the speed of the electron is constant.

If $\mathbf{H}$ is independent of position, show that

$$\dot{\mathbf{r}} = -(e/mc)\mathbf{r}\wedge\mathbf{H}+\mathbf{V},$$

and deduce that the particle moves in a helix.

29. Let $\mathbf{r}$ be the position vector of a point on a plane curve whose arc measured from some reference point to the point is s. Show that $\mathbf{t} = d\mathbf{r}/ds$ is a unit vector along the tangent at $\mathbf{r}$ to the curve. If ρ is the radius of curvature of the curve at the point $\mathbf{r}$, show that

$$\frac{d\mathbf{t}}{ds} = \frac{\mathbf{n}}{\rho},$$

where $\mathbf{n}$ is a unit vector along the normal to the curve at the point $\mathbf{r}$.

Show that $\qquad \dot{\mathbf{r}} = \dot{s}\mathbf{t}, \qquad \ddot{\mathbf{r}} = \ddot{s}\mathbf{t}+(\dot{s}^2/\rho)\mathbf{n},$

and deduce the equations of motion § 61 (9) and (10).

30. A simple pendulum of length l performs small oscillations in latitude θ. If the origin is in its equilibrium position, and the x- and y-axes are chosen as in § 64, Ex. 2, show that $\zeta = x+iy$ satisfies

$$\ddot{\zeta}+2i\omega\dot{\zeta}\sin\theta+n^2\zeta = 0,$$

where $n^2 = g/l$. Show that the path of the pendulum is an ellipse which rotates with constant angular velocity $-\omega\sin\theta$.

31. A particle is projected with velocity u at an angle α to the horizontal in a direction β to the north of east. Show that its deviation to the left of the plane of projection at time t is

$$\{(u\sin\alpha-\tfrac{1}{3}gt)\sin\beta\cos\theta-u\cos\alpha\sin\theta\}\omega t^2,$$

where θ is the latitude and ω the earth's angular velocity.

32. Assuming that, if p is the atmospheric pressure, the components of force in the x and y directions on an element of mass $\rho\,\delta x\delta y$ of air are $-(\partial p/\partial x)\,\delta x\delta y$ and $-(\partial p/\partial y)\,\delta x\delta y$, respectively, where ρ is the density of the air, show that the components of the velocity of the steady wind due to this pressure distribution are

$$-\frac{1}{2\rho\omega\sin\theta}\frac{\partial p}{\partial y} \quad\text{and}\quad \frac{1}{2\rho\omega\sin\theta}\frac{\partial p}{\partial x},$$

where θ is the latitude and ω is the earth's angular velocity. That is (neglecting friction) the wind tends to blow so as to keep the low pressure to its left in the northern hemisphere and to its right in the southern.

VIII

RIGID DYNAMICS

65. Moments and products of inertia

SUPPOSE we have a rigid body and a set of fixed rectangular axes OX, OY, OZ. Let OR be a line through the origin whose direction cosines are (l, m, n). We suppose the body to be divided up into small elements of mass, of which a typical one is m at the point P, (x, y, z), of Fig. 52. The moment of inertia I_{OR} of the body about the axis OR is defined as

$$I_{OR} = \sum mPQ^2, \qquad (1)$$

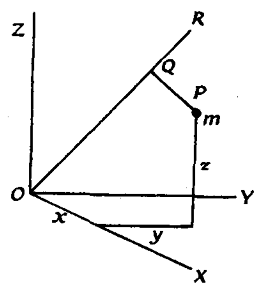


FIG. 52.

where PQ is the perpendicular distance from the particle m to the axis OR, and $\sum$ denotes, here and throughout this chapter, a summation over the elements of mass comprising the body: if the body is a continuous one these sums have to be evaluated by integration by methods described in text-books on the calculus.

If M is the mass of the body, the length k defined by

$$k^2 = I_{OR}/M$$

is called the *radius of gyration* of the body about the axis OR.

For the chosen axes OX, OY, OZ we define six fundamental quantities, namely

$$A = \sum m(y^2+z^2), \quad B = \sum m(z^2+x^2), \quad C = \sum m(x^2+y^2), \qquad (2)$$

$$F = \sum myz, \quad G = \sum mzx, \quad H = \sum mxy. \qquad (3)$$

A, B, C are by (1) the moments of inertia of the body about the axes OX, OY, OZ, respectively. F, G, H are called the products of inertia of the body for the axes OX, OY, OZ. We proceed to show that, if these six quantities are known, the moment of inertia I_{OR} about any axis can be expressed in terms of them.

From (1)†

$$
\begin{aligned}
I_{OR} &= \sum m(OP^2-OQ^2) \\
&= \sum m\{x^2+y^2+z^2-(lx+my+nz)^2\} \\
&= \sum m\{x^2(1-l^2)+y^2(1-m^2)+z^2(1-n^2)- \\
&\qquad\qquad -2mnyz-2nlzx-2lmxy\} \\
&= \sum m\{x^2(m^2+n^2)+y^2(n^2+l^2)+z^2(l^2+m^2)- \\
&\qquad\qquad -2mnyz-2nlzx-2lmxy\} \\
&= l^2\sum m(y^2+z^2)+m^2\sum m(z^2+x^2)+n^2\sum m(x^2+y^2)- \\
&\qquad\qquad -2mn\sum myz-2nl\sum mzx-2lm\sum mxy \\
&= Al^2+Bm^2+Cn^2-2Fmn-2Gnl-2Hlm. \qquad (4)
\end{aligned}
$$

This is the general result in three dimensions. It is interesting, and important for many purposes, to study the corresponding two-dimensional problem of the variation of the moment of inertia of a lamina in the xy-plane about an axis in this plane inclined at θ to OX. Since the lamina lies in the xy-plane, $z = 0$ for all its particles, and so $F = G = 0$, and $C = A+B$. Also the direction cosines of an axis in the plane will be $l = \cos\theta$, $m = \sin\theta$, $n = 0$. Thus (4) becomes

$$I_\theta = A\cos^2\theta+B\sin^2\theta-2H\sin\theta\cos\theta. \qquad (5)$$

To see the way in which the moment of inertia varies with θ we may make a polar plot of I_θ against θ, plotting a distance proportional to I_θ in the direction θ. This is done in Fig. 53 for the lamina of angle section shown dotted, and gives the dumb-bell-shaped figure shown by the full line. To find the maxima and minima of I_θ, we have from (5)

$$\frac{dI_\theta}{d\theta} = (B-A)\sin 2\theta-2H\cos 2\theta, \qquad (6)$$

and thus I_θ is stationary when

$$\tan 2\theta = \frac{2H}{B-A}. \qquad (7)$$

† Note that m is used here in two senses, namely, as a direction cosine and as the mass of the typical particle. The latter is always written immediately after the sign of summation to avoid confusion.

(7) gives two values of θ at right angles, corresponding to maxima and minima of the moment of inertia. The values of the moments of inertia in these directions are called the *Principal Moments of Inertia* of the lamina, and the directions themselves are called the *Principal Axes of Inertia* of the lamina.

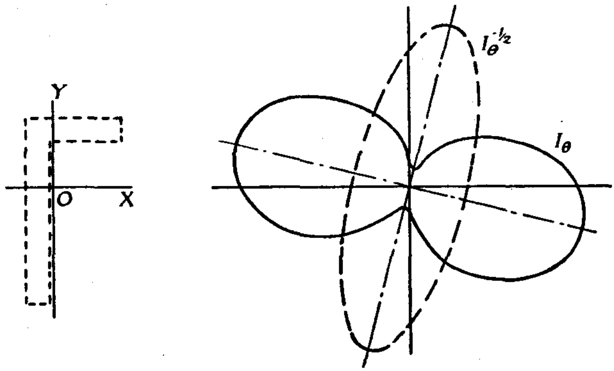


FIG. 53.

These results may be obtained in another way which leads to a very simple geometrical picture. If, instead of plotting I_θ against θ in a polar plot as was done above, we plot $1/\surd(I_\theta)$, then the rectangular coordinates of the point plotted will be

$$X = \frac{k}{\surd(I_\theta)}\cos\theta, \qquad Y = \frac{k}{\surd(I_\theta)}\sin\theta, \tag{8}$$

where k is the constant of proportionality. Substituting these values in (5) it appears that (X, Y) lies on the ellipse

$$AX^2+BY^2-2HXY = k^2. \tag{9}$$

This ellipse is shown dotted in Fig. 53. It is called the *Momental Ellipse* for the lamina. It is known that an ellipse has two principal axes relative to which its equation is

$$A'X'^2+B'Y'^2 = k^2; \tag{10}$$

these axes will be the principal axes of inertia defined previously, and their inclination to the original axes will be given by (7). By inspection of Fig. 53 it appears that the momental ellipse is elongated in the direction of the body, that is, that its major axis, which is the direction of least moment of inertia, lies roughly along the greatest diameter of the body. We may also remark that if the moments of inertia of the lamina in any three directions are equal, the momental ellipse is a circle and therefore the moments of inertia in all directions are equal.

The same theory applies to the three-dimensional case (4). If a point (X, Y, Z) is plotted in the direction (l, m, n) at a distance $k/\sqrt{I}$ from the origin so that

$$X = \frac{kl}{\sqrt{I}}, \qquad Y = \frac{km}{\sqrt{I}}, \qquad Z = \frac{kn}{\sqrt{I}}, \tag{11}$$

then X, Y, Z lies on the ellipsoid

$$AX^2+BY^2+CZ^2-2FYZ-2GZX-2HXY = k^2. \tag{12}$$

This is the *momental ellipsoid* for the body. It has three principal axes at right angles, which are the principal axes of inertia of the body. Relative to these, (12) becomes

$$A'X'^2+B'Y'^2+C'Z'^2 = k^2. \tag{13}$$

The moments of inertia of the body in the directions of the principal axes are called its principal moments of inertia.

So far we have regarded A, B, C, F, G, H as given. For complicated bodies they have to be determined by integration. For simple shapes the results of these integrations are summed up in *Routh's rule*, which states that *the moment of inertia about an axis of symmetry of a body of mass M which has three perpendicular axes of symmetry* is

$$\frac{M}{n}\{\text{the sum of the squares of the semi-axes perpendicular to the one considered}\}, \tag{14}$$

where $n = 3$, for a rectangular parallelepiped,

$n = 4$, for a circular lamina,

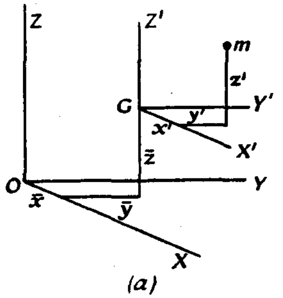
$n = 5$, for a sphere or ellipsoid.

For example, for a rectangular lamina of sides $2a$ and $2b$ the moments of inertia about the axes of symmetry are

$$\tfrac{1}{3}Mb^2, \quad \tfrac{1}{3}Ma^2, \quad \tfrac{1}{3}M(a^2+b^2), \tag{15}$$

the latter being about the line through the centre perpendicular to the plane of the lamina.

The results obtained by Routh's rule (in which the axes, being axes of symmetry, necessarily pass through the centre of mass of the body) may be extended by the use of simple results connecting the moment of inertia of a body about any axis with the moment of inertia about a parallel axis through the centre of mass.



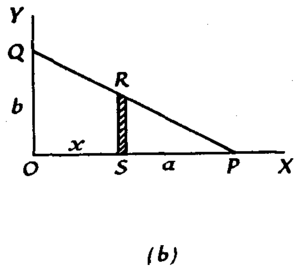


FIG. 54.

Let OX, OY, OZ be any rectangular axes, let $(\bar{x}, \bar{y}, \bar{z})$ and (x, y, z) be the coordinates of the centre of mass G of the body, and the particle of mass m, relative to them. Let GX', GY', GZ' be a set of parallel axes through G, and let (x', y', z') be the coordinates of m relative to them, Fig. 54 (a). Then

$$x = \bar{x}+x', \qquad y = \bar{y}+y', \qquad z = \bar{z}+z'. \tag{16}$$

Also, since the centre of mass of the body is at the origin of the (x', y', z') coordinate system, we have from the definition of the centre of mass, cf. § 66 (18),

$$\sum mx' = \sum my' = \sum mz' = 0. \tag{17}$$

Relative to the axes OX, OY, OZ we have

$$\begin{aligned} A = \sum m(y^2+z^2) &= \sum m\{(\bar{y}+y')^2+(\bar{z}+z')^2\} \\ &= (\bar{y}^2+\bar{z}^2)\sum m+2\bar{y}\sum my'+2\bar{z}\sum mz'+ \\ &\qquad\qquad\qquad\qquad +\sum m(y'^2+z'^2) \\ &= M(\bar{y}^2+\bar{z}^2)+A_{CG}, \end{aligned} \tag{18}$$

by (17), where we have used the fact that $\bar{x}$, $\bar{y}$, $\bar{z}$ are the same for all particles m and so may be taken outside the summations. In (18) M is the total mass of the body, and A_{CG} is the quantity A for the parallel system of axes through the centre of mass.

In the same way

$$\begin{aligned} F = \sum myz &= \sum m(\bar{y}+y')(\bar{z}+z') \\ &= \bar{y}\bar{z}\sum m+\bar{y}\sum mz'+\bar{z}\sum my'+\sum my'z' \\ &= M\bar{y}\bar{z}+F_{CG}. \end{aligned} \tag{19}$$

Similar results hold for B, C, G, H. These results may be expressed by the statement that a moment or product of inertia relative to any system of rectangular axes is obtained by adding to the corresponding quantity for parallel axes through the centre of mass a *transfer term* which is just the moment or product of inertia relative to the original axes of a particle of mass equal to the total mass of the body and placed at its centre of mass.

As an example we find the moments and product of inertia relative to the axes OX, OY (Fig. 54 (b)) of a right-angled triangular lamina of sides a and b.

To find A, the moment of inertia about OX, we consider the strip RS of width δx at x. If ρ is the surface density of the lamina, the moment of inertia of this strip about OX is, by Routh's rule,

$$\tfrac{1}{3}\rho y^3\,\delta x.$$

Here $y = b(a-x)/a$, and, integrating from $x = 0$ to $x = a$ we get

$$A = \frac{\rho b^3}{3a^3}\int\limits_0^a (a-x)^3\,dx = \tfrac{1}{12}\rho ab^3. \tag{20}$$

Interchanging a and b we get

$$B = \tfrac{1}{12}\rho a^3 b. \tag{21}$$

To find the product of inertia we use the fact that, because of its symmetry, the product of inertia of the strip RS about axes parallel to OX and OY through its centre of mass must be zero.

Thus the product of inertia for the strip RS relative to the axes OX, OY consists only of the transfer term

$$\rho y\,\delta x\,x\,\tfrac{1}{2}y.$$

Integrating we get

$$H = \frac{1}{2}\frac{\rho b^2}{a^2}\int\limits_0^a x(a-x)^2\,dx = \tfrac{1}{24}\rho a^2b^2. \tag{22}$$

Thus the momental ellipse (9) for the lamina relative to these axes is

$$b^2x^2+a^2y^2-abxy = \text{constant}. \tag{23}$$

And, by (7), the inclination θ of the principal axes to OX and OY is given by

$$\tan 2\theta = \frac{ab}{a^2-b^2}.$$

66. Fundamental equations

We regard a rigid body as composed of a large number of particles held together by cohesive or *internal* forces acting between them. In addition to these there are *external* forces acting on the particles which may be of two types, either *body* forces such as gravity which act on all particles, or applied forces or reactions which we regard as being applied to certain specified particles. The masses of the particles will always be assumed to be constant.

Let m be the mass of a typical particle of the body and let $\mathbf{r}$ be its position vector relative to a fixed frame of reference with origin O. Let $\mathbf{P}$ and $\mathbf{P}'$, respectively, be the resultant external and internal forces on the particle. Then its equation of motion is

$$\frac{d}{dt}(m\dot{\mathbf{r}}) = \mathbf{P}+\mathbf{P}'. \tag{1}$$

As in § 53 we use $\sum$ to denote a summation over all particles of the body, and write

$$\mathbf{L} = \sum m\dot{\mathbf{r}} \tag{2}$$

for its linear momentum, and

$$\mathbf{H} = \sum \mathbf{r}\wedge m\dot{\mathbf{r}} \tag{3}$$

for its angular momentum about O. Summing (1) over all particles of the body gives

$$\frac{d\mathbf{L}}{dt} = \sum \mathbf{P} + \sum \mathbf{P}'. \tag{4}$$

Also, differentiating (3) and using $\dot{\mathbf{r}}\wedge\dot{\mathbf{r}} = 0$,

$$\frac{d\mathbf{H}}{dt} = \sum \mathbf{r}\wedge\frac{d}{dt}(m\dot{\mathbf{r}}) = \sum \mathbf{r}\wedge\mathbf{P} + \sum \mathbf{r}\wedge\mathbf{P}', \tag{5}$$

using (1).

Some assumption has to be made as to the nature of the internal forces, and here we shall make the simplest possible one, namely, that they consist of forces of attraction or repulsion between the individual particles. Since for each such force on a particle A due to a particle B there is an equal, oppositely directed, reaction force on B due to A, the sum of these, and also the sum of their moments about O, vanishes. And therefore, summing over all particles,

$$\sum \mathbf{P}' = 0, \tag{6}$$

$$\sum \mathbf{r}\wedge\mathbf{P}' = 0. \tag{7}$$

The results (6) and (7) are also true under less restrictive conditions than those assumed above. Using (6) in (4) gives

$$\frac{d\mathbf{L}}{dt} = \sum \mathbf{P}. \tag{8}$$

This is the first fundamental result, and may be stated as:

I. *The rate of change of linear momentum of the body is equal to the vector sum of the external forces acting on it.*

Again, using (7) in (5) gives the second fundamental result:

II. *The rate of change of angular momentum of the body about a fixed origin O is equal to the sum of the moments of the external forces about O; that is,*

$$\frac{d\mathbf{H}}{dt} = \sum \mathbf{r}\wedge\mathbf{P}. \tag{9}$$

If the forces on the body are in static equilibrium and it is initially at rest, $\mathbf{L}$ and $\mathbf{H}$ must remain zero, and (8) and (9) give

$$\sum \mathbf{P} = 0, \qquad \sum \mathbf{r}\wedge\mathbf{P} = 0, \tag{10}$$

which are the conditions of static equilibrium stated in § 53.

If (x,y,z) and (X,Y,Z) are the components of $\mathbf{r}$ and $\mathbf{P}$ relative to right-handed rectangular axes through O, (8) becomes

$$\frac{d}{dt}\sum m\dot{x} = \sum X, \qquad \frac{d}{dt}\sum m\dot{y} = \sum Y, \qquad \frac{d}{dt}\sum m\dot{z} = \sum Z. \tag{11}$$

Also, using § 52 (18), (9) gives

$$\frac{d}{dt}\sum m(y\dot{z}-z\dot{y}) = \sum (yZ-zY), \tag{12}$$

together with two similar equations.

An alternative method of deriving the fundamental equations should also be noted. (1) may be written in the form

$$\mathbf{P}+\mathbf{P}'-m\ddot{\mathbf{r}} = 0. \tag{13}$$

Now in (13), $-m\ddot{\mathbf{r}}$ has the dimensions of a force, and in this sense is referred to as the *reversed effective force* on the particle. The whole of the forces acting on the system of particles which compose the body are now the external, internal, and reversed effective forces, and the equations of motion are obtained by writing down the conditions that these be in static equilibrium. When this is done the internal forces will disappear by (6) and (7), and the result may be stated in the form that the external and reversed effective forces on all particles of the body are in static equilibrium: this is known as *d'Alembert's principle*. In this method the conditions for a set of forces to be in static equilibrium are used, and are assumed to have been derived from statical considerations (cf. Chap. VI, Exs. 11 to 13).

If $M = \sum m$ is the mass of the body, the position vector $\bar{\mathbf{r}}$ of the centre of mass relative to O is defined by

$$M\bar{\mathbf{r}} = \sum m\mathbf{r}. \tag{14}$$

Using this result in (2) gives

$$\mathbf{L} = M\dot{\bar{\mathbf{r}}}, \tag{15}$$

that is, the linear momentum of the body is equal to that of a particle of mass M moving with the velocity of the centre of mass of the body. Using (15) in (8) we get

$$\frac{d}{dt}(M\dot{\bar{\mathbf{r}}}) = \sum \mathbf{P}, \tag{16}$$

which is the third fundamental result and may be stated as:

III. *The centre of mass moves like a particle of mass M placed there and acted on by the vector sum of the external forces on the body.*

Now let $\mathbf{r}'$ be the position vector of the typical particle m relative to the centre of mass, so that

$$\mathbf{r} = \bar{\mathbf{r}} + \mathbf{r}'. \tag{17}$$

From (17) $$\sum m\mathbf{r}' = \sum m\mathbf{r} - \sum m\bar{\mathbf{r}} = 0, \tag{18}$$

using (14). Substituting (17) in (3) we get

$$\begin{aligned} \mathbf{H} &= \sum m(\bar{\mathbf{r}}+\mathbf{r}') \wedge (\dot{\bar{\mathbf{r}}}+\dot{\mathbf{r}}') \\ &= \sum m\bar{\mathbf{r}} \wedge \dot{\bar{\mathbf{r}}} + \sum m\mathbf{r}' \wedge \dot{\bar{\mathbf{r}}} + \sum m\bar{\mathbf{r}} \wedge \dot{\mathbf{r}}' + \sum m\mathbf{r}' \wedge \dot{\mathbf{r}}' \\ &= M\bar{\mathbf{r}} \wedge \dot{\bar{\mathbf{r}}} + \sum m\mathbf{r}' \wedge \dot{\mathbf{r}}', \end{aligned} \tag{19}$$

on taking $\bar{\mathbf{r}}$ and $\dot{\bar{\mathbf{r}}}$ outside the summations over the particles and using (18).

The first term on the right-hand side of (19) is the angular momentum about O of a particle of mass equal to the total mass

M of the body placed at the centre of mass and moving with it: the second term is the angular momentum of the body about its centre of mass. Using (19) and (17) in (9) we get

$$M\frac{d}{dt}(\bar{\mathbf{r}}\wedge\dot{\bar{\mathbf{r}}})+\frac{d}{dt}\sum m\mathbf{r}'\wedge\dot{\mathbf{r}}' = \bar{\mathbf{r}}\wedge\sum \mathbf{P}+\sum \mathbf{r}'\wedge\mathbf{P}.$$

By (16) this reduces to

$$\frac{d}{dt}\sum m\mathbf{r}'\wedge\dot{\mathbf{r}}' = \sum \mathbf{r}'\wedge\mathbf{P}, \tag{20}$$

that is:

IV. *The rate of change of angular momentum about the centre of mass is equal to the sum of the moments of the external forces about the centre of mass.*

The results II, III, IV are those usually needed in solving problems and we shall refer to them shortly as 'motion about a fixed axis', 'motion of the centre of mass', and 'motion about the centre of mass', respectively.

To determine the angular momentum of the body about its centre of mass in (19), suppose that its angular velocity about its centre of mass is $\boldsymbol{\omega}$, so that $\dot{\mathbf{r}}' = \boldsymbol{\omega}\wedge\mathbf{r}'$ and

$$\sum m\mathbf{r}'\wedge\dot{\mathbf{r}}' = \sum m\mathbf{r}'\wedge(\boldsymbol{\omega}\wedge\mathbf{r}') = \boldsymbol{\omega}\sum m(\mathbf{r}'.\mathbf{r}')-\sum m(\boldsymbol{\omega}.\mathbf{r}')\mathbf{r}'. \tag{21}$$

If $(\omega_1, \omega_2, \omega_3)$ and (x', y', z') are the components of $\boldsymbol{\omega}$ and $\mathbf{r}'$ referred to right-handed rectangular axes, those of the angular momentum (21) are

$$(A\omega_1-H\omega_2-G\omega_3,\quad -H\omega_1+B\omega_2-F\omega_3,\quad -G\omega_1-F\omega_2+C\omega_3), \tag{22}$$

where A, B, C, F, G, H are defined in § 65 (2) and (3).

The kinetic energy T of the body may be studied in the same way. It is

$$T = \tfrac{1}{2}\sum m|\dot{\mathbf{r}}|^2 = \tfrac{1}{2}\sum m|\dot{\bar{\mathbf{r}}}+\dot{\mathbf{r}}'|^2 = \tfrac{1}{2}M|\dot{\bar{\mathbf{r}}}|^2+\tfrac{1}{2}\sum m|\dot{\mathbf{r}}'|^2. \tag{23}$$

The first term on the right hand side of (23) is the kinetic energy of a particle of mass M at the centre of mass of the body and moving with it. The second is the kinetic energy of the motion about the centre of mass. With the notation of (22) it is

$$\tfrac{1}{2}\sum m|\dot{\mathbf{r}}'|^2 = \tfrac{1}{2}\sum m\{(\omega_2 z'-\omega_3 y')^2+(\omega_3 x'-\omega_1 z')^2+$$
$$+(\omega_1 y'-\omega_2 x')^2\}$$
$$= \tfrac{1}{2}\{A\omega_1^2+B\omega_2^2+C\omega_3^2-2F\omega_2\omega_3-2G\omega_3\omega_1-$$
$$-2H\omega_1\omega_2\}. \quad (24)$$

The result (24) also gives the kinetic energy of the motion of the body about a fixed origin if $A,\ldots, H$ are the moments and products of inertia relative to fixed axes through this origin, and in the same way (22) gives the angular momentum.

67. Motion about a fixed axis

In this case only the principle II of § 66 is needed. The angular momentum about the axis may be written down from § 66 (22) but we derive it here *ab initio*.

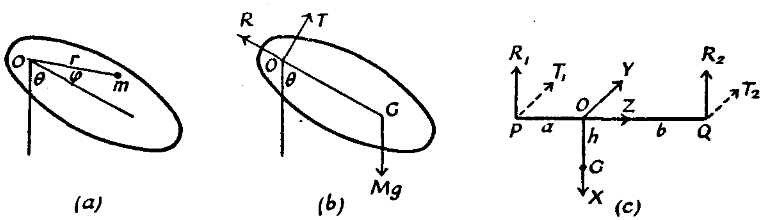


FIG. 55.

Let θ be the angle between a marked plane in the body passing through the axis of rotation and a fixed plane through the axis. Let m be a typical particle of the body, and r its perpendicular distance from the axis. Let the plane through m and the axis make an angle ϕ with the marked plane in the body; cf. Fig. 55 (a).

Then the angular momentum of the body about the axis is

$$\sum mr^2\frac{d}{dt}(\theta+\phi) = \sum mr^2\dot{\theta} = \dot{\theta}\sum mr^2 = I\dot{\theta}, \quad (1)$$

since ϕ is independent of the time, and θ is the same for all particles of the body.

The equation of motion § 66 II thus becomes

$$I\ddot{\theta} = \text{couple}. \tag{2}$$

This is the equation often referred to in Chapter IV.

The compound or rigid body pendulum

It is required to find the motion of a rigid body which can oscillate freely about a horizontal axis. Let θ be the angle which the plane through the axis and the centre of mass G makes with the downward vertical. Then if h is the distance of the centre of mass from the axis, the sum of the moments of the external forces (gravity) about the axis in the direction of θ increasing is $-Mgh\sin\theta$. If I is the moment of inertia of the body about the axis, (2) gives

$$I\ddot{\theta} = -Mgh\sin\theta, \tag{3}$$

or, writing $I = Mk^2$,

$$\ddot{\theta}+(gh/k^2)\sin\theta = 0. \tag{4}$$

This is precisely the same as the equation of motion, § 55 (13), of a simple pendulum of length k^2/h. Thus a simple pendulum of this length and the rigid body pendulum, if started with the same values of θ and $\dot{\theta}$, will keep step exactly. The simple pendulum of length k^2/h is called the *simple equivalent pendulum*.

The solution of (4) has been discussed in § 55.

Next we determine the reactions at the axis of rotation. Consider first the case of a lamina which is freely hinged at a point and which oscillates in the vertical plane through the hinge. We wish to find the reaction at the hinge, and we may either seek its radial and transverse components, R and T, Fig. 55 (b), or its horizontal and vertical components. Here we shall find the former, and in § 68 the latter, as both methods of procedure are important.

We now have to use the principle III of § 66, namely, that the centre of mass moves like a particle of mass M placed there and acted on by the resultant external force on the body, the external forces being in the present case R, T, and Mg. The equations of motion § 61 (9) and (10) give

$$Mh\ddot{\theta} = T-Mg\sin\theta, \tag{5}$$

$$Mh\dot{\theta}^2 = R-Mg\cos\theta. \tag{6}$$

From (5) and (4) it follows that

$$T = Mg\left(1-\frac{h^2}{k^2}\right)\sin\theta. \tag{7}$$

To determine R we need $\dot{\theta}^2$, and thus have to integrate (4). If ω is the value of $\dot{\theta}$ when $\theta = 0$, (4) gives as in § 55

$$\dot{\theta}^2 = \omega^2+\frac{2gh}{k^2}(\cos\theta-1), \tag{8}$$

and from (6)

$$R = Mg\left(1+\frac{2h^2}{k^2}\right)\cos\theta+Mh\left(\omega^2-\frac{2gh}{k^2}\right). \tag{9}$$

We now consider the more general case of any rigid body which can turn freely about a horizontal axis. It may be remarked that the present theory includes that of a fly-wheel whose centre of mass does not lie on its axis, and that the reactions on the bearings may cause important vibrations.

Suppose that the axis of rotation is supported by two bearings P and Q, and that O, the foot of the perpendicular from the centre of mass G on the axis, is distant a from P and b from Q. Let R_1 and T_1 be the radial and transverse components of the reaction at P, and R_2, T_2 those of the reaction at Q; cf. Fig. 55 (c).

The preceding calculation gives the total radial and transverse components of the reactions, that is,

$$R_1+R_2 = R, \tag{10}$$

$$T_1+T_2 = T, \tag{11}$$

where R and T are given by (9) and (7). To determine the reactions completely we need two more equations, and for these we must use § 66 II or IV. A fundamental difficulty arises here which appears with all irregularly shaped bodies: if we take axes fixed in direction, the moments and products of inertia of the body relative to them change as the body moves. Thus we must take a set of axes fixed in the body and allow for the motion of these axes as in § 64. Take axes OX along OG, OZ along the axis of rotation, and OY to make a right-handed system. Let A, B, I, F, G, H be the moments and products of inertia of the body referred to these axes. The components of the angular velocity of the body along these axes are

$$(0, 0, \dot{\theta}). \tag{12}$$

The components of the angular momentum of the body along these axes are by § 66 (22)

$$(-G\dot{\theta}, -F\dot{\theta}, I\dot{\theta}). \tag{13}$$

Using (12) and (13) in § 64 (4), the components of the rate of change of angular momentum of the body about the instantaneous directions of the axes are

$$(-G\ddot{\theta}+F\dot{\theta}^2, -F\ddot{\theta}-G\dot{\theta}^2, I\ddot{\theta}). \tag{14}$$

The sum of the moments of the external forces about the instantaneous directions of the axes are

$$(T_1 a - T_2 b, R_1 a - R_2 b, -Mgh\sin\theta). \tag{15}$$

Equating (14) and (15) by § 66 II we get

$$G\ddot{\theta} - F\dot{\theta}^2 = T_2 b - T_1 a, \tag{16}$$

$$-F\ddot{\theta} - G\dot{\theta}^2 = R_1 a - R_2 b. \tag{17}$$

$\ddot{\theta}$ and $\dot{\theta}^2$ are given by (3) and (8), and (10), (11), (16), (17) are four equations for R_1, R_2, T_1, T_2.

68. Motion in two dimensions

In this section we solve a number of two-dimensional problems.

Ex. 1. *A thin rod of mass M and length $2a$ is initially at rest in the vertical position of unstable equilibrium and rotates freely about its lower end O, Fig. 56 (a). It is required to find the motion and the horizontal and vertical components, F and R, of the reaction at the point of support.*

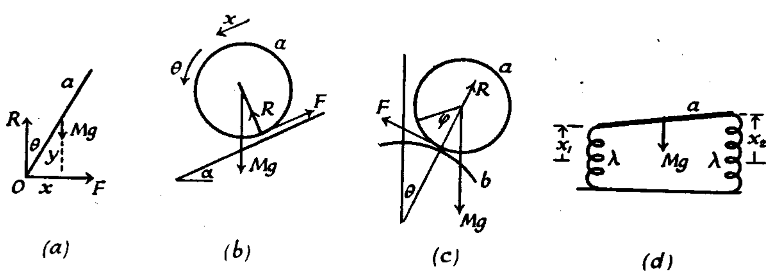


FIG. 56

By Routh's rule, § 65 (14), the square of the radius of gyration, k^2, of the rod about O is $4a^2/3$. Thus the equation of motion of the rod about O, found as in § 67 (3), is

$$\frac{4a^2}{3}\ddot{\theta} = ga\sin\theta. \tag{1}$$

Integrating as in § 55 we have

$$\frac{2a}{3}\dot{\theta}^2 = g(1-\cos\theta), \tag{2}$$

since we are given $\dot{\theta} = 0$ when $\theta = 0$.

We now consider the motion of the centre of mass in rectangular coordinates x, y. We have

$$x = a\sin\theta, \qquad y = a\cos\theta, \tag{3}$$

$$\dot{x} = a\dot{\theta}\cos\theta, \qquad \dot{y} = -a\dot{\theta}\sin\theta, \tag{4}$$

$$\ddot{x} = a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta, \tag{5}$$

$$\ddot{y} = -a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta. \tag{6}$$

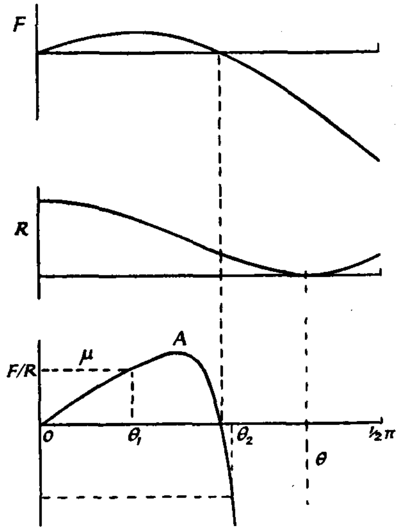


FIG. 57

Thus motion of the centre of mass, § 66 III, gives

$$M(a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta) = F, \tag{7}$$

$$M(-a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta) = R - Mg. \tag{8}$$

Using the values (1) and (2) in these, we get finally

$$F = \frac{3Mg}{4}\sin\theta(3\cos\theta - 2), \tag{9}$$

$$R = \frac{Mg}{4}(3\cos\theta - 1)^2. \tag{10}$$

The way in which F, R, and F/R vary with θ is shown in Fig. 57. F changes sign as θ passes through $\cos^{-1}(2/3)$, so that, for values of θ larger than this, the horizontal component of the reaction is in the opposite direction to the arrow in Fig. 56 (a). The curves will be discussed further in § 69.

Ex. 2. *A cylinder of radius a and moment of inertia Mk^2 rolls down a perfectly rough plane inclined at α to the horizontal* [*Fig. 56* (b)].

Let x be the distance the centre of the cylinder has moved from its initial position, and let θ be the angle through which a marked line on the cylinder has turned from its initial position. Then

$$x = a\theta, \tag{11}$$

and, differentiating,

$$\dot{x} = a\dot{\theta}. \tag{12}$$

(12) might have been written down directly since $\dot{x}-a\dot{\theta}$ is the velocity down the plane of the point of the cylinder instantaneously in contact with the plane, and since the cylinder rolls without slipping this velocity must be zero.

Let F and R be the components of the force on the cylinder at the point of contact; then motion of the centre of mass (§ 66 III) gives

$$M\ddot{x} = Mg\sin\alpha - F, \tag{13}$$

$$0 = R - Mg\cos\alpha. \tag{14}$$

Also motion about the centre of mass (§ 66 IV) gives

$$Mk^2\ddot{\theta} = Fa. \tag{15}$$

Since, by (12), $\ddot{x} = a\ddot{\theta}$, (13) and (15) give

$$\ddot{x} = \frac{ga^2}{k^2+a^2}\sin\alpha, \tag{16}$$

$$F = \frac{Mgk^2}{k^2+a^2}\sin\alpha. \tag{17}$$

The cylinder thus rolls down the plane with constant acceleration.

Ex. 3. *A cylinder of radius a and moment of inertia Mk^2 rolls down the outside of a perfectly rough cylinder of radius b* [*Fig. 56* (c)].

Let θ be the angle the plane containing the axes of the cylinders makes with the vertical, and let ϕ be the angle between a marked plane through the axis of the cylinder of radius a and the plane containing the axes of the cylinders. Then, since the lengths of the arcs of the two circles which have rolled over each other must be equal, we must have

$$b\theta = a\phi + \text{constant},$$

and

$$b\dot{\theta} = a\dot{\phi}. \tag{18}$$

Let F and R be the components of the force on the cylinder of radius a at the point of contact, then, using § 61 (9) and (10), motion of the centre of mass (§ 66 III) gives

$$M(a+b)\ddot{\theta} = Mg\sin\theta - F, \tag{19}$$

$$M(a+b)\dot{\theta}^2 = Mg\cos\theta - R. \tag{20}$$

For motion about the centre of mass (§ 66 IV) we have to specify the position of the marked plane on the cylinder, not relative to the moving plane joining the axes of the cylinders, but relative to a fixed plane such as the vertical with which it makes an angle $(\theta+\phi)$. Thus this equation is

$$Mk^2(\ddot{\theta}+\ddot{\phi}) = Fa. \tag{21}$$

(18), (19), and (21) give

$$(a+b)(a^2+k^2)\ddot{\theta} = ga^2\sin\theta. \tag{22}$$

Ex. 4. *A uniform rod of mass M and length $2a$ is supported in a horizontal position by two equal springs of stiffness λ at its ends. Discuss the small vertical oscillations of the system; cf. Fig. 56 (d).*

Let x_1 and x_2 be the extensions of the springs from their equilibrium positions. The displacement of the centre of mass is $\frac{1}{2}(x_1+x_2)$ so that motion of the centre of mass, § 66 III, gives

$$\tfrac{1}{2}M(\ddot{x}_1+\ddot{x}_2) = -\lambda x_1 - \lambda x_2. \tag{23}$$

The force of gravity is not included in (23) since x_1 and x_2 are measured from their equilibrium positions.

The small angle through which the rod has turned is $(x_2-x_1)/2a$, so that motion about the centre of mass, § 66 IV, gives

$$\tfrac{1}{3}Ma(\ddot{x}_2-\ddot{x}_1) = -\lambda a x_2 + \lambda a x_1. \tag{24}$$

It follows from (23) that x_1+x_2 oscillates with frequency $(2\lambda/M)^{\frac{1}{2}}/2\pi$, and from (24) that x_1-x_2 oscillates with frequency $(6\lambda/M)^{\frac{1}{2}}/2\pi$.

69. Problems of rolling or sliding

When there is an imperfectly rough contact to be considered, we do not know *a priori* whether there is slipping at this contact. We must make one or other of two assumptions, calculate the motion on this basis, and verify that it satisfies the required conditions: if it does not, the other assumption must be used.

(i) *Assume the point of contact to be at rest.* In this case the components of the reaction at the point of contact along and normal to the surface, F and R, can be calculated and we must have $F < \mu R$, where μ is the coefficient of friction. If F becomes equal to μR slipping commences.

(ii) *Assume the point of contact to be moving.* In this case the tangential and normal components of the reaction, F and R, are connected by $F = \mu R$. The velocity of sliding of the point of contact is then calculated: if at any time this becomes zero, calculations must be continued with the assumption (i).

It should be remarked that the coefficients of friction μ in (i) and (ii) are respectively the static and dynamic coefficients and that these are not necessarily the same [cf. § 30 (iii)], but here we shall for simplicity assume that they are equal.

Ex. 1. *A thin rod of mass M and length $2a$ is initially at rest in the position of unstable equilibrium with its lower end resting on a horizontal plane, the coefficient of friction between the rod and the plane being μ.*

If we assume that the point of contact is at rest, the rod turns about it, and the equations of motion are those of § 68, Ex. 1. F and R have been calculated in § 68 (9) and (10), and they and F/R are graphed in Fig. 57. For the point of contact to remain at rest we must have $|F/R| < \mu$. If μ is less than the maximum at A of the curve F/R, Fig. 57, the point of contact will slip backwards at the angle θ_1. But if μ is greater than this maximum, the point of contact will remain at rest all the time that friction acts forwards, and slipping will take place (forwards) at the angle θ_2 at which $-F/R = \mu$. Slipping must occur before θ reaches the value $\cos^{-1}(1/3)$.

Ex. 2. *A cylinder of radius a and moment of inertia Mk^2 is placed at rest on a plane inclined at θ to the horizontal, the coefficient of friction between the cylinder and the plane being μ.*

This has been discussed on the assumption of *rolling*, that is that the point of contact is at rest, in § 68, Ex. 2. F and R are calculated in § 68 (17) and (14), and if the assumption of rolling is correct we must have

$$\frac{F}{R} = \frac{k^2}{k^2+a^2}\tan\alpha < \mu. \tag{1}$$

If this is not the case we must assume that the point of contact slips: since the cylinder is initially at rest, the point of contact must slip down the plane and the frictional force $F = \mu R$ must act up the plane, Fig. 58 (*a*).

x and θ are now independent, and the equations of motion are

$$M\ddot{x} = Mg\sin\alpha - \mu R, \tag{2}$$

$$0 = R - Mg\cos\alpha, \tag{3}$$

$$Mk^2\ddot{\theta} = \mu Ra. \tag{4}$$

(2) and (3) give
$$\ddot{x} = g(\sin\alpha - \mu\cos\alpha),$$
$$\dot{x} = gt(\sin\alpha - \mu\cos\alpha), \tag{5}$$
since $\dot{x} = 0$ when $t = 0$.

(3) and (4) give
$$\ddot{\theta} = \frac{\mu g a}{k^2}\cos\alpha,$$
$$\dot{\theta} = \frac{\mu g a t}{k^2}\cos\alpha, \tag{6}$$
since $\dot{\theta} = 0$ when $t = 0$.

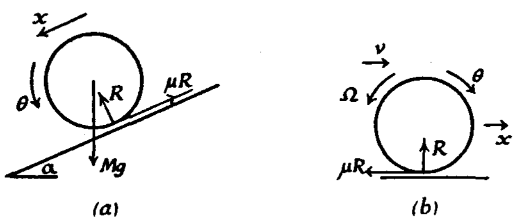


FIG. 58

(5) and (6) give the velocity and angular velocity of the cylinder, both of which increase steadily. The velocity of the point of contact is
$$\dot{x} - a\dot{\theta} = gt\left\{\sin\alpha - \mu\left(1+\frac{a^2}{k^2}\right)\cos\alpha\right\}, \tag{7}$$
and thus is zero if
$$\tan\alpha = \mu\left(1+\frac{a^2}{k^2}\right). \tag{8}$$

This is the transition value between rolling and sliding found in (1).

Ex. 3. *A cylinder of radius a and moment of inertia Mk^2 is placed gently at $t = 0$ on a horizontal table of coefficient of friction μ. It has initial velocity v and back spin Ω [Fig. 58 (b)].*

We measure x in the direction of v from an origin at the initial position of the cylinder, and θ in the direction corresponding to rolling in this direction.

Then when $t = 0$, $\quad \dot{x} = v, \quad \dot{\theta} = -\Omega, \qquad (9)$

and the velocity of the point of contact $\dot{x} - a\dot{\theta}$ is $v + a\Omega$. Since the point of contact initially slips forwards, the frictional force μR must act backwards. The equations of motion are
$$M\ddot{x} = -\mu R, \tag{10}$$
$$Mk^2\ddot{\theta} = \mu R a, \tag{11}$$
$$0 = R - Mg. \tag{12}$$

From (10) with $\dot{x} = v$ when $t = 0$,
$$\dot{x} = v - \mu g t, \tag{13}$$
that is, the velocity of the cylinder decreases linearly.

From (11) with $\dot\theta = -\Omega$ when $t = 0$,

$$k^2\dot\theta = -k^2\Omega+\mu gat, \tag{14}$$

that is, the backspin decreases linearly.

The velocity of the point of contact is

$$\dot x-a\dot\theta = (v+a\Omega)-\mu gt(1+a^2/k^2). \tag{15}$$

This becomes zero, and so rolling commences, when

$$t = \frac{k^2(v+a\Omega)}{\mu g(a^2+k^2)}. \tag{16}$$

At this instant the velocity of the centre is by (13)

$$\frac{va^2-a\Omega k^2}{k^2+a^2}. \tag{17}$$

If $v > \Omega k^2/a$ this is positive and the cylinder rolls forwards; if $v = \Omega k^2/a$ it comes to rest; and if $v < \Omega k^2/a$ it rolls backwards.

70. Impulsive motion

When impulsive forces act on a system of rigid bodies we assume as in § 39 that the time τ during which they act is so small that the changes in position of the bodies during it are negligible. By integrating the equations of motion over the small time τ the changes in velocity due to the blows can be found.

If $\mathbf{P}$ is a typical impulsive force applied at $t = 0$, we write

$$\mathbf{J} = \int\limits_0^\tau \mathbf{P}\,dt \tag{1}$$

for the impulse of the blow.

We now obtain four results corresponding to § 66 I to IV.

Integrating § 66 (8) over the small time τ gives

$$\mathbf{L}_f-\mathbf{L}_i = \sum \mathbf{J}, \tag{2}$$

where $\mathbf{L}_i$ and $\mathbf{L}_f$ are the linear momenta of the body before and after the blow. Thus:

I. *The change in linear momentum of the body is equal to the vector sum of the impulses of the blows applied to it.*

Again, integrating § 66 (9) over the small time τ and remembering that the changes in the $\mathbf{r}$ in this time are negligible, gives

$$\mathbf{H}_f-\mathbf{H}_i = \sum \mathbf{r}\wedge\mathbf{J}, \tag{3}$$

where $\mathbf{H}_i$ and $\mathbf{H}_f$ are the angular momenta of the body about the origin before and after the blow. That is:

II. *The change in angular momentum about a fixed origin is equal to the vector sum of the moments of the impulses of the blows about the origin.*

In the same way from § 66 (16):

III. *M times the change in linear velocity of the centre of mass of the body is equal to the vector sum of the impulses of the blows.*

And finally from § 66 (20):

IV. *The change in angular momentum about the centre of mass is equal to the vector sum of the moments of the impulses of the blows about the centre of mass.*

Usually III and either II or IV are used to determine the change in the motion. If a system is set in motion by blows, we have to determine the initial values of its velocities and angular velocities by the methods of this section and subsequently to study the motion with these initial conditions as in §§ 68, 69. If the Laplace transformation is used, treating a blow of impulse $\mathbf{J}$ as a force $\mathbf{J}\,\delta(t)$, the preliminary calculation of the initial conditions is avoided, but this method is only available when the equations of motion are linear.

Ex. 1. *A lamina can rotate freely in its own plane about a hinge O, its moment of inertia about an axis through O perpendicular to its plane being Mk^2. It is set in motion by a blow of impulse P in a direction perpendicular to the line joining O and the centre of mass G.*

If ω is the angular velocity of the lamina after the blow, and a is the distance of the line of action of P from the hinge, Fig. 59 (a), we have by II

$$Mk^2\omega = Pa. \tag{4}$$

There will be an impulsive reaction at the hinge O: suppose that X and Y are the components of this impulse in the direction of P and perpendicular to it. Then if h is the distance OG, we have by III

$$Mh\omega = P+X, \tag{5}$$

$$0 = Y. \tag{6}$$

Therefore, using (4),

$$X = P\left(\frac{ah}{k^2}-1\right). \tag{7}$$

If $a = k^2/h$, the length of the simple equivalent pendulum, $X = 0$, so that if the blow is struck at this point (called the

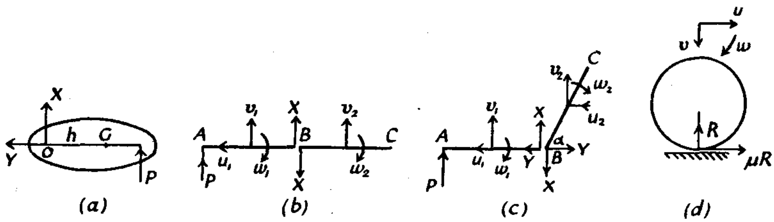


FIG. 59

centre of percussion for this reason) there is no reaction at the hinge. If $a > k^2/h$ the reaction X is positive, that is, in the direction of the blow; if $a < k^2/h$ it is in the opposite direction.

The case of any body free to rotate about an axis which is carried in bearings may be treated as in § 67.

Ex. 2. *Two equal uniform rods AB, BC, each of length $2a$ and mass M, are freely hinged at B and are at rest in a straight line when a blow of impulse P is struck at A in a direction perpendicular to the rods.*

Let v_1 and v_2 be the velocities of the centres of the rods after the blow, and let ω_1 and ω_2 be the angular velocities of the rods, Fig. 59 (*b*). Since the velocity of B, calculated from the assumed velocity and angular velocity of the rod AB, must be the same as that calculated from BC, we must have

$$v_1 - a\omega_1 = v_2 + a\omega_2. \tag{8}$$

If desired, v_2 in Fig. 59 (*b*) could have been replaced by $v_1 - a\omega_1 - a\omega_2$ and the use of (8) avoided; it is desirable to proceed in this way in more complicated problems.

There will be a blow of impulse X on the rod AB at the hinge, and an equal and opposite blow on BC.

From IV we get

$$\tfrac{1}{3}Ma^2\omega_1 = (P-X)a, \tag{9}$$

$$\tfrac{1}{3}Ma^2\omega_2 = -Xa. \tag{10}$$

Also from III

$$Mv_1 = P+X, \tag{11}$$

$$Mv_2 = -X. \tag{12}$$

(8) to (12) are five algebraic equations for v_1, v_2, ω_1, ω_2, X.

Ex. 3. *The rods in Ex. 2 are inclined at an angle α.*

In this case we must assume components of velocity of the rods, and components of the impulse at the hinge, both along and perpendicular to the rod AB [Fig. 59(c)].

III and IV now give
$$Mu_1 = Y,$$
$$Mv_1 = P+X,$$
$$X = -Mv_2 = -M(v_1-a\omega_1-a\omega_2\cos\alpha),$$
$$Y = -Mu_2 = -M(u_1-a\omega_2\sin\alpha),$$
$$\tfrac{1}{3}Ma^2\omega_1 = (P-X)a,$$
$$\tfrac{1}{3}Ma^2\omega_2 = -aX\cos\alpha-aY\sin\alpha.$$

Ex. 4. *A sphere of mass M and radius a, spinning about a horizontal axis with angular velocity $\omega > u/a$ impinges on a horizontal plane with velocity v towards the plane and u parallel to it, Fig. 59(d). The coefficient of resistution is e, and the coefficient of friction is μ.*

If R is the impulsive reaction normal to the plane we have from III
$$R = Mv(1+e). \tag{13}$$
At the instant of contact, the horizontal component of the velocity of the point of contact is
$$u-a\omega,$$
which by hypothesis is negative. We thus assume an impulsive frictional force $\mu R = \mu Mv(1+e)$ acting forwards. Then if U and Ω are the horizontal component of the velocity, and the angular velocity, after the impact
$$M(U-u) = \mu R = \mu Mv(1+e), \tag{14}$$
$$\frac{2Ma^2}{5}(\Omega-\omega) = -\mu Ra = -\mu Mva(1+e). \tag{15}$$
Therefore
$$U = u+\mu v(1+e),$$
$$\Omega = \omega-\frac{5\mu v}{2a}(1+e).$$

Thus the horizontal component of the velocity is increased and the angular velocity decreased.

71. The gyrostat†

A gyrostat consists essentially of a solid of revolution which spins about its axis of symmetry. This axis is freely hinged at a point O of itself. The centre of mass G of the solid is on the

† The name gyrostat is usually used for systems of this type in which gravity has to be considered: the system is also that of a top spinning about a fixed point. In the gyroscope the centre of mass is at the origin, so that gravity exerts no moment about O, and the motion of the system caused by given external couples has to be considered. In the usual mounting in concentric rings or gimbals, θ and ψ specify the positions of the rings and so have a fundamental significance.

axis and distant h from the hinge O, so that the force of gravity has a moment about O. We shall consider the effect of gravity alone; if there are, in addition, externally applied forces they are treated in the same way.

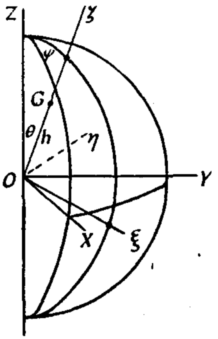


FIG. 60

First we have to specify the position of the solid in space. Let OX, OY, OZ be fixed rectangular axes through the hinge O with OZ vertical. We specify the position of $O\zeta$, the axis of symmetry of the body, by its spherical polar coordinates relative to the axes OX, OY, OZ: that is, θ is the angle between $O\zeta$ and OZ, and ψ is the angle between the planes $ZO\zeta$ and ZOX. We then take a system of axes $O\xi$, $O\eta$, $O\zeta$ such that $O\xi$ lies in the plane $ZO\zeta$ and makes an angle $\frac{1}{2}\pi+\theta$ with OZ, while $O\eta$ is chosen so that $O\xi$, $O\eta$, $O\zeta$ form a right-handed system of rectangular axes. Finally, let ϕ be the angle between a marked plane in the body through its axis $O\zeta$ and the plane $\xi O\zeta$. The position of any point of the body is then known if θ, ψ, ϕ are known; these are called the Eulerian angles and are used in most problems of this type.

There are many ways of deriving the equations of motion of the gyrostat: in this section we shall use moving axes; alternative deductions using Lagrange's equations, and using the energy and momentum equations, are given in §§ 76, 74.

We take the system $O\xi$, $O\eta$, $O\zeta$ as moving axes; cf. § 64. Their components of angular velocity about their instantaneous directions are

$$(-\dot{\psi}\sin\theta, \dot{\theta}, \dot{\psi}\cos\theta). \tag{1}$$

The components of angular velocity of the body itself in the instantaneous directions of the axes are

$$(-\dot{\psi}\sin\theta, \dot{\theta}, \dot{\phi}+\dot{\psi}\cos\theta). \tag{2}$$

Since $O\zeta$ is the axis of symmetry of the body, its moments of inertia about all axes in the plane $\xi O\eta$ will be the same, say A, and all the products of inertia will vanish. If C is the moment of inertia about $O\zeta$, the moments of inertia about the axes $O\xi$, $O\eta$, $O\zeta$ will be A, A, C, respectively, and the components of the angular momentum of the body along the instantaneous directions of these axes will be

$$\{-A\dot{\psi}\sin\theta, A\dot{\theta}, C(\dot{\phi}+\dot{\psi}\cos\theta)\}. \tag{3}$$

The components of the moment of the external force (gravity) about O are

$$(0, Mgh\sin\theta, 0). \tag{4}$$

The equations of motion are now obtained by equating the components of the rate of change of angular momentum along the instantaneous directions of the axes $O\xi$, $O\eta$, $O\zeta$ to the components of the moment of the external force in these directions. That is, using (3) and (1) in § 64 (4),

$$-A\frac{d}{dt}(\dot{\psi}\sin\theta)-A\dot{\theta}\dot{\psi}\cos\theta+C\dot{\theta}(\dot{\phi}+\dot{\psi}\cos\theta) = 0, \tag{5}$$

$$A\frac{d}{dt}(\dot{\theta})-A\dot{\psi}^2\sin\theta\cos\theta+C\dot{\psi}\sin\theta(\dot{\phi}+\dot{\psi}\cos\theta) = Mgh\sin\theta, \tag{6}$$

$$C\frac{d}{dt}(\dot{\phi}+\dot{\psi}\cos\theta) = 0. \tag{7}$$

(7) gives immediately

$$\dot{\phi}+\dot{\psi}\cos\theta = n, \tag{8}$$

where n is a constant (Cn is the angular momentum about the axis of symmetry, and, in the absence of friction, this is constant).

Using (8), (5) and (6) become

$$A\frac{d}{dt}(\dot{\psi}\sin\theta)+A\dot{\theta}\dot{\psi}\cos\theta-Cn\dot{\theta} = 0, \tag{9}$$

$$A\ddot{\theta}-A\dot{\psi}^2\sin\theta\cos\theta+Cn\dot{\psi}\sin\theta = Mgh\sin\theta. \tag{10}$$

Multiplying (9) by $\sin\theta$, it may be written

$$\frac{d}{dt}(A\dot{\psi}\sin^2\theta+Cn\cos\theta) = 0. \tag{11}$$

Therefore, integrating,

$$A\dot{\psi}\sin^2\theta+Cn\cos\theta = H, \tag{12}$$

where H is a constant (which is the constant angular momentum about the vertical). Finally, multiplying (9) by $\dot{\psi}\sin\theta$, (10) by $\dot{\theta}$, and adding gives

$$A\dot{\theta}\ddot{\theta}+A\dot{\psi}\sin\theta\,\frac{d}{dt}(\dot{\psi}\sin\theta)-Mgh\dot{\theta}\sin\theta = 0. \tag{13}$$

Integrating (13) gives

$$\tfrac{1}{2}A(\dot{\theta}^2+\dot{\psi}^2\sin^2\theta)+Mgh\cos\theta = E, \tag{14}$$

where E is a constant. This will be seen in § 76 to be the energy equation.

(8), (12), and (14) are three first integrals of the motion. To study the motion further we eliminate $\dot{\psi}$ from (14) by using (12), and get an equation for θ only. It is a little simpler to work in terms of the new variable

$$x = \cos\theta. \tag{15}$$

Also, for shortness, we write

$$\frac{Cn}{A} = a, \qquad \frac{H}{A} = b, \qquad \frac{Mgh}{A} = c, \qquad \frac{2E}{A} = d, \tag{16}$$

so that (12) and (14) become

$$\dot{\psi} = \frac{b-ax}{1-x^2}, \tag{17}$$

$$\dot{x}^2+\dot{\psi}^2(1-x^2)^2 = (d-2cx)(1-x^2). \tag{18}$$

Using (17) in (18) we get

$$\dot{x}^2 = (d-2cx)(1-x^2)-(b-ax)^2 \tag{19}$$

This is an equation of the form studied in § 55. We remark that, writing $f(x)$ for the right-hand side of (19), $f(x)$ is a cubic in x and so the complete solution of (19) will in general involve elliptic functions [cf. § 55, Ex. 2].

Since $x = \cos\theta$, we are interested in the range $-1 \leqslant x \leqslant 1$ of x, and by (19) both $f(1)$ and $f(-1)$ are negative. But at the point of projection $\dot{x}^2$, and thus $f(x)$, must be positive. Thus $f(x)$ must have two zeros in $-1 \leqslant x \leqslant 1$, and so the motion consists of an oscillation between two fixed values of x or θ.

The most interesting case is that of steady motion in which these fixed values coincide so that $\theta = \alpha$, constant. Putting $\dot{\theta} = 0$ in (9) we find that $\dot{\psi}$ must be constant, and from (10)

$$A\dot{\psi}^2\cos\alpha - Cn\dot{\psi} + Mgh = 0. \tag{20}$$

This equation has real roots if

$$C^2n^2 \geqslant 4MghA\cos\alpha. \tag{21}$$

If this condition is satisfied, *steady precession* of this type is possible with $\theta = \alpha$ and with either of the angular velocities

$$\dot{\psi} = \{Cn \pm \sqrt{(C^2n^2 - 4MghA\cos\alpha)}\}/2A\cos\alpha.$$

Since, usually, the angular velocity of spin about the axis is large, $Cn \gg 4MghA\cos\alpha$ and these angular velocities become approximately

$$Cn/A\cos\alpha, \quad \text{'quick precession'}, \tag{22}$$

$$Mgh/Cn, \quad \text{'slow precession'}. \tag{23}$$

These results, and extensions such as the period of small oscillations about steady precession, can also be found from a further study of (19)—the condition for steady precession at $\cos\alpha = x_0$ is that $f(x)$ should have a double zero at x_0.

The gyro-compass

Suppose the gyroscope is pivoted so that its centre of mass coincides with the hinge O of Fig. 60, and suppose further that its axis is constrained to move in a horizontal plane. We consider the effect of the earth's rotation on its motion when it is in latitude λ.

Suppose the axis $O\zeta$ of the gyroscope makes an angle θ with the axis OY of Fig. 51 (b), that $O\xi$ lies along OZ, and that $O\eta$ makes a right-handed system, Fig. 61, where the axes OX, OY, OZ are those chosen in § 64, Ex. 2.

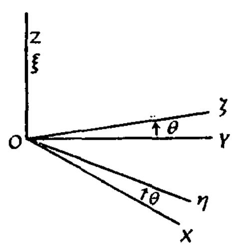


FIG. 61.

The components of the earth's angular velocity in the directions of OX, OY, OZ are by § 64 (7)

$$(0,\ \omega\cos\lambda,\ \omega\sin\lambda). \tag{24}$$

Therefore, in the directions $O\xi$, $O\eta$, $O\zeta$ they are

$$(\omega\sin\lambda,\ \omega\cos\lambda\sin\theta,\ \omega\cos\lambda\cos\theta). \tag{25}$$

The components of the angular velocity of the system $O\xi$, $O\eta$, $O\zeta$ about their instantaneous directions are

$$(\dot{\theta} + \omega\sin\lambda,\ \omega\cos\lambda\sin\theta,\ \omega\cos\lambda\cos\theta). \tag{26}$$

If ϕ and the moments of inertia are defined as before, the components of the angular momentum of the body along the instantaneous directions of $O\xi$, $O\eta$, $O\zeta$ are

$$\{A(\dot\theta+\omega\sin\lambda),\ A\omega\cos\lambda\sin\theta,\ C(\dot\phi+\omega\cos\lambda\cos\theta)\}. \tag{27}$$

There are no couples about the axes $O\xi$ and $O\zeta$. Thus, writing down the rates of change of angular momentum about these axes by using (27) and (26) in § 64 (4), we get

$$A\ddot\theta+C\omega\cos\lambda\sin\theta(\dot\phi+\omega\cos\lambda\cos\theta)-A\omega^2\cos^2\lambda\sin\theta\cos\theta = 0, \tag{28}$$

$$\frac{d}{dt}(\dot\phi+\omega\cos\lambda\cos\theta) = 0. \tag{29}$$

(29) gives $\dot\phi+\omega\cos\lambda\cos\theta = n$, and substituting this in (28) and neglecting the small term in ω^2 gives

$$A\ddot\theta+Cn\omega\cos\lambda\sin\theta = 0. \tag{30}$$

Thus there is equilibrium if $\theta = 0$, that is, if the axis points north, and the period of small oscillations about equilibrium is

$$2\pi\left(\frac{A}{Cn\omega\cos\lambda}\right)^{\frac{1}{2}}. \tag{31}$$

The simple arrangement described above is unsatisfactory owing to the friction introduced by the constraining couple, but the theory of the systems used in practice follows the same lines.

EXAMPLES ON CHAPTER VIII

1. Show that the moments of inertia of a right circular cone of semi-vertical angle α, height h, and mass M are

$$(3/10)Mh^2\tan^2\alpha,\quad (3/20)Mh^2(\tan^2\alpha+4),\quad (3/20)Mh^2(\tan^2\alpha+5\sin^2\alpha)$$

about its axis, a perpendicular to the axis through the vertex, and a slant side, respectively.

2. Show that the moments and products of inertia of a uniform triangle of mass M relative to any axes are the same as those of three particles of mass $M/3$ placed at the mid-points of the sides of the triangle. (Drop a perpendicular from a vertex to the opposite side and take these lines as axes.)

3. A thin wire $ABCD$ whose mass per unit length is ρ is bent in a plane so that $AB = CD = a$, $BC = 2b$, the angles ABC and BCD are 90°, and AB and CD are on opposite sides of BC. Show that the principal axes at the centre of mass are inclined at θ and $\frac{1}{2}\pi-\theta$ to the wire, where

$$\theta = \tfrac{1}{2}\tan^{-1}\frac{3a^2b}{a^3-b^3-3ab^2}.$$

4. l is the length of the simple equivalent pendulum of a rigid body when it oscillates about an axis through O. Show that it will oscillate with the same period about a parallel axis which is distant l from the original axis measured in the direction towards the centre of mass, and

is in the plane of the original axis and the centre of mass. Show also that there are two other axes in this plane about which the period has this value.

5. Show that if the period of small oscillations of a rigid body under gravity about an axis fixed to it is T when the axis is horizontal, it is $T(\sec\theta)^{\frac{1}{2}}$ when the axis is inclined at θ to the horizontal.

6. A fly-wheel of mass M and moment of inertia Mk^2 about its axis has its centre of mass distant h from the axis. Show that, if it is rotating freely with maximum angular velocity ω about a horizontal axis, the horizontal component of the reaction on its bearings is

$$-\frac{3Mgh^2}{k^2}\sin\theta\cos\theta - Mh\left(\omega^2 - \frac{2gh}{k^2}\right)\sin\theta,$$

where θ is the angle the plane containing the axis and the centre of mass makes with the downward vertical.

7. A uniform thin circular disk of mass M and radius a is mounted on a shaft through its centre which makes an angle ϕ with its plane. The shaft is carried in two bearings, each distant b from the centre of the disk, and rotates with angular velocity ω. Show that the reaction of the bearings is a couple of moment

$$\tfrac{1}{4}Ma^2\omega^2\sin\phi\cos\phi$$

in the plane of the shaft and the perpendicular to the disk.

8. In a piston engine AO is the crank of length a, AB is the connecting rod of length b, and the crank rotates uniformly so that the angle AOB is ωt. If the angle ABO is ϕ, show that, neglecting terms in $(a/b)^4$,

$$\cos\phi = 1-(a^2/2b^2)\sin^2\omega t.$$

If the reciprocating parts are of mass M and the connecting rod is uniform and of mass m, show that the reaction on the crankpin in the direction OB is

$$(M+m)\omega^2 a\cos\omega t + (a^2\omega^2/b)(M+\tfrac{1}{2}m)\cos 2\omega t,$$

neglecting friction, and find the reaction in the perpendicular direction.

9. A mass M is hung by two single pulley blocks of mass M'; the wheel in each is of radius a and moment of inertia mk^2. Show that if the rope is allowed to run out freely, the mass M will descend with acceleration

$$\frac{(M+M')g}{M+M'+5mk^2/a^2},$$

neglecting friction and the mass of the rope. Find the corresponding result if friction is included.

10. A solid cylinder of radius b rolls without slipping on the inside of a fixed, hollow, horizontal cylinder of radius a. Discuss the motion, and show that the period of small oscillations about the lowest point is

$$2\pi\{3(a-b)/2g\}^{\frac{1}{2}}.$$

11. A rod OP of length $a+b$, mass M_1, and moment of inertia about O of $M_1 k_1^2$, is freely hinged at O. At P it carries in a frictionless bearing a gear of radius a, mass M, and moment of inertia Mk^2, which meshes with a fixed gear of radius b and centre O. If torque T about O is applied to the rod OP, show that its angular acceleration is

$$T/\{M_1 k_1^2 + M(a+b)^2(k^2+a^2)/a^2\}.$$

12. A straight uniform rod slides down in a vertical plane perpendicular to two smooth planes, one of which is vertical and the other horizontal. Find its motion if it is initially at rest at an angle α to the horizontal and with its ends in contact with the planes. Show that contact with the vertical plane ceases when the inclination of the rod to the horizontal is

$$\sin^{-1}\{(2/3)\sin\alpha\}.$$

13. A uniform sphere is projected up an imperfectly rough plane inclined at an angle α to the horizontal, the coefficient of friction being μ. If, initially, its velocity is V and it is not rotating, show that the distance traversed up the plane while there is slipping at the point of contact is

$$\frac{2V^2(6\mu\cos\alpha+\sin\alpha)}{g(7\mu\cos\alpha+2\sin\alpha)^2}.$$

14. Two equal uniform rods AB, BC, each of mass m, are freely hinged at B and rest on a smooth horizontal table folded together so that A and C are touching. The end A is pulled away from C by a blow of impulse P. Show that the initial velocity of C is $P/2m$.

15. A circular cylinder of radius a whose centre of mass is at a distance h from its geometrical centre rolls on a rough horizontal plane. Write down the equations of motion and show that the period of small oscillations about the position of stable equilibrium is

$$2\pi(k^2/gh)^{\frac{1}{2}},$$

where k is the radius of gyration of the cylinder about the generator nearest to the centre of mass.

16. A rectangular plate of sides $2a$ and $2b$ and mass M is supported on four equal springs of stiffness λ and performs small oscillations. Show that the normal modes consist of a vertical motion of frequency n/π, and oscillations about the axes through the centre of mass parallel to the sides with frequency $(n\surd 3)/\pi$, where $n^2 = \lambda/M$.

17. A mass $M/6$ is connected by a spring of stiffness λ to one end of a uniform rod of mass M and length $2a$. The rod is freely hinged at its centre, and its other end is connected to a fixed point by a spring of stiffness λ. In the equilibrium position the springs are perpendicular to the rod, and the springs and rod are in the same plane. Show that the natural frequencies of small oscillations are, writing $n^2 = \lambda/M$,

$$(6\pm 3\surd 2)^{\frac{1}{2}}n/2\pi,$$

and find the normal modes of oscillation.

18. A uniform log AB of length $2l$ is pushed with velocity V on to a roller rotating with constant angular velocity ω. The other end of the log rests on a horizontal plane at the same height as the top of the roller, and the direction of the log is perpendicular to that of the roller. If the coefficients of friction between the log and the plane, and the log and the roller, both have the same value μ, show that when the mid-point of the log is on the roller its velocity is

$$\{V^2+2\mu lg(2\ln 2-1)\}^{\frac{1}{2}},$$

provided friction at the roller always acts forwards. Discuss also the cases in which this assumption is not true.

19. A uniform rectangular plate $ABCD$ is freely hinged at A and B, and is struck a blow P normal to its plane at D. Show that the reactions at A and B are $-P/4$ and $3P/4$, respectively.

20. A uniform rod AB of mass M and length a is freely hinged at B. An equal rod is freely hinged at a point on the perpendicular to AB at B and rests against the rod AB, making an angle of 45° with it. The rod AB is struck at a distance b from B by a blow P normal to it and in the plane of the rods; show that, if the contact between them is smooth and the rods remain in contact, the initial angular velocity of AB is $3Pb/2Ma^2$. Discuss the validity of the assumptions made.

21. A gyroscope is pivoted so that it rotates about its centre of mass. Find its equations of motion if couples G_ξ and G_η are applied to it about the axes $O\xi$ and $O\eta$. Show that, if $\dot{\psi}$ is kept constant, G_ξ is proportional to $\dot{\theta}$, and thus the gyroscope can be used to generate the differential coefficient of a function mechanically.

22. Discuss the motion of a nearly vertical gyrostat as follows. Replace $\sin\theta$ by θ and $\cos\theta$ by 1 in the equations of motion, § 71 (9) and (10). By adding i times the first of these to the second, show that $\zeta = \theta e^{i\psi}$ satisfies

$$A\ddot{\zeta}-Cni\dot{\zeta}-Mgh\zeta = 0.$$

Show that the path of a point on the axis of the gyrostat is an ellipse which rotates with angular velocity $Cn/2A$.

23. If a ship is sailing with a velocity whose northerly component is v, show that a gyro-compass will point to the west of north by an amount

$$\frac{v}{\omega a\cos\theta},$$

where a is the earth's radius and θ is the latitude of the ship.

24. If $\mathbf{H}$ is the angular momentum of a body relative to an origin O' which moves with velocity $\mathbf{v}$ relative to a fixed origin O, and $\mathbf{L}$ is the linear momentum of the body, show that

$$\dot{\mathbf{H}} = \mathbf{G}-\mathbf{v}\wedge\mathbf{L},$$

where $\mathbf{G}$ is the sum of the moments of the external forces about O'. Discuss the motion of a cylinder rolling down an inclined plane by taking the point of contact as the origin O'.

25. A rigid body moves about a fixed point O. Taking the principal axes of inertia at O as a system of moving axes, show that if ω_1, ω_2, ω_3 are the components of the angular velocity of this system about their instantaneous directions, and G_1, G_2, G_3 are the components of the moment of the external forces about O in these directions, the equations of motion (Euler's equations) are

$$A\dot{\omega}_1-(B-C)\omega_2\omega_3 = G_1, \text{ etc.}$$

IX

THE ENERGY EQUATION AND LAGRANGE'S EQUATIONS

72. Potential energy

WE consider first a single particle in a field of force. The field is supposed to be a vector field, that is, at each point whose position vector is $\mathbf{r}$ there is a force $\mathbf{P}$ on the particle, where $\mathbf{P}$ is a given function of $\mathbf{r}$. Let (x, y, z) and (X, Y, Z) be the components of $\mathbf{r}$ and $\mathbf{P}$ relative to fixed rectangular axes.

If the particle is at $\mathbf{r}$ and is given a small displacement $\delta\mathbf{r}$, the forces of the field do work

$$\mathbf{P}.\delta\mathbf{r} = X\,\delta x + Y\,\delta y + Z\,\delta z \tag{1}$$

on the particle. This conception needs stating a little more precisely. We think of the particle as being placed in the field and held there by some external agency which applies to it a force which just balances the forces exerted by the field on it. When the particle is 'given a small displacement', it is implied that the forces due to the external agency are relaxed slightly so that the particle can move a small distance, but they are always maintained almost balancing the field forces, and the process is carried out infinitely slowly so that the particle gains no momentum. The amount of work (1) is then done by the field forces, and it is absorbed by the agency which holds the particle in position.

Now suppose that p_1 is some path joining two points A and B, let δs be the element of arc of this path, and let $\mathbf{t}$ be a unit vector along its tangent at any point. Then the displacement $\delta\mathbf{r}$ in (1) is $\mathbf{t}\,\delta s$, and the work done by the forces of the field on the particle when it is made to move in the manner described above from A to B along the path p_1 may be written in any of the forms

$$\int_{p_1} \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds = \int_{p_1} \mathbf{P}.\mathbf{t}\,ds = \int_{p_1} \mathbf{P}.d\mathbf{r}, \tag{2}$$

where the integrals are taken from A to B along the path p_1. If this path is retraced from B to A, the same amount of work would have to be done by the external agency which holds the particle in position, since it has to exert force $-\mathbf{P}$ at the point $\mathbf{r}$. Thus in going from A to B and returning by the same path, neither the field forces nor those of the external agency do any net amount of work.

Clearly this need not be the case if the particle returns from B to A by a different path p_2; in this case the net amount of work done by the field forces in the round trip is

$$\int_{p_1} \mathbf{P}.d\mathbf{r} - \int_{p_2} \mathbf{P}.d\mathbf{r}, \tag{3}$$

and this amount of work is available to the external agency. By repeating the cycle an indefinitely large amount of work can be obtained.

The field of forces is called conservative if this cannot be done. Clearly from (3) the condition for this is that the integral (2) be independent of the path from A to B for all points A and B. That is, the work done by the field forces on the particle in going from A to B must depend on the points A and B only. Let S be some point chosen as a standard position, then this work is

$$\int_A^B \mathbf{P}.d\mathbf{r} = \int_A^B \left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)ds = V(A)-V(B), \tag{4}$$

where $$V(A) = \int_A^S \mathbf{P}.d\mathbf{r} = \int_A^S \left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)ds. \tag{5}$$

This quantity $V(A)$ is called the potential energy at the point A in the field. It is the work done by the forces of the field in taking the particle by any path from the point A to the standard position S; it is also equal to the work which has to be done by an external agency in taking the particle from S to A.

When the potential energy is known, the forces of the field can be calculated. For by (4) the work done by the forces of the field in going from (x,y,z) to $(x+\delta x, y+\delta y, z+\delta z)$ is

$$V(x,y,z)-V(x+\delta x, y+\delta y, z+\delta z)$$
$$= -\frac{\partial V}{\partial x}\,\delta x-\frac{\partial V}{\partial y}\,\delta y-\frac{\partial V}{\partial z}\,\delta z-..., \quad (6)$$

and by (1) it is $\quad X\,\delta x+Y\,\delta y+Z\,\delta z.$

Thus we have

$$X = -\frac{\partial V}{\partial x}, \qquad Y = -\frac{\partial V}{\partial y}, \qquad Z = -\frac{\partial V}{\partial z}. \quad (7)$$

It will appear below that it is often simpler to calculate the potential energy of a system and then find the forces by differentiating in this way than to calculate the forces directly. Also the same method holds in other coordinate systems; for example, if the position of the particle is specified by plane polar coordinates (r,θ) the potential energy is $V(r,\theta)$, and if R, Θ are the radial and transverse components of force on the particle the work done by these in a small displacement $(\delta r, r\,\delta\theta)$ is

$$R\,\delta r+r\Theta\,\delta\theta = V(r,\theta)-V(r+\delta r, \theta+\delta\theta)$$
$$= -\frac{\partial V}{\partial r}\,\delta r-\frac{\partial V}{\partial \theta}\,\delta\theta...,$$

and thus $\quad R = -\dfrac{\partial V}{\partial r}, \qquad \Theta = -\dfrac{\partial V}{r\,\partial\theta}. \quad (8)$

For an assemblage of particles, or a rigid body, the potential energy is the sum of the potential energies of the several particles. In this connexion it should be remarked that a number of common types of force do no work in a displacement of the system. These are:

(i) the reaction at a smooth surface;
(ii) the reaction at a frictionless hinge;
(iii) tension or compression in an inextensible rod connecting two bodies;
(iv) the reaction at a rolling contact.

In (i) the reaction is perpendicular to the displacement, while in (iv) the displacement is zero; thus no work is done in either case. In (ii) and (iii) there are equal, oppositely directed, actions and reactions on the two bodies and the displacements are the same so the net work is zero.

We now calculate the potential energy in some systems of practical interest.

(i) *Gravity.* Taking the z-axis vertically upwards and the reference position as $z = 0$ we have

$$Z = -mg,$$

$$V = \int_z^0 (-mg)\,dz = mgz. \tag{9}$$

(ii) *A spring of stiffness* λ. Considering displacements along the x-axis, and taking the reference position at the point $x = 0$ where the spring is unstrained

$$X = -\lambda x,$$

$$V = \int_x^0 (-\lambda x)\,dx = \tfrac{1}{2}\lambda x^2. \tag{10}$$

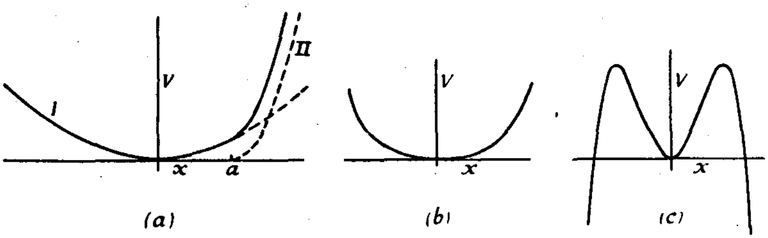


FIG. 62.

(iii) *A spring of stiffness* λ *which comes into contact with another spring of stiffness* λ_1 *when* $x = a$. If $x \leqslant a$ the potential energy is given by (10). If $x > a$ an amount $\tfrac{1}{2}\lambda_1(x-a)^2$, corresponding to the energy stored in the second spring, has to be added. These quantities are shown in curves I and II of Fig. 62 (a).

(iv) *The anharmonic oscillator* § *55 (28) in which* $X = -n^2x - bx^3$. Taking the reference position as $x = 0$ we have

$$V = \int_x^0 (-n^2x - bx^3)\,dx = \tfrac{1}{2}n^2x^2 + \tfrac{1}{4}bx^4. \tag{11}$$

If $b > 0$ the potential energy increases steadily as $|x|$ increases, Fig. 62 (b). If $b < 0$ it vanishes when $x = (-2n^2/b)^{\frac{1}{2}}$ and has a maximum $(-n^4/4b)$ when $x = (-n^2/b)^{\frac{1}{2}}$; cf. Fig. 62 ($c$).

(v) *The anharmonic oscillator* § *55 (29) for which* $X = -n^2x - ax^2$. Here

$$V = \tfrac{1}{2}n^2x^2 + \tfrac{1}{3}ax^3. \tag{12}$$

If, for example, $a > 0$, the curve of V increases steadily for positive x. For negative x there is a zero at $x = -3n^2/2a$, and a maximum of $n^6/6a^2$ at $x = -n^2/a$ (Fig. 63 (a)).

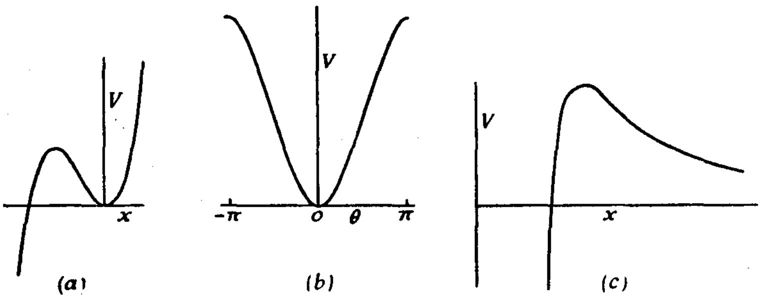


FIG. 63.

(vi) *The simple pendulum.* Taking the reference level as the lowest point of the swing, we have $z = l(1-\cos\theta)$ in (9) and so, cf. Fig. 63 (b),

$$V = mgl(1-\cos\theta). \tag{13}$$

(vii) *Inverse square attraction or repulsion.* Suppose the particle is attracted to the point $x = 0$ by force μ/x^2. The reference position is taken as $x = \infty$. Then

$$V = \int_x^\infty \left(-\frac{\mu}{x^2}\right) dx = -\frac{\mu}{x}. \tag{14}$$

For repulsion the sign is changed.

(viii) *Inverse square repulsion and inverse fifth-power attraction.* The force

$$X = \frac{\mu}{x^2} - \frac{a}{x^5} \tag{15}$$

is attractive at small distances and repulsive at large ones.

Taking the reference position as $x = \infty$,

$$V = \int_x^\infty \left(\frac{\mu}{x^2} - \frac{a}{x^5}\right) dx = \frac{\mu}{x} - \frac{a}{4x^4}. \tag{16}$$

The potential energy has a maximum when $x = (a/\mu)^{\frac{1}{3}}$. It is sketched in Fig. 63 (c). Curves of this sort have been used to represent atomic nuclei.

(ix) *The potential energy of a particle of mass m in the field of a spherical shell of surface density σ and radius a, each element of area dS of which attracts the particle with force $\gamma m\sigma\, dS/R^2$, where R is the distance between the element of area and the particle, and γ is a constant.*

Suppose m is distant $r > a$ from the centre of the shell. The potential energy of m due to the stripe of the shell between the angles θ and $\theta+\delta\theta$, all of which is at distance

$$R = \{a^2+r^2-2ar\cos\theta\}^{\frac{1}{2}}$$

from m, is by (14)
$$-\frac{2\pi a^2 m\sigma\gamma\sin\theta\,\delta\theta}{\{a^2+r^2-2ar\cos\theta\}^{\frac{1}{2}}}.$$

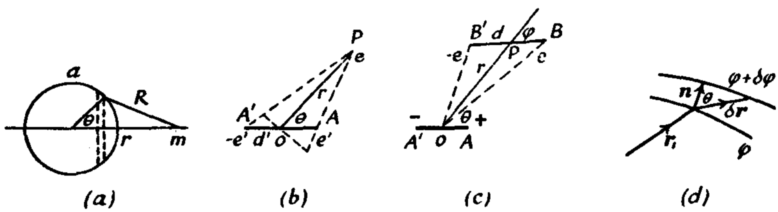


FIG. 64.

Thus the potential energy of m in the field of the whole shell is

$$V = -2\pi a^2 m\sigma\gamma\int\limits_0^\pi \frac{\sin\theta\,d\theta}{\{a^2+r^2-2ar\cos\theta\}^{\frac{1}{2}}} = -\frac{2\pi am\sigma\gamma}{r}[\{a^2+r^2-2ar\cos\theta\}^{\frac{1}{2}}]_0^\pi$$

$$= -\frac{2\pi am\sigma\gamma}{r}\{(r+a)-(r-a)\} \quad (17)$$

$$= -\frac{4\pi a^2\sigma\gamma m}{r} = -\frac{\gamma Mm}{r}, \quad (18)$$

where M is the mass of the shell. Thus the potential energy of the mass m is the same as if the mass of the shell were concentrated at its centre. The same result holds for a solid sphere.

If the mass m is inside the shell, so that $r < a$, the calculation remains the same to (17) in which $r-a$ is to be replaced by $a-r$, and we get finally in place of (18)

$$-\frac{\gamma Mm}{a}. \quad (19)$$

(x) *Potential energy in the field of a dipole.* An electric dipole of moment μ' consists of a charge e' at A and a charge $-e'$ at A', the distance $AA' = 2d'$ being very small and the product $2e'd' = \mu'$ being finite, Fig. 64 (*b*). We calculate the potential energy of a charge e in the field of this dipole: it is attracted to A' with force $ee'/A'P^2$, and repelled from A with force ee'/AP^2, so by (14) its potential energy is

$$V = \frac{ee'}{AP} - \frac{ee'}{A'P}. \quad (20)$$

We may specify the position of e by its distance $r = OP$ from the dipole, and its angular position θ from the direction $A'A$ of the dipole.

Since d' is very small we may neglect d'^2 when it occurs, and so, to this approximation,

$$AP = r-d'\cos\theta; \qquad A'P = r+d'\cos\theta.$$

Then from (20)

$$V = \frac{ee'}{r-d'\cos\theta}-\frac{ee'}{r+d'\cos\theta} = \frac{2ee'd'\cos\theta}{r^2} = \frac{e\mu'\cos\theta}{r^2}. \tag{21}$$

By (8) the radial and transverse components of the force on e are

$$R = \frac{2e\mu'\cos\theta}{r^3}, \qquad \Theta = \frac{e\mu'\sin\theta}{r^3}. \tag{22}$$

In rectangular Cartesians with the z-axis in the direction of the dipole we have $\cos\theta = z/r$ in (21), and the components of the force on e are

$$X = -\frac{\partial V}{\partial x} = \frac{3e\mu' zx}{r^5}, \qquad Y = \frac{3e\mu' zy}{r^5}, \qquad Z = \frac{e\mu'(3z^2-r^2)}{r^5}. \tag{23}$$

(xi) *The potential energy of a dipole of moment μ in the field of a dipole of moment μ'.* Suppose the second dipole is at P, (r,θ), relative to the first, and that its direction makes an angle ϕ with OP. We may suppose it to be composed of charges e at B and $-e$ at B', such that

$$PB = PB' = d \quad \text{and} \quad 2de = \mu,$$

d being very small; cf. Fig. 64 (c).

Then by (21) the potential energy is

$$\begin{aligned} V &= \frac{e\mu'\cos BOA}{BO^2} - \frac{e\mu'\cos B'OA}{B'O^2} \\ &= \frac{e\mu'\{\cos\theta+(d/r)\sin\phi\sin\theta\}}{(r+d\cos\phi)^2} - \frac{e\mu'\{\cos\theta-(d/r)\sin\phi\sin\theta\}}{(r-d\cos\phi)^2} \\ &= \frac{\mu\mu'}{r^3}(\sin\theta\sin\phi-2\cos\theta\cos\phi), \end{aligned} \tag{24}$$

where, throughout, we have neglected terms in d^2.

Finally we define the gradient of a scalar function of position and discuss its relation to the theory given above. Suppose ϕ is a single-valued continuous scalar function of position $\mathbf{r}$, that is, ϕ has a given numerical value at each point $\mathbf{r}$, and if $\phi+\delta\phi$ is its value at $\mathbf{r}+\delta\mathbf{r}$, $\delta\phi \to 0$ as $\delta\mathbf{r} \to 0$. If this is the case, a set of *equipotential surfaces* can be drawn on each of which the value of ϕ is constant; there is one such surface through each point.

Suppose that ϕ and $\phi+\delta\phi$ are the values of the function at $\mathbf{r}_1$ and $\mathbf{r}_1+\delta\mathbf{r}$, and that $\mathbf{n}$ is a unit vector normal to the equipotential surface through $\mathbf{r}_1$, Fig. 64 (d). If $\partial\phi/\partial n$ is the rate of change of ϕ at $\mathbf{r}_1$ in the direction of $\mathbf{n}$, the gradient of ϕ is defined as the vector

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial n}\mathbf{n}. \tag{25}$$

The rate of change of ϕ in the direction of $\delta\mathbf{r}$ is

$$\frac{\partial\phi}{\partial r} = \lim_{\delta r\to 0}\frac{\delta\phi}{\delta r} = \frac{\partial\phi}{\partial n}\cos\theta,$$

where θ is the angle between $\mathbf{n}$ and $\delta\mathbf{r}$. It follows that the rate of change of ϕ is greatest in the direction of $\mathbf{n}$, and also that the magnitude of the component of the vector (25) in the direction of $\delta\mathbf{r}$ is $\partial\phi/\partial r$. In particular, $\partial\phi/\partial x$, $\partial\phi/\partial y$, and $\partial\phi/\partial z$ are the magnitudes of the components of $\operatorname{grad}\phi$ in the directions of the x-, y-, and z-axes of a rectangular system, and if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in the directions of these axes,

$$\operatorname{grad}\phi = \mathbf{i}\frac{\partial\phi}{\partial x}+\mathbf{j}\frac{\partial\phi}{\partial y}+\mathbf{k}\frac{\partial\phi}{\partial z}. \tag{26}$$

The potential energy V in a conservative field of force is a scalar function of position, and by (7) the force $\mathbf{P}$ at any point is connected with it by

$$\mathbf{P} = -\operatorname{grad}V. \tag{27}$$

73. The energy equation: applications

The equation of motion for a particle of mass m moving in the direction of the x-axis under force $f(x)$ in that direction is

$$m\ddot{x} = f(x). \tag{1}$$

In § 55 (5) a first integral of this equation was found to be

$$\tfrac{1}{2}mv^2 = \int^{x} f(x)\,dx + C, \tag{2}$$

where v is the velocity of the particle, and C is a constant.

Now $\frac{1}{2}mv^2$ is the kinetic energy T of the particle, and by § 72 (5) its potential energy V referred to a standard position S is

$$V = \int_{x}^{S} f(x)\,dx. \tag{3}$$

Thus (2) may be written

$$T+V = \text{constant}, \tag{4}$$

or if T_0 and V_0 are the initial values of T and V,

$$T+V = T_0+V_0. \tag{5}$$

This is the energy equation for the particle. It could have been written down immediately in many of the problems previously discussed and the equations of motion found by differentiating it. But perhaps its most important use is to give a simple picture of the nature of the motion.

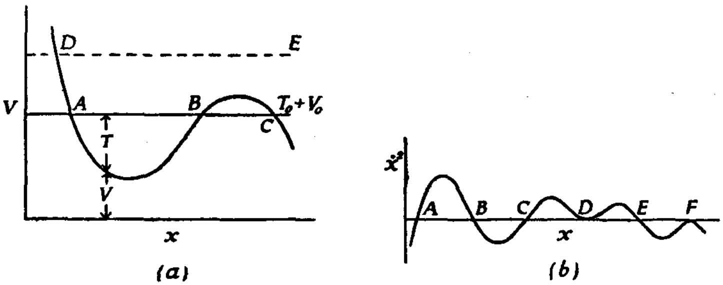


FIG. 65.

Suppose the curve of potential energy V as a function of x is drawn, and we plot on it a horizontal line of ordinate T_0+V_0, e.g. the line ABC in Fig. 65 (a). Then by (5) the amount by which this line is above the curve of V at any value of x is the kinetic energy of the particle at that point. Thus between A and B the kinetic energy is positive; at A and B the kinetic energy is zero, and the particle comes to rest. The particle cannot penetrate into the region to the left of A, or into the region BC, since its total energy is less than the potential energy in these regions. Thus if it is set in motion at a point between A and B with this total energy it will move between A and B (the nature of the motion will be studied further below) and it cannot get into the region to the right of C where motion is also possible with this total energy. On the other hand, if the total energy is given by DE, Fig. 65 (a), there is positive kinetic energy, and motion is possible, for all values of x to the right of D. The types of motion possible with the potential energy curves of Figs. 62 and 63 may be seen easily from these considerations.

To study the motion more closely it is a little more convenient to use (5) in the form $T = T_0+V_0-V$, which gives $\dot{x}^2$ as a function of x. Suppose this curve is as shown in Fig. 65 (b). Motion is only possible in the regions in which $\dot{x}^2$ is positive, such as AB, CDE. We consider now what happens at points such as A, B, at which the velocity is zero.

Suppose that at B, $x = b$, $\dot{x}^2$ has a simple zero as in Fig. 65 (b). Then

$$\dot{x}^2 = (b-x)\phi(x), \tag{6}$$

where $\phi(b) > 0$. From (6), by differentiating,

$$2\dot{x}\ddot{x} = -\dot{x}\,\phi(x)+(b-x)\phi'(x)\dot{x},$$

$$\ddot{x} = -\tfrac{1}{2}\phi(x)+\tfrac{1}{2}(b-x)\phi'(x). \tag{7}$$

It follows from (7) that when $x = b$, $\ddot{x} = -\frac{1}{2}\phi(b)$ and so is finite and negative. Thus at the point b, where the velocity is zero, there is a negative acceleration and the particle commences to move backwards. Similarly when it comes to rest at A there is a forwards acceleration. Thus the particle oscillates between A and B.

Next we consider the nature of the motion near a point D, $x = d$, Fig. 65 (b), where $\dot{x}^2$ has a double zero. In this case

$$\dot{x}^2 = (d-x)^2\psi(x), \tag{8}$$

where $\psi(d) > 0$. Differentiating (8)

$$2\dot{x}\ddot{x} = -2(d-x)\psi(x)\dot{x}+(d-x)^2\psi'(x)\dot{x},$$

$$\ddot{x} = -(d-x)\psi(x)+\tfrac{1}{2}(d-x)^2\psi'(x). \tag{9}$$

Thus $\ddot{x} = 0$ when $x = d$. Also from (8)

$$\frac{dx}{dt} = (d-x)[\psi(x)]^{\frac{1}{2}},$$

and thus the time taken to reach the point $x = d$ is

$$t = \int\limits_x^d \frac{dx}{(d-x)[\psi(x)]^{\frac{1}{2}}}. \tag{10}$$

The integral (10) is divergent, so the particle would take an infinite time to reach $x = d$. This case occurs, for example, when a pendulum has just sufficient energy to reach the upward vertical [cf. Fig. 63 (b)]. In the critical case of a double zero the two regions CD and DE of Fig. 65 (b) are thus effectively separated; if the total energy of the particle is increased slightly, $\dot{x}^2$ will become positive at D and motion takes place in the whole region CE.

Finally, a point such as F, Fig. 65 (b), corresponds to the particle at rest in a position of equilibrium.

The above discussion, based on the behaviour of $\dot{x}^2$ as a function of x, is quite general and not confined to the one-dimensional motion of a particle for which it was made. In more complicated systems specified by several parameters an equation of type $\dot{x}^2 = f(x)$ is usually obtained after some integrations and the elimination of some of the parameters in favour of a chosen one, x, and this equation is discussed as above. For example, an equation of this type was found in § 71 (19) for the motion of a gyroscope, and many properties of the gyroscope can be found from it, e.g. the condition for steady precession which is the condition that § 71 (19) should have a point such as F, Fig. 65 (b).

The energy equation has been derived above for a particle moving in one dimension. We now derive it for the general motion of a system of particles or rigid bodies subject to the assumption that any constraints imposed on the system are independent of the time.† Let m at $\mathbf{r}$ be the typical particle of § 66, then as in § 66 (1) the equations of motion of this particle are

$$m\ddot{\mathbf{r}} = \mathbf{P}+\mathbf{P}'. \tag{11}$$

Taking the scalar product of both sides with $\dot{\mathbf{r}}$, and summing over all particles of the system gives

$$\sum m\ddot{\mathbf{r}}.\dot{\mathbf{r}} = \sum \mathbf{P}.\dot{\mathbf{r}} + \sum \mathbf{P}'.\dot{\mathbf{r}}. \tag{12}$$

† The discussion below is not valid if the system is subject to moving constraints, for example, for a particle in a rotating tube or a body rolling on a rotating plane. The reason for this is that in calculating potential energy the displacements have to be consistent with the constraints; for example, for a particle in a tube the displacement must be along and not perpendicular to the tube. If the tube is rotating, the velocity of the particle will have a component perpendicular to the tube because of its rotation, and the displacements in (13) are taken to be proportional to the velocity.

Integrating (12) with respect to the time gives

$$\tfrac{1}{2}\sum m\dot{\mathbf{r}}.\dot{\mathbf{r}} = \sum\int \mathbf{P}.\dot{\mathbf{r}}\,dt + \sum\int \mathbf{P}'.\dot{\mathbf{r}}\,dt$$

$$= \sum\int \mathbf{P}.d\mathbf{r} + \sum\int \mathbf{P}'.d\mathbf{r}. \tag{13}$$

The first term on the right-hand side of (13) is, except for an arbitrary constant, $-V$, where V is the potential energy of the system in the field of force. The second term vanishes, since, as remarked in § 72, the internal forces between two particles, etc., taken together, do no work in a small displacement. Thus (13) gives the energy equation

$$T+V = \text{constant}. \tag{14}$$

The result that the total energy of a system moving under conservative forces is constant is referred to as the principle of conservation of energy. There are many important cases in which it does not hold:

(i) If the forces are not conservative.

(ii) If the system is subject to moving constraints.

(iii) If there is resistance to motion depending on the velocity. In this case it is shown in § 77 that the total energy diminishes steadily.

(iv) If energy is supplied to the system from outside. For example, it was seen in § 59 that in relaxation oscillations the total energy in the system oscillates periodically, instead of remaining constant as in oscillations under conservative forces.

74. The use of conservation of energy and conservation of momentum

As remarked in § 73, a first integral of the equations of motion can be written down by the use of the energy equation if it is applicable.

Also, if there is no component of external force on a system in a fixed direction, the rate of change of momentum of the system in this direction is zero, and thus the momentum of the system in this direction is constant.

Again, if there is no component of external couple on a system about a fixed axis, the rate of change of angular momentum about this axis is zero, and so the angular momentum of the system about this axis is constant.

These results are known as the principles of conservation of momentum and conservation of angular momentum, and are immediate consequences of the equations of motion. By using them it is often possible to avoid writing down and integrating the equations of motion of the system: thus the integrals § 71 (8) and § 71 (12) of the equations of motion of the gyroscope could have been written down by conservation of angular momentum, and the integral § 71 (14) by conservation of energy. But the process of writing down the fundamental equations of motion and integrating them is very little longer and gives more complete and logically connected information about the problem.

The principles of conservation of energy and momentum are particularly useful when a complete solution is not desired, and also when the motion of a system is suddenly changed by blows.

Ex. *A uniform cylinder of mass M and radius a, rolling with velocity v along a horizontal plane, comes to a kerb of height b ($< a$) parallel to its axis. Find the velocity necessary for it to surmount the kerb.*

We assume that the generator of the cylinder which meets the edge of the kerb becomes fixed there, and that the cylinder rotates about this line as axis. The generator is fixed by blows applied to it, and, since these have no moment about it, the angular momentum about it is unchanged.

The moment of inertia of the cylinder about its axis is $\frac{1}{2}Ma^2$, so its angular momentum about the kerb before the blows is by § 66 (19)

$$Mv(a-b)+\tfrac{1}{2}Mav. \tag{1}$$

If the angular velocity of the cylinder about the edge of the kerb after the blows is Ω, its angular momentum about this axis is

$$\tfrac{3}{2}Ma^2\Omega. \tag{2}$$

Equating (1) and (2),

$$\Omega = \frac{v(3a-2b)}{3a^2}. \tag{3}$$

The cylinder will surmount the kerb if its new kinetic energy,

$$\tfrac{3}{4}Ma^2\Omega^2 = \frac{Mv^2(3a-2b)^2}{12a^2},$$

is greater than the required potential energy Mgb, that is, if

$$v^2 > \frac{12ga^2b}{(3a-2b)^2}. \tag{4}$$

To complete the solution it is necessary to verify that the cylinder remains in contact with the kerb during the motion, and for this the reaction at the point of contact must be determined as in § 67.

75. Generalized coordinates

Suppose $q_1,\ldots, q_n$ are n independent quantities in terms of which it is possible to specify the position of every particle of a dynamical system. $q_1,\ldots, q_n$ are called *generalized coordinates*, and n is called the number of *degrees of freedom* of the system.

For example a single particle needs three coordinates to specify its position so has three degrees of freedom. Cartesian coordinates (x, y, z), spherical polar coordinates (r, θ, ϕ), or any other system may be chosen as the generalized coordinates.

A rigid body needs six coordinates to specify its position, say three to fix the position of a point of it, two to specify the direction of an axis through that point, and one to specify a rotation about that axis. Alternatively its position may be specified by the positions of three points; for this nine coordinates are required, but there are three relations between these since the distances between the points are fixed so that three of the nine coordinates can be eliminated leaving six as before. We shall always suppose that such an elimination has been carried out† so that $q_1,\ldots, q_n$ are independent.

Clearly a system of equations of motion expressed in terms of the generalized coordinates is needed. These equations, Lagrange's equations, will be established in § 76; we first derive some properties of generalized coordinates which are needed for the proof.

Suppose that (x, y, z) are the coordinates of a particular particle of the system referred to fixed rectangular axes, then x is a function of $q_1,\ldots, q_n$ which we shall write in the functional form

$$x = x(q_1,\ldots, q_n), \tag{1}$$

with similar expressions for y and z.

† There are important cases in which this cannot be done because the extra coordinates are connected by differential and not algebraic equations. Such systems are called non-holonomic; they arise in problems involving rolling contacts. The present theory does not apply to them although it is easily extended to do so.

Differentiation of (1) with respect to the time gives

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \dots + \frac{\partial x}{\partial q_n}\dot{q}_n. \tag{2}$$

$\dot{q}_1,\dots, \dot{q}_n$ are called *generalized velocities*. From (2)

$$\frac{\partial \dot{x}}{\partial \dot{q}_r} = \frac{\partial x}{\partial q_r} \quad (r = 1,\dots, n). \tag{3}$$

Also

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_r}\right) = \frac{\partial^2 x}{\partial q_r \partial q_1}\dot{q}_1 + \dots + \frac{\partial^2 x}{\partial q_r \partial q_n}\dot{q}_n$$

$$= \frac{\partial}{\partial q_r}\left\{\frac{\partial x}{\partial q_1}\dot{q}_1 + \dots + \frac{\partial x}{\partial q_n}\dot{q}_n\right\} = \frac{\partial \dot{x}}{\partial q_r}. \tag{4}$$

As usual if we are dealing with a rigid body, we suppose (x, y, z) to be the rectangular coordinates of an element of mass m and use $\sum$ to denote a summation over all such elements. The kinetic energy T is

$$T = \tfrac{1}{2}\sum m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$= \frac{1}{2}\sum m\left\{\left(\frac{\partial x}{\partial q_1}\dot{q}_1 + \dots + \frac{\partial x}{\partial q_n}\dot{q}_n\right)^2 + \right.$$

$$\left. + \left(\frac{\partial y}{\partial q_1}\dot{q}_1 + \dots\right)^2 + \left(\frac{\partial z}{\partial q_1}\dot{q}_1 + \dots\right)^2\right\}. \tag{5}$$

Multiplying out, this may be written in the form

$$T = a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2 + \dots$$

$$= \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}\dot{q}_r\dot{q}_s, \tag{6}$$

where the coefficients a_{rs} are functions of $q_1,\dots, q_n$ which can be written down from (5). If they are written out in full it appears that $a_{rs} = a_{sr}$.

The quantity p_r defined by

$$p_r = \frac{\partial T}{\partial \dot{q}_r} = \sum_{s=1}^{n} a_{rs}\dot{q}_s \tag{7}$$

is called the generalized momentum corresponding to the coordinate q_r. This definition may be regarded as being suggested by the result that, for linear motion of a particle of mass m with velocity v, the linear momentum

$$mv = \frac{d}{dv}(\tfrac{1}{2}mv^2). \tag{8}$$

Since the equations of motion of the particle in (8) involve $d(mv)/dt$, we may expect to have to study d/dt of the quantities in (7), that is,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right). \tag{9}$$

To do this consider

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r}(\tfrac{1}{2}\dot{x}^2)\right\} &= \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_r}\right) \\ &= \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_r}\right) &(10)\\ &= \ddot{x}\frac{\partial x}{\partial q_r}+\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_r}\right) \\ &= \ddot{x}\frac{\partial x}{\partial q_r}+\dot{x}\frac{\partial \dot{x}}{\partial q_r} &(11)\\ &= \ddot{x}\frac{\partial x}{\partial q_r}+\frac{\partial}{\partial q_r}(\tfrac{1}{2}\dot{x}^2), &(12)\end{aligned}$$

where we have used (3) in (10), and (4) in (11).

Adding the corresponding equations for y and z, multiplying by the mass m of the typical particle, and summing over all particles, we get from (12)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r} = \sum m\left(\ddot{x}\frac{\partial x}{\partial q_r}+\ddot{y}\frac{\partial y}{\partial q_r}+\ddot{z}\frac{\partial z}{\partial q_r}\right). \tag{13}$$

76. Lagrange's equations

Suppose a conservative dynamical system is specified as in § 75 by generalized coordinates $q_1,\ldots, q_n$. As before, let (x, y, z) be the position of the element of mass m, and let (X, Y, Z) be the total force on this particle (including both external and internal forces, cf. § 66). Then its equations of motion are

$$m\ddot{x} = X, \qquad m\ddot{y} = Y, \qquad m\ddot{z} = Z. \tag{1}$$

Multiplying these by $\partial x/\partial q_r$, $\partial y/\partial q_r$, $\partial z/\partial q_r$, respectively, adding, and summing over all elementary particles, gives

$$\sum m\left(\ddot{x}\frac{\partial x}{\partial q_r}+\ddot{y}\frac{\partial y}{\partial q_r}+\ddot{z}\frac{\partial z}{\partial q_r}\right) = \sum\left(X\frac{\partial x}{\partial q_r}+Y\frac{\partial y}{\partial q_r}+Z\frac{\partial z}{\partial q_r}\right). \tag{2}$$

The left-hand side of this has been transformed in § 75 (13). We now have to consider the right-hand side. Suppose a small change δq_r is made in q_r, all the other q being left unchanged (this is possible since by hypothesis the q are independent); the resulting changes in x, y, z will be

$$\frac{\partial x}{\partial q_r}\delta q_r,\quad \frac{\partial y}{\partial q_r}\delta q_r,\quad \frac{\partial z}{\partial q_r}\delta q_r,$$

and the total work done will be

$$\sum\left(X\frac{\partial x}{\partial q_r}+Y\frac{\partial y}{\partial q_r}+Z\frac{\partial z}{\partial q_r}\right)\delta q_r, \tag{3}$$

on summing over all particles of the system.† Now this work is also the decrease in potential energy of the system, that is

$$-\frac{\partial V}{\partial q_r}\delta q_r. \tag{4}$$

Therefore, equating (3) and (4),

$$\sum\left(X\frac{\partial x}{\partial q_r}+Y\frac{\partial y}{\partial q_r}+Z\frac{\partial z}{\partial q_r}\right) = -\frac{\partial V}{\partial q_r}, \tag{5}$$

and, using (5) and § 75 (13) in (2), we get finally

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r} = -\frac{\partial V}{\partial q_r}. \tag{6}$$

r is unspecified in this, and the set of n equations (6) for $r = 1,\ldots, n$ are *Lagrange's equations*.

The form (6) will be sufficient for the present applications, but there are simple generalizations‡ to: (i) cases in which the forces are not conservative; (ii) cases in which the time appears

† The internal forces, which have been specifically included in (2), disappear in the summation since equal and opposite contributions come from the two particles between which any internal force acts. Also forces such as reactions at smooth surfaces or hinges, or at rolling contacts, do no work.

‡ Cf. Whittaker, *Analytical Dynamics* (Cambridge, 1927).

explicitly, i.e. $x = x(q_1,..., q_n, t)$; (iii) systems with redundant coordinates connected by algebraic equations; (iv) systems with redundant coordinates connected by differential equations (non-holonomic systems).

If the system is not conservative we define the generalized force corresponding to the coordinate q_r by

$$\sum\left(X\frac{\partial x}{\partial q_r}+Y\frac{\partial y}{\partial q_r}+Z\frac{\partial z}{\partial q_r}\right)\delta q_r = Q_r\,\delta q_r,$$

and (6) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r} = Q_r. \tag{7}$$

A great advantage of Lagrange's equations for the solution of dynamical problems is that forces which do no work, such as reactions at smooth hinges, do not appear. In writing down equations of motion in the ordinary way these usually have to be considered.

The equations may be extended to cover the case in which there is resistance to motion k times the velocity. In this case (1) are replaced by

$$m\ddot{x}+k\dot{x} = X, \qquad m\ddot{y}+k\dot{y} = Y, \qquad m\ddot{z}+k\dot{z} = Z, \tag{8}$$

and (2) is replaced by

$$\sum m\left(\ddot{x}\frac{\partial x}{\partial q_r}+...\right)+\sum k\left(\dot{x}\frac{\partial x}{\partial q_r}+\dot{y}\frac{\partial y}{\partial q_r}+\dot{z}\frac{\partial z}{\partial q_r}\right) = \sum\left(X\frac{\partial x}{\partial q_r}+...\right). \tag{9}$$

Now by § 75 (3)

$$\sum k\left(\dot{x}\frac{\partial x}{\partial q_r}+\dot{y}\frac{\partial y}{\partial q_r}+\dot{z}\frac{\partial z}{\partial q_r}\right) = \sum k\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_r}+\dot{y}\frac{\partial \dot{y}}{\partial \dot{q}_r}+\dot{z}\frac{\partial \dot{z}}{\partial \dot{q}_r}\right)$$

$$= \frac{\partial F}{\partial \dot{q}_r}, \tag{10}$$

where

$$F = \tfrac{1}{2}\sum k(\dot{x}^2+\dot{y}^2+\dot{z}^2). \tag{11}$$

Proceeding as before, Lagrange's equations become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r}+\frac{\partial F}{\partial \dot{q}_r} = -\frac{\partial V}{\partial q_r} \quad (r = 1,..., n). \tag{12}$$

The function F defined in (11) is called the *dissipation function*. Using § 75 (2) it appears that it has the form

$$F = \sum_{r=1}^{n}\sum_{s=1}^{n} b_{rs}\,\dot{q}_r\,\dot{q}_s, \tag{13}$$

where the coefficients b_{rs} are functions of $q_1,..., q_n$, and $b_{rs} = b_{sr}$. It follows from (13), either by writing out the expressions explicitly or by using Euler's theorem on homogeneous functions, that

$$\sum_{r=1}^{n} \frac{\partial F}{\partial \dot{q}_r} \dot{q}_r = 2F. \tag{14}$$

Also, it follows in the same way from § 75 (6) that

$$\sum_{r=1}^{n} \frac{\partial T}{\partial \dot{q}_r} \dot{q}_r = 2T. \tag{15}$$

We now proceed to study the way in which the total energy of the system diminishes because of the resistance to motion. Multiplying the equations (12) by $\dot{q}_1,\dots, \dot{q}_n$, respectively, and adding gives

$$\sum_{r=1}^{n} \dot{q}_r \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \sum_{r=1}^{n} \frac{\partial T}{\partial q_r} \dot{q}_r + \sum_{r=1}^{n} \frac{\partial F}{\partial \dot{q}_r} \dot{q}_r + \sum_{r=1}^{n} \frac{\partial V}{\partial q_r} \dot{q}_r = 0. \tag{16}$$

Now $$\frac{dT}{dt} = \sum_{r=1}^{n} \frac{\partial T}{\partial q_r} \dot{q}_r + \sum_{r=1}^{n} \frac{\partial T}{\partial \dot{q}_r} \ddot{q}_r. \tag{17}$$

Using (17) and (14) in (16) gives

$$\sum_{r=1}^{n} \dot{q}_r \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \sum_{r=1}^{n} \frac{\partial T}{\partial \dot{q}_r} \ddot{q}_r - \frac{dT}{dt} + 2F + \frac{dV}{dt} = 0.$$

Therefore $$\frac{d}{dt} \sum_{r=1}^{n} \left(\frac{\partial T}{\partial \dot{q}_r} \dot{q}_r\right) - \frac{dT}{dt} + \frac{dV}{dt} + 2F = 0.$$

Therefore, using (15), $$\frac{d}{dt}(T+V) = -2F. \tag{18}$$

Thus $2F$ measures the rate at which the total energy of the system is being dissipated by friction. If $F = 0$, (18) is the equation of conservation of energy.

Ex. 1. *The gyroscope.*

The system is specified in § 71. The components of the angular velocity of the body along the instantaneous directions of the axes are by § 71 (2)

$$(-\dot{\psi} \sin\theta,\ \dot{\theta},\ \dot{\phi} + \dot{\psi}\cos\theta). \tag{19}$$

The moments of inertia of the body about these axes are A, A, C. Thus, by § 66 (24)

$$T = \tfrac{1}{2}A(\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2}C(\dot{\phi} + \dot{\psi}\cos\theta)^2. \tag{20}$$

Also the potential energy is $V = Mgh\cos\theta$.

Thus Lagrange's equations for θ, ψ, and ϕ respectively are

$$\frac{d}{dt}(A\dot{\theta})-A\dot{\psi}^2\sin\theta\cos\theta+C\dot{\psi}(\dot{\phi}+\dot{\psi}\cos\theta)\sin\theta = Mgh\sin\theta, \tag{21}$$

$$\frac{d}{dt}\{A\dot{\psi}\sin^2\theta+C\cos\theta(\dot{\phi}+\dot{\psi}\cos\theta)\} = 0, \tag{22}$$

$$\frac{d}{dt}(\dot{\phi}+\dot{\psi}\cos\theta) = 0. \tag{23}$$

(23) gives

$$\dot{\phi}+\dot{\psi}\cos\theta = n. \tag{24}$$

(21) to (23) are the same as § 71 (6), (11), (7). Also, using (24) in (20), the energy equation becomes

$$\tfrac{1}{2}A(\dot{\psi}^2\sin^2\theta+\dot{\theta}^2)+Mgh\cos\theta = \text{constant}, \tag{25}$$

which is § 71 (14).

Ex. 2. *The motion of two free, attracting particles.*

Let m_1 and m_2 be the masses of the particles, and let (x_1, y_1, z_1), (x_2, y_2, z_2), and (X, Y, Z) be respectively the coordinates of the two particles and their centre of mass, relative to fixed rectangular axes. Write

$$x = x_2-x_1, \qquad y = y_2-y_1, \qquad z = z_2-z_1. \tag{26}$$

Then, writing

$$\mu_1 = m_1/(m_1+m_2), \qquad \mu_2 = m_2/(m_1+m_2),$$
$$x_1 = X-\mu_2 x, \qquad x_2 = X+\mu_1 x,$$

with similar expressions for y_1, z_1, y_2, z_2. The kinetic energy T is

$$T = \tfrac{1}{2}m_1\{(\dot{X}-\mu_2\dot{x})^2+(\dot{Y}-\mu_2\dot{y})^2+(\dot{Z}-\mu_2\dot{z})^2\}+$$
$$+\tfrac{1}{2}m_2\{(\dot{X}+\mu_1\dot{x})^2+(\dot{Y}+\mu_1\dot{y})^2+(\dot{Z}+\mu_1\dot{z})^2\}. \tag{27}$$

The potential energy V is a function of x, y, and z. Taking X, Y, Z, x, y, z as coordinates, Lagrange's equations give

$$\ddot{X} = \ddot{Y} = \ddot{Z} = 0,$$

that is, the centre of mass moves with constant velocity. The other three equations are

$$m_1\mu_2\ddot{x} = -\frac{\partial V}{\partial x}; \qquad m_1\mu_2\ddot{y} = -\frac{\partial V}{\partial y}; \qquad m_1\mu_2\ddot{z} = -\frac{\partial V}{\partial z}, \tag{28}$$

that is, the motion of m_2 relative to m_1 is the same as if m_1 were fixed and the potential energy were

$$\frac{m_1+m_2}{m_1}V. \tag{29}$$

Ex. 3. *The problem of § 36.*

Using the result § 72 (10) we have

$$T = \tfrac{1}{2}M_1\dot{x}_1^2+\tfrac{1}{2}M_2\dot{x}_2^2,$$
$$V = \tfrac{1}{2}\lambda_1 x_1^2+\tfrac{1}{2}\lambda_2(x_2-x_1)^2.$$

Therefore Lagrange's equations give

$$M_1\ddot{x}_1 = -\lambda_1 x_1+\lambda_2(x_2-x_1),$$
$$M_2\ddot{x}_2 = -\lambda_2(x_2-x_1),$$

in agreement with § 36 (1) and (2).

Ex. 4. *Two equal uniform rods OA, AB of length $2a$ and mass M are freely hinged together at A, and OA is freely hinged at a fixed point O. They oscillate in a plane under gravity.*

Let θ and ϕ be the angles which the rods make with the vertical. Then the horizontal and vertical displacements of G, the centre of mass of AB, are

$$x = 2a\sin\theta+a\sin\phi,$$
$$y = 2a\cos\theta+a\cos\phi.$$

Therefore

$$\dot{x}^2+\dot{y}^2 = (2a\dot{\theta}\cos\theta+a\dot{\phi}\cos\phi)^2+(2a\dot{\theta}\sin\theta+a\dot{\phi}\sin\phi)^2$$
$$= 4a^2\dot{\theta}^2+a^2\dot{\phi}^2+4a^2\dot{\theta}\dot{\phi}\cos(\theta-\phi).$$

The kinetic energy of OA is $(2Ma^2/3)\dot{\theta}^2$, and that of the motion of AB about its centre of mass is $(Ma^2/6)\dot{\phi}^2$. Using § 66 (23) for the kinetic energy of AB, we have for the kinetic energy T of the system

$$T = \frac{2Ma^2}{3}\dot{\theta}^2+\frac{Ma^2}{6}\dot{\phi}^2+\tfrac{1}{2}M\{4a^2\dot{\theta}^2+a^2\dot{\phi}^2+4a^2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\}$$

$$= \frac{8Ma^2}{3}\dot{\theta}^2+\frac{2Ma^2}{3}\dot{\phi}^2+2Ma^2\dot{\theta}\dot{\phi}\cos(\theta-\phi). \tag{30}$$

Also the potential energy, measured from O, is

$$V = -Mga\{3\cos\theta+\cos\phi\}. \tag{31}$$

Lagrange's equations then give

$$\frac{d}{dt}\left\{\frac{16Ma^2}{3}\dot{\theta}+2Ma^2\dot{\phi}\cos(\theta-\phi)\right\}+2Ma^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) = -3Mga\sin\theta, \tag{32}$$

$$\frac{d}{dt}\left\{\frac{4Ma^2}{3}\dot{\phi}+2Ma^2\dot{\theta}\cos(\theta-\phi)\right\}-2Ma^2\dot{\theta}\dot{\phi}\sin(\theta-\phi) = -Mga\sin\phi. \tag{33}$$

Ex. 5. *Lagrange's equations and electric circuit theory.*

The equations § 41 (5) and (4) for an L, R, C circuit with no applied voltage are

$$L\dot{I}+RI+\frac{1}{C}Q = 0, \tag{34}$$

$$\dot{Q} = I. \tag{35}$$

Multiplying (34) by I and using (35) gives

$$\frac{d}{dt}\left(\tfrac{1}{2}LI^2+\frac{1}{2C}Q^2\right)+RI^2 = 0. \tag{36}$$

Now RI^2 is the rate of dissipation of energy in the circuit, so (36) may be regarded as the energy equation if we identify

$$\tfrac{1}{2}LI^2+\frac{1}{2C}Q^2$$

with the total energy in the circuit. $\frac{1}{2}LI^2$ is regarded as kinetic energy associated with current I in an inductance, and $(1/2C)Q^2$ as the potential energy associated with charge Q on a condenser.

Further, we may regard Q as a generalized coordinate specifying the electrical state of the circuit and $\dot{Q} = I$ as the corresponding generalized velocity. Taking

$$T = \tfrac{1}{2}LI^2, \qquad V = \frac{1}{2C}Q^2, \qquad F = \tfrac{1}{2}RI^2, \tag{37}$$

Lagrange's equations (12) give

$$\frac{d}{dt}(LI)+RI+\frac{1}{C}Q = 0,$$

in agreement with (34).

The general equations of electric circuit theory may be obtained in this way. It has the advantage that in systems such as electric motors which contain moving masses the kinetic and potential energies of these may be included and the electrical and mechanical parts of the system considered together.

77. Small oscillations about statical equilibrium

Lagrange's equations of motion, § 76 (6), are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r} = -\frac{\partial V}{\partial q_r} \quad (r = 1, 2,..., n). \tag{1}$$

At the position of equilibrium these must be satisfied with $\dot{q}_1 = ... = \dot{q}_n = 0$, so that all the terms on the left-hand sides vanish and (1) require

$$\frac{\partial V}{\partial q_1} = ... = \frac{\partial V}{\partial q_n} = 0. \tag{2}$$

These are the conditions to be satisfied at a position of equilibrium. They also express the fact that the potential energy V is to be stationary at such a point. If it has a minimum there the equilibrium will be stable. For if the potential energy has a minimum V_0 at the position of equilibrium, and the system is given a small amount of kinetic energy T_0, the motion must be confined to the small region about the position of equilibrium in which $V < T_0+V_0$. That is, the system executes a small oscillation about the position of equilibrium and so is stable. This argument is essentially the extension to n dimensions of the ideas of § 73.

Ex. 1. *The dipole μ of § 72 (xi) is freely hinged at a point on the axis of the dipole μ' and distant r from it.*

In § 72 (24) we have $\theta = 0$ and so

$$V = -\frac{2\mu\mu'}{r^3}\cos\phi. \tag{3}$$

There is equilibrium when $dV/d\phi = 0$, that is, when $\phi = 0$ and when $\phi = \pi$. Now

$$\frac{d^2V}{d\phi^2} = \frac{2\mu\mu'}{r^3}\cos\phi. \tag{4}$$

When $\phi = 0$, (4) is positive, V has a minimum, and so the equilibrium is stable.

When $\phi = \pi$, (4) is negative, V has a maximum, and the equilibrium is unstable.

Suppose, now, that a position of equilibrium has been found by solving the set of equations (2) and that we wish to study small oscillations of the system about it. It is convenient to change the origin of coordinates to the position of equilibrium so that $q_1,\ldots, q_n$ vanish in the position of equilibrium and are small throughout the motion. The potential energy V can then be written in the form

$$V = \sum_{r=1}^{n}\sum_{r=1}^{n} c_{rs}q_r q_s \tag{5}$$

which contains only terms of the second degree in the coordinates. There will be no terms of the first degree since $\partial V/\partial q_r = 0$ when $q_1 = \ldots = q_n = 0$, and a constant term can be neglected since it will not affect the equations (1). Terms of the third and higher degrees in the q_r are neglected since we are assuming that these are small. In the same way the kinetic energy T, § 75 (6), becomes

$$T = \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}\dot{q}_r\dot{q}_s, \tag{6}$$

where now the a_{rs} may be taken to be constants with the values for the equilibrium position, since allowing for their variation with position would introduce terms of the third degree in the q and $\dot{q}$ and these are negligible.

Thus the equations (1) give

$$\sum_{s=1}^{n}(a_{rs}\ddot{q}_s+c_{rs}q_s) = 0 \quad (r = 1,\dots, n), \tag{7}$$

which are a system of n linear differential equations in n unknowns.

If there is resistance to motion proportional to velocity there will be a dissipation function F as in § 76 (13) given by

$$F = \sum_{r=1}^{n}\sum_{s=1}^{n} b_{rs}\dot{q}_r\dot{q}_s, \tag{8}$$

where the b_{rs} are constants with the values at the equilibrium position. Lagrange's equations in the form § 76 (12) then give

$$\sum_{r=1}^{n}(a_{rs}\ddot{q}_s+b_{rs}\dot{q}_s+c_{rs}q_s) = 0 \quad (r = 1,\dots, n). \tag{9}$$

Ex. 2. *Small oscillations of the system of § 76, Ex. 4, about the vertical.* Retaining only terms of the second degree in the small quantities θ, ϕ, $\dot{\theta}$, $\dot{\phi}$ in § 76 (30) and (31) these become

$$T = \frac{8Ma^2}{3}\dot{\theta}^2+\frac{2Ma^2}{3}\dot{\phi}^2+2Ma^2\dot{\theta}\dot{\phi}, \tag{10}$$

$$V = Mga(-4+\tfrac{3}{2}\theta^2+\tfrac{1}{2}\phi^2). \tag{11}$$

Therefore, by Lagrange's equations (1),

$$\frac{16Ma^2}{3}\ddot{\theta}+2Ma^2\ddot{\phi}+3Mga\theta = 0,$$

$$\frac{4Ma^2}{3}\ddot{\phi}+2Ma^2\ddot{\theta}+Mga\phi = 0.$$

Or, writing $n^2 = g/a$,

$$(16D^2+9n^2)\theta+6D^2\phi = 0,$$

$$6D^2\theta+(4D^2+3n^2)\phi = 0,$$

a pair of simultaneous linear differential equations.

EXAMPLES ON CHAPTER IX

1. A mass m is attached to a fixed point by a spring of stiffness λ. When its displacement is a it comes in contact with another spring of stiffness λ'. If it is set in motion from its equilibrium position by a blow of impulse P, show that it will come to rest when

$$\lambda x^2+\lambda'(x-a)^2 = P^2/m,$$

if $P > a(\lambda m)^{\frac{1}{2}}$.

Which root of the equation is to be taken?

2. A rod of mass m is freely hinged to a fixed point on a horizontal plane. Its centre of mass is distant l from the hinge, and its moment

of inertia about the hinge is I. The rod is held in a nearly vertical position by four equal springs, each of stiffness k and unstrained length $\sqrt{(a^2+b^2)}$, attached to it at a distance a from the hinge, and attached symmetrically to the plane at distances b from the hinge. Show that the period of small oscillations about the vertical is

$$2\pi I^{\frac{1}{2}}\left\{\frac{2ka^2b^2}{a^2+b^2}-mgl\right\}^{-\frac{1}{2}}.$$

3. A uniform rod is hung in a horizontal position by two parallel strings of length l (bifilar suspension). Show that the period of small oscillations of the rod in which its centre moves in a vertical straight line is

$$2\pi\sqrt{(l/3g)}.$$

4. A body rests in equilibrium on a rough surface, its centre of mass being a distance h vertically above the point of contact of the surfaces, and their common tangent plane being horizontal and the bodies on opposite sides of it. If ρ_1 and ρ_2 are the radii of curvature of the surfaces at the point of contact show that the equilibrium is stable if

$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

5. A particle of mass m is at the point (x,y) in the field of a uniform thin rod $-a < x < a$, every element δx of which attracts the mass m with force $\gamma m\rho\,\delta x/r^2$, where γ is a constant, ρ is the density of the rod, and r is the distance between m and the element δx. Show that the potential energy of m is

$$\gamma m\rho \ln\frac{(x-a)+[(x-a)^2+y^2]^{\frac{1}{2}}}{(x+a)+[(x+a)^2+y^2]^{\frac{1}{2}}}.$$

6. If heat is supplied at the constant rate of Q units per unit time at a point in an infinite medium of thermal conductivity K, the steady temperature at a point distant r from the point of supply is known to be $Q/2\pi Kr$.

Show that if heat is supplied at the constant rate q units per unit time per unit area over a square of side $2a$, the steady temperature at the centre of the square is

$$\frac{4qa}{K\pi}\ln(1+\sqrt{2}),$$

and the average temperature over the square is

$$\frac{4qa}{K\pi}\left\{\frac{1-\sqrt{2}}{3}+\ln(1+\sqrt{2})\right\}.$$

7. A, B, C are the principal moments of inertia of a body of mass M referred to its centre of mass O as origin. r is the distance of the typical particle of mass m from O, and x is the projection of r along any axis OX. Show that

$$\sum mr^2 = \tfrac{1}{2}(A+B+C),$$
$$\sum mx^2 = \tfrac{1}{2}(A+B+C)-I,$$

where I is the moment of inertia of the body about OX.

Deduce that the potential energy V of a particle of mass M' at a point on OX distant R from O, where R is large compared with the linear dimensions of the body, is

$$V = -\gamma M' \sum m(R^2+r^2-2Rx)^{-\frac{1}{2}}$$
$$= -\gamma M' \left\{\frac{M}{R}+\frac{A+B+C-3I}{2R^3}+...\right\}.$$

8. Four small magnets, each of moment μ, are placed at the corners of a square of side $2a$ and all point in the direction of a side of it. Show that a small magnet of moment μ' placed at the centre of the square will point in the same direction, and if its moment of inertia is I it will oscillate about this position with frequency

$$(\mu\mu'/Ia^3\sqrt{2})^{\frac{1}{2}}/2\pi.$$

9. Four small magnets are pivoted at the corners of a square of side $2a$ and oscillate under the influence they exert on each other. If their magnetic moments are μ and their moments of inertia are I, show that the natural frequencies of the system are

$$\frac{1}{2\pi}\left\{\frac{3\mu^2(2+2^{-\frac{1}{2}})}{8Ia^3}\right\}^{\frac{1}{2}}, \quad \frac{1}{2\pi}\left\{\frac{\mu^2(3-2^{-\frac{1}{2}})}{8Ia^3}\right\}^{\frac{1}{2}}, \quad \frac{1}{2\pi}\left\{\frac{3\mu^2}{16Ia^3 2^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$$

10. A particle of mass m is connected to the four corners of a square by four equal springs of stiffness λ. If x and y are its displacements from the centre of the square in the directions of the diagonals, and z is its displacement perpendicular to the plane of the square, show that its potential energy for small displacements is

$$2\lambda(x^2+y^2)(1-k)+2\lambda z^2(1-2k)+\text{constant},$$

where k is the ratio of the unstretched length of a spring to a diagonal of the square. Find the natural frequencies of small oscillations of the mass.

11. A uniform rod of length $2l$ has one end attached to a fixed point O by a light string of length $5l/12$. Show that the natural frequencies of small oscillations in a vertical plane through O are

$$(3g/5l)^{\frac{1}{2}}/2\pi \quad \text{and} \quad (3g/l)^{\frac{1}{2}}/\pi.$$

12. A truck of total mass M has two axles carrying wheels of radius a, the moment of inertia of each axle and its wheels being I. Show that the acceleration of the truck when rolling down a plane inclined at α to the horizontal is

$$\frac{Mg\sin\alpha}{M+2I/a^2}.$$

13. A particle moves from A to B along any path in either free or constrained motion in a field of force of potential V, there being no

resistance to the motion. Show that if V_A and V_B are the values of its potential energy at A and B, and v_A and v_B are its speeds there,

$$\tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2 = V_A - V_B.$$

Show that if the particle is charged the same result still holds if there is a magnetic field present.

14. Discuss the nature of the motion under a central force by treating § 62 (22) as in § 73. Show that for the attractive force μr^{-n}, where $n > 3$, a circular orbit is unstable.

15. Show that the kinetic energy of a gyroscope mounted in gimbals may be put in the form

$$2T = (A+A_1)\dot{\theta}^2 + \{(A+C_1)\sin^2\theta + A_1\cos^2\theta + I\}\dot{\psi}^2 + C(\dot{\phi}+\dot{\psi}\cos\theta)^2,$$

where I and A_1 are the moments of inertia of the outer and inner rings about their axes, and C_1 is the moment of inertia of the inner ring about a perpendicular to its plane, and the other symbols have their usual meanings.

Deduce the equations of motion of the system.

16. Discuss the motion of a gyrostat by treating § 71 (19) as in § 73. Writing $f(x)$ for the right-hand side of § 71 (19), show that the condition for steady precession with $x = x_0$ is

$$f(x_0) = f'(x_0) = 0,$$

and deduce § 71 (20). Show that the period of small oscillations about steady precession at x_0 with angular velocity ω is

$$2\pi\{-\tfrac{1}{2}f''(x_0)\}^{-\frac{1}{2}} = 2\pi\{(a-2\omega x_0)^2+\omega^2(1-x_0^2)\}^{-\frac{1}{2}}.$$

17. A uniform rod OA of length $2a$ is freely hinged at the point O and oscillates about it under gravity, θ being its inclination to the vertical and ψ the angle between the vertical plane through the rod and a fixed plane. Find the equations of motion of the rod, and show that a steady motion with $\theta = \alpha$, $\dot{\psi} = \omega$, where $\omega^2 = (3g/4a)\sec\alpha$ is possible.

Discuss small oscillations about steady motion by putting $\theta = \alpha + x$, $\dot{\psi} = \omega + y$, and neglecting squares and products of x, y, $\dot{x}$, $\dot{y}$. Show that the period of these oscillations is

$$2\pi\left\{\frac{4a\cos\alpha}{3g(1+3\cos^2\alpha)}\right\}^{\frac{1}{2}}.$$

18. An engine governor consists of the rod OA of Ex. 17 which is hung from a system of moment of inertia I which rotates about a vertical axis. In order to keep the angular velocity of the system constant at the value ω of Ex. 17, a couple $k(\alpha-\theta)$ is applied about the vertical axis when the inclination of OA to the vertical is θ. Show that the equations of motion of the system are

$$\ddot{\theta} - \dot{\psi}^2\sin\theta\cos\theta = -(3g/4a)\sin\theta,$$

$$(4ma^2/3)\{\ddot{\psi}\sin^2\theta + 2\dot{\psi}\dot{\theta}\sin\theta\cos\theta\} + I\ddot{\psi} = k(\alpha-\theta).$$

Study the effect of oscillations about steady motion, and show that the frequency equation for these is a cubic with one real positive root, so that the system is unstable (as discussed here neglecting friction).

19. One end A of a uniform rod of length $2a$ and mass M is rotated with constant angular velocity ω in a horizontal circle of centre O and radius b ($< a$). The rod is hinged at A so that it can move freely in the plane of OA and the vertical. If θ is the angle between the rod and the downward vertical at A, measured in the direction towards the downward vertical at O, show that the kinetic energy of the rod is

$$M\{4a^2\dot{\theta}^2+3\omega^2(b-a\sin\theta)^2+a^2\omega^2\sin^2\theta\}/6.$$

Deduce that there is always a position of equilibrium of the rod in the range $\frac{1}{2}\pi < \theta < \pi$ and another in the range $3\pi/2 < \theta < 2\pi$, and that if ω is sufficiently large there are two in the range $0 < \theta < \frac{1}{2}\pi$. Show that the position of equilibrium in the range $\frac{1}{2}\pi < \theta < \pi$ is unstable (Lagrange's equations may be used).

20. Show that the components of grad V in the directions r, θ, z of cylindrical coordinates (the 'direction θ' is the direction in which the point specified by (r, θ, z) moves when θ is increased, r and z being kept constant, etc., see Fig. 91 (a)) are

$$\left(\frac{\partial V}{\partial r}, \quad \frac{1}{r}\frac{\partial V}{\partial \theta}, \quad \frac{\partial V}{\partial z}\right).$$

Show that the components of grad V in the directions r, θ, ϕ of spherical polar coordinates (cf. Fig. 91 (b)) are

$$\left(\frac{\partial V}{\partial r}, \quad \frac{1}{r}\frac{\partial V}{\partial \theta}, \quad \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\right).$$

X

BOUNDARY VALUE PROBLEMS

78. Introductory

THE problems of dynamics and electric circuit theory are initial value problems in which we have to find the solution of a set of differential equations which is valid for all times $t > 0$ and which satisfies certain initial conditions at $t = 0$.

In this chapter we study some boundary value problems for ordinary linear differential equations in which we have to find a solution of a differential equation which is valid in a region, say $0 < x < l$, and which has to satisfy conditions at both ends of the region. The region may be the infinite one $x > 0$, but in this case there will be conditions to be satisfied as $x \to \infty$ (e.g. that the solution remain finite), whereas in initial value problems there are no conditions on the behaviour of the solution as $t \to \infty$: this is determined solely by the differential equation and the conditions at $t = 0$.

In problems involving partial differential equations, Chapter XIII, it will appear that in many cases these have to be solved with both initial and boundary conditions.

In this chapter we shall discuss the boundary value problems arising in the theory of the deflexion of beams. These illustrate very well the new types of phenomenon which arise. In §§ 79, 80 a brief discussion of the differential equation and boundary conditions is given, and in the subsequent sections various types of boundary value problem arising from them are studied.

79. Bending moment and shear force

We shall usually consider beams which when undeflected are straight and horizontal. The x-axis will be taken along the beam, and the y-axis vertically downwards† so that deflexions and forces are positive when in a downward direction. Usually

† This corresponds to using left-handed axes which is not very desirable, but it is obviously convenient to have deflexion and load positive when measured in their commonest direction. Right-handed axes with OY upwards are also used.

the beam will be of length l and will be supported in some way at its ends $x = 0$ and $x = l$.

We assume that all forces on the beam act in a vertical direction: the extension to horizontal forces is made in § 86. The loads on the beam may be of two types: (i) concentrated loads W applied at definite points (the reactions at the supports come into this category), and (ii) distributed loads w per unit length, where w is a prescribed function of x.

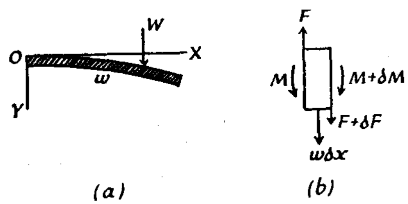


FIG. 66.

We now define two fundamental quantities, shear force and bending moment.

The *shear force* F at a point x of the beam is the resultant of all the forces on the beam to the right of the point x, measured positively in the direction OY. The shear force is thus discontinuous at a concentrated load: this fact will often be used later to calculate the reactions at the supports of a beam.

The *bending moment* M at a point x of the beam is the sum of the moments about x (in the direction from OX to OY) of all the forces to the right of x.

All the forces to the right of the point x are then statically equivalent to a force F and a couple M applied at x.

If the beam has a distributed load w in a region, there are important relations connecting w, F, and M. To derive these, consider the equilibrium of the element of length δx of the beam between x, where the shear force and bending moment are F and M, and $x+\delta x$, where they are $F+\delta F$ and $M+\delta M$. The load on the element is $w\,\delta x$. The forces on the element are shown in Fig. 66 (b); the forces on the beam to the right of $x+\delta x$ exert force $F+\delta F$ and couple $M+\delta M$ at $x+\delta x$ on the element, while, since the forces to the right of x exert force F and couple M

at x, the portion of the beam to the left of x exerts equal and opposite reactions on the element. The conditions of equilibrium give, neglecting squares of small quantities,

$$\delta F + w\,\delta x = 0,$$

$$\delta M + F\,\delta x = 0.$$

In the limit as $\delta x \to 0$ these become

$$\frac{dF}{dx} = -w, \tag{1}$$

$$\frac{dM}{dx} = -F, \tag{2}$$

$$\frac{d^2M}{dx^2} = w. \tag{3}$$

These hold in any portion of a beam free from concentrated loads.

The graphs of F and M against x are known as shear force and bending moment diagrams. In simple cases these may be calculated by the methods of pure statics (for example, when a beam is freely hinged at its ends at the same level), but in more complicated cases, such as a beam with its ends clamped or a beam which runs continuously over several supports, the reactions at the supports and thus the shear and bending moment diagrams cannot be determined by statics alone since the elastic properties of the beam enter into the problem. It will be seen in § 80 that a knowledge of M is sufficient for the simpler design problems of engineering, and it appears that in many cases the theory of deflexion of beams has to be used to calculate M although the deflexion itself is not of great interest to engineers.

Finally it should be remarked that, since the equations of this section and the next are linear, the principle of superposition holds, that is, the shear, bending moment, deflexion, etc., of a beam due to a number of superposed loads are the sum of the values for the separate loads. In particular, if a beam carries a distributed and several concentrated loads, we may make calculations for the distributed and concentrated loads separately and add the results.

In Fig. 67 bending moment and shear force diagrams for four cases are shown. In Figs. 67 (*a*) and (*b*) the beam is freely hinged at its ends, $x = 0$ and $x = l$; it carries a concentrated load W at $x = a$ in Fig. 67 (*a*), and a uniform load w per unit length in Fig. 67 (*b*). Calculations for these cases are made in Exs. 1 and 2 below. In Fig. 67 (*d*) and (*c*) the corresponding

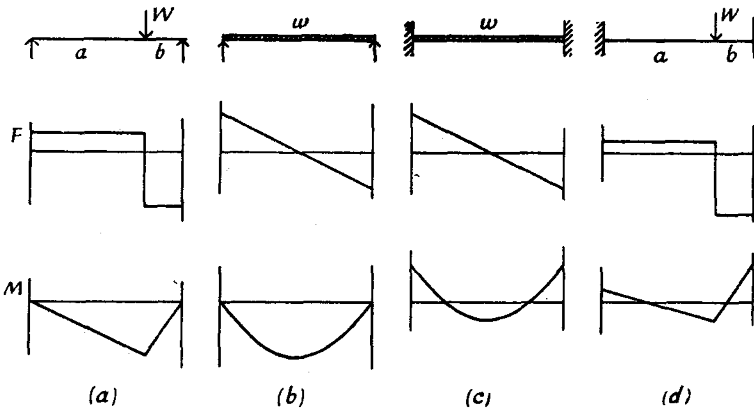


FIG. 67.

cases for a beam which is clamped horizontally at its ends at the same level are given—these cannot be calculated by the methods of the present section and are discussed in §§ 82 and 81, respectively.

Ex. 1. *The beam of length l is freely hinged at its ends and carries a concentrated load W at $x = a$.*

Taking moments about the points $x = l$ and $x = 0$ gives for the reactions R_1 and R_2 at the supports

$$R_1 = W(l-a)/l, \qquad R_2 = Wa/l. \tag{4}$$

The shear is

$$F = -R_2 = -Wa/l \qquad (a < x < l),$$
$$F = W-R_2 = W(l-a)/l \quad (0 < x < a).$$

The bending moment is

$$M = -R_2(l-x) = -Wa(l-x)/l \qquad (a < x < l),$$
$$M = W(a-x)-R_2(l-x) = -W(l-a)x/l \quad (0 < x < a).$$

F and M are graphed in Fig. 67 (*a*).

Ex. 2. *The beam of length l is freely hinged at its ends and carries a uniform load w per unit length.*

Clearly we could determine the reactions, and proceed as in Ex. 1. As an alternative which is useful for distributed loads, we use (3) and the fact that $M = 0$ at the ends $x = 0$ and $x = l$ where the beam is freely hinged. We have

$$\frac{d^2M}{dx^2} = w. \tag{5}$$

Integrating,
$$\frac{dM}{dx} = wx+A, \tag{6}$$

where A is an unknown constant. Integrating (6) gives

$$M = \tfrac{1}{2}wx^2+Ax+B. \tag{7}$$

The conditions $M = 0$ when $x = 0$ and $x = l$ give

$$B = 0, \qquad A = -\tfrac{1}{2}wl,$$

and finally
$$M = -\tfrac{1}{2}wx(l-x). \tag{8}$$

Then by (2)
$$F = \tfrac{1}{2}w(l-2x). \tag{9}$$

F and M are shown in Fig. 67 (b).

80. The differential equation for the deflexion of a beam

To calculate the deflexion we need further information which comes from the theory of elasticity. We consider a portion of the beam and suppose it to be bent to a large radius of curvature by couples M applied to its ends, Fig. 68 (a).

The beam is regarded as being composed of fibres which exert no influence on their neighbours, and it is assumed that a plane section of the beam remains plane after bending. Let AB and $A'B'$ be two sections of the beam which before bending were perpendicular to its direction, and which after bending intersect at a small angle θ in a line through C, parallel to the direction of the couples M and perpendicular to the plane of the paper in Fig. 68 (a). The fibres at AA' will have been extended and those at BB' will have been compressed, so there must be some intermediate fibres OO' whose length is unchanged: the surface in the beam containing these fibres is called the *neutral axis*.

Suppose Fig. 68 (b) is a section through the beam in the plane through AB parallel to the direction of the couples M, and suppose that we take OX and OY as axes in this plane, OX being through O, Fig. 68 (a), and parallel to the direction of the couples M. Then all fibres in the line OX will have their lengths unchanged.

Let R be the radius of curvature of the fibres OO' of the neutral axis, so that $OO' = R\theta$, and this is the unstretched length of all fibres between the planes AB and $A'B'$. Now

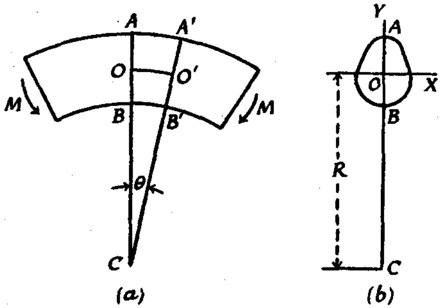


FIG. 68.

consider the fibre at (x, y) in the plane XOY. The stretched length is $(R+y)\theta$ so that its extension is $y\theta$. Therefore, if E is Young's modulus, the tension in it is

$$E\frac{y\theta}{R\theta} = \frac{Ey}{R}. \tag{1}$$

The forces across the cross-section may be determined by combining the effects (1) of the individual fibres.

The total force across the cross-section in the direction of the beam is

$$\frac{E}{R}\iint y\,dydx, \tag{2}$$

and since by hypothesis this is zero we must have

$$\iint y\,dxdy = 0, \tag{3}$$

and thus the 'centre of gravity' of the cross-section must lie in the neutral axis OX.

The sum of the moments about OX of the forces across the cross-section is

$$\frac{E}{R}\iint y^2\,dydx = \frac{EI}{R}, \tag{4}$$

where I is the 'moment of inertia' of the cross-section about OX. Since the sum of the moments must be equal to M, this gives the fundamental relation

$$\frac{EI}{R} = M. \tag{5}$$

Finally, the sum of the moments about OY of the forces across the cross-section is

$$\frac{E}{R}\iint xy\,dxdy. \tag{6}$$

This vanishes if OX is an axis of symmetry. Now the position of O in OX has never been defined, so that if the cross-section has a vertical axis of symmetry we may take this to be OY and again (6) vanishes. If the cross-section has neither a horizontal nor a vertical axis of symmetry, (6) does not vanish. Now there is no couple on the beam about OY, so for the theory to hold (6) should be zero: in the unsymmetrical cases the couple (6) causes the beam to twist and the simple theory is inadequate.

If y_0 is the greatest distance of any fibre from the neutral axis, and f_0 is the stress in this extreme fibre, we have from (1) and (5)

$$f_0 = \frac{Ey_0}{R} = \frac{My_0}{I}, \tag{7}$$

and so f_0 is determined in terms of M. A knowledge of f_0 is required for the design of beams.

The theory above is known as the Bernoulli–Euler theory of flexure and, while it is approximate, its results are sufficiently near to those deduced from the accurate theory of elasticity for most practical purposes.

(5) gives the radius of curvature of a beam bent by couples M applied to its ends, so that M is constant along the beam. In an actual beam in which M is a varying function of x, (5) will give the radius of curvature at a point of the beam in terms of the bending moment M at that point. The radius of curvature R is given in terms of the deflexion by

$$\frac{1}{R} = \frac{d^2y/dx^2}{\{1+(dy/dx)^2\}^{\frac{3}{2}}}, \tag{8}$$

and in the problems we shall consider the slope of the beam is small and $(dy/dx)^2$ negligible so that (8) may be replaced by

$$\frac{1}{R} = \frac{d^2y}{dx^2}. \tag{9}$$

For problems involving large deflexions the complete expression (8) must be used and an awkward non-linear equation results, which, however, can be solved exactly in some simple cases (cf. Ex. 21, p. 302).

From (9) and (5) we get finally

$$EI\frac{d^2y}{dx^2} = M, \tag{10}$$

which is the differential equation for the deflexion. E and I are known, and M is supposed to have been determined by the methods of § 79.

For a distributed load w, (10) may be combined with § 79 (3) to give

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = w, \tag{11}$$

or, if E and I are independent of x,

$$EI\frac{d^4y}{dx^4} = w. \tag{12}$$

The differential equation to be solved for y is thus one of (10) to (12). The connexion between the bending moment and deflexion is given by (10), and, at points free from concentrated loads, the shear F is by § 79 (2)

$$F = -EI\frac{d^3y}{dx^3}, \tag{13}$$

provided E and I are independent of x.

The differential equations have to be solved with boundary conditions depending on the nature of the supports at the ends. The most common such conditions are:

(i) *A freely hinged end.* Here y is prescribed (usually zero) and $M = 0$. That is, by (10),

$$\frac{d^2y}{dx^2} = 0, \qquad y \text{ given}. \tag{14}$$

(ii) *A clamped end.* Here y and dy/dx are prescribed (both usually zero). That is

$$\frac{dy}{dx}\text{ given,} \qquad y \text{ given.} \tag{15}$$

(iii) *A free end.* Here $M = 0$, and $F = 0$ unless there is a concentrated load at the end. That is, by (10) and (13),

$$\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0. \tag{16}$$

If there is a concentrated load at the end, F is prescribed.

(iv) *An elastic support.* Suppose the beam is freely hinged and supported by a reaction k times the deflexion y. Since the shear at the end is $-R$, where R is the reaction, the boundary conditions are

$$M = 0, \qquad R = ky, \tag{17}$$

or

$$\frac{d^2y}{dx^2} = 0, \qquad EI\frac{d^3y}{dx^3} - ky = 0. \tag{18}$$

81. Distributed loads

Problems of distributed loads on uniform beams are usually best solved by integrating § 80 (12) with the appropriate boundary conditions.

As an example we consider first *a uniform beam of length l, carrying a uniform load w per unit length, which is freely hinged at $x = 0$ and $x = l$.*

Writing D for d/dx we have to solve

$$EID^4y = w, \tag{1}$$

with, by § 80 (14)

$$y = D^2y = 0, \quad \text{when } x = 0 \text{ and } x = l.$$

Integrating (1) gives

$$EID^3y = wx + A,$$

where A is an unknown constant. Integrating again,

$$EID^2y = \tfrac{1}{2}wx^2 + Ax + B, \tag{2}$$

and since $D^2y = 0$ when $x = 0$, $B = 0$, so that (2) becomes

$$EID^2y = \tfrac{1}{2}wx^2 + Ax. \tag{3}$$

Integrating this gives

$$EIDy = \tfrac{1}{6}wx^3+\tfrac{1}{2}Ax^2+C, \tag{4}$$

where C is an unknown constant, and integrating again we get

$$EIy = \tfrac{1}{24}wx^4+\tfrac{1}{6}Ax^3+Cx+H. \tag{5}$$

The condition $y = 0$ when $x = 0$ requires $H = 0$. The remaining constants A and C are found from the conditions $y = D^2y = 0$ when $x = l$. These require

$$\tfrac{1}{24}wl^4+\tfrac{1}{6}Al^3+Cl = 0,$$
$$\tfrac{1}{2}wl+A = 0.$$

Solving for A and C and substituting in (5) we get finally

$$y = \frac{wx}{24EI}(x^3-2lx^2+l^3). \tag{6}$$

Ex. 1. *A uniform beam of length l, carrying a uniform load w per unit length, is clamped horizontally at the same level at its ends $x = 0$ and $x = l$.*

Here we have to solve (1) with the conditions

$$y = Dy = 0, \quad \text{when } x = 0, \tag{7}$$
$$y = Dy = 0, \quad \text{when } x = l. \tag{8}$$

Integrating (1) four times as before and using (7), by virtue of which the additive constants of the last two integrations vanish, we get

$$EIy = \tfrac{1}{24}wx^4+\tfrac{1}{6}Ax^3+\tfrac{1}{2}Bx^2, \tag{9}$$

where A and B are arbitrary constants. The conditions (8) then give

$$\tfrac{1}{24}wl^4+\tfrac{1}{6}Al^3+\tfrac{1}{2}Bl^2 = 0,$$
$$\tfrac{1}{6}wl^3+\tfrac{1}{2}Al^2+Bl = 0.$$

Therefore $$A = -\tfrac{1}{2}wl, \qquad B = \tfrac{1}{12}wl^2, \tag{10}$$

and so $$y = \frac{wx^2(l-x)^2}{24EI}. \tag{11}$$

We can now find the bending moment and shear force at any point of the beam; as remarked in § 79, this cannot be done from purely statical considerations.

The bending moment is

$$\begin{aligned} M &= EID^2y \\ &= \tfrac{1}{2}wx^2+Ax+B \\ &= \tfrac{1}{12}wl^2-\tfrac{1}{2}wx(l-x). \end{aligned} \tag{12}$$

The shear force at any point is by § 80 (13)

$$\begin{aligned} F &= -EID^3y \\ &= -wx-A \\ &= w(\tfrac{1}{2}l-x). \end{aligned} \tag{13}$$

When $x = l$, $F = -\frac{1}{2}wl$, and therefore the reaction at l is $\frac{1}{2}wl$: this could have been inferred by symmetry and (13) deduced. Bending moment and shear force diagrams for this problem are shown in Fig. 67 (c).

Ex. 2. *A uniform beam of length l, carrying a uniform load w per unit length, is clamped horizontally at $x = 0$, and at $x = l$ it is freely hinged to a yielding support which provides reaction k times the deflexion.*

We have to solve (1) with $y = Dy = 0$ when $x = 0$, and, by § 80 (17) and (18),

$$D^2y = 0, \quad EID^3y - ky = 0, \qquad \text{when } x = l. \tag{14}$$

As in Ex. 1, (9) we find

$$EIy = \tfrac{1}{24}wx^4 + \tfrac{1}{6}Ax^3 + \tfrac{1}{2}Bx^2, \tag{15}$$

where A and B are to be determined from (14). This requires

$$\tfrac{1}{2}wl^2 + Al + B = 0,$$

$$EI(wl + A) - k(\tfrac{1}{24}wl^4 + \tfrac{1}{6}Al^3 + \tfrac{1}{2}Bl^2) = 0.$$

Solving these for A and B and substituting in (15) gives y.

Ex. 3. *A cantilever of length l has its end $x = 0$ clamped horizontally and its end $x = l$ free. It carries a uniform load w per unit length. Its cross-section is constant in $0 < x < a$, and has a different constant value in $a < x < l$.*

Suppose that

$$\frac{1}{EI} = \alpha \qquad (0 < x < a),$$

$$\frac{1}{EI} = \alpha + \beta \qquad (a < x < l).$$

These may be written as in § 17

$$\frac{1}{EI} = \alpha + \beta H(x - a). \tag{16}$$

Since I is variable, the differential equation must be taken in the form § 80 (10), that is

$$D^2y = \{\alpha + \beta H(x-a)\}M, \tag{17}$$

where the bending moment M, calculated as in § 79, is

$$M = \tfrac{1}{2}w(l-x)^2. \tag{18}$$

Using (18) in (17) we have to solve

$$\begin{aligned} D^2y &= \tfrac{1}{2}\alpha w(l-x)^2 + \tfrac{1}{2}\beta w(l-x)^2H(x-a) \\ &= \tfrac{1}{2}\alpha w(l-x)^2 + \tfrac{1}{2}\beta w\{(x-a)^2 + 2(x-a)(a-l) + (l-a)^2\}H(x-a), \end{aligned} \tag{19}$$

with $y = Dy = 0$ when $x = 0$.

Integrating (19), using § 17 (5) for the integral of the Heaviside function, gives

$$\begin{aligned} Dy = {}& \tfrac{1}{2}\alpha w(l^2x - lx^2 + \tfrac{1}{3}x^3) + \\ & + \tfrac{1}{2}\beta w\{\tfrac{1}{3}(x-a)^3 + (x-a)^2(a-l) + (x-a)(l-a)^2\}H(x-a), \end{aligned}$$

where the additive constant is zero since $Dy = 0$ when $x = 0$.

Integrating again, and using $y = 0$ when $x = 0$, we get finally

$$y = \tfrac{1}{2}\alpha w(\tfrac{1}{2}l^2x^2 - \tfrac{1}{3}lx^3 + \tfrac{1}{12}x^4) +$$
$$+\tfrac{1}{2}\beta w\{\tfrac{1}{12}(x-a)^4 + \tfrac{1}{3}(x-a)^3(a-l) + \tfrac{1}{2}(x-a)^2(l-a)^2\}H(x-a)$$
$$= \frac{\alpha w x^2}{24}(6l^2 - 4lx + x^2) +$$
$$+\frac{\beta w(x-a)^2}{24}\{6(l-a)^2 - 4(l-a)(x-a) + (x-a)^2\}H(x-a). \quad (20)$$

The use of the Heaviside function in this way avoids the necessity of treating the two parts $0 < x < a$ and $a < x < l$ of the beam separately. It may be used in the same way when the load changes discontinuously at a point.

82. Concentrated loads. The Green's function

Concentrated loads are a little more complicated to study than distributed loads. To illustrate the difficulties and the new ideas involved we consider in detail the case of a light uniform beam $0 < x < l$, clamped horizontally at the same level at its ends, and carrying a concentrated load W at $x = a$.

As in § 81, Ex. 1, we cannot determine the bending moment and shear by statical considerations, so we have to use the differential equation § 80 (12), but now the two regions

$$0 < x < a \quad \text{and} \quad a < x < l$$

must be treated separately.

Since we are neglecting the weight of the beam, $w = 0$ in § 80 (12) and this becomes

$$D^4y = 0 \quad (0 < x < a), \tag{1}$$

with

$$y = Dy = 0, \quad \text{when } x = 0. \tag{2}$$

The solution of (1) satisfying (2), found as in § 81, is

$$y = \tfrac{1}{6}Ax^3 + \tfrac{1}{2}Bx^2, \tag{3}$$

where A and B are unknown constants.

Also in $a < x < l$ the differential equation is

$$D^4y = 0 \quad (a < x < l), \tag{4}$$

with

$$y = Dy = 0, \quad \text{when } x = l. \tag{5}$$

The solution of (4) which satisfies (5) is

$$y = \tfrac{1}{6}C(l-x)^3 + \tfrac{1}{2}H(l-x)^2, \tag{6}$$

where C and H are unknown constants. The four unknowns

A, B, C, H are to be found from the conditions at $x = a$. Firstly, at $x = a$ the values of the deflexion y, slope Dy, and bending moment $M = EID^2y$ of the beam, calculated from (3), must be equal to their values calculated from (6). These conditions give

$$\tfrac{1}{6}Aa^3+\tfrac{1}{2}Ba^2 = \tfrac{1}{6}C(l-a)^3+\tfrac{1}{2}H(l-a)^2, \tag{7}$$

$$\tfrac{1}{2}Aa^2+Ba = -\tfrac{1}{2}C(l-a)^2-H(l-a), \tag{8}$$

$$Aa+B = C(l-a)+H. \tag{9}$$

Also, as we pass through the point $x = a$ from right to left, the shear force F increases by W. And, by § 80 (13),

$$F = -EID^3y. \tag{10}$$

Thus the value of (10) calculated from (3) when $x = a$ must be greater by W than the value calculated from (6), that is

$$-EIA = EIC+W. \tag{11}$$

(7), (8), (9), and (11) are four equations for A, B, C, H. Solving we get

$$A = -\frac{W(l-a)^2(l+2a)}{EIl^3}, \qquad B = \frac{Wa(l-a)^2}{EIl^2},$$

and $$y = \frac{Wx^2(l-a)^2}{6EIl^3}\{3al-x(l+2a)\} \quad (0 < x < a). \tag{12}$$

In the same way, solving for C and H we find

$$y = \frac{Wa^2(l-x)^2}{6EIl^3}\{(3l-2a)x-la\} \quad (a < x < l). \tag{13}$$

(13) may be obtained from (12) by interchanging a and x.

The bending moment $M = EID^2y$ is given by

$$\left.\begin{aligned} M &= \frac{W(l-a)^2}{l^3}\{al-x(l+2a)\} \quad (0 < x < a) \\ M &= \frac{Wa^2}{l^3}\{al-2l^2+x(3l-2a)\} \quad (a < x < l) \end{aligned}\right\}. \tag{14}$$

The shear force $F = -EID^3y$ is given by

$$\left.\begin{aligned} F &= \frac{W(l-a)^2(l+2a)}{l^3} \quad (0 < x < a) \\ F &= -\frac{Wa^2(3l-2a)}{l^3} \quad (a < x < l) \end{aligned}\right\}. \tag{15}$$

Bending moment and shear force diagrams for this problem are shown in Fig. 67 (*d*).

The solution for a concentrated load with given boundary conditions is of fundamental theoretical importance, for by using it the solution for any distributed load with the same boundary conditions can be written down immediately. Write $G(a,x)$ for the deflexion at x due to a unit concentrated load at $x = a$ given by (12) and (13): the deflexion of the same beam with the same boundary conditions and a distributed load $w(x)$ may be obtained by superposing the effects of concentrated loads $w(a)\,\delta a$ in the region $a < x < a+\delta a$, so that it is

$$\int_0^l G(a,x)w(a)\,da. \tag{16}$$

The same remark applies to any other boundary conditions. $G(a,x)$ is called the Green's function for the given equation and boundary conditions, and these results are special cases of a very general theory discussed further in § 102. It may be remarked that it follows from this theory that the result $G(x,a) = G(a,x)$ which appeared in (12) and (13) is true in general: from the present point of view this states that the deflexion at a due to a load W at x is equal to the deflexion at x due to a load W at a.

83. Concentrated loads. Use of the δ function

In § 17 it was remarked that a concentrated load W at $x = a$ could be treated as a distributed load

$$w = W\,\delta(x-a). \tag{1}$$

We now discuss the problem of § 82 in this way. We have to solve

$$EID^4y = W\,\delta(x-a) \quad (0 < x < l), \tag{2}$$

with

$$y = Dy = 0, \quad \text{when } x = 0 \text{ and } x = l. \tag{3}$$

Integrating (2) and using § 17 (10) gives

$$EID^3y = WH(x-a)+A, \tag{4}$$

where A is an arbitrary constant. Integrating (4), using § 17 (4), gives

$$EID^2y = W(x-a)H(x-a)+Ax+B. \tag{5}$$

Integrating (5), using § 17 (5), we get

$$EIDy = \tfrac{1}{2}W(x-a)^2H(x-a)+\tfrac{1}{2}Ax^2+Bx+C. \tag{6}$$

The constant C must be zero since (3) requires $Dy = 0$ when $x = 0$, and $H(x-a) = 0$ when $x = 0$.

Finally, integrating again gives

$$EIy = \tfrac{1}{6}W(x-a)^3H(x-a)+\tfrac{1}{6}Ax^3+\tfrac{1}{2}Bx^2, \tag{7}$$

the arbitrary constant being zero by (3).

The conditions $y = Dy = 0$ when $x = l$ require by (6) and (7)

$$\tfrac{1}{2}W(l-a)^2+\tfrac{1}{2}Al^2+Bl = 0,$$

$$\tfrac{1}{6}W(l-a)^3+\tfrac{1}{6}Al^3+\tfrac{1}{2}Bl^2 = 0.$$

Therefore

$$A = -\frac{W(l-a)^2(l+2a)}{l^3}, \qquad B = \frac{Wa(l-a)^2}{l^2}.$$

And, finally,

$$EIy = \tfrac{1}{6}W(x-a)^3H(x-a)+\frac{Wx^2(l-a)^2}{6l^3}\{3al-(l+2a)x\}, \tag{8}$$

or, writing out the function $H(x-a)$ explicitly,

$$EIy = \frac{Wx^2(l-a)^2}{6l^3}\{3al-(l+2a)x\} \quad (0 < x < a), \tag{9}$$

$$EIy = \frac{Wa^2(l-x)^2}{6l^3}\{(3l-2a)x-la\} \quad (a < x < l). \tag{10}$$

These are the results § 82 (12) and (13).

84. The beam on an elastic foundation

Suppose that deflexion of the beam in the direction OY of Fig. 66 (a) is resisted by a force ky per unit length of the beam,† and that the beam carries a load w per unit length. The effect of the elastic foundation is to add a term $-ky$ to the load, and, for a uniform beam in a region free from concentrated loads, the differential equation § 80 (12) for the deflexion becomes

$$EI\frac{d^4y}{dx^4}+ky = w. \tag{1}$$

† This implies that the support exerts a tension if y becomes negative. Frequently this is not the case.

Writing $$4\omega^4 = k/EI, \tag{2}$$
this may be written
$$(D^4+4\omega^4)y = \frac{w}{EI}. \tag{3}$$

As an example we consider a uniform semi-infinite beam $x > 0$ resting on such a foundation with the end $x = 0$ freely hinged at zero deflexion.

The boundary conditions are
$$y = D^2y = 0, \quad \text{when } x = 0, \tag{4}$$
$$y \text{ to remain finite as } x \to \infty. \tag{5}$$

As in § 13, Ex. 5, the general solution of (3) with constant w is
$$y = e^{\omega x}(A\cos\omega x+B\sin\omega x)+$$
$$+e^{-\omega x}(C\cos\omega x+H\sin\omega x)+w/k. \tag{6}$$

In order that y may remain finite as $x \to \infty$ we must have $A = B = 0$.

The conditions at $x = 0$ give
$$C+w/k = 0,$$
$$H = 0,$$
and thus the solution is
$$y = \frac{w}{k}\{1-e^{-\omega x}\cos\omega x\}. \tag{7}$$

85. A continuous beam resting on several supports at the same level

In the other sections of this chapter the beams have been assumed to be supported at two points only. The problem of a uniform continuous beam resting on a number of frictionless supports at the same level and carrying a uniformly distributed load w per unit length can also be treated fairly simply.

Let A, B, C,... be consecutive supports and let $AB = a$, $BC = b$. Take the origin at B with the x-axis along BC. Then in BC we have to solve
$$EID^4y = w, \tag{1}$$
with $$y = 0 \quad \text{when } x = 0 \text{ and when } x = b. \tag{2}$$

Integrating (1) four times in the usual way and using $y = 0$ when $x = 0$ we get

$$EIy = \tfrac{1}{24}wx^4+\tfrac{1}{6}Px^3+\tfrac{1}{2}Qx^2+Rx, \tag{3}$$

where P, Q, and R are arbitrary constants. The condition $y = 0$ when $x = b$ gives

$$\tfrac{1}{24}wb^4+\tfrac{1}{6}Pb^3+\tfrac{1}{2}Qb^2+Rb = 0. \tag{4}$$

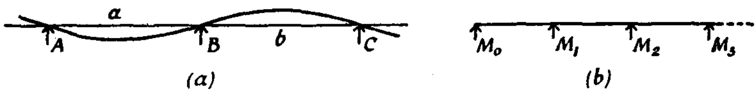


FIG. 69.

Also, if M is the bending moment at x,

$$M = EID^2y = \tfrac{1}{2}wx^2+Px+Q. \tag{5}$$

Now let M_A, M_B, M_C,... be the unknown bending moments at the supports. Putting $x = 0$ and $x = b$ in (5) gives

$$M_B = Q, \tag{6}$$

$$M_C = \tfrac{1}{2}wb^2+Pb+Q. \tag{7}$$

Also, if i_B is the slope of the beam at $x = 0$,

$$\begin{aligned} EIi_B &= R, \\ &= -\tfrac{1}{24}wb^3-\tfrac{1}{6}Pb^2-\tfrac{1}{2}Qb, \\ &= \tfrac{1}{24}wb^3-\tfrac{1}{6}bM_C-\tfrac{1}{3}bM_B, \end{aligned} \tag{8}$$

where we have used (4), and then (6) and (7).

Now suppose, taking B as origin again, that we take the x-axis in the direction BA. The bending moments calculated in this way will be equal to those calculated with the axis in the other direction, so that (8) still holds with the appropriate changes of notation, *except* that, as can be seen from Fig. 69 (a), the sign of i_B is changed. Therefore we get from (8)

$$-EIi_B = \tfrac{1}{24}wa^3-\tfrac{1}{6}aM_A-\tfrac{1}{3}aM_B. \tag{9}$$

Adding (8) and (9) gives finally

$$aM_A+2(a+b)M_B+bM_C = \tfrac{1}{4}w(a^3+b^3). \tag{10}$$

This is Clapeyron's *theorem of three moments.* It is an algebraic equation connecting the bending moments at three consecutive supports. Using it and the conditions at the first and last of the supports, sufficient equations can be written down to determine the bending moments at all the supports. Since P, Q, and R in (3) can be expressed in terms of the bending moments by (4), (6), and (7), the deflexion at any point of the beam then follows. The reaction at the support B, which is the difference between the shears to the right and left of B, is

$$\tfrac{1}{2}w(a+b)+\left(\frac{1}{a}+\frac{1}{b}\right)M_B-\frac{1}{a}M_A-\frac{1}{b}M_C. \tag{11}$$

(10) and (11) and their many generalizations to more complicated systems are of the greatest importance in practice.

Ex. *A uniform semi-infinite beam rests on supports at*

$$x = na \quad (n = 0, 1,...).$$

Let M_n be the bending moment at the nth support, Fig. 69 (*b*), then (10) gives

$$M_{n+2}+4M_{n+1}+M_n = \tfrac{1}{2}wa^2, \tag{12}$$

to be solved with

$$M_0 = 0, \tag{13}$$

and the condition that M_n remains finite as $n \to \infty$.

We seek a solution of (12) of the form

$$M_n = A+Bk^n. \tag{14}$$

Substituting (14) in (12) we must have

$$A = \tfrac{1}{12}wa^2, \tag{15}$$

$$k^2+4k+1 = 0. \tag{16}$$

The solution of (16) is

$$k = -2\pm\sqrt{3}. \tag{17}$$

For M_n to remain finite as $n \to \infty$ we must have $|k| < 1$, and thus must choose the value $-2+\sqrt{3}$ of (17). Therefore a solution of (12) is

$$M_n = \frac{wa^2}{12}+B(-2+\sqrt{3})^n. \tag{18}$$

The condition $M_0 = 0$ gives B, and we get finally

$$M_n = \frac{wa^2}{12}\{1-(-2+\sqrt{3})^n\}. \tag{19}$$

86. A beam with transverse loads and axial tension or compression

Suppose that a uniform beam lies along the x-axis, as in Fig. 70, and is freely hinged at its ends $x = 0$ and $x = l$. Suppose that it carries a uniformly distributed load w per unit length in the direction of OY, and that in addition there is tension T along the x-axis.

We take the differential equation in the form § 80 (10),

$$EID^2y = M, \tag{1}$$

to be solved with $y = 0$ when $x = 0$ and $x = l$. The bending moment M at the point x contains a term Ty, due to the axial tension, as well as the term $-\frac{1}{2}wx(l-x)$, calculated as in § 79 (8) transverse load.

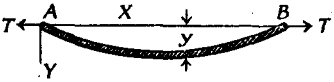


FIG. 70.

Therefore $$M = Ty - \tfrac{1}{2}wx(l-x),$$

and the differential equation (1) becomes

$$D^2y - \frac{T}{EI}y = -\frac{w}{2EI}x(l-x). \tag{2}$$

The general solution of this, found as in § 14 (ii), is

$$y = A\sinh\alpha x + B\sinh\alpha(l-x) - \frac{w}{\alpha^2 T} + \frac{wlx}{2T} - \frac{wx^2}{2T}, \tag{3}$$

where $$\alpha^2 = T/EI. \tag{4}$$

The conditions $y = 0$ when $x = 0$ and $x = l$ give

$$B\sinh\alpha l = w/\alpha^2 T,$$

$$A\sinh\alpha l = w/\alpha^2 T.$$

Therefore, finally,

$$y = \frac{w\{\sinh\alpha x + \sinh\alpha(l-x)\}}{\alpha^2 T\sinh\alpha l} - \frac{w}{\alpha^2 T} + \frac{wx(l-x)}{2T}$$

$$= \frac{w}{\alpha^2 T}\left\{\frac{\cosh\alpha(\frac{1}{2}l-x)}{\cosh\frac{1}{2}\alpha l} - 1\right\} + \frac{wx(l-x)}{2T}. \tag{5}$$

Other cases are treated similarly. If there is axial compression instead of tension, trigonometrical functions appear in place of the hyperbolic functions.

87. Column formulae. Eigenvalue problems

In the theory of columns we consider a beam with axial compression and no transverse loading.

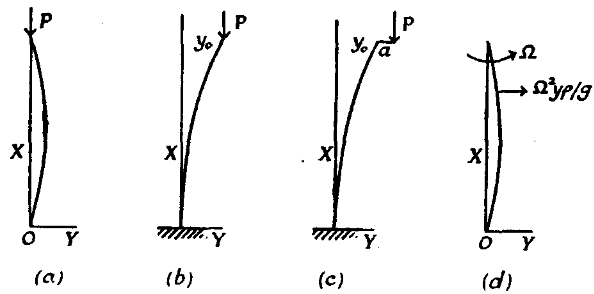


FIG. 71.

The most important case is that of a column of length l, freely hinged at its ends, with axial compression P, Fig. 71 (a). The bending moment M at the point x of the column is $-Py$, as in § 86, and the differential equation for the deflexion becomes

$$EID^2y = -Py,$$

or
$$(D^2+\omega^2)y = 0, \tag{1}$$

where
$$\omega^2 = P/EI. \tag{2}$$

This has to be solved with $y = D^2y = 0$, when $x = 0$ and $x = l$.

The general solution of (1) is

$$y = A\sin\omega x+B\cos\omega x. \tag{3}$$

The condition $y = D^2y = 0$ when $x = 0$ requires $B = 0$, so that (3) becomes

$$y = A\sin\omega x, \tag{4}$$

and the condition $y = D^2y = 0$ when $x = l$ requires

$$A\sin\omega l = 0. \tag{5}$$

Thus we must have *either* $A = 0$, in which case the solution is $y = 0$ for all x and the beam is undeflected, *or*

$$\sin\omega l = 0, \tag{6}$$

that is,
$$\omega l = n\pi \quad (n = 1, 2,...). \tag{7}$$

Using (2), (7) gives

$$P = \frac{n^2\pi^2 EI}{l^2} \quad (n = 1, 2,...). \tag{8}$$

Thus, unless P has one of the values (8), the only solution of the differential equation and boundary conditions is $y = 0$ for all x. If P has the value $n^2\pi^2 EI/l^2$, another solution is

$$y = A \sin\frac{n\pi x}{l}, \tag{9}$$

where A is arbitrary.

From the practical point of view this result may be interpreted in the following way: if $P < \pi^2 EI/l^2$, the first of the values (8), the only solution is $y = 0$ and the column is undeflected. Further discussion shows that this solution is stable, that is, if the column is slightly bent it will return to the straight form. When P reaches the value $\pi^2 EI/l^2$, another solution also becomes possible in which the form of the column is $A \sin \pi x/l$, and now this solution is stable and the undeflected position unstable: thus if the column is slightly disturbed it will assume a bent position and for practical purposes will collapse. The value $\pi^2 EI/l^2$ is known as the Euler load or crippling load, and the above theory is the Euler theory.

From the theoretical point of view the problem is that of a homogeneous linear differential equation which contains a parameter ω^2 and which has to be solved subject to homogeneous boundary conditions. In such problems there is always a trivial solution which is zero for all values of x, and non-zero solutions can be found only for a discrete set of values of the parameter. These values are called the eigenvalues, the corresponding solutions are the eigenfunctions, and such problems are referred to as eigenvalue problems.

Ex. 1. *A column is clamped vertically at its base $x = 0$, and is free at $x = l$.*

Let y_0 be the deflexion at the top of the column (Fig. 71 (b)).

The bending moment M at the point x is $P(y_0 - y)$, so the differential equation for the deflexion is

$$EID^2y = P(y_0 - y),$$

or

$$(D^2 + \omega^2)y = \omega^2 y_0, \tag{10}$$

where ω^2 is defined in (2). (10) has to be solved with $y = Dy = 0$ when $x = 0$, and $y = y_0$ when $x = l$. Its solution which satisfies the conditions at $x = 0$ is

$$y = y_0(1-\cos\omega x). \tag{11}$$

If, in addition, we are to have $y = y_0$ when $x = l$, we must have

$$\cos\omega l = 0,$$

that is

$$\omega l = \frac{(2n-1)\pi}{2} \quad (n = 1, 2,...),$$

or

$$P = \frac{(2n-1)^2\pi^2 EI}{4l^2} \quad (n = 1, 2,...). \tag{12}$$

The crippling load, corresponding to $n = 1$ in (12), is $\pi^2EI/4l^2$.

It is interesting to consider the problem from another point of view. One objection to the Euler theory is that in practice it is impossible to ensure that the load lies precisely along the axis of the column, and it is not clear what effect a small eccentricity of loading may have. We show that if it is allowed for, Euler's result still applies.

Suppose, then, that the load P is applied a small distance a from the axis of the column, Fig. 71 (c). The only change is that (10) has to be replaced by

$$(D^2+\omega^2)y = \omega^2(y_0+a), \tag{13}$$

which has to be solved as before with $y = Dy = 0$, when $x = 0$, and with $y = y_0$ when $x = l$. The solution of (13) for which $y = Dy = 0$ when $x = 0$ is

$$y = (y_0+a)(1-\cos\omega x), \tag{14}$$

and the condition $y = y_0$ when $x = l$ requires

$$y_0 = (y_0+a)(1-\cos\omega l). \tag{15}$$

Solving (15) for y_0 and substituting in (14), gives

$$y = \frac{a(1-\cos\omega x)}{\cos\omega l}. \tag{16}$$

This problem, of course, is not an eigenvalue problem: there is a finite deflexion for any load. But it follows from (16), because of the denominator $\cos\omega l$, that the deflexion becomes very large as P tends to any of the values (12), and this is effectively the same result as before.

Ex. 2. *A column is freely hinged at its ends. Lateral displacement is resisted by a force per unit length equal to k times the displacement.*

In this case we have as in § 86 (1)

$$EID^2y = -Py+M', \tag{17}$$

where M' is the bending moment at x due to the transverse loading $-ky$ at x. Differentiating (17) twice, and using § 79 (3), gives

$$EID^4y+PD^2y+ky = 0. \tag{18}$$

This has to be solved with

$$y = D^2y = 0, \quad \text{when } x = 0, \tag{19}$$

$$y = D^2y = 0, \quad \text{when } x = l. \tag{20}$$

The problem may be treated as above, but the eigenvalues may be found very simply by noting that

$$\sin\frac{n\pi x}{l} \quad (n = 1, 2,...)$$

satisfies (19) and (20), and it also satisfies (18) if

$$\frac{EIn^4\pi^4}{l^4}-\frac{Pn^2\pi^2}{l^2}+k = 0,$$

that is, if
$$P = \frac{EIn^2\pi^2}{l^2}+\frac{kl^2}{n^2\pi^2}. \tag{21}$$

Ex. 3. *Shaft whirling.*

Another simple and important type of eigenvalue problem arises in the following way. Suppose a uniform, straight shaft rotates with angular velocity Ω about its axis and is carried in bearings at $x = 0$ and $x = l$ which may be regarded as free hinges.

Suppose the shaft is in a bent position as in Fig. 71 (*d*). There is a transverse loading of $\Omega^2 y\rho/g$ per unit length on the shaft caused by the centrifugal or reversed effective forces, where ρ is the mass per unit length of the shaft.†

The differential equation for the deflexion y is

$$EID^4y-\Omega^2\rho y/g = 0,$$

or
$$(D^4-k^4)y = 0, \tag{22}$$

where
$$k^4 = \Omega^2\rho/EIg. \tag{23}$$

(22) has to be solved with $y = D^2y = 0$, when $x = 0$ and when $x = l$. Its general solution is

$$y = A\sin kx+B\cos kx+C\sinh kx+D\cosh kx. \tag{24}$$

The boundary condition at $x = 0$ requires

$$B+D = 0,$$

$$B-D = 0,$$

so that $B = D = 0$. The conditions at $x = l$ now give

$$A\sin kl+C\sinh kl = 0,$$

$$A\sin kl-C\sinh kl = 0.$$

Thus we must have either $A = C = 0$, corresponding to the undeflected position, or
$$\sin kl\sinh kl = 0. \tag{25}$$

This gives
$$kl = n\pi \quad (n = 1, 2,...)$$

or
$$\Omega^2 = \frac{n^4\pi^4EIg}{\rho l^4} \quad (n = 1, 2,...). \tag{26}$$

At these angular velocities deflected positions are possible (and the undeflected positions are unstable) and the shaft 'whirls'.

† The units in which the various quantities are measured have not been specified above. In the usual engineering practice gravitational units are used for the loads, and therefore the reversed effective force is divided by g.

EXAMPLES ON CHAPTER X

In all these examples I refers to the moment of inertia of the cross-section of the beam and E is Young's modulus.

1. A beam of length l, freely hinged at its ends, carries a distributed load which increases linearly from zero at the ends to w per unit length at the mid-point of the beam. Sketch the bending moment and shear diagrams, and show that the maximum bending moment in the beam is $wl^2/12$.

2. A uniform beam of length $2l$ and weight w per unit length is rested symmetrically on two supports distant $2a$ apart. Sketch the bending moment and shear diagrams, and discuss the variation of the bending moments at the middle of the beam and at the supports with a. Find the maximum bending moment in the beam, and show that this is least if $a = (2-\sqrt{2})l$.

3. Show that when two unequal concentrated loads a fixed distance apart are traversed along a beam supported at the ends, the maximum bending moment occurs when the heavier load and the centre of gravity of the two loads are equidistant from the centre of the span, provided that both loads are then on the span.

4. Show that the maximum bending moments which can be withstood by beams of square and circular cross-section of the same area are in the ratio $2\sqrt{\pi}/3$.

5. Find the deflexion of a light uniform cantilever clamped horizontally at $x = 0$ and free at $x = l$ for

(i) a uniformly distributed load w per unit length,

(ii) a concentrated load W at $x = l$.

If the cantilever carries a uniformly distributed load w per unit length and is propped at $x = l$ to the same level as that at $x = 0$, show, by combining the two above results, or directly, that the reaction at the prop is $3wl/8$.

6. A beam of length l is freely hinged at the same level at its ends and carries a uniformly distributed load w per unit length. At its mid-point it rests on a yielding support which provides a reaction of k times the deflexion. Show that the reaction at this support is

$$\frac{5kwl^4}{384EI+8kl^3}.$$

7. A tapered cantilever of rectangular section has constant width but its depth decreases linearly to zero at $x = l$. If w_0 is its weight per unit length, and I_0 the moment of inertia of its cross-section at the origin, show that its deflexion under its own weight is

$$w_0 l^2x^2/12EI_0.$$

8. Show that the potential energy of the portion of the beam between the planes AB, $A'B'$, of Fig. 68 (a) is $EI\theta/2R$, and deduce that the potential energy of the whole beam is

$$\frac{1}{2}\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx.$$

Show by integrating by parts that for a beam $0 < x < l$ this may be put in the form

$$\frac{1}{2}\left[Fy+M\frac{dy}{dx}\right]_0^l+\tfrac{1}{2}\int_0^l wy\,dx.$$

9. A light beam of length $a+b$ is freely hinged at its ends and carries a mass M at the point a. Find the deflexion of the beam at a, and show that the natural frequency of oscillation of M is

$$\left\{\frac{3EI(a+b)g}{Ma^2b^2}\right\}^{\frac{1}{2}}\Big/2\pi.$$

10. A light uniform beam of length l is clamped horizontally at the same level at both ends. It carries a constant load w per unit length for $0 < x < \frac{1}{2}l$ and no load in the region $\frac{1}{2}l < x < l$. Show that the deflexion is given by

$$EIy = -\tfrac{13}{192}wlx^3+\tfrac{11}{384}wl^2x^2+\tfrac{1}{24}wx^4$$

for $0 < x < \frac{1}{2}l$, and find its value in $\frac{1}{2}l < x < l$.

11. A light uniform beam of length l carries a load which is zero at its ends and increases linearly up to w per unit length at its mid-point. The beam is clamped horizontally at $x = 0$ and freely hinged at the same level at $x = l$. Show that the deflexion at its mid-point is

$$\frac{53wl^4}{15360EI}.$$

12. A uniform beam of length l and weight w per unit length is freely hinged at its ends and subject to axial compression P. Show that its deflexion at the point x is, writing $\omega^2 = P/EI$,

$$\frac{EIw}{P^2}\left\{\frac{\cos\omega(\frac{1}{2}l-x)}{\cos\frac{1}{2}\omega l}-1\right\}-\frac{wx(l-x)}{2P}.$$

13. A light uniform beam of length l is freely hinged at its ends and subject to axial compression P. It also carries a concentrated transverse load W at $x = a$. Show that the deflexion is

$$\frac{W\sin\omega(l-a)\sin\omega x}{\omega P\sin\omega l}-\frac{W(l-a)x}{Pl},$$

if $0 < x < a$, where $\omega^2 = P/EI$.

14. A light semi-infinite beam $x > 0$ is attached to an elastic foundation which provides restoring force k times the deflexion. It is free at $x = 0$ and carries a concentrated load W at $x = a$. Show that its deflexion at $x = 0$ is

$$\frac{We^{-\omega a}}{2EI\omega^3}\cos\omega a,$$

where $\omega^4 = k/4EI$.

15. Find the form taken by the equation of three moments when two of the supports approach one another (corresponding to a beam clamped horizontally at a point). Deduce the results of § 81, Ex. 1, in this way.

Show that if a uniform beam of length $2l$ is clamped horizontally at $x = 0$ and is supported at this level at $x = l$ and $x = 2l$, the reaction at the support at $x = l$ is
$$8wl/7.$$

16. A uniform beam of length nl rests on $n+1$ equally spaced supports. Show that the bending moment at the rth support is
$$\tfrac{1}{12}wl^2\left\{1+\frac{k^{1-r}(1-k^n)-k^{r-1}(1-k^{-n})}{k^n-k^{-n}}\right\},$$
where $k = -2+\surd 3$ and w is the weight per unit length of the beam.

17. A light beam of length l is freely hinged at its ends. Find the deflexion produced by unit force applied at its mid-point perpendicular to the beam.

If a mass M is attached to the centre of the beam show that its natural frequency is
$$\left(\frac{48gEI}{Ml^3}\right)^{\frac{1}{2}}\Big/2\pi.$$

If the beam carrying the mass M rotates about its axis, show that the angular velocity of whirling is 2π times the above result.

18. A column is clamped vertically at $x = 0$ and freely hinged at $x = l$. Show that the crippling load is $EI\alpha^2/l^2$, where α is the smallest (non-zero) root of
$$\tan\alpha = \alpha.$$

19. Deduce the result § 87 (21) by studying the solution of the differential equation and boundary conditions.

20. A uniform straight shaft of length l and mass ρ per unit length rotates with angular velocity ω and is subject to axial compression P. Show that, if the ends are freely hinged, the whirling speed is given by
$$\left(\frac{P^2}{4E^2I^2}+\frac{\rho\omega^2}{gEI}\right)^{\frac{1}{2}}+\frac{P}{2EI}=\frac{\pi^2}{l^2}.$$

21. Show that for *large* deflexions of a thin wire caused by tension T, the deflexion y at the point x satisfies
$$\rho y = EI/T,$$
where ρ is the radius of curvature at x. Show that an integral of this equation with $y = 0$ when $\psi = 0$ is
$$y = 2(EI/T)^{\frac{1}{2}}\sin\tfrac{1}{2}\psi,$$
where ψ is the slope of the tangent to the wire at any point.

Deduce that, if the origin of x is chosen at $\psi = \pi$,
$$x = (EI/T)^{\frac{1}{2}}\{\ln\tan\tfrac{1}{4}\psi+2\cos\tfrac{1}{2}\psi\},$$
and sketch the curve.

XI

FOURIER SERIES AND INTEGRALS

88. Introductory. Periodic functions

PERIODIC functions arise in many branches of applied mathematics. In electric circuit theory functions which are periodic in the time occur: for example, the voltage applied to a circuit by an alternator is periodic but usually not sinusoidal, while in modern practice periodic voltages with forms such as 'square wave', 'saw-tooth', etc., cf. Fig. 76, are common.

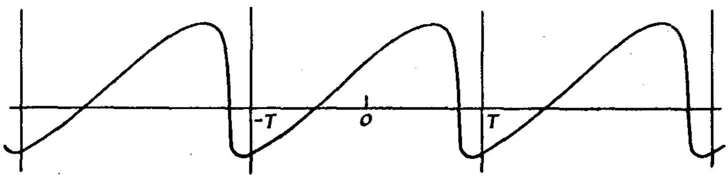


FIG. 72.

Because of the importance of functions which are periodic in the time we shall take t as the independent variable in §§ 88–90 and say $f(t)$ is periodic with period $2T$ if

$$f(t+2rT) = f(t) \quad (r = \pm 1, \pm 2,...). \tag{1}$$

Thus if the value of $f(t)$ is known in any interval of width $2T$ it is known for all t. We shall take $-T \leqslant t \leqslant T$ for the region† in which values of $f(t)$ are given; cf. Fig. 72.

A number of trigonometric functions are known which have period $2T$; these are the set of even‡ functions

$$\cos\frac{n\pi t}{T} \quad (n = 0, 1, 2,...), \tag{2}$$

of which the case $n = 0$ is the constant unity, and the set of odd functions

$$\sin\frac{n\pi t}{T} \quad (n = 1, 2,...). \tag{3}$$

† The period is taken to be $2T$ and the origin in the middle of the region (instead, for example, of the origin at one end of the region which might seem more natural) for convenience in the discussion of odd and even functions in § 89. The final results can easily be put in the appropriate form for functions of period T; cf. Ex. 6.

‡ A function is even if $f(t) = f(-t)$ for all t. It is odd if $f(-t) = -f(t)$.

The complete set of functions (2) and (3) possesses the property that the integral from $-T$ to T of the product of any two different members of the set is zero. This property is known as *orthogonality* over the region $(-T, T)$. To prove it we have

$$\int_{-T}^{T} \cos\frac{n\pi t}{T}\,dt = \int_{-T}^{T} \sin\frac{n\pi t}{T}\,dt = 0 \quad (n = 1, 2, 3,...), \tag{4}$$

$$\int_{-T}^{T} \cos\frac{n\pi t}{T}\sin\frac{m\pi t}{T}\,dt = \frac{1}{2}\int_{-T}^{T} \left\{\sin\frac{(m+n)\pi t}{T} + \sin\frac{(m-n)\pi t}{T}\right\} dt = 0, \tag{5}$$

for all m and n;

$$\int_{-T}^{T} \cos\frac{n\pi t}{T}\cos\frac{m\pi t}{T}\,dt = \frac{1}{2}\int_{-T}^{T} \left\{\cos\frac{(m+n)\pi t}{T} + \cos\frac{(m-n)\pi t}{T}\right\} dt$$

$$= 0, \quad \text{if } m \neq n; \tag{6}$$

and similarly

$$\int_{-T}^{T} \sin\frac{n\pi t}{T}\sin\frac{m\pi t}{T}\,dt = 0, \quad \text{if } m \neq n. \tag{7}$$

The integrals from $-T$ to T of the squares of the functions (2) and (3), however, do not vanish. They are

$$\int_{-T}^{T} \sin^2\frac{n\pi t}{T}\,dt = \frac{1}{2}\int_{-T}^{T} \left\{1 - \cos\frac{2n\pi t}{T}\right\} dt = T \quad (n = 1, 2,...), \tag{8}$$

$$\int_{-T}^{T} \cos^2\frac{n\pi t}{T}\,dt = T \quad (n = 1, 2, 3,...), \tag{9}$$

$$\int_{-T}^{T} dt = 2T. \tag{10}$$

It should be noticed that in (10), which corresponds to (9) with $n = 0$, $2T$ occurs in place of T.

It is natural to inquire whether *any* periodic function $f(t)$ with period $2T$, such as that of Fig. 72, can be represented in terms of the set of trigonometric functions (2) and (3) of period $2T$, that is, whether there exists an expansion

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{T}, \tag{11}$$

where the a_n and b_n are constants.

We first show that, if such an expansion exists and it is permissible to integrate the infinite series in it term by term, the coefficients a_0, a_n, and b_n can be found very simply. First, if we integrate (11) with respect to t from $-T$ to T, all the integrals on the right-hand side except the first vanish by (4), and we get

$$\int_{-T}^{T} f(t)\,dt = a_0 \int_{-T}^{T} dt = 2Ta_0. \tag{12}$$

Therefore
$$a_0 = \frac{1}{2T} \int_{-T}^{T} f(t')\,dt'. \tag{13}$$

In (13), t' has been written for the variable of integration in the definite integral in place of t as in (12). This will also be done in (14) and (15) below in order to avoid confusion with t in subsequent work.

To find a_m we multiply both sides of (11) by $\cos m\pi t/T$ and integrate from $-T$ to T. All terms on the right-hand side will vanish except one, by (4), (5), and (6), and we get

$$\int_{-T}^{T} f(t)\cos\frac{m\pi t}{T}\,dt = a_m \int_{-T}^{T} \cos^2\frac{m\pi t}{T}\,dt = Ta_m,$$

using (9). Therefore, again replacing t by t' in the definite integral

$$a_m = \frac{1}{T} \int_{-T}^{T} f(t')\cos\frac{m\pi t'}{T}\,dt' \quad (m = 1, 2, 3,\ldots). \tag{14}$$

Similarly

$$b_m = \frac{1}{T} \int_{-T}^{T} f(t')\sin\frac{m\pi t'}{T}\,dt' \quad (m = 1, 2, 3,\ldots). \tag{15}$$

In the same way, if we square both sides of (11) and integrate with respect to t from $-T$ to T, the integrals of all products of different functions on the right-hand side are zero, and, using (8), (9), and (10) for the integrals of squares of trigonometric functions, we get

$$\frac{1}{2T}\int\limits_{-T}^{T}[f(t)]^2\,dt = a_0^2+\tfrac{1}{2}\sum_{r=1}^{\infty}a_r^2+\tfrac{1}{2}\sum_{r=1}^{\infty}b_r^2. \tag{16}$$

These quantities a_0, a_m, b_m can be found for any integrable function $f(t)$ and are called its *Fourier constants*. It will appear from later results that they provide a complete alternative specification of the function, that is, if the Fourier constants of the function $f(t)$ are known it is just as completely specified as by the usual statement of the value of $f(t)$ for each value of t in $(-T, T)$. In some problems it is more useful to specify a function by its Fourier constants than in the ordinary way.

The process of finding the Fourier constants for a function given numerically or graphically is called Fourier analysis or harmonic analysis: there are many ways of doing this arithmetically, also mechanical devices have been invented for finding the Fourier constants of a function whose graph is given, and for finding the graph of a function whose Fourier constants are given.

The series on the right-hand side of (11) with the values (13), (14), and (15) of a_0, a_n, and b_n inserted is known as the *Fourier series* for the function $f(t)$. It may be written as

$$\frac{1}{2T}\int\limits_{-T}^{T}f(t')\,dt'+\frac{1}{T}\sum_{n=1}^{\infty}\cos\frac{n\pi t}{T}\int\limits_{-T}^{T}f(t')\cos\frac{n\pi t'}{T}\,dt'+$$
$$+\frac{1}{T}\sum_{n=1}^{\infty}\sin\frac{n\pi t}{T}\int\limits_{-T}^{T}f(t')\sin\frac{n\pi t'}{T}\,dt', \tag{17}$$

or, combining the last two series, as

$$\frac{1}{2T}\int\limits_{-T}^{T}f(t')\,dt'+\frac{1}{T}\sum_{n=1}^{\infty}\int\limits_{-T}^{T}f(t')\cos\frac{n\pi(t-t')}{T}\,dt'. \tag{18}$$

The above argument has not *proved* that (17) or (18) is equal to $f(t)$. It has merely been a 'plausibility' argument† to intro-

† Such arguments are often not true and must always be supplemented by careful pure-mathematical discussion. For example, the result would have been equally plausible if the constant term a_0 in (11) had been omitted.

duce the Fourier constants and the series (17). To prove the result suggested in (11) it is necessary to show that the series (17) is convergent and that its sum is $f(t)$. Whether this will be the case or not depends on the nature of the function $f(t)$, and it is important to allow $f(t)$ to be fairly general: in particular, since discontinuous functions appear so often in applications, these must be considered. The restrictions usually placed on $f(t)$ are

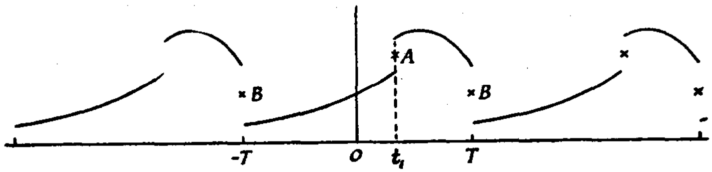


FIG. 73.

that it satisfy 'Dirichlet's conditions', a simple form of which, adequate for our purpose, is that $f(t)$ should be continuous in $(-T, T)$, except at a finite number of ordinary discontinuities, and that it should have only a finite number of maxima and minima in this region.

Fourier's theorem then states that, if $f(t)$ satisfies Dirichlet's conditions, the series (17) *is convergent and its sum is*

(i) $f(t)$ *at all points where this function is continuous,* (19)

(ii) $\frac{1}{2}\{f(t+0)+f(t-0)\}$ *at points where $f(t)$ is discontinuous,*† (20)

(iii) $\frac{1}{2}\{f(T-0)+f(-T+0)\}$ *at* $t = \pm T$. (21)

Thus if $f(t)$ is the function shown in Fig. 73, the sum of the series has the value A when $t = t_1$, and the value B when $t = \pm T$, and for all other values of t is given by the graph. It is in this sense that the expansion (11) is true.

The proof of Fourier's theorem is rather long‡ and will not be given here.

We also state without proof the 'uniqueness theorem' that if

† We write $f(t+0)$ for $\lim_{\tau \to t+0} f(\tau)$ and $f(t-0)$ for $\lim_{\tau \to t-0} f(\tau)$.

‡ Cf. Carslaw, *Fourier's Series and Integrals* (ed. 3, Macmillan, 1930); Churchill, *Fourier Series and Boundary Value Problems* (McGraw-Hill, 1941).

two Fourier series can be found for the same function the coefficients in these must be equal.

Finally, it should be remarked that series of the general type

$$\sum_{n=1}^{\infty} \left\{a_n \cos\frac{n\pi t}{T} + b_n \sin\frac{n\pi t}{T}\right\}, \tag{22}$$

are called trigonometric series and are perhaps, next to power series $\sum a_n t^n$, the most important type of series in mathematics.

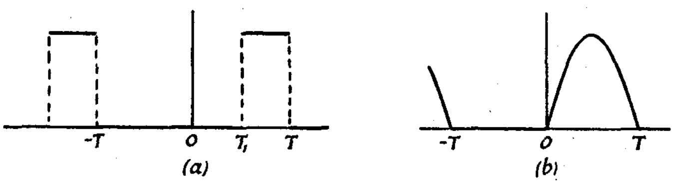


FIG. 74.

If a_n and b_n in (22) are the Fourier constants (14) and (15) of some function $f(t)$, the series becomes a Fourier series, but there are many trigonometric series of type (22) which are not the Fourier series of any function.

Ex. 1. *The repeated pulse of width* $(T-T_1)$ *of Fig.* 74(*a*).
In this case

$$f(t) = 0 \quad (-T < t < T_1),$$
$$f(t) = 1 \quad (T_1 < t < T).$$

Here (13), (14), and (15) give

$$a_0 = \frac{1}{2T}\int\limits_{T_1}^{T} dt = \frac{T-T_1}{2T},$$

$$a_m = \frac{1}{T}\int\limits_{T_1}^{T} \cos\frac{m\pi t'}{T}\,dt' = -\frac{1}{m\pi}\sin\frac{m\pi T_1}{T},$$

$$b_m = \frac{1}{T}\int\limits_{T_1}^{T} \sin\frac{m\pi t'}{T}\,dt' = \frac{1}{m\pi}\left\{(-1)^{m+1}+\cos\frac{m\pi T_1}{T}\right\}.$$

Thus the Fourier series for $f(t)$ is

$$\frac{T-T_1}{2T} - \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{m\pi T_1}{T}\cos\frac{m\pi t}{T} +$$
$$+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left\{(-1)^{m+1}+\cos\frac{m\pi T_1}{T}\right\}\sin\frac{m\pi t}{T}. \tag{23}$$

By Fourier's theorem, (19)–(21), the sum of this series is 0 if $-T < t < T_1$; 1 if $T_1 < t < T$; and $\frac{1}{2}$ when $t = T_1$ or $t = \pm T$.

Ex. 2. *The half-wave rectified sine wave, Fig. 74 (b).*

Here
$$f(t) = 0 \quad (-T \leqslant t \leqslant 0),$$
$$f(t) = \sin\frac{\pi t}{T} \quad (0 \leqslant t \leqslant T).$$

By (13) and (14), respectively, we have
$$a_0 = \frac{1}{2T}\int\limits_0^T \sin\frac{\pi t'}{T}\,dt' = \frac{1}{\pi},$$
$$a_m = \frac{1}{T}\int\limits_0^T \sin\frac{\pi t'}{T}\cos\frac{m\pi t'}{T}\,dt'$$
$$= \frac{1}{2T}\int\limits_0^T \left\{\sin\frac{(m+1)\pi t'}{T} - \sin\frac{(m-1)\pi t'}{T}\right\}dt'$$
$$\left.\begin{aligned} &= \frac{1}{2\pi}\left\{\frac{1-\cos(m+1)\pi}{m+1} - \frac{1-\cos(m-1)\pi}{m-1}\right\}, && \text{if } m > 1 \\ &= 0, && \text{if } m = 1 \end{aligned}\right\}. \tag{24}$$

It follows from (24) that $a_m = 0$ if m is odd, while if m is even, say $m = 2r$,
$$a_{2r} = \frac{1}{2\pi}\left\{\frac{2}{2r+1} - \frac{2}{2r-1}\right\} = -\frac{2}{\pi(4r^2-1)}. \tag{25}$$

Similarly by (15)
$$b_m = \frac{1}{T}\int\limits_0^T \sin\frac{\pi t'}{T}\sin\frac{m\pi t'}{T}\,dt'$$
$$= \frac{1}{2T}\int\limits_0^T \left\{\cos\frac{(m-1)\pi t'}{T} - \cos\frac{(m+1)\pi t'}{T}\right\}dt',$$
so that $b_m = \frac{1}{2}$ if $m = 1$, and is zero if $m > 1$.

Therefore, finally,
$$f(t) = \frac{1}{\pi} + \tfrac{1}{2}\sin\frac{\pi t}{T} - \frac{2}{\pi}\sum_{r=1}^{\infty}\frac{1}{(4r^2-1)}\cos\frac{2r\pi t}{T}. \tag{26}$$

89. Odd and even functions: Fourier sine and cosine series

If the function $f(t)$ of § 88 is either odd or even, important simplifications occur.

Suppose, first, that $f(t)$ *is even* so that $f(-t) = f(t)$ for all t,

Fig. 75 (*a*), and, in addition, of course, $f(t)$ is periodic with period $2T$. It follows that $f(T-0) = f(T+0)$, so that the function must† be continuous at $\pm T$.

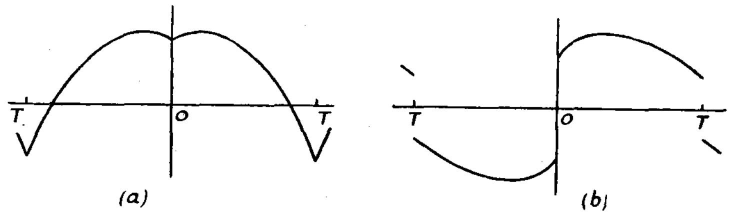


FIG. 75.

In this case from § 88 (13), (14), and (15), using the fact that $f(t)$ is even,

$$a_0 = \frac{1}{2T}\int\limits_{-T}^{T} f(t')\,dt' = \frac{1}{T}\int\limits_{0}^{T} f(t')\,dt', \tag{1}$$

$$a_m = \frac{1}{T}\int\limits_{-T}^{T} f(t')\cos\frac{m\pi t'}{T}\,dt' = \frac{2}{T}\int\limits_{0}^{T} f(t')\cos\frac{m\pi t'}{T}\,dt', \tag{2}$$

$$b_m = \frac{1}{T}\int\limits_{-T}^{T} f(t')\sin\frac{m\pi t'}{T}\,dt' = 0, \tag{3}$$

and the series

$$\frac{1}{T}\int\limits_{0}^{T} f(t')\,dt' + \frac{2}{T}\sum_{n=1}^{\infty}\cos\frac{n\pi t}{T}\int\limits_{0}^{T} f(t')\cos\frac{n\pi t'}{T}\,dt', \tag{4}$$

is convergent, its sum being $f(t)$ at points where the function is continuous and $\frac{1}{2}\{f(t+0)+f(t-0)\}$ at points of discontinuity.

The series (4) is called a *Fourier cosine series*; its coefficients a_0 and a_n could have been determined directly by the method of § 88 instead of quoting the results of that section.

† Except, of course, for the trivial possibility $f(T+0) = f(T-0) \neq f(T)$, which we exclude.

If the function $f(t)$ is odd, so that $f(-t) = -f(t)$, Fig. 75 (*b*), we have from § 88 (13), (14), and (15)

$$a_m = 0 \quad (m = 0, 1, 2,...), \tag{5}$$

$$b_m = \frac{2}{T}\int\limits_0^T f(t')\sin\frac{m\pi t'}{T}\,dt', \tag{6}$$

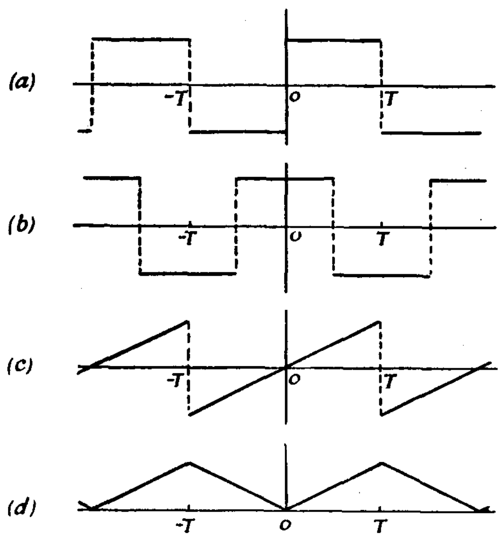


FIG. 76.

and the series

$$\frac{2}{T}\sum_{n=1}^{\infty}\sin\frac{n\pi t}{T}\int\limits_0^T f(t')\sin\frac{n\pi t'}{T}\,dt', \tag{7}$$

is convergent, its sum being $f(t)$ at points where this function is continuous and $\frac{1}{2}\{f(t+0)+f(t-0)\}$ at points where it is discontinuous. In particular, if $f(t)$ is discontinuous at $t = 0$, the sum of the series must be zero there since $f(+0) = -f(-0)$; also, if $f(t)$ is discontinuous when $t = T$, the sum of the series must be zero when $t = T$ since

$$f(T-0) = -f(-T+0) = -f(T+0),$$

cf. Fig. 75 (*b*). The series (7) is called a *Fourier sine series*, and the remark above establishes that the sum of such a series is zero when $t = 0$ and $t = T$.

Ex. 1. *The 'square wave' of Fig. 76 (a) defined by*

$$f(t) = 1 \qquad (0 < t < T),$$
$$= -1 \quad (-T < t < 0).$$

This is an odd function, and so by (6),

$$b_m = \frac{2}{T}\int\limits_0^T \sin\frac{m\pi t'}{T}\,dt' = \frac{2}{m\pi}\{1-(-1)^m\}. \tag{8}$$

Thus b_m is zero if m is even, and if m is odd, say $m = 2r+1$,

$$b_{2r+1} = \frac{4}{\pi(2r+1)}.$$

Thus, by (7) the Fourier series for $f(t)$ is

$$\frac{4}{\pi}\sum_{r=0}^{\infty}\frac{1}{2r+1}\sin\frac{(2r+1)\pi t}{T}. \tag{9}$$

The sum of the series (9) is zero when $t = 0$ or T, and unity if $0 < t < T$.

It is of some importance to study the way in which a Fourier series such as (9) converges to its sum. When such a series is used in practice it is usually hoped that the results derived from the first few terms will give an adequate approximation to the result. In Fig. 77 the sums of 1, 3, and 6 terms of the series (9) are graphed: it appears that 6 terms give a reasonable approximation to the function *except*† near the ends of the interval where it has a maximum which is rather large. The example chosen is rather an unfavourable one, but it serves to illustrate the fact that the convergence of the series is often rather slow for practical purposes, particularly when discontinuous functions are involved.

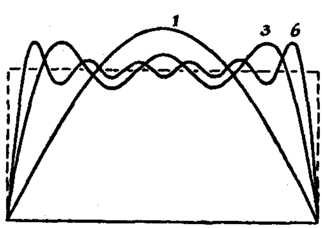


FIG. 77.

Ex. 2. *The function of Fig. 76 (b) defined by*

$$f(t) = 1, \quad -\tfrac{1}{2}T < t < \tfrac{1}{2}T,$$
$$f(t) = -1, \quad \textit{when } -T < t < -\tfrac{1}{2}T \textit{ and } \tfrac{1}{2}T < t < T.$$

In this case $f(t)$ is an even function of t, and by (1) and (2)

$$a_0 = 0,$$

$$a_m = \frac{2}{T}\int\limits_0^{\frac{1}{2}T}\cos\frac{m\pi t'}{T}\,dt' - \frac{2}{T}\int\limits_{\frac{1}{2}T}^{T}\cos\frac{m\pi t'}{T}\,dt' = \frac{4}{m\pi}\sin\tfrac{1}{2}m\pi.$$

† This maximum moves towards the ends of the interval as the number of terms is increased but it does not disappear. This is an illustration of the Gibbs phenomenon, caused by non-uniform convergence of the series.

Therefore $a_{2r} = 0$, and

$$a_{2r+1} = \frac{4(-1)^r}{(2r+1)\pi} \quad (r = 0, 1,...).$$

Thus the Fourier series for $f(t)$ is

$$\frac{4}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)}\cos\frac{(2r+1)\pi t}{T}. \tag{10}$$

The function $f(t)$ of this example and the last are the same except for a shift of origin. This illustrates the fact that, since any point may be taken as origin, many apparently different expressions for the same periodic function may be obtained. (9) reduces to (10) on putting $t = \frac{1}{2}T+t'$.

Ex. 3. *The saw-tooth wave, Fig. 76 (c),*

$$f(t) = t \quad (-T < t < T).$$

Here $f(t)$ is an odd function of t, and from (6)

$$\begin{aligned} b_m &= \frac{2}{T}\int_0^T t'\sin\frac{m\pi t'}{T}\,dt' \\ &= \frac{2}{T}\left[-\frac{Tt'}{m\pi}\cos\frac{m\pi t'}{T}\right]_0^T + \frac{2}{m\pi}\int_0^T \cos\frac{m\pi t'}{T}\,dt' \\ &= \frac{2T}{m\pi}(-1)^{m-1}. \end{aligned}$$

Therefore the Fourier series for $f(t)$ is

$$\frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi t}{T}. \tag{11}$$

Ex. 4. *The function of Fig. 76 (d)*

$$f(t) = t \qquad (0 < t < T),$$
$$f(t) = -t \quad (-T < t < 0).$$

This is an even function, and proceeding in the usual way we find

$$f(t) = \frac{T}{2} - \frac{4T}{\pi^2}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^2}\cos\frac{(2r+1)\pi t}{T}. \tag{12}$$

Ex. 5. *Summation of important infinite series.*

The sums of a number of important series may be found by inserting particular values in Fourier series.

For example, the sum of the series (9) when $t = \frac{1}{2}T$ is 1, so putting $t = \frac{1}{2}T$ in (9) gives

$$\sum_{r=0}^{\infty}\frac{(-1)^r}{2r+1} = \frac{\pi}{4}. \tag{13}$$

The sum of the series (12) when $t = 0$ is zero, so

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}. \tag{14}$$

Putting $t = 0$ and $t = \frac{1}{2}T$, respectively, in § 88 (26) gives

$$\sum_{r=1}^{\infty} \frac{1}{4r^2-1} = \frac{1}{2}, \tag{15}$$

and

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{4r^2-1} = \frac{1}{2}-\frac{\pi}{4}. \tag{16}$$

90. The Fourier series of a function defined in $(-T, T)$ or $(0, T)$

In §§ 88 and 89 we have discussed the representation of periodic functions by Fourier series. But in the theory the fact

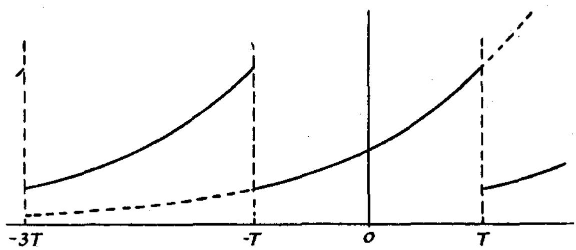


FIG. 78.

that the function was periodic was never used; in particular, in the integrals § 88 (13)–(15) for the Fourier constants only the values of $f(t)$ in the region $(-T, T)$ appear. Thus Fourier's theorem § 88 (19)–(21) may be regarded as a statement about a function $f(t)$ defined in $(-T, T)$: namely that the series § 88 (17) is convergent for all values of t between $-T$ and T, and that its sum is $f(t)$ at points where the function is continuous and $\frac{1}{2}\{f(t+0)+f(t-0)\}$ at points where it is discontinuous. But since the terms of the series § 88 (17) are all periodic with period $2T$, the sum of the series is periodic also, and thus repeats the set of values of $f(t)$ in the range $(-T, T)$, irrespective of whether $f(t)$ has any values, or different values, outside this range..

For example, if $f(t) = e^t$, $-T < t < T$, its Fourier series repeats periodically the portion of the graph of e^t between $-T$ and T as in Fig. 78.

In the Fourier cosine and sine series of § 89 (4) and (7) only the values of $f(t)$ in the region $(0, T)$ are needed. The sums of these series are respectively even and odd functions of t as well as being periodic with period $2T$.

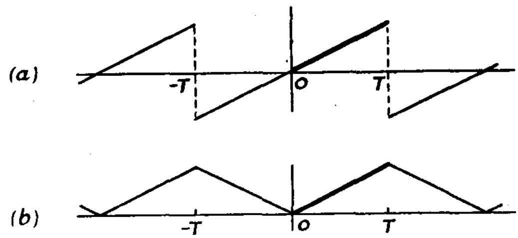


FIG. 79.

Thus suppose we form the cosine series of the function defined by $f(t) = t$ in $0 < t < T$, the sum of the series, being an even function, must be symmetrical about $t = 0$ giving Fig. 79 (b). If we form the sine series of the same function, it gives Fig. 79 (a).

91. Fourier series in electric circuit theory

If a periodic voltage $V(t)$ of period $2\pi/\omega$ is applied to a circuit, we express it by its Fourier series

$$V(t) = \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \tag{1}$$

$$= \sum_{n=1}^{\infty} A_n \sin(n\omega t + \phi_n), \tag{2}$$

as in §§ 88, 89, and treat each term separately. Thus, for example, using the result of § 44 (11), the steady current in an L, R, C circuit due to the voltage (2) is

$$\sum_{n=1}^{\infty} \frac{A_n}{Z_n} \sin(n\omega t + \phi_n - \theta_n), \tag{3}$$

where
$$Z_n = \left\{\left(nL\omega - \frac{1}{nC\omega}\right)^2 + R^2\right\}^{\frac{1}{2}}, \tag{4}$$

$$\theta_n = \tan^{-1}\left\{\left(nL\omega - \frac{1}{nC\omega}\right)\Big/R\right\}. \tag{5}$$

It is frequently necessary to know the mean value of the square of a periodic quantity, or the mean value of the product of two such quantities. These can be written down as in § 88 (16).

Suppose V is the voltage applied to a circuit, and I the current at its point of application, and that

$$V = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{T}, \tag{6}$$

$$I = a'_0 + \sum_{n=1}^{\infty} a'_n \cos\frac{n\pi t}{T} + \sum_{n=1}^{\infty} b'_n \sin\frac{n\pi t}{T}. \tag{7}$$

Then by § 88 (16)

$$\frac{1}{2T}\int_{-T}^{T} V^2\,dt = a_0^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r^2+b_r^2). \tag{8}$$

In the same way

$$\frac{1}{2T}\int_{-T}^{T} VI\,dt = a_0 a'_0 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r a'_r + b_r b'_r), \tag{9}$$

which is the mean rate at which energy is being supplied to the circuit.

Similar analysis occurs in the theory of rectifiers. Suppose that voltage given by (6) is applied to a rectifier with a non-linear characteristic

$$I = f(V). \tag{10}$$

Using (6) in (10) gives by Taylor's theorem

$$I = f\left\{a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{T} + b_n\sin\frac{n\pi t}{T}\right)\right\}$$

$$= f(a_0) + f'(a_0)\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{T} + b_n\sin\frac{n\pi t}{T}\right) +$$

$$+\tfrac{1}{2}f''(a_0)\left\{\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{T} + b_n\sin\frac{n\pi t}{T}\right)\right\}^2 + \dots.$$

$f(a_0)$ is the current due to the steady component a_0 of the voltage V, so writing I' for the change of current, $I - f(a_0)$, due to the periodic part of (6), we get for the mean change of current, using § 88 (4)–(9),

$$\frac{1}{2T}\int\limits_{-T}^{T} I'\,dt = \tfrac{1}{4}f''(a_0)\sum_{n=1}^{\infty}(a_n^2+b_n^2)+\dots,$$
$$= \tfrac{1}{2}(\overline{v^2})f''(a_0)+\dots,$$

where $(\overline{v^2})$ is by (6) and (8) the mean of the square of $V-a_0$, the departure of the voltage V from its steady value.

92. Fourier series in mechanical problems

As an example we treat the slider-crank mechanism whose theory is fundamental for the study of reciprocating engines.

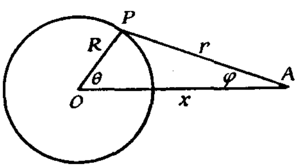


Fig. 80.

Suppose that a crank OP of length R rotates with constant angular velocity ω, and that the point A, which is connected to P by a connecting rod of length r, is constrained to move in a straight line through O. If the connecting rod is very long the motion of A is very nearly simple harmonic, but for shorter connecting rods the departure from this is of importance in problems of engine balancing.

Writing $\omega t = \theta$, measuring θ from the line OA, we see that $OA = x$ is an even function of θ with period 2π and thus may be expanded in the Fourier cosine series

$$x = a_0+\sum_{n=1}^{\infty}a_n\cos n\theta. \tag{1}$$

From the triangle OAP it follows that

$$\sin\phi = k\sin\theta, \tag{2}$$

where
$$k = R/r. \tag{3}$$

Also
$$x = R\cos\theta+r\cos\phi = R\cos\theta+r\sqrt{(1-k^2\sin^2\theta)}. \tag{4}$$

The coefficients a_0, a_1,... in (1) may now be found from § 89 (1) and (2). Thus

$$a_0 = \frac{1}{\pi}\int\limits_0^{\pi}\{R\cos\theta+r\sqrt{(1-k^2\sin^2\theta)}\}\,d\theta = \frac{2r}{\pi}\int\limits_0^{\frac{1}{2}\pi}\sqrt{(1-k^2\sin^2\theta)}\,d\theta, \tag{5}$$

$$a_1 = \frac{2}{\pi}\int\limits_0^{\pi}R\cos^2\theta\,d\theta+\frac{2}{\pi}\int\limits_0^{\pi}r\cos\theta\sqrt{(1-k^2\sin^2\theta)}\,d\theta = R. \tag{6}$$

Also, for $m > 1$,

$$a_m = \frac{2R}{\pi}\int\limits_0^{\pi}\cos\theta\cos m\theta\,d\theta+\frac{2r}{\pi}\int\limits_0^{\pi}\cos m\theta\sqrt{(1-k^2\sin^2\theta)}\,d\theta,$$

so that
$$a_{2n+1} = 0 \quad (n = 1, 2,...), \tag{7}$$

$$a_{2n} = \frac{4r}{\pi}\int\limits_0^{\frac{1}{2}\pi} \cos 2n\theta\sqrt{(1-k^2\sin^2\theta)}\, d\theta \quad (n = 1, 2,...). \tag{8}$$

Thus, finally, replacing θ by ωt we have

$$x = a_0 + R\cos\omega t + \sum_{n=1}^{\infty} a_{2n}\cos 2n\omega t, \tag{9}$$

where a_0 and a_{2n} are given by (5) and (8). It appears that only even harmonics are present.

There are no simple expressions for a_0 and a_{2n} in terms of elementary functions, for example a_0 involves the elliptic integral $E(k, \frac{1}{2}\pi)$; cf. § 55 (20). But since $k < 1$ we may expand the square root in (8) by the binomial theorem and interchange the orders of integration and summation: since in most practical systems $k < \frac{1}{4}$ the series obtained in this way is rapidly convergent. Thus we have from (8)

$$a_{2n} = \frac{4r}{\pi}\int\limits_0^{\frac{1}{2}\pi} \cos 2n\theta\{1-\tfrac{1}{2}k^2\sin^2\theta - \tfrac{1}{8}k^4\sin^4\theta...\}\, d\theta$$

$$= -\frac{2rk^2}{\pi}\int\limits_0^{\frac{1}{2}\pi} \cos 2n\theta\{\sin^2\theta + \tfrac{1}{4}k^2\sin^4\theta + ...\}\, d\theta. \tag{10}$$

Therefore
$$a_2 = \tfrac{1}{4}rk^2 + \tfrac{1}{16}rk^4 + \dots. \tag{11}$$

In general, since

$$\left.\begin{aligned} \int\limits_0^{\frac{1}{2}\pi} \cos 2n\theta \sin^{2m}\theta\, d\theta &= 0 && (m < n) \\ &= (-1)^n \frac{\pi}{2^{2n+1}} && (m = n) \end{aligned}\right\}, \tag{12}$$

$$a_{2n} = (-1)^{n+1}\frac{1.3...(2n-3)}{2^{3n-1}n!} rk^{2n} + ..., \tag{13}$$

and thus the amplitude of the term in $\cos 2n\omega t$ is proportional to k^{2n}.

For example, if k is so small that k^6 is negligible, the departure of A from its mean position is

$$R\left\{\cos\omega t + (\tfrac{1}{4}k + \tfrac{1}{16}k^3)\cos 2\omega t - \frac{k^3}{64}\cos 4\omega t\right\}.$$

93. Fourier series in boundary value problems

In this section we give some examples of the use of Fourier series in deflexion of beams. Further examples in connexion with partial differential equations will be given in § 111.

Ex. 1. *A uniform beam is freely hinged at $x = 0$ and $x = l$, and carries a load $f(x)$.*

We have to solve
$$D^4 y = \frac{1}{EI} f(x) \tag{1}$$
with
$$y = D^2 y = 0 \quad \text{when } x = 0 \text{ and } x = l. \tag{2}$$

If we assume a sine series for y,
$$y = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}, \tag{3}$$
the boundary conditions (2) at $x = 0$ and $x = l$ will all be satisfied, since it was shown in § 89 that the sum of a sine series is zero when $x = 0$ and $x = l$. Now suppose that the sine series for $f(x)$ is
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \tag{4}$$
where, by § 89 (6),
$$b_n = \frac{2}{l}\int_0^l f(x')\sin\frac{n\pi x'}{l}\,dx'. \tag{5}$$

Substituting (3) and (4) in (1) requires
$$\sum_{n=1}^{\infty} \frac{n^4\pi^4}{l^4} a_n \sin\frac{n\pi x}{l} = \frac{1}{EI}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}. \tag{6}$$

Comparing coefficients† of $\sin n\pi x/l$ on the two sides gives
$$a_n = \frac{l^4}{EIn^4\pi^4} b_n. \tag{7}$$

Thus the deflexion y is given by
$$y = \frac{2l^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\int_0^l f(x')\sin\frac{n\pi x'}{l}\,dx'. \tag{8}$$

If $f(x) = w$, constant, (8) becomes
$$y = \frac{4wl^4}{EI\pi^5}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^5}\sin\frac{(2r+1)\pi x}{l}. \tag{9}$$

If $f(x) = W\delta(x-a)$, a concentrated load W at $x = a$, (8) becomes
$$y = \frac{2Wl^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi a}{l}. \tag{10}$$

† This is justified by the uniqueness theorem stated towards the end of § 88. It should be remarked also that arguments involving the differentiation of Fourier series need pure-mathematical justification: this can be supplied for the cases considered here.

The solutions (9) and (10) are extremely useful in practice because they converge very rapidly.

Ex. 2. *The problem of Ex. 1 except that in addition the beam rests on an elastic foundation giving restoring force per unit length of k times the displacement.*

The differential equation (1) is now replaced by

$$EID^4y = -ky+f(x),$$

or

$$(D^4+\omega^4)y = \frac{1}{EI}f(x), \tag{11}$$

where $\omega^4 = k/EI$.

Assuming the sine series (3) and (4) for y and $f(x)$, respectively, and substituting in (11) gives

$$\sum_{n=1}^{\infty}\left(\omega^4+\frac{n^4\pi^4}{l^4}\right)a_n\sin\frac{n\pi x}{l} = \frac{1}{EI}\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}.$$

Hence

$$a_n = \frac{l^4}{EI(l^4\omega^4+n^4\pi^4)}b_n,$$

and, finally,

$$y = \frac{2l^3}{EI}\sum_{n=1}^{\infty}\frac{1}{(l^4\omega^4+n^4\pi^4)}\sin\frac{n\pi x}{l}\int_0^l f(x')\sin\frac{n\pi x'}{l}\,dx'. \tag{12}$$

94. Double and multiple Fourier series

Such series are often required in the solution of partial differential equations: for example, a double Fourier series is used in the theory of deflexion of rectangular plates in much the same way that an ordinary Fourier series was used in the theory of the deflexion of beams in § 93; cf. Ex. 18.

Here we give only a brief sketch of the theory of the double sine series: other types of series involving both sines and cosines may be obtained in the same way, or the whole theory may be developed *ab initio* along the lines of §§ 88, 89.

Suppose $f(x,y)$ is defined in the region $0 < x < a$, $0 < y < b$, then, as in § 89, we can expand $f(x,y)$ in a sine series in x

$$f(x,y) = \sum_{n=1}^{\infty}b_n(y)\sin\frac{n\pi x}{a}, \tag{1}$$

in which the coefficients

$$b_n(y) = \frac{2}{a}\int_0^a f(x',y)\sin\frac{n\pi x'}{a}\,dx', \tag{2}$$

are functions of y.

Since $b_n(y)$ is a function of y in $0 < y < b$ we can expand it in the sine series

$$b_n(y) = \sum_{m=1}^{\infty} c_{m,n} \sin\frac{m\pi y}{b}, \tag{3}$$

where

$$c_{m,n} = \frac{2}{b}\int_0^b b_n(y')\sin\frac{m\pi y'}{b}\,dy'$$

$$= \frac{4}{ab}\int_0^b \sin\frac{m\pi y'}{b}\,dy' \int_0^a f(x',y')\sin\frac{n\pi x'}{a}\,dx' \tag{4}$$

$$= \frac{4}{ab}\int_0^b\int_0^a f(x',y')\sin\frac{m\pi y'}{b}\sin\frac{n\pi x'}{a}\,dx'dy'. \tag{5}$$

Thus, finally, with this value of $c_{m,n}$ we have the double sine series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{m,n}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \tag{6}$$

whose sum is $f(x,y)$ at every point in the rectangle $0 < x < a$, $0 < y < b$. As in § 89 it follows that the sum is zero on the boundaries of the rectangle and that, outside it, it is periodic and odd in x and y.

Ex. *To expand $f(x,y) = 1$ in $0 < x < a$, $0 < y < b$ in a double sine series.*

As in § 89, Ex. 1, the sine series for 1 in $0 < x < a$ is

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{a} = 1.$$

Expanding each of the coefficients (again constants) in a sine series in $0 < y < b$ we have

$$\frac{16}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(2m+1)(2n+1)}\sin\frac{(2n+1)\pi x}{a}\sin\frac{(2m+1)\pi y}{b} = 1. \tag{7}$$

95. Fourier integrals

Fourier series are trigonometric series for a function $f(t)$ defined in $(-T, T)$: if this interval becomes infinite, so that the function is defined in $(-\infty, \infty)$, they tend to Fourier integrals which are trigonometric double integrals.

To see the form of these, consider the Fourier series, § 88 (18), for $f(t)$

$$\frac{1}{2T}\int_{-T}^{T} f(t')\,dt' + \frac{1}{T}\sum_{n=1}^{\infty}\int_{-T}^{T} f(t')\cos\frac{n\pi(t-t')}{T}\,dt', \tag{1}$$

and suppose that $T \to \infty$ in this. First we must assume that

$$\int_{-\infty}^{\infty} f(t')\,dt' \tag{2}$$

is convergent, and if this is the case the first term of (1) tends to zero as $T \to \infty$.

Next we have to consider what becomes of the series in (1) as $T \to \infty$. Writing

$$h = \pi/T \tag{3}$$

for the small quantity π/T, the series in (1) can be written

$$\frac{1}{\pi}\left\{h\int_{-T}^{T} f(t')\cos h(t-t')\,dt' + h\int_{-T}^{T} f(t')\cos 2h(t-t')\,dt' + \right.$$
$$\left. + h\int_{-T}^{T} f(t')\cos 3h(t-t')\,dt' + \ldots\right\}, \tag{4}$$

and as $T \to \infty$, $h \to 0$. As $T \to \infty$, all the integrals from $-T$ to T become integrals from $-\infty$ to ∞, also remembering that in the limit as $h \to 0$

$$h\{\phi(h)+\phi(2h)+\phi(3h)+\ldots\} \to \int_{0}^{\infty} \phi(\omega)\,d\omega,$$

it appears that, as $T \to \infty$ and $h \to 0$, (4) tends to the double integral

$$\frac{1}{\pi}\int_{0}^{\infty} d\omega \int_{-\infty}^{\infty} f(t')\cos\omega(t-t')\,dt'. \tag{5}$$

This is Fourier's integral for the function $f(t)$. As before, the above discussion is illustrative only, but the exact theory shows that *if* $f(t)$ *satisfies Dirichlet's conditions, and the integral*

$$\int_{-\infty}^{\infty} |f(t)|\,dt \tag{6}$$

exists, the double integral (5) *has the value* $f(t)$ *at points where this function is continuous, and the value*

$$\tfrac{1}{2}\{f(t+0)+f(t-0)\}$$

at points where $f(t)$ *is discontinuous.*

The Fourier integral representation of a non-periodic function plays the same part in the theory of aperiodic phenomena that the Fourier series does in that of periodic phenomena. But it has one important restriction, namely that $f(t)$ must be such that the integral (6) is convergent: this is not the case for such common functions as $\sin\omega t$ or a constant, though the theory can be extended to include them.

If $f(t)$ is an odd function of t, (5) simplifies into

$$f(t) = \frac{2}{\pi}\int\limits_0^\infty \sin\omega t\,d\omega \int\limits_0^\infty f(t')\sin\omega t'\,dt', \tag{7}$$

while if $f(t)$ is an even function of t, it simplifies to

$$f(t) = \frac{2}{\pi}\int\limits_0^\infty \cos\omega t\,d\omega \int\limits_0^\infty f(t')\cos\omega t'\,dt', \tag{8}$$

these being known as Fourier's sine and cosine integrals respectively. In (7) and (8), as always, $f(t)$ is to be replaced on the left by $\tfrac{1}{2}\{f(t+0)+f(t-0)\}$ at points of discontinuity.

96. Fourier transforms. Applications

For simplicity we consider only a continuous function $f(t)$ satisfying § 95 (6). Then, by § 95 (5),

$$f(t) = \frac{1}{\pi}\int\limits_0^\infty d\omega \int\limits_{-\infty}^\infty f(t')\cos\omega(t-t')\,dt'$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^\infty d\omega \int\limits_{-\infty}^\infty f(t')\cos\omega(t-t')\,dt', \tag{1}$$

since the inner integral is an even function of ω.

Also $$\frac{i}{2\pi}\int\limits_{-\infty}^{\infty} d\omega \int\limits_{-\infty}^{\infty} f(t')\sin\omega(t-t')\,dt' = 0, \tag{2}$$

since the inner integral is an odd function of ω.

Adding (1) and (2) gives

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{i\omega t}\,d\omega \int\limits_{-\infty}^{\infty} e^{-i\omega t'}f(t')\,dt' = f(t). \tag{3}$$

This is called the complex form of Fourier's integral theorem. In applications it is most convenient to express it in a slightly different way. Let

$$F(\omega) = \int\limits_{-\infty}^{\infty} e^{-i\omega t'}f(t')\,dt', \tag{4}$$

then (3) states that

$$f(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{i\omega t}\,F(\omega)\,d\omega. \tag{5}$$

$F(\omega)$, defined in (4), is called the *Fourier transform*† of $f(t)$; it is obtained from it by multiplying $f(t)$ by $e^{-i\omega t}$ and integrating from $-\infty$ to ∞, just as the Fourier constants of a function defined in $(-T, T)$ are obtained by multiplying by $\sin n\pi t/T$ or $\cos n\pi t/T$ and integrating over the region in which the function is defined. By (5), if the Fourier transform $F(\omega)$ of a function $f(t)$ is known, the function can be found by a similar integration.

(4) and (5) are the usual form in which Fourier's integral theorem is applied to boundary value problems. In initial value problems a further simplification is often possible, since usually $f(t) = 0$ if $t < 0$. If this is the case (4) becomes

$$F(\omega) = \int\limits_{0}^{\infty} e^{-i\omega t'}f(t')\,dt', \tag{6}$$

† An extensive table of such transforms is given by Campbell and Foster, *Fourier Integrals for Practical Applications*, Bell System Technical Monograph B–584 (1931). The Fourier transform is often defined as $(2\pi)^{-\frac{1}{2}}F(\omega)$.

while (5) gives

$$\left.\begin{aligned}\frac{1}{2\pi}\int\limits_0^\infty \{e^{i\omega t}F(\omega)+e^{-i\omega t}F(-\omega)\}\,d\omega &= f(t), \qquad \text{if } t>0\\ &= 0, \qquad \text{if } t<0\end{aligned}\right\}. \tag{7}$$

If, in addition, $f(t)$ is real, it follows from (6) that $F(-\omega)$ is the conjugate of $F(\omega)$, and (7) becomes

$$\left.\begin{aligned}\frac{1}{\pi}\mathbf{R}\int\limits_0^\infty e^{i\omega t}F(\omega)\,d\omega &= f(t), \quad \text{if } t>0\\ &= 0, \quad \text{if } t<0\end{aligned}\right\}. \tag{8}$$

The similarity of (6) to the Laplace transform § 18 (1) is evident, $i\omega$ simply appearing in place of p. But it must be remembered that the integral (6) is often not convergent for common functions. We may state that, *if the Fourier transform $F(\omega)$ of a function $f(t)$ exists it is just $\bar{f}(i\omega)$* and thus may be written down from the table of Laplace transforms.

For example, if

$$f(t) = e^{-at}, \qquad \bar{f}(p) = \frac{1}{p+a} \quad \text{and} \quad F(\omega) = \frac{1}{a+i\omega}. \tag{9}$$

Similarly if

$$f(t) = e^{-at}\sin bt, \qquad \bar{f}(p) = \frac{b}{(p+a)^2+b^2},$$

$$F(\omega) = \frac{b}{(a+i\omega)^2+b^2}. \tag{10}$$

As an example of the use of the Fourier transform we consider the response of a circuit to an aperiodic voltage $f(t)$ applied for $t>0$.

First the Fourier transform $F(\omega)$ of $f(t)$ has to be calculated, and then (8) may be regarded as expressing $f(t)$ in terms of vibrations of all possible frequencies, the complex voltage of the component of frequency $\omega/2\pi$ being

$$\frac{1}{\pi}F(\omega)\,d\omega. \tag{11}$$

If this voltage is applied to a circuit of impedance $z(i\omega)$ the steady complex current of period $2\pi/\omega$ in it is

$$\frac{F(\omega)\,d\omega}{\pi z(i\omega)}, \tag{12}$$

and the whole current is

$$\frac{1}{\pi}\mathbf{R}\int_0^\infty e^{i\omega t}\frac{F(\omega)\,d\omega}{z(i\omega)}. \tag{13}$$

This procedure is exactly analogous to that of § 91 for a periodic voltage—the effects of all the harmonics are calculated separately and then combined. It has an advantage over the Laplace transformation technique of § 51 because of this analogy, but results obtained by it are rather harder to evaluate.

Ex. *The ideal low-pass filter.*

This is defined as having

$$z(i\omega) = \infty, \text{ if } \omega > \omega_0, \text{ and } z(i\omega) = Ze^{i\pi\omega/\omega_0}, \quad \omega < \omega_0,$$

where Z is a constant. In this case the current I given by (13) is

$$I = \frac{1}{\pi Z}\mathbf{R}\int_0^{\omega_0} e^{i\omega(t-\pi/\omega_0)}F(\omega)\,d\omega. \tag{14}$$

Suppose we wish to calculate the response of this filter to a unit pulse of applied voltage,

$$\left.\begin{aligned} f(t) &= 0, \quad \text{for } t < 0 \text{ and } t > T \\ f(t) &= 1, \quad 0 < t < T \end{aligned}\right\}. \tag{15}$$

Then by (6)

$$F(\omega) = \int_0^T e^{-i\omega t'}\,dt' = (2/\omega)e^{-\frac{1}{2}i\omega T}\sin\tfrac{1}{2}\omega T,$$

and by (14)

$$I = \frac{2}{\pi Z}\mathbf{R}\int_0^{\omega_0} e^{i\omega(t-\frac{1}{2}T-\pi/\omega_0)}\sin\tfrac{1}{2}\omega T\,\frac{d\omega}{\omega}$$

$$= \frac{2}{\pi Z}\int_0^{\omega_0}\cos\omega(t-\tfrac{1}{2}T-\pi/\omega_0)\sin\tfrac{1}{2}\omega T\,\frac{d\omega}{\omega}$$

$$= \frac{1}{\pi Z}\int_0^{\omega_0}\{\sin\omega(t-\pi/\omega_0)+\sin\omega(T-t+\pi/\omega_0)\}\,\frac{d\omega}{\omega}$$

$$= \frac{1}{\pi Z}\{\mathrm{Si}(\omega_0 t-\pi)+\mathrm{Si}(\omega_0 T-\omega_0 t+\pi)\}, \tag{16}$$

where

$$\mathrm{Si}\,x = \int_0^x \frac{\sin u}{u}\,du \tag{17}$$

is a tabulated function.

The effect of unit voltage applied for $t > 0$ may be found by letting $T \to \infty$ in (15) and (16). Using the result $\mathrm{Si}(\infty) = \frac{1}{2}\pi$, (16) becomes in this case

$$\frac{1}{Z}\left\{\frac{1}{2}+\frac{1}{\pi}\mathrm{Si}(\omega_0 t-\pi)\right\}. \tag{18}$$

It should be noticed that this result cannot be obtained by taking $f(t) = 1$ for $t > 0$ since this does not satisfy § 95 (6).

EXAMPLES ON CHAPTER XI

1. If $f(t) = |\sin \omega t|$, full wave rectified alternating current, show that

$$f(t) = \frac{2}{\pi}-\frac{4}{\pi}\sum_{r=1}^{\infty}\frac{1}{(4r^2-1)}\cos 2r\omega t.$$

2. If

$$f(t) = 1 \qquad (0 < t < T_1),$$
$$f(t) = 0 \qquad (T_1 < t < T-T_1),$$
$$f(t) = -1 \qquad (T_1-T < t < T),$$

show that the cosine series for $f(t)$ is

$$\frac{4}{\pi}\sum_{r=0}^{\infty}\frac{1}{2r+1}\sin\frac{(2r+1)\pi T_1}{T}\cos\frac{(2r+1)\pi t}{T}.$$

3. Show that the sine series for $\sin^2\omega t$ in $0 < t < \pi/\omega$ is

$$\frac{8}{\pi}\sum_{r=0}^{\infty}\frac{1}{(2r+1)\{4-(2r+1)^2\}}\sin(2r+1)\omega t.$$

4. By expanding $\sin mx$ and $\cos mx$ in sine and cosine series, respectively, in $0 < x < \pi$, show that

$$\sin mx = \frac{2}{\pi}\sin m\pi\sum_{r=1}^{\infty}\frac{(-1)^{r-1}r\sin rx}{r^2-m^2},$$

$$\cos mx = \frac{2}{\pi}\sin m\pi\left\{\frac{1}{2m}+\sum_{r=1}^{\infty}\frac{(-1)^{r-1}m\cos rx}{r^2-m^2}\right\}.$$

5. By expanding $x+x^2$ in a series of sines and cosines in $-\pi < x < \pi$, show that

$$x+x^2 = \frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left\{\frac{4}{n^2}\cos nx-\frac{2}{n}\sin nx\right\} \quad (-\pi < x < \pi),$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

6. Show that the series of sines and cosines which represents $f(x)$ in $0 < x < l$ is

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{l} + b_n \sin\frac{2n\pi x}{l}\right),$$

where $$a_0 = \frac{1}{l}\int\limits_0^l f(x')\,dx', \qquad a_n = \frac{2}{l}\int\limits_0^l f(x')\cos\frac{2n\pi x'}{l}\,dx',$$

$$b_n = \frac{2}{l}\int\limits_0^l f(x')\sin\frac{2n\pi x'}{l}\,dx'.$$

7. If $$V = (1-e^{-\alpha t}) \quad (0 < t < T),$$

and V is periodic with period T (this is the voltage drop over the condenser in the circuit of Fig. 49 (a)), show that

$$V = 1 + \frac{1}{\alpha T}(e^{-\alpha T}-1) + $$
$$+ (e^{-\alpha T}-1)\sum_{n=1}^{\infty} \frac{2\alpha T\cos(2n\pi t/T) + 4n\pi\sin(2n\pi t/T)}{\alpha^2T^2+4n^2\pi^2}.$$

8. Show that the sine series for the function $\delta(t-t_0)$ in $0 < t < T$ is

$$\delta(t-t_0) = \frac{2}{T}\sum_{n=1}^{\infty} \sin\frac{n\pi t}{T}\sin\frac{n\pi t_0}{T}.$$

This series, of course, is not convergent, but correct results may often be obtained very simply by using it. Deduce § 93 (10) in this way.

9. Show that if p and q are positive integers less than n, and $p \neq q$,

$$\sum_{r=1}^{n-1} \sin\frac{rp\pi}{n}\sin\frac{rq\pi}{n} = 0,$$

and that if $p = q$ the sum has the value $\frac{1}{2}n$.

The values $y_1, y_2,..., y_{n-1}$ of a function at the points $t = sT/n$ ($s = 1, 2,..., n-1$), are known; show that

$$y = \sum_{r=1}^{n-1} b_r \sin\frac{r\pi t}{T},$$

where $$b_r = \frac{2}{n}\sum_{s=1}^{n-1} y_s \sin\frac{rs\pi}{n}$$

passes through all these points. (The condition that y should pass through the $(n-1)$ given points gives $n-1$ equations for $b_1,..., b_{n-1}$ which are solved by using the first result given.) This gives a sine series of period $2T$ which passes through $n-1$ given points.

10. Show that if

$$y = a_0 + \sum_{r=1}^{n} \left(a_r \cos\frac{2r\pi t}{T} + b_r \sin\frac{2r\pi t}{T}\right)$$

passes through the points $(0, y_0), (\alpha, y_1), \ldots, (2n\alpha, y_{2n})$, where $\alpha = T/(2n+1)$,

$$a_0 = \frac{1}{(2n+1)} \sum_{s=0}^{2n} y_s, \qquad a_r = \frac{2}{2n+1} \sum_{s=0}^{2n} y_s \cos\frac{2\pi rs}{(2n+1)},$$

$$b_r = \frac{2}{2n+1} \sum_{s=1}^{2n} y_s \sin\frac{2\pi rs}{(2n+1)}.$$

11. A uniform beam of length l is freely hinged at its ends. Show that the bending moment M in it is

$$-\frac{4wl^2}{\pi^3} \sum_{r=0}^{\infty} \frac{1}{(2r+1)^3} \sin\frac{(2r+1)\pi x}{l}$$

if the beam carries a uniformly distributed load w per unit length, and

$$-\frac{2Wl}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi x}{l} \sin\frac{n\pi a}{l}$$

if it carries a concentrated load W at $x = a$.

12. Assuming the result

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2},$$

deduce that

$$\begin{aligned} \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{\sin \omega t}{\omega}\, d\omega &= 1, \quad \text{if } t > 0, \\ &= \tfrac{1}{2}, \quad \text{if } t = 0, \\ &= 0, \quad \text{if } t < 0. \end{aligned}$$

This gives a representation of the unit function $H(t)$.

13. Show that if $\quad f(t) = 0 \quad \text{if } t < 0,$

$$f(t) = e^{-\alpha t} \quad \text{if } t > 0,$$

where $\alpha > 0$, Fourier's integral theorem § 96 (8) gives

$$\begin{aligned} \frac{1}{\pi} \int_0^{\infty} \frac{\alpha \cos \omega t + \omega \sin \omega t}{\alpha^2 + \omega^2}\, d\omega &= e^{-\alpha t} \quad (t > 0), \\ &= 0 \quad (t < 0). \end{aligned}$$

Deduce that

$$\int_0^{\infty} \frac{\omega \sin \omega t}{\alpha^2 + \omega^2}\, d\omega = \int_0^{\infty} \frac{\alpha \cos \omega t}{\alpha^2 + \omega^2}\, d\omega = \frac{\pi}{2} e^{-\alpha t}.$$

14. Show that the Fourier transform of the function $f(t)$ which is zero for $t < 0$ and has the value $te^{-\alpha t}$ for $t > 0$ is

$$\frac{1}{(\alpha+i\omega)^2}.$$

If this voltage is applied to an L, R, C circuit, show that the amplitude of the component of frequency $\omega/2\pi$ of the current is the real part of

$$\frac{1}{\pi(\alpha+i\omega)^2\{(Li\omega-i/C\omega)+R\}}.$$

15. If
$$f(t) = \cos\omega_0 t \quad (|t| < T),$$
$$= 0 \quad (|t| > T),$$

show from either § 96 (5) or § 95 (8) that

$$f(t) = \frac{1}{\pi}\int\limits_0^\infty \left\{\frac{\sin(\omega_0-\omega)T}{\omega_0-\omega}+\frac{\sin(\omega_0+\omega)T}{\omega_0+\omega}\right\}\cos\omega t\,d\omega.$$

This shows that if radiation of frequency $\omega_0/2\pi$ is emitted for time $2T$, the amplitude of the component of frequency $\omega/2\pi$ is

$$\frac{1}{\pi}\left\{\frac{\sin(\omega_0-\omega)T}{\omega_0-\omega}+\frac{\sin(\omega_0+\omega)T}{\omega_0+\omega}\right\}.$$

Discuss the variation of this with T, and show that as T is increased the relative importance of frequencies near ω_0 increases steadily.

16. Voltage $e^{-\alpha t}$ is applied to the ideal low-pass filter of § 96 for $t > 0$. Find the current in it, and deduce § 96 (18) by letting $\alpha \to 0$. [Use the last integral evaluated in Ex. 13.]

17. Show that a function of x which is defined (and satisfies Dirichlet's conditions) in $0 < x < l$ may be expanded in the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{(2n+1)\pi x}{2l},$$

where
$$a_n = \frac{2}{l}\int\limits_0^l f(x')\cos\frac{(2n+1)\pi x'}{2l}\,dx'.$$

[Expand the function defined by $f(x)$ in $0 < x < l$ and $-f(2l-x)$ in $l < x < 2l$ in a cosine series in $0 < x < 2l$.]

18. The deflexion z of a uniform plate carrying a load w per unit area satisfies

$$\frac{\partial^4 z}{\partial x^4}+2\frac{\partial^4 z}{\partial x^2\partial y^2}+\frac{\partial^4 z}{\partial y^4} = \frac{w}{D},$$

where D is a constant depending on the material and thickness of the plate. For a rectangular plate $0 < x < a$, $0 < y < b$, simply supported at its edges, z has to satisfy

$$z = \frac{\partial^2 z}{\partial y^2} = 0, \quad \text{when } y = 0 \text{ and } y = b,$$

$$z = \frac{\partial^2 z}{\partial x^2} = 0, \quad \text{when } x = 0 \text{ and } x = a.$$

If w is constant, show by using § 94 (7) that

$$z = \frac{16w}{\pi^6 D} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin[(2m+1)\pi x/a]\sin[(2n+1)\pi y/b]}{(2m+1)(2n+1)[(2m+1)^2/a^2+(2n+1)^2/b^2]^2}.$$

XII

ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

97. Introductory

In this chapter we consider the general linear equation of the nth order with coefficients which are functions of x,

$$\{f_1(x)D^n+f_2(x)D^{n-1}+\ldots+f_n(x)\}y=f(x). \tag{1}$$

In a few special cases the solution of this equation may be made to depend on the solution of types previously studied. For example the 'Euler' equation

$$\{p_0(a+bx)^nD^n+p_1(a+bx)^{n-1}D^{n-1}+\ldots+p_n\}y=f(x), \tag{2}$$

where $p_0, p_1,\ldots, p_n$ are constants, is reduced to a linear equation by the substitution

$$a+bx=e^t, \tag{3}$$

cf. Ex. 2 on Chapter II.

Again, if the operator can be factorized the solution is reduced to the solution of a chain of first-order linear equations. For example the equation

$$\{xD^2+(x^2+\overset{2}{\cancel{3}})D+2x\}y=0, \tag{4}$$

can be written

$$(D+x)(xD+2)y=0, \tag{5}$$

where it must be remembered that the operators are not commutative. To solve (5) put

$$(xD+2)y=v, \tag{6}$$

so that (5) becomes

$$(D+x)v=0,$$

and the solution of this is

$$v=Ae^{-\frac{1}{2}x^2}. \tag{7}$$

Substituting (7) in (6) gives the first-order equation for y,

$$(xD+2)y=Ae^{-\frac{1}{2}x^2}.$$

Methods such as these are clearly only applicable to limited and rather unimportant classes of equation to which the equations which arise most commonly in applied mathematics do not belong. The most important of these latter are second-order equations in which the coefficients $f_1(x)$, $f_2(x)$, $f_3(x)$ are polynomials in x. The method adopted for the solution of these is to seek a solution in the form of an infinite series

$$y = x^c(a_0+a_1 x+a_2 x^2+\dots). \tag{8}$$

By substituting this series in the equation and equating the coefficients of the successive powers of x to zero we obtain an equation, the *indicial equation*, for c, and a set of equations for the coefficients a_0, a_1, a_2,.... Naturally matters such as the existence of solutions of this type need careful discussion. Here only the process of solution will be given; for its justification the reader is referred to the works on differential equations listed in the footnote † below.

Since the equations we are considering are linear, the general solution of (1) consists of a linear combination of n independent solutions with arbitrary coefficients. In the examples of the second order which we shall consider, two independent solutions are required and usually they correspond to two different values of c in (8): occasionally the method gives only one solution, and a second is obtained by devices due to Frobenius.† In §§ 98 and 99 we shall obtain the series solutions of the two most important equations, Bessel's and Legendre's, and give a brief sketch of their properties.

The equations studied in §§ 98–100 are homogeneous: methods for solving inhomogeneous equations are given in §§ 101, 102. Approximate solutions are discussed in § 103. Finally, in § 104 equations with periodic coefficients are discussed briefly.

98. Bessel's equation‡ of order ν

This is
$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{\nu^2}{x^2}\right)y = 0, \tag{1}$$

where ν is any (real) number, fractional or integral: the usual

† Cf. Piaggio, *Differential Equations* (Bell, 1925); Ince, *Ordinary Differential Equations* (Longmans, 1927).

‡ The standard works on Bessel functions are Watson, *Bessel Functions* (ed. 2, Cambridge, 1944); Gray and Mathews, *Treatise on Bessel Functions* (ed. 2, Macmillan, 1922); McLachlan, *Bessel Functions for Engineers* (Oxford, 1934).

convention is to use ν for a fraction, and to replace it by n if it is integral. ν is called the order of the equation.

To solve (1) we assume that an expression of type

$$y = x^c(a_0+a_1x+a_2x^2+...) \tag{2}$$

satisfies (1), and seek to find c and the successive coefficients $a_0, a_1,...$. Substituting (2) in (1) gives

$$\begin{aligned} c(c-1)a_0x^{c-2}&+(c+1)ca_1x^{c-1}+(c+2)(c+1)a_2x^c+...\\ +ca_0x^{c-2}&+\ (c+1)a_1x^{c-1}+\qquad (c+2)a_2x^c+...\\ &\qquad\qquad + \qquad\qquad a_0x^c+...\\ -\nu^2a_0x^{c-2}&-\nu^2a_1x^{c-1}-\quad \nu^2a_2x^c-... = 0. \end{aligned}$$

Equating the coefficients of x^{c-2}, x^{c-1},... to zero we find

$$(c^2-\nu^2)a_0 = 0, \tag{3}$$

$$\{(c+1)^2-\nu^2\}a_1 = 0, \tag{4}$$

$$\{(c+2)^2-\nu^2\}a_2+a_0 = 0, \tag{5}$$

$$\{(c+3)^2-\nu^2\}a_3+a_1 = 0, \tag{6}$$

$$\{(c+4)^2-\nu^2\}a_4+a_2 = 0, \tag{7}$$

. . . .

We may assume $a_0 \neq 0$, since taking $a_0 = 0$ is equivalent to changing the value of c. Then (3) gives

$$c = \pm\nu. \tag{8}$$

For either of these values of c, (4) gives $a_1 = 0$, then by (6), $a_3 = 0$, and so on, that is

$$a_1 = a_3 = a_5 = ... = 0, \tag{9}$$

so all the odd coefficients vanish. The even coefficients come successively from (5), (7),...

$$a_2 = \frac{a_0}{\nu^2-(c+2)^2},$$

$$a_4 = \frac{a_2}{\nu^2-(c+4)^2} = \frac{a_0}{\{\nu^2-(c+2)^2\}\{\nu^2-(c+4)^2\}},$$

and so on.

Using these results, the solution for $c = \nu$ becomes

$$a_0 x^\nu \left\{1 - \frac{x^2}{(\nu+2)^2-\nu^2} + \frac{x^4}{\{(\nu+2)^2-\nu^2\}\{(\nu+4)^2-\nu^2\}} - \ldots\right\}$$

$$= a_0 x^\nu \left\{1 - \frac{(\frac{1}{2}x)^2}{\nu+1} + \frac{(\frac{1}{2}x)^4}{(\nu+1)(\nu+2)2!} - \ldots\right\}$$

$$= a_0 x^\nu \left\{1 + \sum_{r=1}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{2r}}{(\nu+1)(\nu+2)\ldots(\nu+r)r!}\right\}. \qquad (10)$$

Taking the negative sign, $c = -\nu$, in (8) gives the same result except that ν is replaced by $-\nu$. Thus if ν is not an integer we have found two independent solutions of (1) which behave like x^ν and $x^{-\nu}$, respectively, as $x \to 0$.

If ν is a positive integer n, or is zero, this procedure only leads to one solution. For if $\nu = 0$, (3) only gives one value of c, namely $c = 0$; while if $\nu = n$, taking $c = n$ gives the series (10) with $\nu = n$, that is,

$$a_0 x^n \left\{1 + \sum_{r=1}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{2r}}{(n+1)\ldots(n+r)r!}\right\}, \qquad (11)$$

but if $c = -n$, the coefficient of a_{2n} in the chain of equations (3), (4),... vanishes; this requires that $a_{2n-2} = \ldots = a_2 = 0$, and if the procedure is carried through in detail the result is found merely to be a constant multiple of the result for $c = n$.

Those cases in which the order is zero or a positive integer are in fact much the most important, and their solutions are tabulated for a very wide range of n and x. The function chosen to study and tabulate† is a constant multiple, namely $1/(2^n a_0\, n!)$, of the series (11); the notation $J_n(x)$ is used for it so that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{n+2r}}{r!(n+r)!}. \qquad (12)$$

This is called the Bessel function of the first kind of order n. When $n = 0$ it becomes

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{2r}}{(r!)^2}. \qquad (13)$$

† Tables are given in the works quoted above, also Jahnke-Emde, *Tables of Functions* (Teubner).

When $x = 0$ we have from (12) and (13)

$$J_0(0) = 1, \quad \text{but} \quad J_n(0) = 0 \quad (n = 1, 2, \dots). \tag{14}$$

We now derive some important properties of $J_n(x)$. From (12), assuming that the infinite series may be differentiated term by term, we get for $n \geqslant 1$,

$$\begin{aligned} \frac{d}{dx}\{x^n J_n(x)\} &= \frac{d}{dx}\left\{\sum_{r=0}^{\infty} \frac{(-1)^r x^{2n+2r}}{2^{n+2r} r!(n+r)!}\right\} \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2n+2r-1}}{2^{n+2r-1} r!(n+r-1)!} \\ &= x^n \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2}x)^{2r+(n-1)}}{r!(n+r-1)!} \\ &= x^n J_{n-1}(x). \end{aligned} \tag{15}$$

Therefore, writing $J_n'(x)$ for $(d/dx)J_n(x)$,

$$xJ_n'(x) + nJ_n(x) = xJ_{n-1}(x). \tag{16}$$

In the same way

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x), \tag{17}$$

and so

$$xJ_n'(x) - nJ_n(x) = -xJ_{n+1}(x). \tag{18}$$

Adding and subtracting (16) and (18) gives

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x), \tag{19}$$

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x). \tag{20}$$

(19) and (20) are called the recurrence relations. It appears from (20) that if we know $J_0(x)$ and $J_1(x)$ for any value of x, $J_2(x)$ can be calculated, and then $J_3(x)$ and so on. (19) gives the differential coefficient $J_n'(x)$ if $n \geqslant 1$. If $n = 0$, it follows immediately by differentiating (13) that

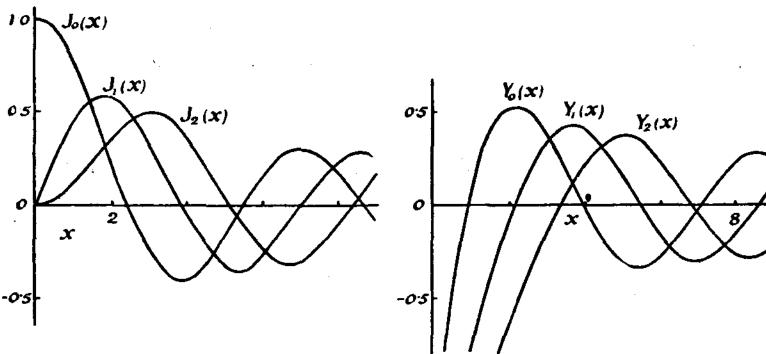
$$J_0'(x) = -J_1(x). \tag{21}$$

For large values of x the Bessel functions can be shown to oscillate steadily with decreasing amplitude; in fact

$$J_n(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi) + \mathbf{O}(x^{-\frac{3}{2}}), \tag{22}$$

where $\mathbf{O}(x^{-\frac{3}{2}})$ is written for a term which decreases like $x^{-\frac{3}{2}}$ when x becomes large.

The graphs of $J_0(x)$, $J_1(x)$, and $J_2(x)$ are shown in Fig. 81 (a).



(a) (b)

FIG. 81.

The zeros of the Bessel functions are of importance in practice and are tabulated. The first few zeros of $J_0(x)$ are 2·4048..., 5·5200..., 8·6537....

Before considering the second solution of Bessel's equation of positive integral order, we return to the solutions (10) of the equation of fractional order. These can be expressed in a form similar to (12) by the use of the gamma function.

The gamma function $\Gamma(\nu)$ is given by

$$\Gamma(\nu) = \int\limits_0^\infty e^{-x}x^{\nu-1}\,dx, \tag{23}$$

if $\nu > 0$. Integrating (23) by parts gives, if $\nu > 1$,

$$\Gamma(\nu) = (\nu-1)\Gamma(\nu-1). \tag{24}$$

$\Gamma(\nu)$ is tabulated in most books of tables for values of ν between 1 and 2: its graph in this region is shown in Fig. 82. By repeated application of (24) the gamma function of any argument may be expressed in terms of the function whose argument lies between 1 and 2.

If $\nu = n$, a positive integer, (24) gives

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)...1.\Gamma(1) = (n-1)!. \tag{25}$$

Thus the gamma function provides a generalization of $n!$ to non-integral n.

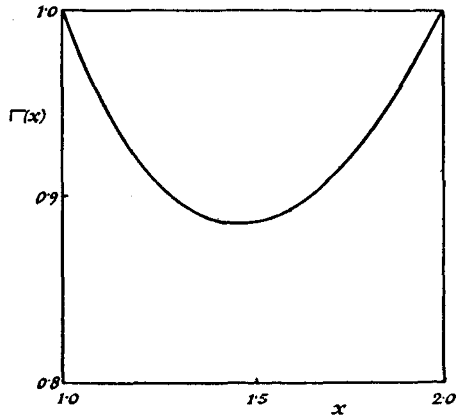


FIG. 82.

Two other properties which are frequently needed may be quoted here; firstly

$$\Gamma(\tfrac{1}{2}) = \pi^{\frac{1}{2}}, \tag{26}$$

and secondly that gamma functions of negative argument may be expressed in terms of those of positive argument by the relation

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin \nu\pi}. \tag{27}$$

As $\nu \to 0$, or to a negative integer, $\Gamma(\nu) \to \infty$. The representation (23) does not hold for gamma functions of negative argument, but (24) remains true.

We now define the Bessel function $J_\nu(x)$ of order ν by

$$J_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{\nu+2r}}{r!\Gamma(\nu+r+1)}. \tag{28}$$

By (25) this definition agrees with (12) if ν is a positive integer n. Also by (24) it can be seen that it is a constant multiple of (10). Thus if ν is fractional, $J_\nu(x)$ and $J_{-\nu}(x)$ are a pair of independent solutions of (1), and so its general solution is

$$y = AJ_\nu(x)+BJ_{-\nu}(x), \tag{29}$$

where A and B are arbitrary constants. Frequently, however, the linear combination $Y_\nu(x)$ of $J_\nu(x)$ and $J_{-\nu}(x)$ defined by

$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}, \tag{30}$$

is taken as the second solution of (1) so that its general solution is

$$y = CJ_\nu(x)+DY_\nu(x). \tag{31}$$

$Y_\nu(x)$ is called the Bessel function of the second kind of order ν. The main reason for its introduction is that it provides a second solution for the equation of integral order. If ν is zero or a positive integer, both the numerator and denominator of (30) vanish, but it can be shown that

$$\lim_{\nu\to n} Y_\nu(x) \tag{32}$$

exists. We write this $Y_n(x)$, and it is the required second solution for functions of integral order. These Bessel functions of the second kind are of less importance than those of the first kind since

$$Y_\nu(x) \to -\infty \quad \text{and} \quad Y_n(x) \to -\infty \quad \text{as} \quad x \to 0, \tag{33}$$

and for this reason they are excluded from the solutions of problems in applied mathematics for regions including the origin $x = 0$. The graphs of $Y_0(x)$, $Y_1(x)$, and $Y_2(x)$, showing this behaviour as $x \to 0$, are given in Fig. 81 (b).

$Y_n(x)$ satisfies the same recurrence relations (19), (20), (21) as $J_n(x)$, and $Y_\nu(x)$ and $J_\nu(x)$ also satisfy these relations with n replaced by ν for all ν. Finally the important formula

$$J_\nu(x)Y'_\nu(x) - Y_\nu(x)J'_\nu(x) = \frac{2}{\pi x} \tag{34}$$

may be quoted.

The most important functions of fractional order are those of orders $\frac{1}{2}$ and $\frac{1}{3}$. By (28) and (24)

$$\begin{aligned} J_{\frac{1}{2}}(x) &= \frac{2^{\frac{1}{2}}}{x^{\frac{1}{2}}\Gamma(\frac{1}{2})}\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right\} \\ &= \sqrt{\left(\frac{2}{\pi x}\right)}\sin x, \end{aligned} \tag{35}$$

using (26) and the series for $\sin x$. Similarly

$$J_{-\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos x, \tag{36}$$

and other functions of half-integral order may be expressed in terms of these by the recurrence relation (20).

The importance of the Bessel functions of order $\frac{1}{3}$ arises from the fact that the solution of the equation

$$\frac{d^2y}{dx^2}+xy = 0 \tag{37}$$

is

$$y = Ax^{\frac{1}{2}}J_{\frac{1}{3}}(\tfrac{2}{3}x^{\frac{3}{2}})+Bx^{\frac{1}{2}}J_{-\frac{1}{3}}(\tfrac{2}{3}x^{\frac{3}{2}}). \tag{38}$$

A number of second-order equations, of which (37) is the most important, can be transformed into Bessel's equation by an appropriate change of variables and thus their solutions can be expressed in terms of tabulated functions.

Finally the equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(1+\frac{\nu^2}{x^2}\right)y = 0, \tag{39}$$

which corresponds to (1) with x replaced by ix, should be mentioned. It is called the modified Bessel equation and is of almost equal importance to Bessel's equation. Its theory may be developed in the same way.

The solution analogous to $J_\nu(x)$, which is called the modified Bessel function of the first kind of order ν and is denoted by $I(x)$, is

$$I_\nu(x) = \sum_{r=0}^{\infty}\frac{(\frac{1}{2}x)^{\nu+2r}}{r!\Gamma(\nu+r+1)}; \tag{40}$$

it is proportional to $J_\nu(ix)$. The modified Bessel function of the second kind, corresponding to $Y_\nu(x)$, is defined by

$$K_\nu(x) = \tfrac{1}{2}\pi\frac{I_{-\nu}(x)-I_\nu(x)}{\sin\nu\pi}. \tag{41}$$

For integral or zero order, the solutions are

$$I_n(x) = \sum_{r=0}^{\infty}\frac{(\frac{1}{2}x)^{n+2r}}{r!(n+r)!} \tag{42}$$

and

$$K_n(x) = \lim_{\nu\to n}K_\nu(x). \tag{43}$$

As $x \to 0$, $I_0(x) \to 1$; $I_\nu(x) \to 0$, $\nu \neq 0$; and $K_\nu(x) \to \infty$. But the behaviour of $I_\nu(x)$ and $K_\nu(x)$ as $x \to \infty$ is fundamentally different from that of $J_\nu(x)$ and $Y_\nu(x)$: it is

$$I_\nu(x) = \frac{e^x}{(2\pi x)^{\frac{1}{2}}}\left\{1+\mathrm{O}\left(\frac{1}{x}\right)\right\}, \qquad (44)$$

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\left\{1+\mathrm{O}\left(\frac{1}{x}\right)\right\}. \qquad (45)$$

99. Legendre's equation†

The differential equation is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y = 0, \qquad (1)$$

where n is zero or any positive integer. The case of fractional n may be treated in the same way.

As before, we seek a solution of type

$$y = x^c(a_0+a_1x+a_2x^2+\dots), \qquad (2)$$

with $a_0 \neq 0$. Substituting in (1) gives

$$\begin{aligned}
&c(c-1)a_0x^{c-2}+\\
&+(c+1)ca_1x^{c-1}+(c+2)(c+1)a_2x^c+(c+3)(c+2)a_3x^{c+1}+\dots-\\
&\qquad\qquad -c(c-1)a_0x^c \qquad -(c+1)ca_1x^{c+1}-\dots-\\
&\qquad\qquad -2ca_0x^c \qquad -2(c+1)a_1x^{c+1}-\dots+\\
&\qquad\qquad +n(n+1)a_0x^c \qquad +n(n+1)a_1x^{c+1}+\dots = 0.
\end{aligned}$$

Equating coefficients of the powers of x to zero we get

$$c(c-1)a_0 = 0, \qquad (3)$$

$$c(c+1)a_1 = 0, \qquad (4)$$

$$(c+1)(c+2)a_2-\{c(c+1)-n(n+1)\}a_0 = 0, \qquad (5)$$

$$(c+2)(c+3)a_3-\{(c+1)(c+2)-n(n+1)\}a_1 = 0, \qquad (6)$$

$$(c+3)(c+4)a_4-\{(c+2)(c+3)-n(n+1)\}a_2 = 0, \qquad (7)$$

. . . .

Since $a_0 \neq 0$ it follows from (3) that c must be zero or unity.

† Macrobert, *Spherical Harmonics* (Methuen, 1927); Hobson, *Spherical and Ellipsoidal Harmonics* (Cambridge, 1931); Byerly, *Fourier's Series and Spherical Harmonics* (Ginn, 1895).

If $c = 0$, (4) is satisfied, so a_1 is unspecified. Putting $c = 0$ in (5), (6),... we get

$$a_2 = -\frac{n(n+1)}{2!}a_0, \qquad a_4 = \frac{(n-2)n(n+1)(n+3)}{4!}a_0, \tag{8}$$

$$a_3 = -\frac{(n-1)(n+2)}{3!}a_1, \qquad a_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1. \tag{9}$$

Thus with $c = 0$ we get for the solution of (1)

$$a_0\left\{1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-...\right\}+$$
$$+a_1\left\{x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5-...\right\}. \tag{10}$$

The solution (10) consists of arbitrary constant multiples of two independent series, and so is the general solution of (1). The other choice $c = 1$ from (3) merely gives the second series in (10).

If n is even, the coefficients of x^{n+2} and the higher powers of x in the first series in (10) are zero, and so the series reduces to a polynomial of degree n in x. If $n = 0$, only the first term, 1, remains; if $n = 2$, the series reduces to $1-3x^2$, and so on. Similarly if n is odd, the second series in (10) reduces to a polynomial.

Thus in either case the solution of (1) consists of one infinite series and one polynomial. We define $P_n(x)$, the *Legendre Polynomial* or Legendre coefficient of degree n, to be a constant multiple of this polynomial, namely

$$P_n(x) = (-1)^{\frac{1}{2}n}\frac{1.3.5...(n-1)}{2.4...n}\times$$
$$\times\left\{1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-...\right\},$$

if n is even, (11)

$$P_n(x) = (-1)^{\frac{1}{2}(n-1)}\frac{1.3.5...n}{2.4...(n-1)}\left\{x-\frac{(n-1)(n+2)}{3!}x^3+...\right\},$$

if n is odd. (12)

If the polynomials (11) and (12) are written in the reverse order, beginning with the terms in x^n, they both take the form

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \dots\right\}. \tag{13}$$

This leads immediately to *Rodrigues's formula*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \tag{14}$$

as may be verified by expanding $(x^2-1)^n$ by the binomial theorem and differentiating n times.

A constant multiple of the infinite series solution of (1) is called $Q_n(x)$, the Legendre function of the second kind of order n. It is of less importance than the Legendre polynomials since $Q_n(x) \to \infty$ as $x \to \pm 1$, and because of this has to be excluded from the solution of many physical problems.

The first few Legendre polynomials are

$$P_0(x) = 1, \tag{15}$$

$$P_1(x) = x, \tag{16}$$

$$P_2(x) = \tfrac{1}{2}(3x^2-1), \tag{17}$$

$$P_3(x) = \tfrac{1}{2}(5x^3-3x); \tag{18}$$

they are obtained easily by Rodrigues's formula.

Next we derive the important result

$$(1-2hx+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x), \tag{19}$$

that is, that the Legendre polynomials $P_n(x)$ defined above are the coefficients of h^n in the expansion of the function on the left of (19). To derive (19), assuming that h is sufficiently small to ensure convergence, we expand the left-hand side of (19) by the binomial theorem and rearrange the resulting series. In this way we get

$$(1-2hx+h^2)^{-\frac{1}{2}} = 1+\tfrac{1}{2}(2x-h)h+\frac{1.3}{2.4}(2x-h)^2h^2+\ldots+$$
$$+\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}(2x-h)^nh^n+\ldots$$
$$= 1+xh+\tfrac{1}{2}(3x^2-1)h^2+\tfrac{1}{2}(5x^3-3x)h^3+\ldots$$
$$= P_0(x)+hP_1(x)+h^2P_2(x)+\ldots,$$

as required.

The importance of (19) in applied mathematics arises from the fact that it gives an expansion of $1/R$ in ascending or descending powers of r, where R is the distance between the points whose polar coordinates are $(r,0)$ and (a,θ), so that

$$\frac{1}{R} = \frac{1}{\{r^2+a^2-2ra\cos\theta\}^{\frac{1}{2}}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n P_n(\mu) \quad (r > a), \tag{20}$$

$$= \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(\mu) \quad (r < a), \tag{21}$$

where $\mu = \cos\theta$.

Many important properties of the Legendre polynomials follow from (19). For example, differentiating it with respect to h gives

$$(x-h)(1-2hx+h^2)^{-\frac{3}{2}} = \sum_{n=1}^{\infty} nh^{n-1}P_n(x).$$

Therefore

$$(x-h)\sum_{n=0}^{\infty} h^nP_n(x) = (1-2hx+h^2)\sum_{n=1}^{\infty} nh^{n-1}P_n(x). \tag{22}$$

Equating coefficients of h^n in (22) gives

$$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x) = 0, \tag{23}$$

which is the recurrence relation connecting the polynomials of degrees $n-1$, n, and $n+1$.

Next we evaluate some important integrals involving Legendre polynomials. Writing D for d/dx, consider

$$2^nn!\int_{-1}^{1} P_n(x)x^m\,dx = \int_{-1}^{1} x^m\{D^n(x^2-1)^n\}\,dx$$

$$= \left[x^mD^{n-1}(x^2-1)^n\right]_{-1}^{1}-m\int_{-1}^{1} x^{m-1}\{D^{n-1}(x^2-1)^n\}\,dx, \tag{24}$$

where Rodrigues's formula (14) has been used. Since

$$D^{n-1}(x^2-1)^n$$

has x^2-1 as a factor, the integrated part in (24) vanishes at both limits. If the process of integrating by parts is continued the final result is zero if $m < n$, and if $m = n$ it is

$$\begin{aligned}(-1)^n n! \int_{-1}^{1} (x^2-1)^n \, dx &= n! \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^{2n+1}\theta \, d\theta \\ &= 2(n!) \frac{2n(2n-2)...2}{(2n+1)(2n-1)...3} \\ &= \frac{2^{2n+1}(n!)^3}{(2n+1)!}. \end{aligned} \tag{25}$$

Since $P_n(x)$ is a polynomial of degree n in x, it follows from these results and (13) that

$$\int_{-1}^{1} P_n(x) P_m(x) \, dx = 0 \quad (m < n), \tag{26}$$

$$\begin{aligned}\int_{-1}^{1} [P_n(x)]^2 \, dx &= \frac{(2n)!}{2^n(n!)^2} \int_{-1}^{1} P_n(x) x^n \, dx \\ &= \frac{2}{2n+1}. \end{aligned} \tag{27}$$

Since m and n in (26) are interchangeable, this result holds also for $m > n$. It is analogous to the results of § 88 for trigonometric functions and may be stated in the form that $P_n(x)$ and $P_m(x)$ are orthogonal in $-1 \leqslant x \leqslant 1$. Using (26) and (27), a function $f(x)$, defined in $-1 \leqslant x \leqslant 1$, may be expanded in a series of Legendre polynomials. Assume

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x). \tag{28}$$

Multiplying (28) by $P_m(x)$, integrating with respect to x from -1 to 1, and using (26) and (27), we get

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) \, dx. \tag{29}$$

This corresponds exactly to the procedure used in § 88 for determining the coefficients in a Fourier series, and, as in that case, the processes used have to be justified carefully from the pure-mathematical point of view.

Finally we consider the equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]y = 0, \tag{30}$$

where m and n are integers.

Putting
$$y = (1-x^2)^{\frac{1}{2}m}z, \tag{31}$$

(30) becomes

$$(1-x^2)\frac{d^2z}{dx^2}-2(m+1)x\frac{dz}{dx}+[n(n+1)-m(m+1)]z = 0, \tag{32}$$

and this equation is satisfied by

$$\frac{d^m v}{dx^m}, \tag{33}$$

where v is any solution of Legendre's equation (1). (30) is known as the associated Legendre equation, and its solutions

$$P_n^m(x) = (1-x^2)^{\frac{1}{2}m}\frac{d^m P_n(x)}{dx^m}, \tag{34}$$

as the associated Legendre functions of the first kind. For the important case in which $x = \cos\theta$ the first few of these are

$$P_0^0 = 1, \qquad P_1^0 = \cos\theta, \qquad P_1^1 = \sin\theta,$$
$$P_2^0 = \tfrac{1}{2}(3\cos^2\theta-1), \qquad P_2^1 = 3\sin\theta\cos\theta, \qquad P_2^2 = 3\sin^2\theta.$$

100. Schrödinger's equation for the hydrogen atom

This is, in effect, a rather more complicated problem on solution in series. The problem is to find the conditions on k under which the differential equation

$$\frac{d^2R}{d\rho^2}+\frac{2}{\rho}\frac{dR}{d\rho}+\left(-\frac{1}{4}+\frac{k}{\rho}-\frac{l(l+1)}{\rho^2}\right)R = 0, \tag{1}$$

where l is zero or a positive integer, will have a solution which remains finite as $\rho \to 0$ and as $\rho \to \infty$. This is an eigenvalue problem: the values of k found will be the eigenvalues, and the corresponding solutions the eigenfunctions. (1) has, of course, solutions for all values of k, but these do not satisfy the required conditions.

First we make the change of variable

$$R = ve^{-\frac{1}{2}\rho}, \tag{2}$$

in (1). This is suggested by the fact that if ρ is so large that $1/\rho$ is negligible (1) becomes

$$\frac{d^2R}{d\rho^2}-\tfrac{1}{4}R = 0,$$

and the solution of this which is finite as $\rho \to \infty$ is $e^{-\frac{1}{2}\rho}$.

Substituting (2) in (1) gives

$$\frac{d^2v}{d\rho^2}+\left(\frac{2}{\rho}-1\right)\frac{dv}{d\rho}+\left\{\frac{k-1}{\rho}-\frac{l(l+1)}{\rho^2}\right\}v = 0. \tag{3}$$

We seek a solution

$$v = \rho^c \sum_{n=0}^{\infty} a_n \rho^n \tag{4}$$

of this. Substituting (4) in (3) gives

$$\begin{aligned} &c(c-1)a_0\rho^{c-2}+(c+1)ca_1\rho^{c-1}+(c+2)(c+1)a_2\rho^c+\ldots+\\ &+2ca_0\rho^{c-2}+2(c+1)a_1\rho^{c-1} \qquad +2(c+2)a_2\rho^c+\ldots-\\ &\qquad\qquad -ca_0\rho^{c-1} \qquad -(c+1)a_1\rho^c-\ldots+\\ &\qquad\qquad +(k-1)a_0\rho^{c-1} \qquad +(k-1)a_1\rho^c+\ldots-\\ &-l(l+1)a_0\rho^{c-2}-l(l+1)a_1\rho^{c-1} \qquad -l(l+1)a_2\rho^c+\ldots = 0. \end{aligned}$$

Equating the coefficients of the powers of ρ to zero we get

$$c(c+1)-l(l+1) = 0, \tag{5}$$

$$\{(c+1)(c+2)-l(l+1)\}a_1-\{c-(k-1)\}a_0 = 0, \tag{6}$$

$$\{(c+2)(c+3)-l(l+1)\}a_2-\{(c+1)-(k-1)\}a_1 = 0, \tag{7}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\{(c+n)(c+n+1)-l(l+1)\}a_n-\{(c+n-1)-(k-1)\}a_{n-1} = 0. \tag{8}$$

(5) gives $c = l$ or $c = -l-1$. In order to have v finite as $\rho \to 0$ we must choose the solution with $c = l$. Then the other equations give

$$a_n = \frac{l+n-k}{n(n+2l+1)}a_{n-1} \quad (n = 1, 2,\ldots), \tag{9}$$

from which the solution (4) can be written down.

Now for large n it follows from (9) that

$$\frac{a_n}{a_{n-1}} = \frac{1}{n}$$

approximately. Thus when n is large the terms of the series (4) behave like those of e^{ρ}, and R, given by (2), behaves like $e^{\frac{1}{2}\rho}$, and so tends to infinity as $\rho \to \infty$ and thus does not satisfy the required conditions. The only exception to this is the case in which k is an integer, $l+n$. Then by (9), a_n and all subsequent coefficients in (4) vanish, so that v becomes a polynomial in ρ of degree $l+n-1$, and $R \to 0$ as $\rho \to \infty$ as required. Thus the conditions are only satisfied if

$$k = l+n, \quad \text{where } n = 1, 2, 3, \ldots.$$

Since in the problem of the hydrogen atom† k is

$$\left\{-\frac{2\pi^2 mZ^2e^4}{Eh^2}\right\}^{\frac{1}{2}}, \tag{10}$$

this leads to the formula for the energy levels of the hydrogen atom

$$E = -\frac{2\pi^2 me^4Z^2}{h^2(l+n)^2} \quad (n = 1, 2, \ldots). \tag{11}$$

101. Inhomogeneous equations. Variation of parameters

The solution of an inhomogeneous linear differential equation with variable coefficients, just as in the case of constant coefficients, consists of the sum of a particular integral and the complementary function. The complementary function is found by methods such as those of the preceding sections. Variation of parameters is a method for finding a particular integral when the complementary function is known.

† Schrödinger's equation for an electron of charge $-e$ and mass m in the field of a nucleus of charge Ze is

$$\nabla^2\psi + \frac{8\pi^2 m}{h^2}\left(E + \frac{Ze^2}{r}\right)\psi = 0,$$

where h is Planck's constant and E is the total energy of the electron. If, as in § 115, we seek a solution of type

$$\psi = R(\rho)P_l^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi,$$

where $\rho = 4\pi r(-2mE/h^2)^{\frac{1}{2}}$, the equation (1) for R results with k given by (10).

Suppose the differential equation is

$$D^2y+P(x)\,Dy+Q(x)y = R(x), \tag{1}$$

and that u and v are two independent solutions of the corresponding homogeneous equation, so that

$$D^2u+P\,Du+Qu = 0, \tag{2}$$

$$D^2v+P\,Dv+Qv = 0. \tag{3}$$

We seek a solution of (1) of the form

$$y = \phi u+\psi v, \tag{4}$$

where ϕ and ψ are functions of x to be determined. Differentiating (4) gives

$$Dy = \phi\,Du+\psi\,Dv+u\,D\phi+v\,D\psi, \tag{5}$$

and if we require†

$$Dy = \phi\,Du+\psi\,Dv, \tag{6}$$

we must have by (5)

$$u\,D\phi+v\,D\psi = 0. \tag{7}$$

Differentiating (6) gives

$$D^2y = \phi\,D^2u+\psi\,D^2v+D\phi Du+D\psi Dv. \tag{8}$$

Substituting (6), (8), and (4) in (1), and using (2) and (3), gives

$$D\phi Du+D\psi Dv = R(x). \tag{9}$$

(7) and (9) are a pair of equations for $D\phi$ and $D\psi$. Solving them we have

$$(u\,Dv-v\,Du)\,D\phi = -R(x)v, \tag{10}$$

$$(u\,Dv-v\,Du)\,D\psi = R(x)u. \tag{11}$$

Integrating (10) and (11) gives ϕ and ψ, and the complete solution of (1) is finally

$$y = Au+Bv+\phi u+\psi v, \tag{12}$$

where A and B are arbitrary constants. The method can be extended to equations of higher order.

102. Inhomogeneous equations with boundary conditions. The Green's function

The method of variation of parameters given in the preceding section is an analytical device for finding a particular integral

† That is, we make Dy have the form it would have if ϕ and ψ were constants. This is often done in problems of this type; cf. § 58 (10) and (11) for another example.

of any inhomogeneous second-order equation. In this section we give a method for finding the complete solution, satisfying given boundary conditions, of an inhomogeneous second-order equation of the form†

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\}-q(x)y = r(x), \tag{1}$$

in terms of a special solution of the corresponding homogeneous equation and the same boundary conditions. This special solution is called the Green's function and it has a simple physical interpretation. Suppose we have to find a solution of (1) with the boundary conditions

$$a\,Dy+by = 0, \quad \text{when } x = 0, \tag{2}$$

$$a'\,Dy+b'y = 0, \quad \text{when } x = l, \tag{3}$$

where a, b, a', b' are constants, and D is written for d/dx.

Let $G(x,\xi)$ be the solution of the homogeneous equation corresponding to (1) which satisfies the boundary conditions (2) and (3), and which is continuous at $x = \xi$ but has a discontinuous first derivative at $x = \xi$ such that

$$\lim_{\epsilon\to 0}[p(x)\,DG]_{x=\xi-\epsilon}^{x=\xi+\epsilon} = -1. \tag{4}$$

$G(x,\xi)$ then satisfies

$$D\{p(x)\,DG\}-q(x)G = 0, \tag{5}$$

$$a\,DG+bG = 0 \quad (x = 0), \tag{6}$$

$$a'\,DG+b'G = 0 \quad (x = l). \tag{7}$$

Multiplying (1) by G, (5) by y, and subtracting, gives

$$GD\{p(x)\,Dy\}-yD\{p(x)\,DG\} = r(x)\,G.$$

Integrating with respect to x from $x = 0$ to $x = l$ gives

$$\int\limits_0^l \{GD[p(x)\,Dy]-yD[p(x)\,DG]\}\,dx = \int\limits_0^l r(x)\,G(x,\xi)\,dx. \tag{8}$$

Because of the discontinuity of DG at $x = \xi$, the integral on the left has to be split into integrals from 0 to $\xi-\epsilon$ and from

† The equations of §§ 98, 99 can be put in this form.

$\xi+\epsilon$ to l. Integrating by parts it becomes

$$[G(x,\xi)p(x)\,Dy-y(x)p(x)\,DG(x,\xi)]_{\xi+\epsilon}^{l}+$$

$$+[G(x,\xi)p(x)\,Dy-y(x)p(x)\,DG(x,\xi)]_{0}^{\xi-\epsilon}=\int\limits_{0}^{l}r(x)G(x,\xi)\,dx. \quad (9)$$

The terms on the left vanish at the limits l and 0 by (2), (3), (6), and (7), and we are left with

$$\lim_{\epsilon\to 0}y(\xi)[p(x)\,DG(x,\xi)]_{\xi-\epsilon}^{\xi+\epsilon}=\int\limits_{0}^{l}r(x)G(x,\xi)\,dx,$$

or, using (4),
$$y(\xi)=-\int\limits_{0}^{l}r(x)\,G(x,\xi)\,dx. \quad (10)$$

Thus when $G(x,\xi)$ has been found, the required value of the solution y at any point is found by simple integration. It can be shown† that the Green's function $G(x,\xi)$ is a symmetrical function of x and ξ, that is,

$$G(x,\xi)=G(\xi,x). \quad (11)$$

The theory can be extended to equations of order n; in this case G and all its derivatives up to $D^{n-2}G$ are to be continuous at $x=\xi$, and $D^{n-1}G$ is to be discontinuous there.

The fourth-order equation for deflexion of beams has been studied in § 82, and $G(x,\xi)$ in that case was found to be the deflexion at ξ due to a unit concentrated load at x.

103. Reduction to the normal form. Approximate solutions

If the change of variables

$$y=z\exp\{-\tfrac{1}{2}\int P(x)\,dx\} \quad (1)$$

is made in the second-order linear equation

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x), \quad (2)$$

† By the type of argument leading to (9) except that in place of $G(x,\xi)$ and y two Green's functions $G(x,\xi)$ and $G(x,\eta)$ are used.

the resulting equation for z is

$$\frac{d^2z}{dx^2}+\left\{Q(x)-\frac{1}{2}\frac{dP(x)}{dx}-\tfrac{1}{4}[P(x)]^2\right\}z = R(x)\exp\left\{\tfrac{1}{2}\int P(x)\,dx\right\}. \tag{3}$$

This is called the *Normal Form* of (1). By using it, it is often possible to decide whether two different equations can be transformed into one another.

Since any second-order equation may be put into this form we are led to consider the equation

$$\frac{d^2z}{dx^2}-\phi z = 0, \tag{4}$$

where ϕ may be a complicated function of x. It is often useful to have an analytical approximation to the solution of this equation. We consider first the case in which ϕ is a slowly varying function of x which does not vanish in the range of x in which we are interested.

Making the substitution

$$z = \exp\left\{\int^x \eta\,dx\right\}, \tag{5}$$

suggested by the form of the solution of the first-order linear equation, we get from (4) the differential equation for η

$$\frac{d\eta}{dx}+\eta^2 = \phi. \tag{6}$$

If ϕ is slowly varying as we have assumed, $d\eta/dx$ will be small, and as a first approximation $\eta = \pm\phi^{\frac{1}{2}}$. Taking the positive sign, and substituting $\eta = \phi^{\frac{1}{2}}$ in the small term $d\eta/dx$ in (6), gives for the second approximation

$$\eta^2 = \phi-\frac{\phi'}{2\phi^{\frac{1}{2}}},$$

where ϕ' is written for $d\phi/dx$. Taking the square root we get

$$\eta = \phi^{\frac{1}{2}}-\frac{\phi'}{4\phi}, \tag{7}$$

to the same approximation, and (5) gives

$$z = \exp\left\{\int^x\left(\phi^{\frac{1}{2}}-\frac{\phi'}{4\phi}\right)dx\right\} = [\phi(x)]^{-\frac{1}{4}}\exp\left\{\int^x[\phi(x)]^{\frac{1}{2}}\,dx\right\}. \tag{8}$$

Approximate solutions of this type have been much used in quantum mechanics.

An entirely different type of solution arises near a zero of $\phi(x)$. Suppose that $\phi(x)$ has a simple zero at $x = a$, so that

$$\phi(x) = (x-a)\phi'(a), \tag{9}$$

near this point. Putting $\xi = x-a$, $\phi'(a) = k$, (4) becomes

$$\frac{d^2z}{d\xi^2}-k\xi z = 0. \tag{10}$$

If $\xi < 0$ and $k > 0$, putting $\xi = -\zeta$ the solution of (10) is by § 98 (38)

$$A\zeta^{\frac{1}{2}}J_{\frac{1}{3}}(\tfrac{2}{3}k^{\frac{1}{2}}\zeta^{\frac{3}{2}})+B\zeta^{\frac{1}{2}}J_{-\frac{1}{3}}(\tfrac{2}{3}k^{\frac{1}{2}}\zeta^{\frac{3}{2}}), \tag{11}$$

while in the same way if $\xi > 0$ and $k > 0$ it is

$$C\xi^{\frac{1}{2}}I_{\frac{1}{3}}(\tfrac{2}{3}k^{\frac{1}{2}}\xi^{\frac{3}{2}})+E\xi^{\frac{1}{2}}K_{\frac{1}{3}}(\tfrac{2}{3}k^{\frac{1}{2}}\xi^{\frac{3}{2}}). \tag{12}$$

These solutions are often required in a study of the behaviour of waves near their point of reflection.

104. Linear differential equations with periodic coefficients

The theory of these equations is relatively difficult and cannot be given here. At the same time a surprisingly large number of mechanical problems involve differential equations of this type, and it is desirable to indicate the new phenomena which arise. This is done below, briefly, and without proof.

We consider Mathieu's equation†

$$\frac{d^2y}{dt^2}+(\lambda+\gamma\cos\omega t)y = 0, \tag{1}$$

in which λ and γ are constants and so the coefficient of y is periodic with period $2\pi/\omega$. We remark first that, since the equation is linear, the general solution will as usual be a sum of arbitrary constant multiples of two linearly independent solutions. Secondly, while it is true that (1) has solutions with period $2\pi/\omega$, it is not true (as might perhaps be supposed) that all solutions of (1) are periodic: in fact here we shall discuss only the

† There is no standard notation for this equation: many different ones have been used. For an account of Mathieu functions see, for example, Whittaker and Watson, *Modern Analysis* (Cambridge); McLachlan, *Mathieu Functions* (Oxford, 1949).

non-periodic solutions. The equation (1) also arises in the solution of Laplace's and Maxwell's equations for an elliptic boundary, and in such problems it is the periodic solutions which are of interest.

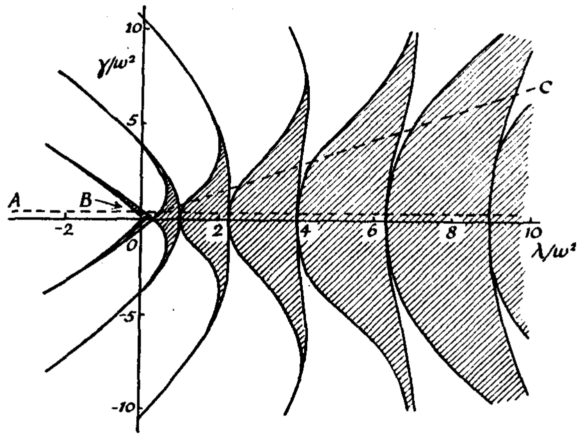


FIG. 83.

There is a theorem (Floquet's theorem) which states that the general solution of (1) has the form

$$y = A\,\phi(t)e^{\mu t} + B\,\phi(-t)e^{-\mu t}, \tag{2}$$

where A and B are arbitrary constants, $\phi(t)$ is a periodic function of t with period $2\pi/\omega$, and μ is a constant depending on λ and γ.

The way in which μ depends on these quantities is shown in Fig. 83 in which the (λ, γ)-plane is divided into shaded and unshaded regions.

If the point whose abscissa is λ/ω^2 and whose ordinate is γ/ω^2 falls in a shaded region, μ is pure imaginary, and the solution (2) is finite for all values of t. These shaded regions are called the *stable* regions for the equation.

On the other hand, if the point $(\lambda/\omega^2, \gamma/\omega^2)$ falls in an unshaded region, μ is complex, and thus one or other of the terms in (2) will increase without limit as t increases; the solution is unstable, and the unshaded regions in Fig. 83 are called the unstable regions for the equation.

A great deal of information can be obtained from a study of this figure. For example, as ω^2 is increased from 0 to ∞, λ/ω^2 will decrease from ∞ to 0 if λ is positive, and thus the point $(\lambda/\omega^2, \gamma/\omega^2)$ will pass through a number of unstable regions: for instance, if $\lambda/\omega^2 = \frac{1}{4}$, so that $\omega = 2\sqrt{\lambda}$ and therefore the frequency $\omega/2\pi$ is twice the natural frequency of the system with $\gamma = 0$, the system is usually unstable. These points are discussed more fully in the examples below.

Equation (1) may be regarded as the equation of motion of a mass attached to a spring whose stiffness varies harmonically; similar results occur if the stiffness varies in any periodic fashion, or if the mass or damping coefficient varies periodically.

Ex. 1. *The inverted pendulum with vertical harmonic motion of the support.*

Suppose the pendulum to consist of a mass m at the end of a light rigid rod of length l. Let θ be the inclination of the rod to the upward vertical, and ξ the upward displacement of the point of support.

The equations for vertical and horizontal motion of the mass m are

$$m\,\frac{d^2}{dt^2}\{\xi+l\cos\theta\} = -mg+P\cos\theta,$$

$$m\,\frac{d^2}{dt^2}(l\sin\theta) = P\sin\theta,$$

where P is the stress in the rod. These give

$$\ddot{\xi}-l\ddot{\theta}\sin\theta-l\dot{\theta}^2\cos\theta = -g+(P/m)\cos\theta,$$

$$l\ddot{\theta}\cos\theta-l\dot{\theta}^2\sin\theta = (P/m)\sin\theta.$$

Eliminating P we get

$$\ddot{\theta}-\left(\frac{g}{l}+\frac{\ddot{\xi}}{l}\right)\sin\theta = 0. \tag{3}$$

For small oscillations we replace $\sin\theta$ by θ.

If $\xi = a\cos\omega t$, (3) becomes

$$\ddot{\theta}-\left(n^2-\frac{a\omega^2}{l}\cos\omega t\right)\theta = 0, \tag{4}$$

where $n^2 = g/l$.

This corresponds to (1) with $\lambda = -n^2$ and $\gamma = a\omega^2/l$.

Suppose, for example, that a/l, the ratio of the amplitude of the vibration to the length of the pendulum, is $\frac{1}{2}$. Then $\gamma/\omega^2 = \frac{1}{2}$, and this is the ordinate of the line AB in Fig. 83. λ/ω^2, the abscissa in Fig. 83, is $-n^2/\omega^2$, and as ω is increased the point $(\gamma/\omega^2, \lambda/\omega^2)$ moves to the right along the line AB and finally enters a stable region at the point B. Thus if the frequency of oscillation is increased sufficiently, an inverted pendulum can be stabilized.

Ex. 2. *The hanging pendulum with vertical harmonic motion of the support.*

If the pendulum hangs in the usual way and its point of support is given a downwards vertical motion $a\cos\omega t$, the equation of motion, found as above, is

$$\ddot{\theta}+\left(n^2+\frac{a\omega^2}{l}\cos\omega t\right)\theta = 0. \tag{5}$$

This corresponds to (1) with $\lambda = n^2$, $\gamma = a\omega^2/l$. We have a positive ordinate as before, but $\lambda/\omega^2 = n^2/\omega^2$ is now positive, and is large for small ω. As ω is increased this decreases, and, for the ordinate shown, passes mostly through stable regions, but crosses narrow bands of instability near $n^2/\omega^2 = \frac{1}{4}, 1, \frac{9}{4},\ldots$, corresponding to $\omega = 2n, n, \frac{2}{3}n,\ldots$. In these regions the motion is unstable, and oscillations of large amplitude can be excited.

Ex. 3. *Forced oscillations in an L, C circuit due to a harmonically varying capacitance.*

The differential equation § 41 (6) for the charge on the condenser is

$$L\frac{d^2Q}{dt^2}+\frac{Q}{C} = 0. \tag{6}$$

Suppose the capacitance is varied in such a way that

$$\frac{1}{C} = \frac{1}{C_0}+\frac{1}{C_1}\cos\omega t.$$

Then (6) becomes

$$\frac{d^2Q}{dt^2}+\left(\frac{1}{LC_0}+\frac{1}{LC_1}\cos\omega t\right)Q = 0. \tag{7}$$

Writing $\lambda = 1/LC_0$, $\gamma = 1/LC_1$, we regain (1). Suppose now that the ratio $C_0/C_1 = k$ is fixed so that $\gamma = k\lambda$, and that ω is steadily increased from zero. The point $(\lambda/\omega^2, \gamma/\omega^2)$ then travels in towards the origin along the line CO of Fig. 83, passing through a number of unstable regions. When ω is such that the point lies in one of these, the circuit will oscillate.

EXAMPLES ON CHAPTER XII

1. Show that

$$\frac{d}{dx}\{x^\nu J_\nu(x)\} = x^\nu J_{\nu-1}(x), \qquad \frac{d}{dx}\{x^{-\nu}J_\nu(x)\} = -x^{-\nu}J_{\nu+1}(x),$$

$$\int x^{\nu+1}J_\nu(x)\,dx = x^{\nu+1}J_{\nu+1}(x), \qquad \int x^{1-\nu}J_\nu(x)\,dx = -x^{1-\nu}J_{\nu-1}(x).$$

2. Show that

$$J_{\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left\{\frac{\sin x}{x}-\cos x\right\},$$

$$J_{-\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left\{-\sin x-\frac{\cos x}{x}\right\}.$$

3. Show that $x^{\frac{1}{2}}J_1(2x^{\frac{1}{2}})$ and $x^{\frac{1}{2}}Y_1(2x^{\frac{1}{2}})$ satisfy

$$x\frac{d^2y}{dx^2}+y = 0.$$

4. Show that $x^{\frac{1}{2}}J_{1/2\beta}(\gamma x^\beta)$ and $x^{\frac{1}{2}}Y_{1/2\beta}(\gamma x^\beta)$ satisfy

$$\frac{d^2y}{dx^2}+\beta^2\gamma^2x^{2\beta-2}y = 0,$$

where β and γ are positive constants.

5. Kelvin's ber and bei functions (which occur in the theory of the skin effect in alternating currents) are defined by

$$\operatorname{ber} x+i\operatorname{bei} x = I_0(x\sqrt{i}).$$

Show that

$$\operatorname{ber} x = 1-\frac{(\frac{1}{2}x)^4}{(2!)^2}+\frac{(\frac{1}{2}x)^8}{(4!)^2}-\dots,$$

$$\operatorname{bei} x = (\tfrac{1}{2}x)^2-\frac{(\frac{1}{2}x)^6}{(3!)^2}+\frac{(\frac{1}{2}x)^{10}}{(5!)^2}-\dots.$$

6. If $P(x)$ and $Q(x)$ are any two solutions of Bessel's equation, show that

$$P(x)Q'(x)-P'(x)Q(x) = C/x,$$

where C is a constant.

[Write down the equations satisfied by P and Q, multiply the former by Q and the latter by P, and subtract; use the form of Bessel's equation given in Ex. 10.] § 98 (34) is found in this way, the value of the constant C being found from the series expansions of $J_\nu(x)$ and $Y_\nu(x)$. Deduce from § 98 (34) that

$$J_{\nu+1}(x)Y_\nu(x)-J_\nu(x)Y_{\nu+1}(x) = 2/\pi x.$$

7. Show that

$$\exp\tfrac{1}{2}x(t-t^{-1}) = J_0(x)+\sum_{n=1}^{\infty}\{t^n+(-t)^{-n}\}J_n(x).$$

[Expand the left-hand side in the power series and collect terms in t^n.] The function is called the generating function; compare the similar result § 99 (19) for Legendre polynomials.

8. By putting $t = e^{i\phi}$ in the result of Ex. 7, show that

$$\cos(x\sin\phi) = J_0(x)+2J_2(x)\cos 2\phi+2J_4(x)\cos 4\phi+\dots,$$

$$\sin(x\sin\phi) = 2J_1(x)\sin\phi+2J_3(x)\sin 3\phi+\dots.$$

Replacing ϕ in these by $\frac{1}{2}\pi-\phi$, deduce two similar formulae, and also

$$\exp(ix\cos\phi) = J_0(x)+2\sum_{s=1}^{\infty}i^sJ_s(x)\cos s\phi.$$

9. By multiplying results of Ex. 8 by $\cos n\phi$ and $\sin n\phi$, and assuming that the series can be integrated term by term, show that if r is zero or

any positive integer

$$\int_0^\pi \cos n\phi \cos(x\sin\phi)\,d\phi = \pi J_n(x) \qquad (n = 2r),$$
$$= 0 \qquad (n = 2r+1),$$
$$\int_0^\pi \sin n\phi \sin(x\sin\phi)\,d\phi = 0 \qquad (n = 2r),$$
$$= \pi J_n(x) \qquad (n = 2r+1),$$
$$\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi = \pi J_n(x).$$

The results of this example and the last are fundamental in the theory of frequency modulation.

10. Show that $J_n(\alpha r)$ satisfies

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{dJ_n(\alpha r)}{dr}\right\}+\left(\alpha^2-\frac{n^2}{r^2}\right)J_n(\alpha r) = 0,$$

if α is any constant. Show that

$$\int_0^a rJ_n(\alpha r)J_n(\beta r)\,dr = 0,$$

where α and β are any two different roots of $J_n(ax) = 0$. [Multiply the equation for $J_n(\alpha r)$ by $J_n(\beta r)$, multiply the corresponding equation for $J_n(\beta r)$ by $J_n(\alpha r)$, subtract, and integrate by parts.] Prove that the above result holds also if α and β are roots of $J_n'(ax) = 0$ or of

$$xJ_n'(ax)+hJ_n(ax) = 0,$$

where h is a positive constant.

11. Writing $u = J_n(\alpha r)$ for shortness, show, by multiplying the first equation of Ex. 10 by $2r^2(du/dr)$, that

$$\frac{d}{dr}\left(r\frac{du}{dr}\right)^2+\alpha^2r^2\frac{du^2}{dr}-n^2\frac{du^2}{dr} = 0.$$

By integrating this equation from 0 to a show that

$$\int_0^a r\{J_n(\alpha r)\}^2\,dr = \tfrac{1}{2}a^2[J_n'(\alpha a)]^2$$

if α is a root of $J_n(ax) = 0$.

12. Assuming that $f(r)$ can be expanded in the series

$$f(r) = \sum_{n=1}^{\infty} a_n J_0(r\alpha_n),$$

where $\alpha_1, \alpha_2,\ldots$ are the positive roots of $J_0(ax) = 0$, show, by using the results of Exs. 10 and 11, that

$$a_n = \frac{2}{a^2J_1^2(a\alpha_n)}\int_0^a rJ_0(r\alpha_n)f(r)\,dr.$$

The series is called a Fourier–Bessel series; cf. Chap. XI and § 99 (28) for similar results.

13. Show that
$$1 = \frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(r\alpha_n)}{\alpha_n J_1(a\alpha_n)},$$
where $\alpha_1, \alpha_2,\ldots$ are the positive roots of $J_0(ax) = 0$.

14. Prove that if n is a positive integer
$$nP_n(x) = x\frac{dP_n(x)}{dx} - \frac{dP_{n-1}(x)}{dx}.$$
[Equate the expansions for $(1-2hx+h^2)^{-\frac{3}{2}}$ obtained by differentiating § 99 (19) with respect to x, and with respect to h.]

Using this result, and differentiating § 99 (23), show that
$$(2n+1)P_n(x) = \frac{dP_{n+1}(x)}{dx} - \frac{dP_{n-1}(x)}{dx},$$
$$\int P_n(x)\,dx = \frac{1}{2n+1}\{P_{n+1}(x) - P_{n-1}(x)\}.$$

15. Using § 99 (19), show that if n is a positive integer
$$P_n(1) = 1,$$
$$P_n(-1) = 1 \quad \text{(if } n \text{ is even)},$$
$$= -1 \quad \text{(if } n \text{ is odd)}.$$
$$P_n(0) = 0 \quad \text{(if } n \text{ is odd)},$$
$$P_{2r}(0) = (-1)^r\frac{1.3\ldots(2r-1)}{2.4\ldots(2r)}.$$
Sketch the graphs of the first few $P_n(x)$ for $-1 < x < 1$.

16. Show that if $f(\theta)$ is defined for $0 \leqslant \theta \leqslant \pi$
$$f(\theta) = \tfrac{1}{2}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\,d\theta.$$
If
$$f(\theta) = 1 \quad (0 < \theta < \alpha),$$
$$= 0 \quad (\alpha < \theta < \pi),$$
show that (using the result of Ex. 14)
$$f(\theta) = \tfrac{1}{2}(1-\cos\alpha) + \tfrac{1}{2}\sum_{n=1}^{\infty}\{P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)\}P_n(\cos\theta).$$
And if $f(\theta) = \delta(\theta-\alpha)$, show that
$$f(\theta) = \tfrac{1}{2}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)P_n(\cos\alpha)\sin\alpha.$$

17. Show that $P_n(\cos\theta)$ has n zeros between $\theta = 0$ and $\theta = \pi$. Thus if it is represented on a sphere this is divided into $n+1$ zones by the zeros of $P_n(\cos\theta)$; for this reason this function is called a zonal harmonic. Show that if $P_n^m(\cos\theta)\cos m\phi$ and $P_n^m(\cos\theta)\sin m\phi$ are represented in the

same way, the sphere is divided into tesserae bounded by great circles through $\theta = 0$ and $\theta = \pi$ and small circles. Sketch the patterns for the cases $n = 1, 2, 3$. These functions are called tesseral harmonics.

18. By writing

$$(1-2h\cos\theta+h^2)^{-\frac{1}{2}} = (1-he^{i\theta})^{-\frac{1}{2}}(1-he^{-i\theta})^{-\frac{1}{2}}$$

and expanding both expressions on the right by the binomial theorem, show that

$$\tfrac{1}{2}P_n(\cos\theta) = \frac{1.3...(2n-1)}{2.4...2n}\cos n\theta+\frac{1}{2}\,\frac{1.3...(2n-3)}{2.4...(2n-2)}\cos(n-2)\theta+....$$

Deduce that $|P_n(\cos\theta)| \leqslant 1$ for all θ.

19. The rth Laguerre polynomial $L_r(x)$ is defined by

$$L_r(x) = e^x\,\frac{d^r}{dx^r}(x^r e^{-x});$$

verify that it satisfies the differential equation

$$x\,\frac{d^2y}{dx^2}+(1-x)\,\frac{dy}{dx}+ry = 0.$$

Show that its sth derivative

$$L_r^{(s)}(x) = \frac{d^s}{dx^s}L_r(x)$$

satisfies

$$x\,\frac{d^2y}{dx^2}+(s+1-x)\,\frac{dy}{dx}+(r-s)y = 0.$$

Show that § 100 (3) has the polynomial solution

$$\rho^l L_{k+l}^{(2l+1)}(\rho),$$

if k is any integer greater than or equal to $l+1$.

20. If

$$H_n(x) = (-1)^n e^{x^2}\,\frac{d^n}{dx^n}(e^{-x^2}),$$

show that $y = H_n(x)$ satisfies

$$\frac{d^2y}{dx^2}-2x\,\frac{dy}{dx}+2ny = 0.$$

Show also that

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty}\frac{1}{n!}H_n(x)t^n.$$

These are the Hermite polynomials which occur in Schrödinger's equation for the harmonic oscillator in the same way that the Laguerre polynomials of Ex. 19 occurred in the theory of the hydrogen atom.

21. If $\quad T_0(x) = 1, \qquad T_n(x) = 2^{1-n}\cos(n\cos^{-1}x),$

show that

$$\frac{1-t^2}{1-2tx+t^2} = \sum_{n=0}^{\infty}(2t)^n T_n(x),$$

and that $y = T_n(x)$ satisfies

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y = 0.$$

These are the Tchebycheff polynomials which are of importance, e.g. in the theory of filter circuits.

22. The Laguerre, Hermite, and Tchebycheff polynomials are all examples of 'orthogonal polynomials' in the sense that if two different ones are multiplied by an appropriate function and integrated over an appropriate range the integral vanishes (cf. § 88 (6), § 99 (26), and Ex. 10 above for similar results).

Show that

$$\int_{-1}^{1} T_n(x)T_m(x)(1-x^2)^{-\frac{1}{2}}\,dx = 0 \quad (n \neq m),$$

$$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)\,dx = 0 \quad (n \neq m),$$

$$\int_{0}^{\infty} e^{-x}L_n(x)L_m(x)\,dx = 0 \quad (n \neq m).$$

23. A particle of mass m is attached to the mid-point of a string of length l stretched to harmonically varying tension

$$T = T_0(1+k\cos\omega t).$$

Show that the system is unstable if $\omega = 2n/r$, approximately, where $r = 1, 2, 3,\ldots$, and $n^2 = 4T_0/ml$.

24. The point of support of a simple pendulum is rotated with constant angular velocity ω in a vertical circle of small radius a. Show that the motion is unstable if

$$\omega = 2n/r \quad (r = 1, 2, 3,\ldots),$$

approximately, where $n^2 = g/l$.

XIII

PARTIAL DIFFERENTIAL EQUATIONS

105. Introductory

In this chapter we shall give a brief account of the most important simple types of linear partial differential equations.† Naturally nothing more than a sketch can be given, but it is useful to know how such equations arise and how the methods given earlier may be used to solve them.

First we derive the most important equations in two variables, namely

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = 0, \tag{1}$$

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = 0, \tag{2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \tag{3}$$

where κ and c are constants. The first of these is *the diffusion equation*, the second *the wave equation*, and the third *Laplace's equation*. In (1) and (2), t is the time, and the equations have to be solved with *initial conditions* at $t = 0$ and *boundary conditions* at certain values of x. In (3), x and y are both space variables, and the equation has to be solved within a region in the (x, y)-plane, and with conditions at the boundary of the region.

The first point to notice is that the nature of the boundary conditions under which they can be solved, the methods of solution applicable to them, and the properties of their solutions, differ widely between the three types. The general linear second-order partial differential equation in x and y with constant coefficients is

$$a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial^2 v}{\partial y^2} + 2h\frac{\partial^2 v}{\partial x \partial y} + 2g\frac{\partial v}{\partial x} + 2f\frac{\partial v}{\partial y} + cv = 0. \tag{4}$$

† The standard works are Bateman, *Partial Differential Equations* (Cambridge, 1932); Frank–von Mises, *Differentialgleichungen der Physik* (Vieweg, 1930); Webster, *Partial Differential Equations of Mathematical Physics* (Teubner, 1927).

This is said to be of elliptic, parabolic, or hyperbolic type according as the conic

$$ax^2+by^2+2hxy+2gx+2fy+c = 0 \tag{5}$$

is an ellipse, parabola, or hyperbola, that is, according as $ab-h^2 \gtreqless 0$. This classification is intimately connected with the detailed theory of the nature of the solutions of the equation, and the three types have very different properties and methods of solution. It appears that (1) is of parabolic type, (2) of hyperbolic type, and (3) of elliptic type.

In §§ 106, 107, 110 the equations (1)–(3) are derived. Each of them has some simple special solutions which follow immediately from the equation and because of their simplicity and generality are of great physical importance. These are given with the equations. The equations for the uniform transmission line, which are of hyperbolic type and contain (1) and (2) as special cases, are given in § 109.

The general methods of solution are next discussed. Fourier series are applicable to all three equations if the range of the independent variable concerned is finite; if it is infinite, Fourier integrals are used in the same way. The Laplace transformation may be applied to (1) and (2) but is not very suitable for (3). But the method of conformal representation allows (3) to be solved for a wide variety of bounding surfaces.

In §§ 114 to 116 the equations corresponding to (1)–(3) but in two and three space dimensions are discussed briefly. Normally these can be solved only for regions bounded by the surfaces of some simple coordinate system, rectangular, spherical polar, cylindrical polar, elliptic, etc., in which case they can be split up into a number of equations in the separate coordinates. Fourier series and integrals and analogous expansions in Legendre and Bessel functions are used in the process of solution.

Finally the question of uniqueness must be mentioned. For each of the equations considered a *uniqueness theorem* can be proved which states that (subject, of course, to pure-mathematical restrictions on the nature of the functions involved) there is only one solution of a completely stated problem on a linear differential equation and boundary conditions. This might be regarded as obvious from the physical point of view. The importance of it is that it allows us to assert that if we can find a solution by any method, this is in fact the unique solution of the problem.

106. The equation of linear flow of heat. Simple solutions

We suppose heat to be flowing in the direction of the x-axis, the temperature being the same over any plane $x =$ constant. Let v be the temperature at the point x, and let K, ρ, and c be the thermal conductivity, density, and specific heat of the medium, which are assumed to be constant.

The fundamental assumption of the theory of conduction of heat is that the rate of flow of heat, per unit time per unit area, across the plane x is

$$-K\frac{\partial v}{\partial x}. \tag{1}$$

The differential equation is found by considering a region of unit area of the medium between the planes x and $x+\delta x$. Heat flows into this region across the plane x at the rate (1). It flows out of it across the plane $x+\delta x$ at the rate

$$-K\frac{\partial v}{\partial x}-\frac{\partial}{\partial x}\left(K\frac{\partial v}{\partial x}\right)\delta x. \tag{2}$$

Thus the region gains heat by flow across its surfaces at the rate

$$K\frac{\partial^2 v}{\partial x^2}\,\delta x \tag{3}$$

per unit time. This gain of heat causes a rise of temperature in the region, and since the thermal capacity of the region is $\rho c\,\delta x$, its rate of rise of temperature $\partial v/\partial t$ is

$$\frac{\partial v}{\partial t}=\frac{K\,\delta x}{\rho c\,\delta x}\frac{\partial^2 v}{\partial x^2}.$$

That is,

$$\frac{\partial^2 v}{\partial x^2}-\frac{1}{\kappa}\frac{\partial v}{\partial t}=0, \tag{4}$$

where

$$\kappa = K/\rho c \tag{5}$$

is called the diffusivity of the medium.

(4) is the required differential equation. It has to be solved

in some region such as $0 < x < l$, $x > 0$, or $-\infty < x < \infty$, with a given initial value of $v(x,t)$ at the instant $t = 0$, and with boundary conditions at the ends of the region. The usual boundary conditions are:

(i) Prescribed temperature v. This may be constant or a given function of the time.

(ii) Prescribed rate of flow of heat. In this case, by (1), $\partial v/\partial x$ is prescribed. If there is no flow of heat, $\partial v/\partial x = 0$.

(iii) The rate of flow of heat proportional to the temperature difference between the solid and its surroundings which are at v_0, that is

$$K\frac{\partial v}{\partial x}+H(v-v_0) = 0, \tag{6}$$

where H is a constant.

(iv) If the region extends to infinity the temperature must be finite there.

The differential equation (4) and the boundary conditions above are all linear, and it is this fact which makes it possible to go so far with the theory. In practice non-linear boundary conditions of type

$$K\frac{\partial v}{\partial x}+H(v-v_0)^n = 0 \tag{7}$$

often arise, but little can be done with these.

In this section we give some simple solutions of (4) for the infinite region $-\infty < x < \infty$, and the semi-infinite region $x > 0$, which are of great practical importance. They depend on the fact that

$$v = t^{-\frac{1}{2}}e^{-(x-x')^2/4\kappa t}, \tag{8}$$

where x' is a constant, satisfies the equation (4). This may be verified immediately by differentiation. Thus from (8)

$$\frac{\partial v}{\partial t} = \left\{-\frac{1}{2t^{\frac{3}{2}}}+\frac{(x-x')^2}{4\kappa t^{\frac{5}{2}}}\right\}e^{-(x-x')^2/4\kappa t},$$

$$\frac{\partial v}{\partial x} = -\frac{(x-x')}{2\kappa t^{\frac{3}{2}}}e^{-(x-x')^2/4\kappa t},$$

$$\frac{\partial^2 v}{\partial x^2} = \left\{-\frac{1}{2\kappa t^{\frac{3}{2}}}+\frac{(x-x')^2}{4\kappa^2 t^{\frac{5}{2}}}\right\}e^{-(x-x')^2/4\kappa t} = \frac{1}{\kappa}\frac{\partial v}{\partial t},$$

as required.

Thus (8) satisfies the equation of conduction of heat for all values of x'. We next find a physical interpretation for it. The total quantity of heat per unit area in the region $-\infty < x < \infty$ when the temperature is given by (8) is

$$Q = \rho c \int_{-\infty}^{\infty} v\,dx$$

$$= \frac{\rho c}{t^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-(x-x')^2/4\kappa t}\,dx = 2\rho c \kappa^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\xi^2}\,d\xi = 2\rho c(\kappa\pi)^{\frac{1}{2}}, \quad (9)$$

and thus is independent of the time.

Also, from (8), if $x \neq x'$, $v \to 0$ as $t \to 0$; but if $x = x'$, $v = t^{-\frac{1}{2}}$ and $v \to \infty$ as $t \to 0$. Therefore, multiplying (8) by a constant determined by (9), we get the result that

$$\frac{Q}{2\rho c(\pi\kappa t)^{\frac{1}{2}}} e^{-(x-x')^2/4\kappa t} \quad (10)$$

is a solution of the equation of conduction of heat which corresponds to releasing instantaneously when $t = 0$ a quantity of heat Q per unit area on the plane $x = x'$, the solid being at zero temperature when $t = 0$.

From this elementary solution many important results can be derived. For example, suppose that the solid $-\infty < x < \infty$ has temperature $f(x)$ when $t = 0$, and we wish to find its temperature subsequently. This initial temperature may be produced by liberating an amount of heat $Q = \rho c\, f(x')\,\delta x'$ per unit area in the region between each pair of planes $(x', x'+\delta x')$. Putting this value of Q in (10) and integrating with respect to x', the temperature at x in the solid at time t is found to be

$$\frac{1}{2(\pi\kappa t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x')e^{-(x-x')^2/4\kappa t}\,dx'. \quad (11)$$

As a second example on these elementary solutions we note that, since (8) satisfies (4), all its derivatives and integrals will do so, and in fact they all have a fundamental significance in the theory. Thus

$$\int^{x} e^{-x^2/4\kappa t}\frac{dx}{2\sqrt{(\kappa t)}} = \int^{x/2(\kappa t)^{\frac{1}{2}}} e^{-\xi^2}\,d\xi \tag{12}$$

will satisfy (4).

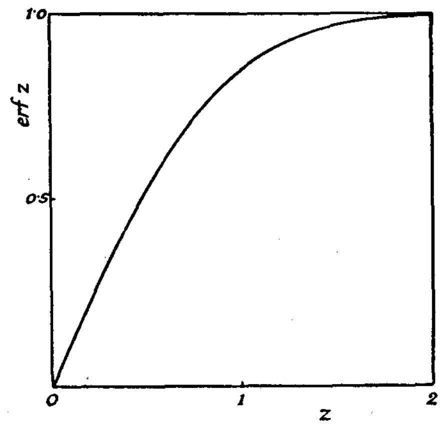


FIG. 84.

The function $$\operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}\,d\xi \tag{13}$$

is called the error function and is tabulated; its graph is shown in Fig. 84. Its principal properties are

$$\operatorname{erf} z \to 1 \quad \text{as } z \to \infty, \tag{14}$$

$$\operatorname{erf} 0 = 0. \tag{15}$$

It follows from (12) that

$$v = V_0 \operatorname{erf}\frac{x}{2(\kappa t)^{\frac{1}{2}}}, \tag{16}$$

where V_0 is a constant, is a solution of the equation of conduction of heat. By (14) and (15) this solution has the properties

$$v = 0, \quad \text{when } x = 0, \quad \text{for all } t > 0,$$

$$v = V_0, \quad \text{when } t = 0, \quad \text{for all } x > 0.$$

Thus (16) is the solution of the problem of *the region* $x > 0$ *with initial temperature* V_0 *and with the surface* $x = 0$ *kept at zero temperature for* $t > 0$.

In the same way, the solution of the problem of *the region* $x > 0$ *with zero initial temperature and with the surface* $x = 0$ *kept at constant temperature* V_0 *for* $t > 0$ *is*

$$v = V_0\left(1-\operatorname{erf}\frac{x}{2\sqrt{(\kappa t)}}\right). \tag{17}$$

As a final example of a different type we study the *steady periodic oscillations in the temperature in the semi-infinite solid* $x > 0$ *due to periodic surface temperature.* The results have important applications to the annual and diurnal fluctuations of soil temperature, and to temperatures in the cylinder walls of reciprocating engines.

Following the usual procedure for finding steady periodic solutions, we seek a solution of (4) of the form

$$v = V(x)e^{i\omega t}, \tag{18}$$

where $V(x)$ is a function of x only. Substituting (18) in (4) gives the differential equation for V,

$$\frac{d^2V}{dx^2}-\frac{i\omega}{\kappa}V = 0. \tag{19}$$

The general solution of (19) is

$$V = Ae^{(1+i)x(\omega/2\kappa)^{\frac{1}{2}}}+Be^{-(1+i)x(\omega/2\kappa)^{\frac{1}{2}}},$$

and, since V must be finite as $x \to \infty$, A must be zero. Therefore

$$v = Be^{-(1+i)x(\omega/2\kappa)^{\frac{1}{2}}+i\omega t}.$$

Taking the imaginary part we find that

$$v = V_0\, e^{-x(\omega/2\kappa)^{\frac{1}{2}}}\sin\left\{\omega t-x\left(\frac{\omega}{2\kappa}\right)^{\frac{1}{2}}\right\} \tag{20}$$

is the required solution for harmonic surface temperature

$$v = V_0 \sin \omega t.$$

The temperature oscillations at depth x diminish in amplitude as x increases, and they lag in phase by an increasing amount behind the surface oscillations.

Many important extensions of (4) may easily be derived in the same way. If heat is produced in the solid at a rate $A(x,t)$ per unit time per unit volume, (4) is replaced by

$$\frac{\partial^2 v}{\partial x^2}-\frac{1}{\kappa}\frac{\partial v}{\partial t} = -\frac{A(x,t)}{K}. \tag{21}$$

Clearly (4) and (21) hold for a rod with no loss of heat from its surface. If the rod is so thin that its temperature is uniform across its cross-section and it loses heat from its surface at a rate proportional to its temperature, (21) is replaced by

$$\frac{\partial^2 v}{\partial x^2}-\frac{\nu}{\kappa}v-\frac{1}{\kappa}\frac{\partial v}{\partial t} = -\frac{A(x,t)}{K}, \tag{22}$$

where ν is a constant depending on the size and material of the rod.

Differential equations of type (4) arise in many other connexions, notably in the theory of laminar motion of viscous fluid and in the theory of diffusion. In the latter case, if c is the concentration of a dissolved substance, the fundamental assumption is that the rate at which this substance crosses any plane is

$$-D\frac{\partial c}{\partial x}, \tag{23}$$

where D is the diffusion constant. Then in the same way the differential equation for c is found to be

$$\frac{\partial^2 c}{\partial x^2}-\frac{1}{D}\frac{\partial c}{\partial t} = 0. \tag{24}$$

For this reason the differential equation (4) is often called the diffusion equation.

107. The wave equation in one dimension. Simple solutions

This equation appears in many connexions with different notations, for example in the longitudinal vibrations of rods, transverse vibrations of stretched strings, sound waves, water waves, etc.

We derive it first for the longitudinal vibrations of a bar of uniform cross-section. Let u be the displacement of the plane of the bar whose normal position is at x, and let $u+\delta u$ be the displacement of the plane $x+\delta x$. Let X be the stress in the bar across the plane whose normal position is at x, then, by Hooke's law,

$$X = E\,\frac{\delta u}{\delta x},$$

where E is Young's modulus. That is, in the limit as $\delta x \to 0$

$$X = E\,\frac{\partial u}{\partial x}. \tag{1}$$

We now find the equation of motion of the element of the bar whose normal position is between the planes x and $x+\delta x$. The displacement of this element is u, and its mass is $\rho\,\delta x$ per unit area of the bar, where ρ is the density of the material of the bar. The forces per unit area on the element are $-X$ on the face x, and

$$X+\frac{\partial X}{\partial x}\,\delta x$$

on the face $x+\delta x$. Thus its equation of motion is

$$\rho\,\delta x\,\frac{\partial^2 u}{\partial t^2} = \frac{\partial X}{\partial x}\,\delta x,$$

or, using (1),

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{2}$$

where

$$c^2 = E/\rho. \tag{3}$$

The common boundary conditions are:

(i) prescribed displacement u;

(ii) prescribed stress $E\,\partial u/\partial x$;

(iii) the bar attached to a mass ma, where a is the area of the bar. In this case the equation of motion of this mass gives a boundary condition

$$m\,\frac{d^2 u}{dt^2} = -X = -E\,\frac{\partial u}{\partial x}. \tag{4}$$

Before discussing the solution of (2) we derive the corresponding equation for transverse vibrations of a stretched string. Let T be the tension in the string, σ its line of density, y the displacement at the point x, and ψ the slope of the tangent at this point.

Consider an element of the string of length δs at x. The forces on this in the direction of y increasing are

$$T\sin(\psi+\delta\psi)-T\sin\psi = T\,\delta\psi, \tag{5}$$

since ψ is small. Therefore its equation of motion is

$$\sigma\,\delta s\,\frac{\partial^2 y}{\partial t^2} = T\,\delta\psi,$$

or in the limit as $\delta s \to 0$

$$\sigma\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}, \tag{6}$$

where ρ, the radius of curvature of the string, is $\partial s/\partial\psi$. Also, since the string is nearly straight,

$$\frac{1}{\rho} = \frac{\partial^2 y}{\partial x^2},$$

very nearly, and (6) becomes

$$\frac{\partial^2 y}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0, \tag{7}$$

where
$$c^2 = T/\sigma. \tag{8}$$

We now consider some simple solutions of (2) or (7). These depend on the fact that, if $f(x)$ is any differentiable function of x, it follows immediately by differentiating that

$$f(x+ct) \tag{9}$$

satisfies (7). Physically it represents a disturbance, whose form when $t = 0$ is $f(x)$, travelling to the left with undisturbed shape and with velocity c. In the same way

$$F(x-ct) \tag{10}$$

is also a solution, and represents a wave travelling to the right with velocity c.

Thus the general solution of (7) may be regarded as a combination of motions to the left and right given by (9) and (10).

These results enable us to find the solution of (7) for an infinite string $-\infty < x < \infty$ set in motion at $t = 0$ with initial displacement $y = \phi(x)$ and initial velocity $\partial y/\partial t = \psi(x)$. Taking for the general solution

$$y = f(x+ct)+F(x-ct), \tag{11}$$

the initial conditions require

$$f(x)+F(x) = \phi(x), \tag{12}$$

$$cf'(x)-cF'(x) = \psi(x). \tag{13}$$

Integrating (13) gives

$$cf(x)-cF(x) = \int^{x} \psi(x)\,dx. \tag{14}$$

From (12) and (14),

$$f(x) = \tfrac{1}{2}\phi(x)+\frac{1}{2c}\int^{x} \psi(x)\,dx,$$

$$F(x) = \tfrac{1}{2}\phi(x)-\frac{1}{2c}\int^{x} \psi(x)\,dx.$$

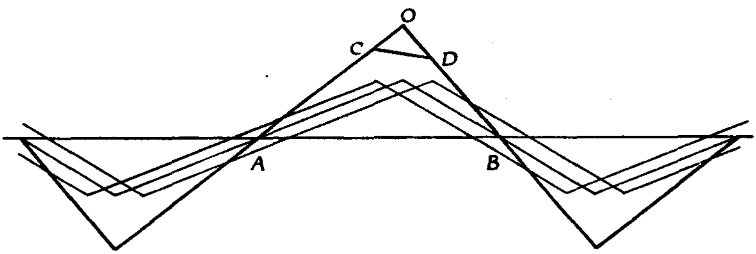


FIG. 85.

Using these values in (11), we get finally

$$y = \tfrac{1}{2}\{\phi(x+ct)+\phi(x-ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x)\,dx. \tag{15}$$

For the case in which $\psi(x) = 0$, the solution corresponds to two waves, each of half the original wave form, propagated to the left and right respectively.

This type of solution for the infinite string may be extended to give the solution for a finite string of length l.

Suppose that the string is fixed at A and B, and is plucked at one point O as in Fig. 85. Suppose that we repeat this pattern indefinitely to make an odd periodic function of period $2l$, and regard this as an initial displacement of an infinite string. Then the general solution (15) consists of half the displacement moving to the right and half moving to the left. From the figure it appears that the sums of these displacements at A and B are both zero, so that the solution obtained in this way is a solution of (7) with $y = 0$ at A and B, and thus is the solution of our problem. The form of the string is shown in Fig. 85; it consists of three straight portions, AC, CD, DB.

Finally, a fundamental distinction between the results of this section and the last should be pointed out. Solutions of the wave equation are propagated with finite velocity c, so that if portion of the medium is initially undisturbed it remains so until the wave front reaches it. In diffusion problems, on the other hand, there is theoretically a disturbance at all points at all times, though for large distances this will be negligibly small.

108. The wave equation. Natural frequencies

It was found in Chapter IV, when studying the vibrations of a system consisting of a finite number of masses, that such a system had a finite number of natural frequencies, that with each frequency there was associated a normal mode of vibration, and that the most general motion consisted of a sum of vibrations of these types with coefficients determined by the initial conditions.

The natural frequencies and normal modes were found by seeking solutions of the equations of motion in which all quantities were proportional to $e^{i\omega t}$. The same procedure applies to the wave equation in one or more dimensions. As an example consider the vibrations of a stretched string whose ends, $x = 0$ and $x = l$, are fixed. The differential equation § 107 (7) is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0 \quad (0 < x < l), \tag{1}$$

with
$$y = 0, \text{ when } x = 0 \text{ and } x = l. \tag{2}$$

We seek a solution of this of the form

$$y = Y(x)e^{i\omega t}, \tag{3}$$

and, substituting in (1) and (2), Y must satisfy

$$\frac{d^2 Y}{dx^2} + \frac{\omega^2}{c^2} Y = 0, \tag{4}$$

with
$$Y = 0, \quad \text{when } x = 0, \tag{5}$$
$$Y = 0, \quad \text{when } x = l. \tag{6}$$

The general solution of (4) is
$$Y = A\sin\frac{\omega x}{c}+B\cos\frac{\omega x}{c}, \tag{7}$$
and (5) requires $B = 0$. Thus by (6) we must have
$$A\sin\frac{\omega l}{c} = 0, \tag{8}$$
so that either $A = 0$, which gives the trivial solution $Y = 0$, or
$$\sin\frac{\omega l}{c} = 0,$$
that is,
$$\omega = \frac{n\pi c}{l} \quad (n = 1, 2, 3,...). \tag{9}$$

The problem is an eigenvalue problem (cf. § 87). The eigenvalues (9) give the natural frequencies $(nc/2l)$, $n = 1, 2,...$, and the corresponding eigenfunctions,
$$A_n \sin\frac{n\pi x}{l} \quad (n = 1, 2,...), \tag{10}$$
are the normal modes of vibration. A combination of these, namely
$$\sum_{n=1}^{\infty}\left\{A_n\sin\frac{n\pi ct}{l}+B_n\cos\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}, \tag{11}$$
gives the most general solution of (1). The A_n and B_n in (11) are determined from the initial conditions by the use of Fourier series as in § 111. But, just as in the case of the vibrations of a finite number of masses, the natural frequencies and normal modes are usually easy to find and frequently supply as much information as is needed.

The normal modes (10) are shown in Fig. 86, which may be compared with Fig. 17 for a finite number of masses.

As an example of the determination of the natural frequencies in a more complicated case we consider *transverse vibrations of a uniform beam of length l freely hinged at its ends.*

Let y be the displacement of the point x of the beam, and ρ the mass per unit length of the beam. Then the reversed effective force, § 66 (13), at x due to the motion of the beam is

$$-\frac{\rho}{g}\frac{\partial^2 y}{\partial t^2} \tag{12}$$

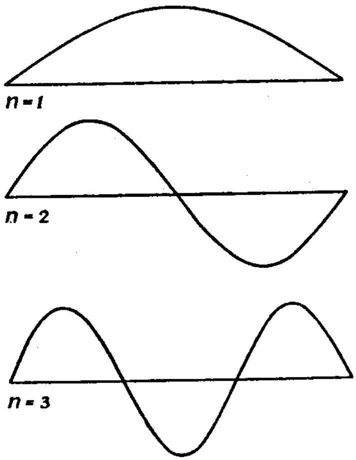


FIG. 86.

per unit length. Treating this as a load on the beam, and neglecting any static loads, § 80 (12) gives for the equation of transverse vibrations of the beam

$$EI\frac{\partial^4 y}{\partial x^4}+\frac{\rho}{g}\frac{\partial^2 y}{\partial t^2} = 0. \tag{13}$$

As before, we seek solutions of (13) of the form

$$y = Y(x)e^{i\omega t},$$

then Y has to satisfy

$$\frac{d^4Y}{dx^4}-k^4Y = 0, \tag{14}$$

where $k^4 = \omega^2\rho/EIg$, and

$$Y = \frac{d^2Y}{dx^2} = 0, \quad \text{when } x = 0 \text{ and } x = l. \tag{15}$$

This eigenvalue problem has been discussed in § 87, Ex. 3, and, using the results of that section, it appears that the eigenvalues are given by

$$kl = n\pi \quad (n = 1, 2,...). \tag{16}$$

Therefore the natural frequencies, $\omega/2\pi$, are

$$\frac{n^2\pi}{2l^2}\left(\frac{EIg}{\rho}\right)^{\frac{1}{2}} \quad (n = 1, 2,...). \tag{17}$$

Incidentally it appears that they are identical with the critical frequencies for whirling of the shaft.

109. The equations for the uniform transmission line

Suppose the line has inductance L, resistance R, capacitance C, and leakage conductance G per unit length. Let I be the current in the line and V the voltage drop across it at x, and let

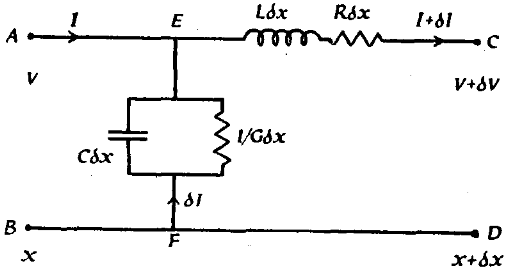


FIG. 87.

$I+\delta I$ and $V+\delta V$ be the corresponding quantities at $x+\delta x$. We may replace the portion of the line between x and $x+\delta x$ by the four-terminal network shown in Fig. 87. The circuit relations for this are

$$L\,\delta x\frac{\partial I}{\partial t}+R\,\delta x I = -\delta V, \tag{1}$$

$$C\,\delta x\frac{\partial V}{\partial t}+G\,\delta x V = -\delta I. \tag{2}$$

Dividing (1) and (2) by δx and taking the limit as $\delta x \to 0$ gives the equations

$$L\frac{\partial I}{\partial t}+RI = -\frac{\partial V}{\partial x}, \tag{3}$$

$$C\frac{\partial V}{\partial t}+GV = -\frac{\partial I}{\partial x}. \tag{4}$$

These are a pair of simultaneous linear partial differential equations of the first order for V and I. They have to be solved with given initial values of the current and voltage drop in the line as functions of x, and with boundary conditions at the ends of the line such as:

(i) prescribed voltage;
(ii) prescribed current;
(iii) a relation between I and V when the line is connected to a terminal impedance.

Either I or V may be eliminated from (3) and (4) and a second-order equation obtained. For example, eliminating I, the equation for V is

$$\frac{\partial^2 V}{\partial x^2}-LC\frac{\partial^2 V}{\partial t^2}-(LG+RC)\frac{\partial V}{\partial t}-RGV = 0. \tag{5}$$

In special cases this reduces to equations which have been considered earlier:

(i) For the 'lossless' line in which $R = G = 0$, it becomes

$$\frac{\partial^2 V}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0, \tag{6}$$

with $c^2 = 1/LC$, which is the wave equation of § 107.

(ii) For the ideal submarine cable in which $L = G = 0$ it becomes

$$\frac{\partial^2 V}{\partial x^2}-\frac{1}{\kappa}\frac{\partial V}{\partial t} = 0, \tag{7}$$

with $\kappa = 1/RC$, which is the diffusion equation of § 106.

(iii) For Heaviside's distortionless line in which $R/L = G/C$ it becomes

$$\frac{\partial^2 V}{\partial x^2}-LC\left(\frac{\partial}{\partial t}+\frac{R}{L}\right)^2 V = 0. \tag{8}$$

In these cases solutions of transient problems may be obtained by the methods of §§ 106, 107, 111, 112. The complete solution for the general equation (5) is complicated, and here we shall consider only the steady state behaviour for this case.

Suppose that complex voltage E', in the notation of § 44, is applied to the line at $x = 0$. Let V' be the complex voltage drop at the point x of the line and let I' be the complex current there:

these are now functions of x. $V'e^{i\omega t}$ and $I'e^{i\omega t}$ will be solutions of (3), (4), and (5), and of these (5) gives

$$\frac{d^2V'}{dx^2}-\gamma^2 V' = 0, \tag{9}$$

where

$$\gamma = \{(R+L\omega i)(G+C\omega i)\}^{\frac{1}{2}}, \tag{10}$$

and, for definiteness, the square root in (10) is chosen so that its real part is positive. The quantity γ is called the propagation constant of the line.

As a first example suppose that the line extends to infinity.

In the general solution

$$V' = Ae^{-\gamma x}+Be^{\gamma x} \tag{11}$$

of (9), B must be zero since V' must remain finite as $x \to \infty$. Also since $V' = E'$ when $x = 0$, $A = E'$ and we get finally

$$V' = E'e^{-\gamma x} \tag{12}$$

for the complex voltage drop at the point x of the line. The complex current I' is from (3)

$$I' = -\frac{1}{R+L\omega i}\frac{dV'}{dx} \tag{13}$$

$$= E'\left\{\frac{G+C\omega i}{R+L\omega i}\right\}^{\frac{1}{2}} e^{-\gamma x}. \tag{14}$$

The input impedance z_0 of the line, which is the ratio of E' to I' when $x = 0$, is given by

$$z_0 = \left(\frac{R+L\omega i}{G+C\omega i}\right)^{\frac{1}{2}}. \tag{15}$$

This quantity is called the characteristic impedance of the line.

Next we consider the case of the *line* $0 < x < l$, *with complex voltage* E' *applied at* $x = 0$ *as before, but terminated at* $x = l$ *by a complex impedance* z.

The boundary conditions are now

$$V' = E', \quad \text{when } x = 0, \tag{16}$$

$$zI' = V', \quad \text{when } x = l. \tag{17}$$

The solution of (9) is

$$V' = A\sinh\gamma x + B\cosh\gamma x. \tag{18}$$

The condition (16) gives $B = E'$, and (17) gives, using (13) and the notation (15),

$$-\frac{z}{z_0}\{A\cosh\gamma l + B\sinh\gamma l\} = A\sinh\gamma l + B\cosh\gamma l.$$

Solving for A and substituting in (18) gives finally

$$V' = \frac{E'\{z_0\sinh\gamma l + z\cosh\gamma l\}\cosh\gamma x - E'\{z\sinh\gamma l + z_0\cosh\gamma l\}\sinh\gamma x}{z_0\sinh\gamma l + z\cosh\gamma l}. \tag{19}$$

The input impedance is

$$\left[\frac{V'}{I'}\right]_{x=0} = z_0\frac{z_0\sinh\gamma l + z\cosh\gamma l}{z_0\cosh\gamma l + z\sinh\gamma l}. \tag{20}$$

If $z = 0$, i.e. short-circuit at $x = l$, the input impedance z_s is by (20)

$$z_s = z_0\tanh\gamma l, \tag{21}$$

while, if $z = \infty$, open circuit at $x = l$, the input impedance z_{op} is

$$z_{op} = z_0\coth\gamma l. \tag{22}$$

From (21) and (22) $\quad z_s z_{op} = z_0^2.$

110. Laplace's equation in two dimensions. Simple solutions

Laplace's equation arises in a very large number of contexts in mathematical physics. Perhaps the most fundamental of these is the expression of continuity in steady flow. As an example of this we consider the steady flow of heat in two dimensions.

Take rectangular axes OX, OY, and consider the rectangle bounded by the planes x, $x+\delta x$ and y, $y+\delta y$. Since the flow is steady the temperature is independent of time, and therefore the net flow of heat into this rectangle must be zero.

The rates of flow into the region over the faces x, $x+\delta x$, y, and $y+\delta y$ are, respectively, by § 106 (1),

$$-K\frac{\partial v}{\partial x}\,\delta y,$$

$$-\left\{-K\frac{\partial v}{\partial x}\,\delta y-\frac{\partial}{\partial x}\left[K\frac{\partial v}{\partial x}\,\delta y\right]\delta x\right\},$$

$$-K\frac{\partial v}{\partial y}\,\delta x,$$

$$-\left\{-K\frac{\partial v}{\partial y}\,\delta x-\frac{\partial}{\partial y}\left[K\frac{\partial v}{\partial y}\,\delta x\right]\delta y\right\}.$$

Adding these we have, if K is independent of x and y,

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0. \tag{1}$$

This is Laplace's equation in two dimensions in rectangular Cartesian coordinates.

Before discussing it, we derive the corresponding equation in polar coordinates. This can be done by transforming (1) into polars r, θ, or it may be obtained by applying the argument used above to the element of area bounded by the circles r and $r+\delta r$, and the rays θ and $\theta+\delta\theta$. The expression of the fact that the net rate of flow of heat into the region is zero is

$$-K\frac{\partial v}{\partial r}r\,\delta\theta-\left\{-K\frac{\partial v}{\partial r}r\,\delta\theta-\frac{\partial}{\partial r}\left[Kr\frac{\partial v}{\partial r}\,\delta\theta\right]\delta r\right\}-$$

$$-K\frac{\partial v}{r\,\partial\theta}\,\delta r-\left\{-K\frac{\partial v}{r\,\partial\theta}\,\delta r-\frac{\partial}{r\,\partial\theta}\left[-K\frac{\partial v}{r\,\partial\theta}\,\delta r\right]r\,\delta\theta\right\}=0.$$

That is,
$$r\frac{\partial}{\partial r}\left[r\frac{\partial v}{\partial r}\right]+\frac{\partial^2 v}{\partial\theta^2}=0, \tag{2}$$

or
$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}=0. \tag{3}$$

Laplace's equation in the form (1) has simple polynomial solutions
$$1,\ x,\ y,\ xy,\ x^2-y^2,\ x^3-3xy^2,\ldots.$$

In the form (2) it is satisfied by

$$r^n \frac{\cos}{\sin} n\theta$$

for any n, and also by $\log r$.

Ex. *Radial flow of heat in the hollow cylinder $a < r < b$. $r = a$ kept at v_1, and $r = b$ at v_2.*

The differential equation (2) gives, since the solution is to be independent of θ,

$$\frac{d}{dr}\left(r\frac{dv}{dr}\right) = 0,$$

$$r\frac{dv}{dr} = A,$$

$$v = A\ln r + B,$$

where A and B are unknown constants to be found from the conditions at $r = a$ and $r = b$. These give

$$v_1 = A\ln a + B,$$

$$v_2 = A\ln b + B,$$

and therefore
$$v = \frac{v_1\ln(b/r)+v_2\ln(r/a)}{\ln(b/a)}. \tag{4}$$

The rate of flow of heat through the cylinder per unit length is

$$-2\pi rK\frac{dv}{dr} = \frac{2\pi K(v_1-v_2)}{\ln b/a}. \tag{5}$$

Laplace's equation in two dimensions occurs also in the flow of current electricity in plane sheets, in the flow of viscous fluid between parallel planes, in the theory of torsion of shafts, in the deflexion of a soap film or a sheet of rubber, and in two-dimensional problems in hydrodynamics, electrostatics, and the flow of incompressible fluid through a porous medium.

The inhomogeneous equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = f(x,y) \tag{6}$$

is Poisson's equation in two variables. It arises, for example, in the steady flow of heat in a medium in which heat is being generated. Thus if heat is generated at a rate $f(x,y)$ per unit time per unit area in the medium, the net rate of loss of heat calculated in (1) is to be equated to this and we get

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = -\frac{1}{K}f(x,y). \tag{7}$$

In other fields the relation between Laplace's and Poisson's equations is the same: Poisson's equation occurs in regions where there are sources of heat or current, charges, etc., and reduces to Laplace's equation in regions free from such sources.

111. The use of Fourier series

Fourier series are applied in the same way to all the equations of §§ 106–10. A simple solution of the differential equation and some of the boundary conditions which is a product of trigonometrical or hyperbolic functions is written down; the solution of the problem is assumed to be a series of such terms, the coefficients in which are found from a Fourier series determined by the remaining boundary conditions.

To illustrate the method we first consider Laplace's equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0, \tag{1}$$

and seek a solution of this in the rectangle $0<x<a, 0<y<b$ which satisfies the boundary conditions

$$v=0, \quad \text{when } x=0 \quad (0<y<b), \tag{2}$$

$$v=0, \quad \text{when } x=a \quad (0<y<b), \tag{3}$$

$$v=0, \quad \text{when } y=b \quad (0<x<a), \tag{4}$$

$$v=f(x), \quad \text{when } y=0 \quad (0<x<a). \tag{5}$$

We notice first that

$$\sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a} \quad (n=1,2,...) \tag{6}$$

satisfies the differential equation (1) and the boundary conditions (2), (3), and (4). This suggests that the series

$$\sum_{n=1}^{\infty} A_n \frac{\sin n\pi x}{a}\sinh\frac{n\pi(b-y)}{a}, \tag{7}$$

where the A_n are unknown, will also satisfy them. When $y=0$, (7) reduces to

$$\sum_{n=1}^{\infty} A_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a}. \tag{8}$$

Now suppose that $f(x)$ is expanded in the sine series, § 89 (7),

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}, \tag{9}$$

where
$$b_n = \frac{2}{a}\int_0^a f(x')\sin\frac{n\pi x'}{a}\,dx'. \tag{10}$$

Comparing coefficients between (8) and (9) gives

$$A_n \sinh\frac{n\pi b}{a} = b_n,$$

and thus, finally, the solution of the differential equation and boundary conditions is

$$v = \sum_{n=1}^{\infty} b_n \frac{\sin(n\pi x/a)\sinh n\pi(b-y)/a}{\sinh n\pi b/a}, \tag{11}$$

where b_n is given by (10).

For example, if $v = 1$ on $y = 0$, $0 < x < a$, the solution is, using § 89 (9),

$$v = \frac{4}{\pi}\sum_{r=0}^{\infty} \frac{\sin[(2r+1)\pi x/a]\sinh[(2r+1)\pi(b-y)/a]}{(2r+1)\sinh[(2r+1)\pi b/a]}. \tag{12}$$

Of course the above argument is not rigorous and needs pure-mathematical justification; this is easily supplied for all the examples of this section.

If the writing down of (6) is considered too abrupt, it may be obtained by the following method, which will also be used in § 115 for the study of equations in three variables. We seek a solution of (1) which is the product of a function of x and a function of y, say
$$v = X(x)Y(y). \tag{13}$$

Substituting (13) in (1) we get

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2} = 0. \tag{14}$$

This is satisfied if

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2, \tag{15}$$

$$\frac{1}{Y}\frac{d^2Y}{dx^2} = k^2, \tag{16}$$

where k^2 is any number.

The general solution of (15) is

$$X = A\sin kx + B\cos kx. \tag{17}$$

If (17) is to satisfy the boundary conditions (2) and (3) we must have

$$B = 0, \tag{18}$$

$$A\sin ka = 0. \tag{19}$$

From (19), either $A = 0$, giving the trivial solution $v = 0$, or

$$k = \frac{n\pi}{a} \quad (n = 1, 2,...), \tag{20}$$

so that in (15) and (16), k must be $n\pi/a$ and X must have the form $A_n \sin n\pi x/a$ $(n = 1, 2,...)$. The numbers (20) are in fact the eigenvalues of the differential equation (15) with the boundary conditions (2) and (3), and the corresponding values of X are its eigenfunctions (cf. § 87).

The general solution of (16) with one of the values (20) of k may be written

$$Y = C\sinh\frac{n\pi(b-y)}{a} + D\cosh\frac{n\pi(b-y)}{a}, \tag{21}$$

and the condition (4) gives $D = 0$. Thus, finally, the solutions of (1)–(4) of type $v = X(x)Y(y)$ must be of type

$$v = A_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a} \quad (n = 1, 2,...), \tag{22}$$

in agreement with (6).

The same argument may be used to derive (28) and (38) below.

Ex. 1. *Conduction of heat in the region $0 < x < l$. The ends $x = 0$ and $x = l$ kept at zero temperature for $t > 0$. The initial temperature $f(x)$.*

We have to solve § 106 (4), namely

$$\frac{\partial^2 v}{\partial x^2}-\frac{1}{\kappa}\frac{\partial v}{\partial t}=0 \quad (0<x<l), \tag{23}$$

with $\qquad v=0, \quad \text{when } x=0 \text{ and } x=l, \quad t>0, \qquad (24)$

and with $\qquad v=f(x), \quad \text{when } t=0, \quad 0<x<l. \qquad (25)$

We suppose $f(x)$ to be expanded in the sine series § 89 (7)

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \tag{26}$$

where

$$b_n=\frac{2}{l}\int_0^l f(x')\sin\frac{n\pi x'}{l}\,dx'. \tag{27}$$

Now

$$e^{-\kappa n^2\pi^2 t/l^2}\sin\frac{n\pi x}{l} \tag{28}$$

satisfies (23) and the boundary conditions (24). Thus

$$\sum_{n=1}^{\infty} A_n e^{-\kappa n^2\pi^2 t/l^2}\sin\frac{n\pi x}{l} \tag{29}$$

does so also. When $t=0$, (29) has the value

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}. \tag{30}$$

Comparing coefficients between (26) and (30) we find $A_n=b_n$, and the solution is finally

$$v=\frac{2}{l}\sum_{n=1}^{\infty} e^{-\kappa n^2\pi^2 t/l^2}\sin\frac{n\pi x}{l}\int_0^l f(x')\sin\frac{n\pi x'}{l}\,dx'. \tag{31}$$

If the initial temperature is constant, V_0, this becomes

$$v=\frac{4V_0}{\pi}\sum_{r=0}^{\infty} e^{-\kappa(2r+1)^2\pi^2 t/l^2}\frac{\sin(2r+1)\pi x/l}{2r+1}. \tag{32}$$

Ex. 2. *The problem of Ex. 1 except that there is no flow of heat at $x=0$.*

The only change is that the boundary condition at $x=0$ is $\partial v/\partial x=0$. To satisfy this we take as the elementary solution

$$e^{-\kappa(2n+1)^2\pi^2 t/4l^2}\cos\frac{(2n+1)\pi x}{2l}, \tag{33}$$

and expand $f(x)$ in the cosine series of Chapter XI, Ex. 17.

Ex. 3. *A string is stretched between the points $x=0$ and $x=l$ and is set in motion at $t=0$ with initial displacement $f(x)$ and zero initial velocity.*

The differential equation, § 107 (7), is

$$\frac{\partial^2 y}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}=0 \quad (0<x<l), \tag{34}$$

to be solved with

$$y=0, \quad \text{when } x=0 \text{ and } x=l, \quad t>0, \tag{35}$$

$$\frac{\partial y}{\partial t}=0, \quad 0<x<l, \quad t=0, \tag{36}$$

$$y=f(x), \quad 0<x<l, \quad t=0. \tag{37}$$

(34) and (35) are satisfied by

$$\sin\frac{n\pi x}{l}\left\{A_n\cos\frac{n\pi ct}{l}+B_n\sin\frac{n\pi ct}{l}\right\} \quad (n=1,2,\ldots). \tag{38}$$

This has already been derived and its significance discussed in § 108. To satisfy (36) we must have $B_n = 0$ in (38). Thus (34), (35), and (36) are satisfied by

$$\sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \tag{39}$$

and when $t = 0$ this reduces to

$$\sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{l}. \tag{40}$$

If we expand $f(x)$ in the sine series (26) and (27), comparing coefficients with (40) gives $A_n = b_n$, and the final result is

$$y=\frac{2}{l}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}\int_0^l f(x')\sin\frac{n\pi x'}{l}\,dx'. \tag{41}$$

Suppose, as an example, that initially the string is plucked a distance d at $x = b$, so that

$$f(x)=dx/b \quad (0<x<b), \tag{42}$$

$$f(x)=d(l-x)/(l-b) \quad (b<x<l). \tag{43}$$

Evaluating the integral in (41) we get

$$y=\frac{2dl^2}{\pi^2 b(l-b)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi b}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}. \tag{44}$$

Ex. 4. *Laplace's equation in the infinite strip* $0<y<l$, $-\infty<x<\infty$, *with* $v=f(x)$ *on* $y=0$, *and* $v=0$ *on* $y=l$.

In this case the region is infinite instead of finite, and a Fourier integral must be used instead of a Fourier series. The solution is sketched below merely to show the correspondence.

We have to solve
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0, \tag{45}$$

with the boundary conditions stated above.

We notice that
$$\sinh\omega(l-y)e^{i\omega x} \tag{46}$$
satisfies (45), is finite as $x \to \pm\infty$ and vanishes when $y = l$ for all ω. Therefore
$$\int_{-\infty}^{\infty} \phi(\omega)\sinh\omega(l-y)e^{i\omega x}\,d\omega \tag{47}$$
does so also. To satisfy the boundary condition at $y = 0$ we must have
$$\int_{-\infty}^{\infty} \phi(\omega)\sinh l\omega e^{i\omega x}\,d\omega = f(x).$$
And, by § 96 (4) and (5), this gives
$$2\pi\phi(\omega)\sinh\omega l = \int_{-\infty}^{\infty} e^{-i\omega x'}f(x')\,dx', \tag{48}$$
and the solution is
$$v = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh\omega(l-y)}{\sinh\omega l}e^{i\omega x}\,d\omega \int_{-\infty}^{\infty} e^{-i\omega x'}f(x')\,dx'. \tag{49}$$

112. The use of the Laplace transformation

This is one of the most powerful methods for solving equations of parabolic and hyperbolic types. We suppose the equations have to be solved for $t > 0$ with initial conditions at $t = 0$. Suppose that $v(x,t)$ is the solution; its Laplace transform with respect to t will be
$$\bar{v} = \int_0^{\infty} e^{-pt}v\,dt, \tag{1}$$
which is a function of p and x. The Laplace transform of $\partial v/\partial t$ is
$$\int_0^{\infty} \frac{\partial v}{\partial t}e^{-pt}\,dt = [ve^{-pt}]_0^{\infty}+p\int_0^{\infty} ve^{-pt}\,dt$$
$$= -v_0(x)+p\bar{v}, \tag{2}$$
just as in § 18 (21), except that the initial value, $v_0(x)$, is now a function of x.

Similarly the Laplace transform of $\partial^2 v/\partial t^2$ is
$$\int_0^{\infty} \frac{\partial^2 v}{\partial t^2}e^{-pt}\,dt = -pv_0(x)-v_1(x)+p^2\bar{v}, \tag{3}$$
where $v_1(x)$ is the value of $\partial v/\partial x$ when $t = 0$.

Finally, the Laplace transform of $\partial^2 v/\partial x^2$ is

$$\int\limits_0^\infty e^{-pt}\frac{\partial^2 v}{\partial x^2}\,dt = \frac{d^2}{dx^2}\int\limits_0^\infty e^{-pt}v\,dt = \frac{d^2\bar{v}}{dx^2}, \tag{4}$$

assuming that the orders of differentiation and integration in (4) may be interchanged.

Now suppose we multiply our partial differential equation by e^{-pt} and integrate with respect to t from 0 to ∞. For the equation § 105 (1) we get, using (2) and (4),

$$\frac{d^2\bar{v}}{dx^2} - \frac{p}{\kappa}\bar{v} = -\frac{1}{\kappa}v_0(x). \tag{5}$$

From the equation § 105 (2) we get

$$\frac{d^2\bar{v}}{dx^2} - \frac{p^2}{c^2}\bar{v} = -\frac{p}{c^2}v_0(x) - \frac{1}{c^2}v_1(x). \tag{6}$$

And in the same way from § 109 (3) and (4) we get

$$\frac{d\bar{V}}{dx} + (Lp+R)\bar{I} = LI_0(x), \tag{7}$$

$$\frac{d\bar{I}}{dx} + (Cp+G)\bar{V} = CV_0(x), \tag{8}$$

where $I_0(x)$ and $V_0(x)$ are the values of I and V when $t = 0$.

(5), (6), and (7) and (8) are the *subsidiary equations* corresponding to the partial differential equations and their initial conditions. They have to be solved with boundary conditions which are the Laplace transforms of the given boundary conditions. When this has been done, the Laplace transform $\bar{v}$ of the solution v has been found. To find v from $\bar{v}$ two methods are available; (i) an extension of the Table of Transforms of § 18 together with the development of more theorems of the type of § 18, Theorem 1; (ii) what is in effect an extension of § 18 (28) which allows series of Fourier type for v to be written down from $\bar{v}$. To go into these points would take too long† and we merely solve one example to illustrate the way the method applies to an interesting type of problem.

† For further details see Carslaw and Jaeger, *Operational Methods in Applied Mathematics* (Oxford, 1948); Churchill, *Modern Operational Mathematics in Engineering* (McGraw-Hill, 1944); Gardner and Barnes, *Transients in Linear Systems* (Wiley, 1942).

Ex. *A bar of length l with its end $x = l$ fixed is at rest and unstrained, when at $t = 0$ the end $x = 0$ is given a small displacement a.*

The differential equation, § 107 (2),

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad (0 < x < l,\ t > 0), \tag{9}$$

has to be solved with

$$u = 0, \quad \text{when } t = 0 \quad (0 < x < l), \tag{10}$$

$$\frac{\partial u}{\partial t} = 0, \quad \text{when } t = 0 \quad (0 < x < l), \tag{11}$$

$$u = 0, \quad \text{when } x = l \quad (t > 0), \tag{12}$$

$$u = a, \quad \text{when } x = 0 \quad (t > 0). \tag{13}$$

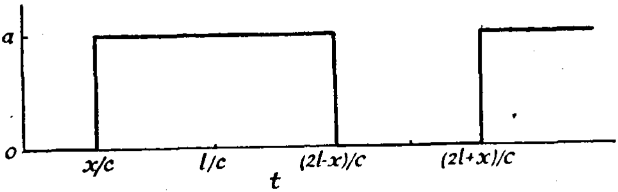


FIG. 88.

Writing $\bar{u}$ for the Laplace transform of u, the subsidiary equation for (9) with initial conditions (10) and (11) is by (6)

$$\frac{d^2\bar{u}}{dx^2} - \frac{p^2}{c^2}\bar{u} = 0. \tag{14}$$

Taking the Laplace transforms of (12) and (13), using § 18 (7), this has to be solved with

$$\bar{u} = 0, \quad \text{when } x = l, \tag{15}$$

$$\bar{u} = \frac{a}{p}, \quad \text{when } x = 0. \tag{16}$$

The solution of (14) satisfying (15) and (16) is

$$\bar{u} = \frac{a \sinh p(l-x)/c}{p \sinh pl/c}. \tag{17}$$

To find u from $\bar{u}$ we expand (17) in a series of negative exponentials as follows

$$\bar{u} = \frac{a\{e^{-px/c} - e^{-p(2l-x)/c}\}}{p\{1 - e^{-2pl/c}\}}$$

$$= \frac{a}{p}\{e^{-px/c} - e^{-p(2l-x)/c}\}\{1 + e^{-2pl/c} + e^{-4pl/c} + \ldots\}$$

$$= \frac{a}{p}\{e^{-px/c} - e^{-p(2l-x)/c} + e^{-p(2l+x)/c} - \ldots\}.$$

Then, by § 18 (9),

$$u = a\left\{H\left(t - \frac{x}{c}\right) - H\left(t - \frac{2l-x}{c}\right) + H\left(t - \frac{2l+x}{c}\right) - \ldots\right\}. \tag{18}$$

The graph of u as a function of t is shown in Fig. 88.

113. The use of conformal representation

While the use of Fourier series as in § 111 gave solutions of the fundamental problems on the wave equation and the diffusion equation, it only provided a solution of Laplace's equation for a region with rectangular boundaries. By the use of the theory of functions of a complex variable many two-dimensional regions can be transformed into regions with rectangular boundaries and in this way Laplace's equation solved in them.

Suppose that $\zeta = \xi + i\eta$ is a function $f(z)$ of a complex variable $z = x+iy$. We represent z by its rectangular coordinates (x, y) in the 'z-plane', and in the same way ζ by its coordinates (ξ, η) in the 'ζ-plane'. Then the relation

$$\zeta = f(z) \tag{1}$$

defines a correspondence between points in the z- and ζ-planes, and we shall suppose this to be one to one, that is, to each point of a region A in the z-plane there corresponds one point of a region B in the ζ-plane, and vice versa. The region A is then said to be mapped on the region B.

The function (1) is called an *analytic* function in a region if ζ has a definite differential coefficient with respect to z at each point of the region. That is, if

$$\lim_{\delta z \to 0} \frac{\delta \zeta}{\delta z}$$

exists and is independent of the way in which $\delta z = \delta x + i\,\delta y$ tends to zero in the (x, y)-plane. Now

$$\frac{\delta \zeta}{\delta z} = \frac{\delta(\xi + i\eta)}{\delta(x+iy)} = \frac{\left(\dfrac{\partial \xi}{\partial x} + i\,\dfrac{\partial \eta}{\partial x}\right)\delta x + \left(\dfrac{\partial \xi}{\partial y} + i\,\dfrac{\partial \eta}{\partial y}\right)\delta y}{\delta x + i\,\delta y}. \tag{2}$$

If this is to have a limit independent of $\delta y/\delta x$, the coefficient of δy in the numerator must be i times the coefficient of δx, that is

$$\frac{\partial \xi}{\partial y} + i\,\frac{\partial \eta}{\partial y} = i\left(\frac{\partial \xi}{\partial x} + i\,\frac{\partial \eta}{\partial x}\right). \tag{3}$$

Equating real and imaginary parts of (3) gives

$$\frac{\partial\xi}{\partial y} = -\frac{\partial\eta}{\partial x}, \tag{4}$$

$$\frac{\partial\eta}{\partial y} = \frac{\partial\xi}{\partial x}. \tag{5}$$

These are the Cauchy–Riemann differential equations. If they are satisfied

$$\frac{d\zeta}{dz} = \frac{\partial\xi}{\partial x} + i\frac{\partial\eta}{\partial x}. \tag{6}$$

It follows from (4) and (5) that

$$\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} = \frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} = 0, \tag{7}$$

and

$$\frac{\partial\xi}{\partial x}\bigg/\frac{\partial\xi}{\partial y} = -\frac{\partial\eta}{\partial y}\bigg/\frac{\partial\eta}{\partial x}. \tag{8}$$

Also

$$\left|\frac{d\zeta}{dz}\right| = \left\{\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2\right\}^{\frac{1}{2}} = \left\{\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2\right\}^{\frac{1}{2}}. \tag{9}$$

The relation (7) states that the real and imaginary parts of any analytic function of a complex variable satisfy Laplace's equation. Also the property of satisfying Laplace's equation is preserved by the transformation (1), that is, we shall prove that if

$$\frac{\partial^2 v}{\partial\xi^2} + \frac{\partial^2 v}{\partial\eta^2} = 0, \tag{10}$$

then

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{11}$$

To verify this we have

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial v}{\partial\eta}\frac{\partial\eta}{\partial x},$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial\xi^2}\left(\frac{\partial\xi}{\partial x}\right)^2 + \frac{\partial^2 v}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{\partial v}{\partial\xi}\frac{\partial^2\xi}{\partial x^2} + \frac{\partial v}{\partial\eta}\frac{\partial^2\eta}{\partial x^2} + 2\frac{\partial^2 v}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}. \tag{12}$$

Similarly

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial\xi^2}\left(\frac{\partial\xi}{\partial y}\right)^2 + \frac{\partial^2 v}{\partial\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2 + \frac{\partial v}{\partial\xi}\frac{\partial^2\xi}{\partial y^2} + \frac{\partial v}{\partial\eta}\frac{\partial^2\eta}{\partial y^2} + 2\frac{\partial^2 v}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}. \tag{13}$$

Adding (12) and (13), and using (4), (5), (7), and (10), gives (11).

Since if $f(z)$ is an analytic function of z, $d\zeta/dz$ has a unique value independent of the way in which $\delta z \to 0$, an infinitesimal figure in the z-plane will be similar to the one which corresponds to it in the ζ-plane; in particular the angle at which two curves cut in the z-plane will be equal to the angle at which the corresponding curves in the ζ-plane cut. The word conformal is used

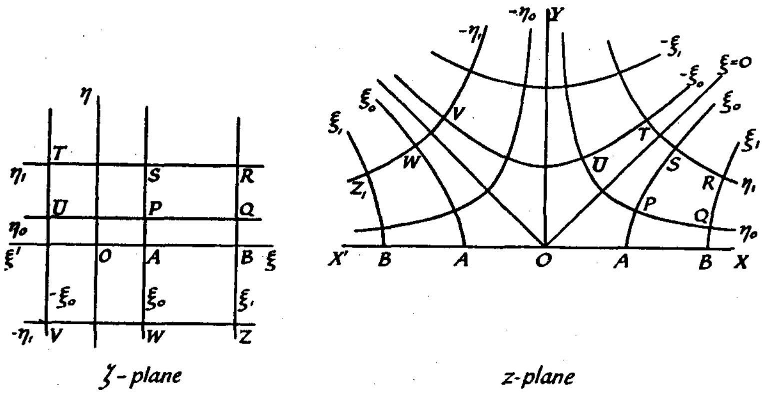


FIG. 89.

to denote this property, and the transformation is said to give a conformal representation of portion of one plane on the other.

In particular, if ξ_0, ξ_1, η_0, η_1 are constants, the region bounded by the curves $\xi = \xi_0$, $\xi = \xi_1$, $\eta = \eta_0$, $\eta = \eta_1$ in the z-plane is transformed into a rectangle in the ζ-plane.

The nature of the correspondence set up by a transformation is best appreciated by the detailed study of examples. The theory of two-dimensional hydrodynamics and electrostatics is largely founded on the study of such transformations, and an account of the useful ones† is given in works on those subjects.

Ex. 1. $$\zeta = z^2. \tag{14}$$

$$\xi + i\eta = (x+iy)^2 = x^2 - y^2 + 2ixy,$$

$$\xi = x^2 - y^2, \qquad \eta = 2xy. \tag{15}$$

If $\theta = \arg z$, and $\phi = \arg \zeta$, we have $\phi = 2\theta$, and thus the upper half $0 \leqslant \theta \leqslant \pi$ of the z-plane is mapped on the whole $0 \leqslant \phi \leqslant 2\pi$ of the ζ-plane. The positive half of the x-axis, $\theta = 0$, and its negative half,

† Cf. also Churchill, *Introduction to Complex Variables and Applications* (McGraw-Hill, 1948).

$\theta = \pi$, both correspond to the positive half of the ξ-axis. The positive half of the y-axis, $\theta = \frac{1}{2}\pi$, corresponds to the negative half of the ξ-axis, $\phi = \pi$.

The line $\xi = \xi_0$ becomes the hyperbola

$$x^2 - y^2 = \xi_0, \tag{16}$$

in the z-plane. $\xi = 0$ becomes the pair of lines $y = \pm x$.

The line $\eta = \eta_0$ becomes the hyperbola

$$2xy = \eta_0, \tag{17}$$

and the line $\eta = 0$ becomes the pair of axes OX, OY.

The rectangle $PQRS$ in the ζ-plane corresponds to the figure $PQRS$ in the z-plane bounded by four hyperbolas, cf. Fig. 89,

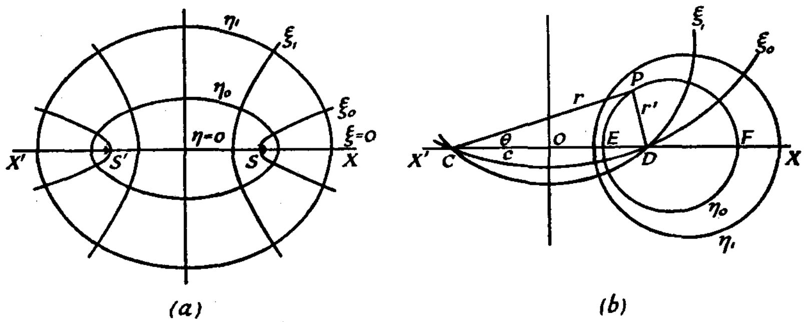


FIG. 90.

Ex. 2. $$z = \cos\zeta. \tag{18}$$

$$x + iy = \cos(\xi + i\eta) = \cos\xi\cosh\eta + i\sin\xi\sinh\eta.$$

$$x = \cos\xi\cosh\eta, \tag{19}$$

$$y = \sin\xi\sinh\eta. \tag{20}$$

Therefore

$$\frac{x^2}{\cos^2\xi} - \frac{y^2}{\sin^2\xi} = 1, \tag{21}$$

$$\frac{x^2}{\cosh^2\eta} + \frac{y^2}{\sinh^2\eta} = 1. \tag{22}$$

The line $\xi = \xi_0$ in the ζ-plane corresponds to an hyperbola (21) in the z-plane, and the line $\eta = \eta_0$ in the ζ-plane to an ellipse (22); cf. Fig. 90 (a).

These ellipses and hyperbolas are confocal: $\eta = 0$ corresponds to the degenerate ellipse consisting of the segment SS' joining the foci, while $\xi = 0$ corresponds to the degenerate hyperbola consisting of the lines SX and $S'X'$ running outwards from the foci. Thus the transformation will be useful for studying either an isolated strip or a plane with a slit in it (for example, the hyperbolas are the stream lines for flow through a slit in a plane).

The detailed correspondence between points in the z- and ζ-planes is best followed by fixing (say) ξ in (19) and (20) and following the variations of x and y as η varies. It will be found that the whole of the z-plane is mapped on the strip $0 \leqslant \xi \leqslant \pi$, $-\infty < \eta < \infty$ of the ζ-plane.

Ex. 3.
$$\zeta = i \ln \frac{z+c}{z-c}. \tag{23}$$

Since the logarithm of a complex number z is given by

$$\ln z = \ln |z| + i \arg z,$$

it follows from (23) that
$$\xi = \theta' - \theta, \tag{24}$$
$$\eta = \ln r/r', \tag{25}$$

where r and r' are the distances of the point P, (x,y) from the points $(\mp c, 0)$, and θ and θ' are the angles PCX and PDX, Fig. 90 (b). The curves $\eta = \eta_0$ in the z-plane belong to a system of coaxial circles with C and D as limiting points, and the circles $\xi = \xi_0$ are a system of circles through C and D.

Now suppose that we wish to solve

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0, \tag{26}$$

in a region bounded by the curves $\xi = \xi_0$, $\xi = \xi_1$, $\eta = \eta_0$, $\eta = \eta_1$, in the (x, y)-plane which are determined by a transformation $\zeta = f(z)$, and that v is to have specified values on the boundaries. We solve

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = 0 \tag{27}$$

in the rectangle $\xi_0 < \xi < \xi_1$, $\eta_0 < \eta < \eta_1$, giving v at each point of the boundary the prescribed value at the corresponding point in the z-plane. This can be done as in § 111. By (10) and (11), this solution, expressed as a function of x and y, satisfies (26), and since it has the required value at each point of the boundary it is the required solution.

In most practical problems much less than this is needed. Interest is usually centred on the equipotentials on which v is constant. In such cases we take v itself to be the imaginary (or real) part of an analytic function

$$w = u + iv = f(z)$$

of z. Then, as in (7), both u and v satisfy Laplace's equation; the curves u = constant, which by (8) are orthogonal to the equipotentials, also have a fundamental significance in the theory (e.g. in electrostatics they are the lines of force). As an example, suppose we consider

$$w = z^2$$

already studied in Ex. 1. The lines v = constant are the hyperbolas

$$xy = \text{constant}$$

of Fig. 89: they are the equipotentials when the planes OX and OY are held at constant potential, or the lines of flow of a perfect, incompressible fluid in a right-angled corner.

114. The wave and diffusion equations in two dimensions

The theory of § 106 extends immediately to give

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-\frac{1}{\kappa}\frac{\partial v}{\partial t}=0 \tag{1}$$

for the equation of conduction of heat in rectangular Cartesian coordinates in two dimensions.

The wave equation in two dimensions is

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}=0 \tag{2}$$

in rectangular Cartesian coordinates, or

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}-\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}=0 \tag{3}$$

in polar coordinates.

Equations (2) and (3) hold, for example, for the vibrations of a membrane, for water waves, and in many problems arising from Maxwell's equations. As in § 111, these equations are treated by forming series of elementary solutions, and determining their coefficients by Fourier or other series or integrals.

Ex. 1. *Symmetrical vibrations of a circular membrane of radius a.*

Since we are concerned with a circular boundary we take the wave equation in the form (3), and since we are given that the vibrations are symmetrical (that is, independent of θ) this reduces to

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}=0. \tag{4}$$

Suppose we seek a solution of (4) of the form

$$v = R(r) \begin{matrix}\sin\\ \cos\end{matrix}\, \omega t, \tag{5}$$

then $R(r)$ must satisfy

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\frac{\omega^2}{c^2}R = 0. \tag{6}$$

This is Bessel's equation of order zero, and the solution of it which remains finite when $r \to 0$ is as in § 98

$$J_0\left(\frac{\omega r}{c}\right). \tag{7}$$

If the membrane is fixed at $r = a$, the displacement at $r = a$ must be zero, and we must have

$$J_0\left(\frac{\omega a}{c}\right) = 0,$$

that is

$$\omega = c\alpha_n/a \quad (n = 1, 2,\ldots), \tag{8}$$

where the α_n are the roots of $J_0(\alpha) = 0$; the first few of these are given in § 98. The values (8) of ω give the natural frequencies of the membrane, and the general solution of (4) is

$$\sum_{n=1}^{\infty} J_0\left(\frac{\alpha_n r}{a}\right)\left\{A_n \cos\frac{\alpha_n ct}{a}+B_n \sin\frac{\alpha_n ct}{a}\right\}; \tag{9}$$

the coefficients A_n and B_n, for given initial conditions, are then found from a Fourier–Bessel series analogous to a Fourier series (cf. Ex. 12 on Chapter XII).

Ex. 2. *Maxwell's equations for a rectangular wave guide.*

The wave guide is parallel to the z-axis and consists of the rectangle bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = b$. A solution of Maxwell's equations of the form

$$e^{i\omega t-\gamma z}\phi(x,y) \tag{10}$$

is sought, where $\omega/2\pi$ is the frequency of the radiation and γ is a constant. It is found that ϕ has to satisfy

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\left(\frac{\omega^2}{c^2}+\gamma^2\right)\phi = 0, \tag{11}$$

and has to vanish on the boundaries of the rectangle. Clearly

$$\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}, \tag{12}$$

$n = 1, 2,\ldots$; $m = 1, 2,\ldots$, vanishes on the boundaries of the region. Also it satisfies (11) if

$$\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)\pi^2-\frac{\omega^2}{c^2}-\gamma^2 = 0. \tag{13}$$

This determines γ in terms of m and n, and the required general solution is a sum of terms of type

$$A_{n,m}\, e^{i\omega t-\gamma z} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}, \tag{14}$$

with γ given by (13). This corresponds to a disturbance propagated along the z-axis only if γ is imaginary, that is, if γ^2 in (13) is negative, i.e. if

$$\frac{\omega^2}{\pi^2 c^2} > \frac{n^2}{a^2}+\frac{m^2}{b^2}. \tag{15}$$

In practice the frequency and dimensions are usually arranged so that this is only true for $n = 1$, $m = 0$, or $m = 1$, $n = 0$, so that only this solution is propagated. For all other values of m and n, γ^2 given by (13) is positive and the corresponding solution (14) dies away exponentially.

115. Laplace's equation in three dimensions

In rectangular Cartesian coordinates this is

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2} = 0. \tag{1}$$

It could be derived, as in § 110, for steady flow of heat or electricity in three dimensions. It is also the fundamental equation for the potential in electrostatics in a region free from electric charges, and arises in the same way in magnetostatics and the theory of potential generally.

As in § 113, the form (1) is suitable only when the equation has to be solved in a region with rectangular boundaries. An elementary solution of it is

$$\sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sinh\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)^{\frac{1}{2}}\pi z, \tag{2}$$

and, by superposing such solutions and using the theory of double Fourier series, (1) may be solved in the rectangular parallelepiped bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, $z = c$ with assigned values of v on the boundaries.

In the same way, solutions of (1) can be obtained for a region bounded by the surfaces of any orthogonal coordinate system by transforming it into the appropriate coordinates. We shall consider only cylindrical polar coordinates and spherical polar coordinates.

In cylindrical polar coordinates, Fig. 91 (*a*), the point P is specified by (r, θ, z) and the surfaces of the coordinate system are cylinders, $r =$ constant; axial planes, $\theta =$ constant; and planes perpendicular to the axis, $z =$ constant. In these coordinates (1) becomes†

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial z^2} = 0. \tag{3}$$

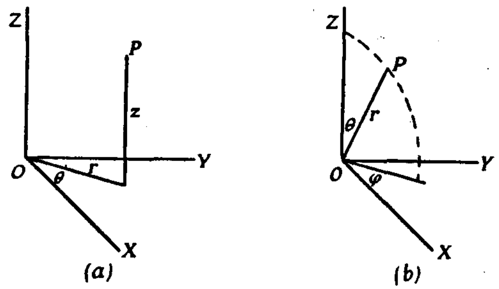


FIG. 91.

We seek an elementary solution of this of the form

$$v = R(r)\Theta(\theta)Z(z), \tag{4}$$

then, substituting (4) in (3), we get

$$\frac{1}{R}\left\{\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right\}+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{5}$$

If
$$\frac{1}{Z}\frac{d^2Z}{dz^2}-m^2 = 0, \tag{6}$$

so that
$$Z = \frac{\sinh}{\cosh} mz, \tag{7}$$

and
$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}+n^2 = 0, \tag{8}$$

so that
$$\Theta = \frac{\sin}{\cos} n\theta, \tag{9}$$

(5) becomes
$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(m^2-\frac{n^2}{r^2}\right)R = 0. \tag{10}$$

† This result and (13) may be derived either by direct change of variables in (1) or as in Ex. 29 at the end of the chapter.

This is Bessel's equation of order n, and as in § 98 its general solution is

$$AJ_n(mr)+BY_n(mr). \tag{11}$$

Therefore

$$\begin{matrix}\cosh\\ \sinh\end{matrix}\, mz \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta\{AJ_n(mr)+BY_n(mr)\} \tag{12}$$

satisfies (3) for any values of m and n. There are restrictions on the possible values of m and n caused by the boundary conditions (cf. Ex. 30).

In spherical polar coordinates, Fig. 91 (b), a point is specified by r, θ, ϕ, and the surfaces of the coordinate system are spheres, cones, and axial planes. Equation (1) becomes

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}=0. \tag{13}$$

As before, we seek an elementary solution of this of the form

$$v = R(r)\Theta(\theta)\Phi(\phi), \tag{14}$$

and (13) becomes

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0. \tag{15}$$

This is satisfied if

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\alpha, \tag{16}$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+m^2=0, \tag{17}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}+\alpha=0, \tag{18}$$

where m and α are any numbers.

(16) may be written

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-\alpha R=0.$$

It is satisfied by $R = Ar^\nu$, where A is a constant, if

$$\nu(\nu+1)-\alpha = 0.$$

If we write

$$\alpha = n(n+1),$$

which can always be done, we have $\nu = n$ or $\nu = -(n+1)$, and

$$R = (Ar^n + Br^{-n-1}), \tag{19}$$

where A and B are arbitrary constants.

(17) is satisfied by $\quad \Phi = \frac{\cos}{\sin} m\phi.$

Finally, putting $\cos\theta = \mu$, (18) becomes

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta}{d\mu}\right\} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0, \tag{20}$$

which is the associated Legendre equation § 99 (30).

In the above, m and n may have any values, but these will be restricted when the region in which the equation has to be solved is known. Thus if it is the interior of the sphere $r = a$, m must be integral, since if we add 2π to ϕ the value of the solution must not be affected. Also n must be an integer, since if n is not integral the solutions of (20) tend to infinity as $\mu \to -1$.

Thus the solution of (15) appropriate to the interior of a sphere is

$$Ar^n P_n^m(\mu) \frac{\cos}{\sin} m\phi. \tag{21}$$

Ex. *To find the solution of* (1) *for the region* $0 \leqslant r < a$ *which takes the value* $f(\mu)$ *on the surface of the sphere, where* $\mu = \cos\theta$.

Since the solution is to be independent of ϕ, (21) takes the form

$$Ar^n P_n(\mu),$$

and the general solution of (1) will be

$$\sum_{n=0}^{\infty} A_n r^n P_n(\mu). \tag{22}$$

On the sphere $r = a$ this has the value

$$\sum_{n=0}^{\infty} A_n a^n P_n(\mu). \tag{23}$$

Now if we assume $f(\mu)$ to be expanded in the series § 99 (28), (29),

$$f(\mu) = \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) P_n(\mu) \int_{-1}^{1} f(\mu') P_n(\mu')\, d\mu'. \tag{24}$$

Comparing coefficients between (23) and (24) gives A_n, and we get finally

$$v = \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) \left(\frac{r}{a}\right)^n P_n(\mu) \int_{-1}^{1} f(\mu') P_n(\mu')\, d\mu'. \tag{25}$$

116. The diffusion equation and the wave equation in three dimensions

These are
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}-\frac{1}{\kappa}\frac{\partial v}{\partial t} = 0, \tag{1}$$
and
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = 0. \tag{2}$$

The procedure is much the same for both, and we consider only the latter. In the form (2) it has elementary solutions
$$\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{l\pi x}{a}\right)\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{m\pi y}{b}\right)\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{n\pi z}{d}\right)\begin{matrix}\cos\\ \sin\end{matrix}\,\omega t, \tag{3}$$
where
$$\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{d^2}\right)\pi^2 = \frac{\omega^2}{c^2}. \tag{4}$$

The values of l, m, n will be restricted by the boundary conditions: for example if the solution has to vanish on the planes $x = 0, x = a, y = 0, y = b, z = 0, z = d$ they must be integers. Each set of values of l, m, n determines a value of ω, and $\omega/2\pi$ is the corresponding natural frequency.

In cylindrical polar coordinates, (2) becomes
$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = 0, \tag{5}$$
and elementary solutions of this are
$$\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{n\pi z}{l}\right)\begin{matrix}\cos\\ \sin\end{matrix}\,m\theta\{AJ_m(\alpha r)+BY_m(\alpha r)\}\begin{matrix}\cos\\ \sin\end{matrix}\,\omega t, \tag{6}$$
where
$$\alpha^2+\frac{n^2\pi^2}{l^2} = \frac{\omega^2}{c^2}. \tag{7}$$

Here, again, α and n are determined by the boundary conditions, and for each value of them ω is determined, and $\omega/2\pi$ is the corresponding natural frequency.

In spherical polar coordinates (2) becomes
$$\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial v}{\partial \mu}\right\}+\frac{1}{r^2(1-\mu^2)}\frac{\partial^2 v}{\partial \phi^2}-\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = 0, \tag{8}$$
where μ is written for $\cos\theta$. As in § 114 we seek a solution
$$v = R(r)\Theta(\theta)\Phi(\phi)\begin{matrix}\cos\\ \sin\end{matrix}\,\omega t \tag{9}$$

of this, and for the region inside a sphere we will have

$$\Phi(\phi) = \frac{\cos}{\sin} m\phi,$$

$$\Theta(\theta) = P_n^m(\mu),$$

where m and n are integers, so that the equation for R becomes

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + R\left\{\frac{\omega^2}{c^2} - \frac{n(n+1)}{r^2}\right\} = 0. \tag{10}$$

Putting $R = \left(\frac{\omega r}{c}\right)^{-\frac{1}{2}} Y$, (10) gives

$$\frac{d^2Y}{dr^2} + \frac{1}{r}\frac{dY}{dr} + Y\left\{\frac{\omega^2}{c^2} - \frac{(n+\frac{1}{2})^2}{r^2}\right\} = 0, \tag{11}$$

the solutions of which are $J_{n+\frac{1}{2}}(\omega r/c)$ and $J_{-(n+\frac{1}{2})}(\omega r/c)$. The latter of these is inadmissible in the interior of a sphere since it tends to infinity as $r \to 0$. Thus the required elementary solution of (8) is

$$\left(\frac{\omega r}{c}\right)^{-\frac{1}{2}} J_{n+\frac{1}{2}}\left(\frac{\omega r}{c}\right) P_n^m(\mu) \frac{\cos}{\sin} m\phi \frac{\cos}{\sin} \omega t. \tag{12}$$

As remarked in § 98 (35), the Bessel functions of half-integral order which occur in (12) can be expressed in a simple form.

Ex. *Radial vibrations in the sphere $r < a$ with $v = 0$ when $r = a$.*

For the case in which the solution is independent of θ and ϕ, (12) reduces to

$$\left(\frac{\omega r}{c}\right)^{-\frac{1}{2}} J_{\frac{1}{2}}\left(\frac{\omega r}{c}\right) \frac{\cos}{\sin} \omega t = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{c}{\omega r} \sin\frac{\omega r}{c} \frac{\cos}{\sin} \omega t, \tag{13}$$

by § 98 (35).

In order to have $v = 0$ when $r = a$ we must have

$$\frac{\omega a}{c} = n\pi \quad (n = 1, 2, \ldots),$$

and the general solution of (8) for the case is

$$\sum_{n=1}^{\infty} \frac{A_n}{r} \sin\left(\frac{n\pi r}{a}\right) \frac{\cos}{\sin}\left(\frac{n\pi ct}{a}\right).$$

The natural frequencies are $nc/2a \quad (n = 1, 2, \ldots)$.

117. The divergence of a vector and the equation of continuity

In this chapter, which has dealt mainly with special problems involving one space variable, vectors have not been used. But in developing the general theory of any of these subjects in three dimensions vector ideas are extremely useful, and also they help to emphasize the connexions between different subjects in which much the same mathematics occurs, but in different notations and with rather different points of view.

In deriving the diffusion equation and Laplace's equation in §§ 106, 110 the fundamental calculation was that of the amount of heat flowing into a small region; this determined the rise in temperature in the region. Similarly, the amount of compressible fluid which flows into a small region determines the increase of density of the fluid in that region. Equations based on such considerations are called equations of continuity, and they are most simply expressed in terms of the idea of the divergence of a vector.

Suppose $\mathbf{F}$ is a vector function of position, and that we take a small closed surface S surrounding any point P, (x, y, z), in which we are interested. Let $\mathbf{n}$ be a unit vector in the direction of the outward normal to the surface S at any point, so that $\mathbf{F}.\mathbf{n}$ is the normal component of $\mathbf{F}$ at the point. Then if dS is an element of area of the surface at the point,

$$\iint_S \mathbf{F}.\mathbf{n}\, dS, \tag{1}$$

taken over the whole of the surface S, is sometimes called the flux of the vector $\mathbf{F}$ over the surface. For example, if the vector $\mathbf{F}$ were $\rho\mathbf{v}$, where $\mathbf{v}$ is the velocity and ρ the density of a fluid at any point, the integral (1) gives the (mass) rate at which fluid is flowing out over the surface S.

The divergence of the vector $\mathbf{F}$, $\operatorname{div}\mathbf{F}$, at the point P is defined as the limit of the ratio of the flux given by (1) to the volume δV bounded by S as this surface shrinks on to the point P, that is

$$\operatorname{div}\mathbf{F} = \lim_{\delta V\to 0}\frac{1}{\delta V}\iint_S \mathbf{F}.\mathbf{n}\, dS. \tag{2}$$

If this limit exists it will be independent of the shape of the surface S. As in all such definitions, there will be pure-mathematical restrictions on the nature of the function $\mathbf{F}$ in order that the limit may exist, but these are in fact simply the obvious ones that the components of $\mathbf{F}$ should be differentiable. Since the value of div $\mathbf{F}$ is independent of the shape of the surface, we may calculate it for the simplest surface, namely a rectangular parallelepiped whose sides are the planes $x \pm \delta x$, $y \pm \delta y$, $z \pm \delta z$. If F_x is the value of the component of $\mathbf{F}$ in the x direction at (x,y,z), its value on the face $x+\delta x$ will be

$$F_x + \frac{\partial F_x}{\partial x}\,\delta x,$$

neglecting terms in $(\delta x)^2$, $\delta x\delta y$, etc., and the contribution to the integral in (2) from the face $x+\delta x$ will be

$$4F_x\,\delta y\delta z + 4\,\frac{\partial F_x}{\partial x}\,\delta x\delta y\delta z.$$

The contribution from the face $x-\delta x$ will be

$$-\left\{4F_x\,\delta y\delta z - 4\,\frac{\partial F_x}{\partial x}\,\delta x\delta y\delta z\right\},$$

and there will be similar results for the faces $y \pm \delta y$ and $z \pm \delta z$.

Thus the value of the integral in (2) is

$$8\,\delta x\delta y\delta z\left\{\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right\},$$

and, dividing by the volume $8\,\delta x\delta y\delta z$ of the parallelepiped, we get finally

$$\operatorname{div}\mathbf{F} = \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}. \tag{3}$$

The value of div $\mathbf{F}$ in terms of its components in the directions of cylindrical or spherical polar coordinates may be found in the same way; cf. Ex. 28.

If the vector $\mathbf{F}$ is the gradient of a scalar function of position ϕ, so that

$$\mathbf{F} = \operatorname{grad}\phi, \tag{4}$$

and by § 72 (26)

$$F_x = \frac{\partial\phi}{\partial x}, \qquad F_y = \frac{\partial\phi}{\partial y}, \qquad F_z = \frac{\partial\phi}{\partial z}, \tag{5}$$

(3) gives
$$\operatorname{div}\operatorname{grad}\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}. \tag{6}$$

For shortness the notation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} \tag{7}$$

is often used, and (6) becomes

$$\operatorname{div}\operatorname{grad}\phi = \nabla^2\phi. \tag{8}$$

It is in this way that the quantity $\nabla^2\phi$ which has appeared so much in this chapter usually arises in applied mathematics.

Considering conduction of heat first, if v is the temperature at the point P in a solid, the vector

$$\mathbf{F} = -K\operatorname{grad} v$$

describes the direction and magnitude of the flow of heat at the point P (this is a generalization of § 106 (1) and may be regarded as a fundamental assumption based on experimental evidence). Then the amount of heat flowing out of a small element of volume δV containing P in the small time δt is

$$\delta V\delta t\operatorname{div}\mathbf{F},$$

and since this must be equal to the loss of heat from the region we get

$$-\rho c\,\delta V\delta v = \delta V\delta t\operatorname{div}\mathbf{F}.$$

That is, in the limit as $\delta t \to 0$,

$$\rho c\frac{\partial v}{\partial t} = -\operatorname{div}\mathbf{F} \tag{9}$$

$$= \operatorname{div}(K\operatorname{grad} v). \tag{10}$$

This is the equation of conduction of heat in three dimensions for a medium in which K may be a function of position or temperature. If K is constant, the right-hand side of (10) becomes $K\operatorname{div}\operatorname{grad} v = K\nabla^2 v$, giving § 116 (1).

The equation of continuity for compressible fluid follows in the same way. If $\mathbf{v}$ is the velocity of the fluid at a point P, and

ρ its density there, the mass which flows out of a small volume δV about P in time δt is

$$\delta t \delta V \operatorname{div}(\rho \mathbf{v}),$$

and this must also be $-\delta\rho\delta V$. Equating these we get

$$\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{v}) = 0, \tag{11}$$

which is the equation of continuity.

For incompressible fluid this becomes

$$\operatorname{div}\mathbf{v} = 0. \tag{12}$$

In certain types of motion (irrotational motion) the velocity $\mathbf{v}$ is given by minus the gradient of a scalar ϕ called the velocity potential, that is

$$\mathbf{v} = -\operatorname{grad}\phi, \tag{13}$$

and (12) becomes Laplace's equation

$$\nabla^2\phi = 0.$$

Finally we notice that $\psi = r^{-1}$, where r is the distance of the point (x,y,z) from the point (x',y',z'), satisfies Laplace's equation. For

$$\frac{\partial \psi}{\partial x} = -\frac{x-x'}{r^3}, \qquad \frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{r^3}+\frac{3(x-x')^2}{r^5},$$

and, adding these results,

$$\nabla^2\psi = 0. \tag{14}$$

If V is the potential energy of a particle at P in the field of a number of centres of force which attract or repel according to the inverse square law, it follows from § 72 (14) that

$$V = \sum \frac{\mu_s}{r_s}, \tag{15}$$

the summation being taken over all the centres of force, r_s being the distance of P from the sth centre and μ_s a constant depending on its strength. It follows from (14) that

$$\nabla^2 V = 0, \tag{16}$$

that is, that the potential energy satisfies Laplace's equation. This holds for potential due to gravitational, electric, and magnetic forces. For a continuous distribution of attracting material the sum in (15) is replaced by an integral, and the result still holds *provided* the point P is not within the distribution.

By § 72 (27) the force $\mathbf{P}$ on the particle is

$$\mathbf{P} = -\operatorname{grad} V, \tag{17}$$

and so, by (16),

$$\operatorname{div} \mathbf{P} = -\operatorname{div} \operatorname{grad} V = -\nabla^2 V = 0, \tag{18}$$

provided, again, that the particle is not within a continuous distribution.

EXAMPLES ON CHAPTER XIII

1. The region $-a < x < a$ of the infinite solid $-\infty < x < \infty$ is initially at constant temperature V, and the remainder of the solid is at zero. Show that the temperature at any point x at time t is

$$\tfrac{1}{2}V\left\{\operatorname{erf}\frac{a-x}{2\sqrt{(\kappa t)}}+\operatorname{erf}\frac{a+x}{2\sqrt{(\kappa t)}}\right\}.$$

2. Heat is supplied in the plane $x = 0$ of the infinite solid at the rate Q per unit area per unit time for $t > 0$, the solid being initially at zero temperature. Show that the temperature at the point x at time t is

$$\frac{Q}{K}\left\{\left(\frac{\kappa t}{\pi}\right)^{\frac{1}{2}} e^{-x^2/4\kappa t}-\tfrac{1}{2}x+\tfrac{1}{2}x\operatorname{erf}\frac{x}{2\sqrt{(\kappa t)}}\right\}.$$

[Combine solutions of type § 106 (10) at times from 0 to t.]

If the semi-infinite solid $x > 0$ is heated over the plane $x = 0$ at the rate Q per unit area per unit time, show that its surface temperature is

$$\frac{2Q}{K}\left(\frac{\kappa t}{\pi}\right)^{\frac{1}{2}}.$$

3. A string $0 < x < l$ of line density σ is stretched to tension $T = \sigma c^2$. If it is plucked a distance d at its middle point and then released, show by the method of Fig. 85 that for $0 < t < l/2c$ the form of the string consists of a straight portion of length $2ct$ parallel to the x-axis, which is joined to the points $x = 0$ and $x = l$ by straight portions of slope $\tan^{-1}(2d/l)$, that is, in the original direction of the string. For $t > l/2c$ this form repeats itself on the other side of the x-axis, and so on. This result may also be obtained by the Laplace transformation method of § 112. The Fourier series of § 111 (44) with $b = \frac{1}{2}l$ is the Fourier sine series for the above curve at time t.

4. A particle of mass M is attached to the end $x = l$ of a bar of length l and area a whose end $x = 0$ is fixed. If the system oscillates longitudinally, show that the natural frequencies of the system are

$$\alpha_n\left(\frac{E}{4\pi^2 l^2 \rho}\right)^{\frac{1}{2}},$$

where $\alpha_1, \alpha_2,...$ are the positive roots of

$$\alpha \tan \alpha = al\rho/M.$$

Show that if $m = al\rho$ is the mass of the rod, and $\lambda = aE/l$ is its stiffness regarded as a spring, the lowest natural frequency is approximately

$$\{\lambda/(M+\tfrac{1}{3}m)\}^{\frac{1}{2}}/2\pi,$$

if m/M is small.

5. Show that the natural frequencies of a uniform beam of length l and weight w per unit length, clamped at both ends, are

$$\alpha_n^2(EIg/4\pi^2\rho l^4)^{\frac{1}{2}},$$

where $\alpha_1, \alpha_2,...$ are the positive roots of $\cos\alpha\cosh\alpha = 1$.

Show that if the beam is free at both ends, the natural frequencies are the same as those given above.

Show that for the natural frequencies of a cantilever of length l, the α_n in the above expression are the roots of

$$\cos\alpha\cosh\alpha = -1.$$

Discuss graphically the nature of the roots of these equations.

6. The surface $r = a$ of a hollow cylinder $a < r < b$ is kept at temperature v_1. At $r = b$ the cylinder loses heat into a medium at temperature v_2 at a rate proportional to its temperature excess above v_2, that is

$$\frac{dv}{dr}+h(v-v_2) = 0, \quad r = b.$$

Show that under steady conditions the rate of loss of heat from the cylinder is

$$2\pi K(v_1-v_2)\frac{hb}{1+hb\ln(b/a)}.$$

Discuss the behaviour of this expression as b increases from a, and show that if $ah < 1$ it has a maximum when $b = 1/h$, that is, that in certain circumstances it is possible to increase the heat loss from a cylindrical surface by covering it with insulating material.

7. Writing $\gamma = \alpha+i\beta$, $z_s = R+iX$ in the formulae § 109 (21) for the short-circuit impedance of a uniform transmission line of length l, show that if z_0 is real

$$R = z_0 \sinh\alpha l \cosh\alpha l[\cosh^2\alpha l\cos^2\beta l+\sinh^2\alpha l\sin^2\beta l]^{-1},$$

$$X = z_0 \sin\beta l\cos\beta l[\cosh^2\alpha l\cos^2\beta l+\sinh^2\alpha l\sin^2\beta l]^{-1},$$

and discuss the general case in which z_0 is not real.

Find the input impedance $(R^2+X^2)^{\frac{1}{2}}$ of the line, and show that, if α is small, it has maxima near $\beta l = (n+\frac{1}{2})\pi$ and minima near $\beta l = n\pi$.

Discuss the input impedance of the open-circuited line in the same way.

8. If the points $x = R/z_0$, $y = X/z_0$ are plotted, representing the resistance and reactance of the short-circuited line of Ex. 7, show that the curves of constant αl are circles of centre $(\coth 2\alpha l, 0)$ and radius $\operatorname{cosech} 2\alpha l$, while the curves of constant βl are circles of centre $(0, -\cot 2\beta l)$ and

radius cosec $2\beta l$. Using this result, charts of coaxial circles can be drawn from which the resistance and reactance of any line can be read off.

9. A uniform string of line density σ and length l is stretched to tension $T = \sigma c^2$. It is set in motion at $t = 0$ from its equilibrium position with velocity $\phi(x)$. Show that its displacement at any time is

$$\frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}\int_0^l \phi(x')\sin\frac{n\pi x'}{l}\,dx'.$$

If the string is set in motion by a blow of impulse P applied to a very short length of the string at $x = a$, show that the displacement is

$$\frac{2P}{\pi c\sigma}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}\sin\frac{n\pi a}{l}\sin\frac{n\pi ct}{l}.$$

10. Show that if $\bar{x}(p)$ is the Laplace transform of $x(t)$, then $e^{-ap}\bar{x}(p)$ is the Laplace transform of $x(t-a)H(t-a)$, that is, of the function which is zero up to time $t = a$ and subsequently has the values of $x(t)$ shifted to the right by a distance a.

11. A uniform bar of length l and unit area has its end $x = 0$ fixed. At $t = 0$, when the bar is at rest and unstrained, a constant tension T is applied at $x = l$. Show that the stress at $x = 0$ is

$$2T\left\{H\left(t-\frac{l}{c}\right)-H\left(t-\frac{3l}{c}\right)+H\left(t-\frac{5l}{c}\right)+\dots\right\},$$

and, using the result of Ex. 10, find the displacement at any point of the bar.

12. The bar of Ex. 11 is struck by a blow of impulse P at $x = l$. Show that the stress at $x = 0$ is

$$2P\left\{\delta\left(t-\frac{l}{c}\right)-\delta\left(t-\frac{3l}{c}\right)+\dots\right\}.$$

13. A bar $0 < x < l$ of density ρ is moving along the x-axis with velocity $-V$, when at $t = 0$ the point $x = 0$ is fixed. Show that the stress at $x = 0$ is

$$-\frac{EV}{c}\left\{1-2H\left(t-\frac{2l}{c}\right)+2H\left(t-\frac{4l}{c}\right)-\dots\right\}.$$

This may be regarded as the problem of the collision of two equal rods moving along the x-axis with equal speeds in opposite directions. Show that the rods separate after a time $2l/c$.

14. Voltage $V(t)$ which is any function of the time is applied at $t = 0$ to the end $x = 0$ of the semi-infinite lossless line $x > 0$, cf. § 109 (6), which has zero initial charge and current. Show that the voltage at the point x is zero up to time x/c, and is

$$V(t-x/c) \quad \text{for } t > x/c,$$

that is, it is exactly the voltage applied at $x = 0$ delayed by time x/c.

For the distortionless line § 109 (8) with the same conditions, show that the voltage at x is zero up to time x/c, and is

$$e^{-Rx/Lc}V(t-x/c) \quad \text{for } t > x/c,$$

that is, it has the same form as the applied voltage, but is attenuated by the factor $\exp(-Rx/Lc)$.

15. Given that the Laplace transform of

$$1-\operatorname{erf}[\tfrac{1}{2}x(\kappa t)^{-\frac{1}{2}}]$$

is $(1/p)\exp[-x(p/\kappa)^{\frac{1}{2}}]$, deduce the result § 106 (17).

The region $-l < x < l$ is initially at zero temperature, and for $t > 0$ its surfaces $x = -l$ and $x = l$ are kept at constant temperature V_0. Show that the temperature at the point x at time t is

$$V_0 \sum_{n=0}^{\infty} (-1)^n \left\{2-\operatorname{erf}\frac{(2n+1)l-x}{2\sqrt{(\kappa t)}}-\operatorname{erf}\frac{(2n+1)l+x}{2\sqrt{(\kappa t)}}\right\}.$$

Use the method of § 112 and the Laplace transform given above. A solution of this problem in the form of a Fourier series may be deduced from § 111 (32) or derived independently. The Fourier series converges slowly for small values of the time and the series given above is more useful.

16. Discuss the transformation of § 113, Ex. 3, in greater detail and show that the whole of the z-plane is mapped on the strip $-\pi < \xi < \pi$, $-\infty < \eta < \infty$ of the ζ-plane.

Verify, by writing down its Cartesian equation, that if η_0 is constant

$$r/r' = e^{\eta_0}$$

is a circle, and show, by writing down its values for the points E, F, Fig. 90 (b), that if this circle has radius a and its centre is at the point $(d, 0)$,

$$c^2 = d^2-a^2, \qquad \frac{d+\sqrt{(d^2-a^2)}}{a} = e^{\eta_0}.$$

17. Discuss the nature of the equipotentials $v =$ constant, where v is the real or imaginary part of the functions

(i) $\cos z$, (ii) $\ln\{(z+c)/(z-c)\}$, (iii) $1/z$.

18. Show that the imaginary part of

$$V(z+a^2/z),$$

where V and a are constants, gives a solution of Laplace's equation which vanishes on the circle $r = a$ and tends to the value Vy at large distances from the origin.

19. Show that the 'bilinear' transformation

$$\zeta = \frac{\alpha z+\beta}{\gamma z+\delta},$$

where $\alpha, \beta, \gamma, \delta$ are constants, gives a one-to-one correspondence between the z- and ζ-planes, and that there are just two points which are unchanged by the transformation.

Show that it may be built up from the three successive transformations

$$\zeta = \frac{\alpha}{\gamma}+\frac{\beta\gamma-\alpha\delta}{\gamma}z_1, \qquad z_1 = \frac{1}{z_2}, \qquad z_2 = \gamma z+\delta.$$

Show that each of these transformations possesses the property of transforming circles into circles, so that the bilinear transformation will do so also (straight lines are regarded as limiting cases of circles).

20. Show that the transformation

$$\zeta = -i\ln(z/a)$$

transforms concentric circles and radii through the origin, respectively, into the lines η = constant, and ξ = constant.

Using the result § 111 (12), show that the solution of Laplace's equation in the half-ring bounded by concentric circles of radii a and b, $b < a$, and by the portions $\theta = 0$ and $\theta = \pi$ of the x-axis, which has the value 1 on $r = a$ and vanishes on the other boundaries, is

$$\frac{4}{\pi}\sum_{s=0}^{\infty}\frac{\sin[(2s+1)\theta]\sinh[(2s+1)(\eta_0-\eta)]}{(2s+1)\sinh(2s+1)\eta_0},$$

where $\eta_0 = \ln(a/b)$, $\eta = \ln(a/r)$.

21. The rectangular corner $x > 0$, $y > 0$ is initially at unit temperature, and for $t > 0$ its surfaces $x = 0$ and $y = 0$ are kept at zero temperature. Show that

$$\operatorname{erf}\frac{x}{2\sqrt{(\kappa t)}}\operatorname{erf}\frac{y}{2\sqrt{(\kappa t)}}$$

satisfies the differential equation and boundary conditions.

22. Show that the normal modes of vibration of a rectangular membrane $0 < x < a$, $0 < y < b$, fixed at its edges, are

$$\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\begin{matrix}\cos\\ \sin\end{matrix}\,\omega t,$$

where $$\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right) = \frac{\omega^2}{c^2}.$$

Sketch the nodal lines (lines of zero displacement) for the first few normal modes.

23. In a cylindrical wave guide of radius a, § 114 (11) in cylindrical polars has to be solved with $\phi = 0$ when $r = a$. Show that the solutions independent of θ are of type

$$e^{i\omega t-\gamma z}J_0(r\alpha_n/a),$$

where α_n is a root of $J_0(x) = 0$, and

$$\frac{\alpha_n^2}{a^2}-\gamma^2-\frac{\omega^2}{c^2} = 0.$$

Show that this mode is propagated only if $\omega > c\alpha_n/a$.

24. The temperature of an infinite circular cylinder of radius a has the constant value V when $t = 0$, and for $t > 0$ its surface is kept at zero temperature. Show that the temperature at the radius r at time t is

$$\frac{2V}{a}\sum_{n=1}^{\infty} e^{-\kappa\alpha_n^2 t}\frac{J_0(r\alpha_n)}{\alpha_n J_1(a\alpha_n)},$$

where $\alpha_1, \alpha_2,...$ are the positive roots of $J_0(ax) = 0$. [Use the result of Ex. 13 on Chap. XII.]

25. Show that

$$\frac{Q}{8\rho c(\pi\kappa t)^{\frac{3}{2}}}\exp\left\{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}\right\}$$

satisfies the equation of conduction of heat and represents the temperature at (x,y,z) due to a quantity of heat Q liberated instantaneously at the point (x',y',z') at $t = 0$.

Deduce that if a quantity of heat Q per unit length is liberated at $t = 0$ along the z-axis, the temperature at (x, y) at time t will be

$$\frac{Q}{4\pi Kt}\exp\{-(x^2+y^2)/4\kappa t\}.$$

26. Show that the solution of Laplace's equation in the region $0 < x < a$, $0 < y < b$, $0 < z < c$, with $v = 1$ on the surface $z = 0$ and $v = 0$ on the other surfaces, is [using § 94 (7)]

$$\frac{16}{\pi^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\sinh l(c-z)\sin[(2p+1)\pi x/a]\sin[(2q+1)\pi y/b]}{(2p+1)(2q+1)\sinh cl},$$

where $$l^2 = \frac{(2p+1)^2\pi^2}{a^2}+\frac{(2q+1)^2\pi^2}{b^2}.$$

27. Show that, if ϵ is small, the surface

$$r = a+\epsilon P_1(\cos\theta)$$

is very nearly a sphere of radius a with its centre displaced a small distance ϵ from the origin. Show that a function v which satisfies Laplace's equation, vanishes on this sphere, and has the value unity on the sphere $r = b$, where $b < a$, is

$$\frac{b(a-r)}{r(a-b)}+\frac{\epsilon ab(r^3-b^3)}{r^2(a-b)(a^3-b^3)}P_1(\cos\theta).$$

28. Show that if F_r, F_θ, F_z are the components of a vector $\mathbf{F}$ in cylindrical coordinates (i.e. F_θ is the component in the direction in which the point specified by r, θ, z moves if θ is increased, r and z being kept constant, etc.)

$$\operatorname{div}\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r)+\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}+\frac{\partial F_z}{\partial z}.$$

And if F_r, F_θ, F_ϕ are the components of a vector $\mathbf{F}$ in spherical polar coordinates, show that

$$\operatorname{div}\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta)+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}.$$

29. Using the results of Ex. 28 and of Ex. 20 on Chapter IX, deduce the expressions of § 115 (3) and (13) for

$$\nabla^2 v = \operatorname{div}\operatorname{grad} v$$

in cylindrical and spherical polar coordinates.

30. Show that Laplace's equation in cylindrical coordinates, § 115 (3), has solutions of type

$$\frac{\cos}{\sin} mz \frac{\cos}{\sin} n\theta\{AI_n(mr)+BK_n(mr)\}$$

in addition to those specified in § 115 (12).

Discuss the choice of solutions appropriate to the finite cylinder $0 \leqslant r < a$, $0 < z < l$. Show that the solution of Laplace's equation which takes the value 1 on the curved surface of this cylinder and is zero on the plane ends is

$$\frac{4}{\pi}\sum_{s=0}^{\infty}\frac{1}{(2s+1)}\sin\left(\frac{(2s+1)\pi z}{l}\right)\frac{I_0\{(2s+1)\pi r/l\}}{I_0\{(2s+1)\pi a/l\}}.$$

31. The free vibrations of air in a sphere of radius a involve the solution of the equation of wave motion, § 116 (8), with the boundary condition

$$\frac{\partial v}{\partial r} = 0, \quad \text{when } r = a.$$

Find equations for the natural frequencies, and show that those corresponding to radial vibrations are $c\alpha_n/2\pi a$ where α_n, $n = 1, 2,\ldots$ are the positive roots of

$$\tan\alpha = \alpha.$$

XIV

NUMERICAL METHODS

118. Introductory

IT will be clear from the difficulties which have appeared earlier that explicit solutions can only be obtained for relatively simple problems. In many branches of mathematics there is a need for solutions of important special problems for which exact solutions cannot be found. For these, numerical methods must be used, and at the present time a great deal of attention is being paid to their development.

The methods available fall sharply into two classes, 'digital' and 'analogue'. The analogue methods, which will not be studied here, depend on replacing the problem of which a solution is required by an analogous problem (that is, one whose basic theory involves the same system of equations) whose solution can be found by measurement. The result can thus only be obtained with an error depending on the errors of measurement and of the construction of the system. The analogy between mechanical and electrical systems has already been studied in § 43. Another good illustration comes from the solution of Laplace's equation, § 110, in an irregular region: this equation occurs in the theory of electrostatic potential, flow of heat, flow of electric current, deflexion of a membrane, etc.; of these the deflexion of a membrane is easy to measure and from it solutions of the corresponding problems in the other fields may be obtained (this is the principle of the 'rubber table'); alternatively, the voltage at a point in an electrolyte is easy to measure, so this again may be used to give the solution of the other problems (the 'electrolytic tank').

The most important analogue machines in use are the *differential analysers*.† These contain only mechanical elements and depend on the principle that if θ and ϕ are the rotations of two shafts connected by a continuously variable gear whose gear

† Hartree, *Math. Gazette*, **22** (1938), 342; Crank, *The Differential Analyser* (Longmans, Green, 1947); Bush, *J. Franklin Inst.* **212** (1931), 447.

ratio $f(\theta, \phi)$ is a prescribed function of θ and ϕ, then

$$\frac{d\theta}{d\phi} = f(\theta, \phi).$$

By combining a number of such variable gears, together with elements such as gears of constant ratio for multiplying, and differential gears for adding, the analogues of many ordinary differential equations can be set up. This machine has the great advantage that it applies to non-linear differential equations. It can be used for solving partial differential equations by replacing them by a system of ordinary differential equations by the methods of § 122.

Digital methods involve the carrying out of the processes of numerical mathematics on digital machines. These are essentially machines to which numbers are supplied by a keyboard (hence the name digital) and which perform the basic mathematical processes on them; they range from simple desk machines which form the sum, product, or quotient of two numbers, to the complicated types such as punched card machines and electronic computers, in which long sequences of these processes are carried out automatically. All such machines have the property that the accuracy obtainable from them is limited only by the number of figures they can hold; thus they are suitable for calculating functions to a large number of significant figures at fairly widely separated intervals. The numerical methods described below are required in the process of calculating functions in this way and of using the results when calculated.

The basic theory consists of a number of formulae, essentially due to Newton, for interpolating and extrapolating from a set of tabulated values. From these, formulae for the differential coefficient and integral of a function given in this way can be found. These formulae can then be used in various ways for the solution of ordinary and partial differential equations, the commonest method being to replace the differential equation by a difference equation, which is in effect a set of algebraic equations for the values of the solution at regularly spaced points. Relaxation methods are one way of solving such a set of equations.

It must be emphasized that most of the processes given below are intended to be carried out on a desk calculating machine, the simplest type will suffice: without such a machine the work is disheartening, but with one it is extraordinarily rapid. Needless to say, also, such work abounds in short-cuts and tricks which are well worth learning if a great deal of numerical work has to be done but not otherwise.

119. Interpolation

Suppose we know the values $\ldots, y_{-1}, y_0, y_1, y_2, \ldots$ of a function at regular† intervals h of its argument, say at the points $\ldots, a-h, a, a+h, a+2h, \ldots$. The object of interpolation formulae is to estimate the value of the function at intermediate points, say $a+\theta h$ where θ is fractional, as accurately as possible in terms of the given values.

We define

$$\Delta y_n = y_{n+1}-y_n, \tag{1}$$

$$\Delta^2 y_n = \Delta y_{n+1}-\Delta y_n = y_{n+2}-2y_{n+1}+y_n, \tag{2}$$

$$\Delta^3 y_n = \Delta^2 y_{n+1}-\Delta^2 y_n = y_{n+3}-3y_{n+2}+3y_{n+1}-y_n, \tag{3}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

these are called the first, second, third,... *forward differences* of y_n. To find them we build up a table, called a *difference table*, by first writing down the values of the function, subtracting each of these from the following one to form the first differences, and entering these in the lines between the values of the function; treating the first differences in the same way to form the second differences, and so on. The result is shown schematically in Table I.

It should be noticed also that (1), (2), (3) can also be used to obtain values of the function if the differences are known. For example, if y_0, Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ along the diagonal line in Table I are known, we have $y_1 = y_0+\Delta y_0$, and so on, and can fill in the part of the table below this line down to y_4. If, in addition, the subsequent fourth differences $\Delta^4 y_1$, $\Delta^4 y_2, \ldots$ are

† For further theory see Whittaker and Robinson, *The Calculus of Observations* (Blackie, 1924); Milne, *Numerical Calculus* (Princeton University Press, 1949). They give corresponding formulae for unequal intervals of the argument.

TABLE I

Argument	*Function*	*Differences*			
		First	*Second*	*Third*	*Fourth*
$a-4h$	y_{-4}				
		Δy_{-4}			
$a-3h$	y_{-3}		$\Delta^2 y_{-4}$		
		Δy_{-3}		$\Delta^3 y_{-4}$	
$a-2h$	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$
		Δy_{-2}		$\Delta^3 y_{-3}$	
$a-h$	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$
		Δy_{-1}		$\Delta^3 y_{-2}$	
a	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
$a+h$	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$
		Δy_1		$\Delta^3 y_0$	
$a+2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
$a+3h$	y_3		$\Delta^2 y_2$		
		Δy_3			
$a+4h$	y_4				

known, the values $y_5, y_6, \ldots$ of the function can be found. This process is known as 'building up'.

In Table II, values of $\cos x$ at intervals of $0{\cdot}1$ of x have been taken from tables and formed into a difference table.† In such a table the number of places of decimals is always known (in this case five) and labour is saved, and the results made much easier to read, by omitting the decimal point and any noughts between it and the non-zero figures.

† A number of important practical points may be referred to briefly here.

(i) *Rounding-off errors.* The values in Table II have been taken from a fifteen-place table and 'rounded-off' by taking the nearest five-decimal number to the one tabulated. Thus their last figures may be in error by as much as $0{\cdot}000005$, and the possible errors in the successive differences increase steadily (cf. Ex. 2) and cause the fluctuation in Δ^4 in the table. These errors, called 'rounding-off errors', always make the last figure unreliable, and for this reason calculations are always made with one, or better, two more figures than are required in the result. These extra figures are called 'guarding figures'.

(ii) *Checking tabulated values.* An error in the tabulated values of a function will show up readily in the higher differences (cf. Ex. 3).

(iii) *Checking.* It is not desirable to check a calculation by repeating it since it is easy to repeat the commonest type of error (cf. Ex. 3). If possible an independent check should be devised: for example, Table II should be checked by reversing the process and working backwards from the fourth differences to the values of the function.

TABLE II

x	$\cos x$	Δ	Δ^2	Δ^3	Δ^4
0·0	1·00000				
		−500			
0·1	0·99500		−993		
		−1493		13	
0·2	0·98007		−980		12
		−2473		25	
0·3	0·95534		−955		10
		−3428		35	
0·4	0·92106		−920		9
		−4348		44	
0·5	0·87758		−876		6
		−5224		50	
0·6	0·82534		−826		13
		−6050		63	
0·7	0·76484		−763		3
		−6813		66	
0·8	0·69671		−697		10
		−7510		76	
0·9	0·62161		−621		
		−8131			
1·0	0·54030				

The differences of a polynomial are of great importance in the theory, and we consider first the special polynomial $[\theta]^n$ of degree n in θ called the factorial polynomial which is defined by

$$[\theta]^n = \theta(\theta-1)...(\theta-n+1), \tag{4}$$

with $[\theta]^0 = 1$. This is connected with the general binomial coefficient for non-integral θ by the relation

$$\binom{\theta}{n} = \frac{\theta(\theta-1)...(\theta-n+1)}{n!} = \frac{1}{n!}[\theta]^n. \tag{5}$$

The importance of these polynomials arises from the fact that their differences, for unit differences in the argument θ, have a very simple form. Thus

$$\begin{aligned}\Delta[\theta]^n &= [\theta+1]^n-[\theta]^n \\ &= (\theta+1)\theta(\theta-1)...(\theta-n+2)-\theta(\theta-1)...(\theta-n+1) \\ &= n[\theta]^{n-1}. \end{aligned} \tag{6}$$

The analogy between the result (6) and the formula for the differential coefficient of x^n is the reason for the choice of the notation (4). It follows from (6) that

$$\Delta^r[\theta]^n = n(n-1)...(n-r+1)[\theta]^{n-r}, \tag{7}$$

$$\Delta^n[\theta]^n = n!. \tag{8}$$

Thus the nth differences of $[\theta]^n$ are constant, and since any polynomial of degree n can be expressed in the form

$$\alpha_n[\theta]^n+\alpha_{n-1}[\theta]^{n-1}+...+\alpha_1[\theta]+\alpha_0, \tag{9}$$

where $\alpha_n,..., \alpha_0$ are constants, it follows that the nth differences of any polynomial of degree n are constant.

The Gregory–Newton formula. This is the fundamental formula of interpolation. Suppose that the values y_0, $y_1,...$, y_n of the function at the points $a+\theta h$, $\theta = 0, 1,..., n$, are known, and that we wish to estimate the value of the function at a point $a+\theta h$, where θ is fractional. To do this, we find the polynomial $f(a+\theta h)$ of degree n which passes through the $n+1$ points

$$(a, y_0),\ (a+h, y_1),...,\ (a+nh, y_n)$$

and take the value of this polynomial at the intermediate point $a+\theta h$ as our estimate of the function there. This function $f(a+\theta h)$ will be called the interpolating polynomial.†

Instead of proceeding in the obvious way, that is by assuming a polynomial in θ for $f(a+\theta h)$ and determining its coefficients from the result $f(a+rh) = y_r$, $r = 1, 2,..., n$, it is simpler to take $f(a+\theta h)$ as a linear combination of factorial polynomials as in (9) and to determine the coefficients in this from the relation $\Delta^r f(a) = \Delta^r y_0$, $r = 1,..., n$. Since, as remarked earlier, the values of $y_1,..., y_n$ are determined by the values of y_0, $\Delta y_0,..., \Delta^n y_0$ this leads to the same result.

Taking $f(a+\theta h)$ in the form (9) and differencing this n times, using (6), (7), or (8) and remembering that a step of h in the

† The term interpolation will always be used here in this sense; strictly the process is 'polynomial interpolation' and is the most common but by no means the only method of interpolation; thus, for example, Exs. (9) and (10) of Chapter XI are formulae for trigonometric interpolation.

argument $a+\theta h$ of $f(a+\theta h)$ corresponds to a step of 1 in θ, gives

$$f(a+\theta h) = \alpha_0+\alpha_1[\theta]+\alpha_2[\theta]^2+...+\alpha_n[\theta]^n, \tag{10}$$

$$\Delta f(a+\theta h) = \alpha_1 + 2\alpha_2[\theta]+...+n\alpha_n[\theta]^{n-1}, \tag{11}$$

$$\Delta^2 f(a+\theta h) = 2\alpha_2 +...+n(n-1)\alpha_n[\theta]^{n-2}, \tag{12}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta^n f(a+\theta h) = n!\,\alpha_n. \tag{13}$$

Putting $\theta = 0$ in these we get

$$\left.\begin{array}{ll} \alpha_0 = f(a) = y_0; & \alpha_1 = \Delta f(a) = \Delta y_0; \\ \alpha_2 = \frac{1}{2!}\Delta^2 f(a) = \frac{1}{2!}\Delta^2 y_0; \quad ...; & \alpha_n = \frac{1}{n!}\Delta^n y_0, \end{array}\right\} \tag{14}$$

and using these results and (5) in (10) gives finally

$$f(a+\theta h) = y_0+\theta\Delta y_0+\binom{\theta}{2}\Delta^2 y_0+...+\binom{\theta}{n}\Delta^n y_0. \tag{15}$$

(15) is the Gregory–Newton formula, and its value for fractional θ gives the interpolated value of the function. For example, to calculate $\cos 0{\cdot}22$ from the values in Table II we have, taking $a = 0{\cdot}2$, $\theta = 0{\cdot}2$,

$$\binom{\theta}{1} = 0{\cdot}2, \quad \binom{\theta}{2} = -0{\cdot}08, \quad \binom{\theta}{3} = 0{\cdot}048, \quad \binom{\theta}{4} = -0{\cdot}0336,$$

and so, using the differences in Table II,

$$\begin{aligned} \cos 0{\cdot}22 & \\ &= 0{\cdot}98007-0{\cdot}2\times 0{\cdot}02473+0{\cdot}08\times 0{\cdot}00955+0{\cdot}048\times 0{\cdot}00035 \\ &= 0{\cdot}97590. \end{aligned}$$

The accurate value is $0{\cdot}975897...$. It is usually most convenient, but not essential, to choose the starting value a so that θ is less than unity, as was done above.

The Gregory–Newton formula is fundamental, but it is not always the most convenient since the differences involved in it run down the sloping line in Table I. In practice a table, e.g. Table II, has a beginning and an end; at the beginning, the line of differences runs downwards and the Gregory–Newton formula which involves the first of each column of differences must be used; at the end of the table the differences slope upwards and there is an appropriate formula ('Gregory–Newton backwards'; cf. Ex. 7) which must be used; at any point in the middle of the table, and this, of course, comprises the vast bulk of it, while either of the above formulae still can be used, it clearly would be more convenient to have formulae which involve differences in the same horizontal line.

The first step in this direction consists in deducing a formula involving the differences running in the zigzag dotted line shown in Table I. This is the *Newton–Gauss* formula

$$f(a+\theta h) = y_0+\binom{\theta}{1}\Delta y_0+\binom{\theta}{2}\Delta^2 y_{-1}+ \\ +\binom{\theta+1}{3}\Delta^3 y_{-1}+\binom{\theta+1}{4}\Delta^4 y_{-2}+\dots. \quad (16)$$

The first few terms of (16) may be derived from (15) by substituting

$$\Delta^2 y_0 = \Delta^3 y_{-1}+\Delta^2 y_{-1}, \qquad \Delta^3 y_0 = \Delta^3 y_{-1}+\Delta^4 y_{-1},$$

etc., and using (17). There are many general proofs.

If we eliminate the odd differences in (16) by using

$$\Delta y_0 = y_1-y_0,\ \ \Delta^3 y_{-1} = \Delta^2 y_0-\Delta^2 y_{-1},\ \ \Delta^5 y_{-2} = \Delta^4 y_{-1}-\Delta^4 y_{-2},\dots$$

and use the relation

$$\binom{\theta+1}{r+1} = \binom{\theta}{r+1}+\binom{\theta}{r}, \quad (17)$$

(16) becomes

$$f(a+\theta h) = (1-\theta)y_0+\binom{\theta+1}{3}\Delta^2 y_0+\binom{\theta+2}{5}\Delta^4 y_{-1}+\dots+ \\ +\theta y_1-\binom{\theta}{3}\Delta^2 y_{-1}-\binom{\theta+1}{5}\Delta^4 y_{-2}+\dots. \quad (18)$$

These formulae are rather more simply expressed in the *central difference* notation which we now define. This notation is also the one most commonly used in applications, but it is used in conjunction with the forward difference notation Δ and does not supplant it. The notation is

$$\delta^1 y_{n+\frac{1}{2}} = \Delta y_n = y_{n+1}-y_n, \quad (19)$$

$$\delta^2 y_n = \delta^1 y_{n+\frac{1}{2}}-\delta^1 y_{n-\frac{1}{2}} = \Delta y_n-\Delta y_{n-1} = y_{n+1}-2y_n+y_{n-1}, \quad (20)$$

and so on.

In this notation Table I becomes

TABLE III

$$\begin{array}{ccccccccc}
y_{-2} & & \delta^2 y_{-2} & & \delta^4 y_{-2} \\
& \delta^1 y_{-\frac{3}{2}} & & \delta^3 y_{-\frac{3}{2}} & \\
y_{-1} & & \delta^2 y_{-1} & & \delta^4 y_{-1} \\
& \delta^1 y_{-\frac{1}{2}} & & \delta^3 y_{-\frac{1}{2}} & \\
y_0 & & \delta^2 y_0 & & \delta^4 y_0 \\
& \delta^1 y_{\frac{1}{2}} & & \delta^3 y_{\frac{1}{2}} & \\
y_1 & & \delta^2 y_1 & & \delta^4 y_1 \\
& \delta^1 y_{\frac{3}{2}} & & \delta^3 y_{\frac{3}{2}} & \\
y_2 & & \delta^2 y_2 & & \delta^4 y_2
\end{array}$$

where, now, quantities in the same horizontal line have the same suffix.

In addition the unoccupied places in the table may be filled in with the arithmetic mean of the quantities above and below them. The prefix μ is used to indicate† quantities derived in this way; thus

$$\mu\delta^1 y_0 = \tfrac{1}{2}\{\delta^1 y_{\frac{1}{2}}+\delta^1 y_{-\frac{1}{2}}\} = \tfrac{1}{2}(y_1-y_{-1}), \tag{21}$$
$$\mu\delta^3 y_0 = \tfrac{1}{2}\{\delta^3 y_{\frac{1}{2}}+\delta^3 y_{-\frac{1}{2}}\},$$
$$\mu\delta^2 y_{\frac{1}{2}} = \tfrac{1}{2}\{\delta^2 y_1+\delta^2 y_0\},$$

etc. Using this notation (18) becomes

$$f(a+\theta h) = (1-\theta)y_0-\binom{\theta}{3}\delta^2 y_0-\binom{\theta+1}{5}\delta^4 y_0-...+$$
$$+\theta y_1+\binom{\theta+1}{3}\delta^2 y_1+\binom{\theta+2}{5}\delta^4 y_1+.... \tag{22}$$

Putting $\phi = 1-\theta$ and using (5) this becomes for $0 \leqslant \theta \leqslant 1$

$$f(a+\theta h) = \phi y_0+\frac{\phi(\phi^2-1)}{3!}\delta^2 y_0+\frac{\phi(\phi^2-1)(\phi^2-4)}{5!}\delta^4 y_0+...+$$
$$+\theta y_1+\frac{\theta(\theta^2-1)}{3!}\delta^2 y_1+\frac{\theta(\theta^2-1)(\theta^2-4)}{5!}\delta^4 y_1+.... \tag{23}$$

(23) is Everett's formula, which is the one most commonly used for interpolation from tables when a large number of figures is needed. Numerical values for the coefficients in it are published to facilitate its use. It only requires the even differences in the same horizontal line for two values of the argument.

† In complete treatments of the subject the quantities δ and μ are defined, following Sheppard, as the operators

$$\delta y_n = y_{n+\frac{1}{2}}-y_{n-\frac{1}{2}}, \qquad \mu y_n = \tfrac{1}{2}(y_{n+\frac{1}{2}}+y_{n-\frac{1}{2}}).$$

Thus the values of $\cos x$ given in tables would read

x	$\cos x$	δ^2	δ^4
0·20	0·98006 658	−979 250	9 783
0·30	0·95533 649	−954 541	9 539

And, using these values with $\theta = 0{\cdot}2$, $\phi = 0{\cdot}8$, (23) gives $\cos 0{\cdot}22 = 0{\cdot}97589745$ which is the accurate value.

Finally it should be remarked that though we have always spoken of the Gregory–Newton formula as an interpolation formula it can also be used for extrapolation beyond the beginning of the table—with a corresponding loss of accuracy, of course. Thus changing the sign of θ in (15) gives

$$f(a-\theta h) = y_0-\theta\Delta y_0+\frac{\theta(\theta+1)}{2!}\Delta^2 y_0+\dots+\\ +(-1)^n\frac{\theta(\theta+1)\dots(\theta+n-1)}{n!}\Delta^n y_0. \quad (24)$$

For extrapolation beyond the end of a table, the simplest result is given in Ex. 8.

120. Differentiation and integration

Suppose, as before, that the function is tabulated at the points $a+rh$, $r = 0, \pm 1$, etc., and that we wish to find its derivatives in terms of either the tabular values or their differences. We interpolate between the values by the polynomial $f(a+\theta h)$ of § 119 and find the derivatives of this function by differentiating the formulae of § 119. Thus differentiating the Gregory–Newton formula § 119 (15) gives

$$hf'(a+\theta h) = \Delta y_0+\tfrac{1}{2}(2\theta-1)\Delta^2 y_0+\tfrac{1}{6}(3\theta^2-6\theta+2)\Delta^3 y_0+\dots, \quad (1)$$

$$h^2 f''(a+\theta h) = \Delta^2 y_0+(\theta-1)\Delta^3 y_0+\dots. \quad (2)$$

These formulae are most useful near the beginning of a table. Usually it is the values at the tabulated points that are needed, and these may be expressed in terms of either the differences, or the tabulated values, as desired. Thus

$$hf'(a) = \Delta y_0-\tfrac{1}{2}\Delta^2 y_0+\tfrac{1}{3}\Delta^3 y_0+\dots, \quad (3)$$

$$= \tfrac{1}{2}(-3y_0+4y_1-y_2)+\tfrac{1}{3}\Delta^3 y_0\dots, \quad (4)$$

$$hf'(a+h) = \Delta y_0+\tfrac{1}{2}\Delta^2 y_0-\tfrac{1}{6}\Delta^3 y_0+\dots, \quad (5)$$

$$= \tfrac{1}{2}(-y_0+y_2)-\tfrac{1}{6}\Delta^3 y_0+\dots. \quad (6)$$

For calculating derivatives at points in the body of a table, results expressed in central difference notation are a little more useful. Differentiating the Newton–Gauss formula § 119 (16) gives

$$hf'(a+\theta h) = \Delta y_0+\tfrac{1}{2}(2\theta-1)\Delta^2 y_{-1}+\tfrac{1}{6}(3\theta^2-1)\Delta^3 y_{-1}+ \\ +\tfrac{1}{24}(4\theta^3-6\theta^2-2\theta+2)\Delta^4 y_{-2}+\dots, \quad (7)$$

$$h^2f''(a+\theta h) = \Delta^2 y_{-1}+\theta\Delta^3 y_{-1}+\tfrac{1}{24}(12\theta^2-12\theta-2)\Delta^4 y_{-2}+\dots. \quad (8)$$

Putting $\theta = 0$ in these gives results for the derivatives at the point a which may be put in many different useful forms.

$$\begin{aligned} hf'(a) &= \Delta y_0-\tfrac{1}{2}\Delta^2 y_{-1}-\tfrac{1}{6}\Delta^3 y_{-1}+\tfrac{1}{12}\Delta^4 y_{-2}+\dots \\ &= \tfrac{1}{2}\{\Delta y_0+\Delta y_{-1}\}-\tfrac{1}{12}\{\Delta^3 y_{-1}+\Delta^3 y_{-2}\}+\dots \\ &= \mu\delta^1 y_0-\tfrac{1}{6}\mu\delta^3 y_0+\tfrac{1}{30}\mu\delta^5 y_0-\dots & (9) \\ &= \tfrac{1}{2}(y_1-y_{-1})-\tfrac{1}{12}(\delta^2 y_1-\delta^2 y_{-1})+\tfrac{1}{60}(\delta^4 y_1-\delta^4 y_{-1})-\dots & (10) \\ &= \tfrac{2}{3}(y_1-y_{-1})-\tfrac{1}{12}(y_2-y_{-2})+\tfrac{1}{60}(\delta^4 y_1-\delta^4 y_{-1})-\dots, & (11) \\ h^2f''(a) &= \Delta^2 y_{-1}-\tfrac{1}{12}\Delta^4 y_{-2}+\dots \\ &= \delta^2 y_0-\tfrac{1}{12}\delta^4 y_0+\tfrac{1}{90}\delta^6 y_0-\dots & (12) \\ &= (y_1-2y_0+y_{-1})-\tfrac{1}{12}\delta^4 y_0+\tfrac{1}{90}\delta^6 y_0-\dots. & (13) \end{aligned}$$

For integration, suppose, as usual, the function is specified by its values $\dots, y_{-1}, y_0, y_1,\dots$ at the points $a+rh$. The integral is taken to be the integral of the interpolating polynomial $f(a+\theta h)$ of § 119 between appropriate limits. The most useful results come from taking the limits symmetrical about the point a. Thus, using § 119 (16),

$$\begin{aligned} \int_{a-h}^{a+h} f(a+\theta h)\,dx &= h\int_{-1}^{1} f(a+\theta h)\,d\theta \\ &= 2h\{y_0+\tfrac{1}{6}\Delta^2 y_{-1}-\tfrac{1}{180}\Delta^4 y_{-2}+\dots\} \\ &= 2h\{y_0+\tfrac{1}{6}\delta^2 y_0-\tfrac{1}{180}\delta^4 y_0+\dots\} \\ &= \tfrac{1}{3}h(y_{-1}+4y_0+y_1)-\tfrac{1}{90}h\,\delta^4 y_0+\dots. & (14) \end{aligned}$$

This is Simpson's rule including a correction term involving the fourth central difference. Similarly,

$$\int_{a-2h}^{a+2h} f(a+\theta h)\,dx = \tfrac{4}{3}h(2y_{-1}-y_0+2y_1)+\tfrac{14}{45}h\,\delta^4 y_0-\dots. \quad (15)$$

It should be noticed that (15) does not involve the ordinates y_2 and y_{-2} at the ends of the interval, but that the fourth differences are much more important in it than in (14).

121. Ordinary differential equations

The problem is to solve a given equation with initial or boundary conditions which are given numerically. Suppose, first, that we have to solve the first-order equation

$$\frac{dy}{dx} = g(x,y) \tag{1}$$

with $y = y_0$ when $x = 0$.

We choose an interval h at which we wish to tabulate y, and the process of solution consists of calculating successively the values y_1, y_2,... of y at the points h, $2h$,.... The choice of the interval h is of considerable importance: if small intervals are chosen, relatively simple formulae which ignore higher differences may be used at the expense of the increase of labour involved in calculating many points; on the other hand, if relatively large intervals are used, allowance must be made for high differences and the procedure becomes more involved.

Whatever method is used, the first step consists of calculating the values of y at a number of the early points h, $2h$,... (the number of values needed depends on the method, and varies from one to three). This may be done by forming the Taylor series for y, or if this is not convenient other numerical methods are available. When these values have been calculated, they are extended by an extrapolation formula which may be derived by either of two general methods: (i) replacing the differential coefficient in (1) by an expression in finite differences, or (ii) by integrating equation (1) and using one of the integration formulae of § 120.

There are many methods suitable for serious computation; of these the Milne–Simpson method given below is probably the simplest, and another, the Adams–Bashforth, is given in Ex. 19: these are both of the second type referred to above. We give first a very simple method of the first type to illustrate the ideas involved and the way in which errors arise: this follows from the result § 120 (10), namely

$$h\left(\frac{dy}{dx}\right)_r = \tfrac{1}{2}(y_{r+1}-y_{r-1})-\tfrac{1}{6}\mu\delta^3 y_r+\dots. \tag{2}$$

Neglecting the third differences, this gives

$$\begin{aligned} y_{r+1} &= y_{r-1}+2h\left(\frac{dy}{dx}\right)_r \\ &= y_{r-1}+2h\,g(rh, y_r), \end{aligned} \tag{3}$$

if y satisfies (1). Thus if y_{r-1} and y_r are known, y_{r+1} can be calculated, and so on.

Ex. *To solve* $$\frac{dy}{dx} = y+x \tag{4}$$

with $y = 1$ *when* $x = 0$, *at intervals of* 0·1 *in* x.

Before the process (3) is started the value y_1 of y at $x = 0{\cdot}1$ has to be calculated. As remarked above, this is done by forming the Taylor series for y. It follows by differentiating (4) that

$$\frac{d^2y}{dx^2} = 1+\frac{dy}{dx}, \qquad \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2}, \qquad \dots,$$

and therefore

$$\begin{aligned} y &= y_0+x\left(\frac{dy}{dx}\right)_0+\tfrac{1}{2}x^2\left(\frac{d^2y}{dx^2}\right)_0+\frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0+\dots \\ &= 1+x+x^2+\frac{2}{3!}x^3+\frac{2}{4!}x^4+\dots. \end{aligned} \tag{5}$$

Using this series with $x = 0{\cdot}1$ gives $y_1 = 1{\cdot}1103$, and then (4) gives 1·2103 for the value of dy/dx when $x = 0{\cdot}1$. Then using (3) with $r = 1$ gives $y_2 = 1{\cdot}2421$ and so on. Continuing in this way gives the values entered in Table IV.

The values of the differences of y are also given in the table, and these provide an indication of the error. Third differences were neglected in (3), and since they enter the third place of decimals it is likely that the solution will be in error at least to this extent. The accurate value for $x = 1$ is 3·43656.... One disadvantage of simple methods such as this is that it is difficult to estimate the error.

TABLE IV

r	x	$g(x, y) = x+y$	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0	1	1			
				1103		
1	0·1	1·2103	1·1103		215	
				1318		33
2	0·2	1·4421	1·2421		248	
				1566		17
3	0·3	1·6987	1·3987		265	
				1831		37
4	0·4	1·9818	1·5818		302	
				2133		22
5	0·5	2·2951	1·7951		324	
				2457		44
6	0·6	2·6408	2·0408		368	
				2825		29
7	0·7	3·0233	2·3233		397	
				3222		50
8	0·8	3·4455	2·6455		447	
				3669		40
9	0·9	3·9124	3·0124		487	
				4156		
10	1·0		3·4280			

The Milne–Simpson method

Writing y_r and g_r for the values of y and $g(x,y)$ at the point $x = rh$, and integrating (1) from $x = rh$ to $x = (r+4)h$ gives, using § 120 (15),

$$y_{r+4}-y_r = \tfrac{4}{3}h(2g_{r+1}-g_{r+2}+2g_{r+3})+\tfrac{14}{45}h\, \delta^4 g_{r+2}. \tag{6}$$

Neglecting the fourth differences, this can be used to give y_{r+4} if the values of y up to y_{r+3} are known. This extrapolated value may be made more accurate as follows: using Simpson's rule, § 120 (14) in the same way gives

$$y_{r+4}-y_{r+2} = \tfrac{1}{3}h(g_{r+4}+4g_{r+3}+g_{r+2})-\tfrac{1}{90}h\, \delta^4 g_{r+3}. \tag{7}$$

(7) will give more accurate results than (6) if in both cases we neglect the fourth differences, since the term involving these in (7) is 1/28 of that in (6). But (7) cannot be used immediately to calculate y_{r+4} since it involves the unknown g_{r+4}; however, if we take the value of g_{r+4} calculated from (6) as a first approximation, and use this in the right-hand side of (7), we get a more accurate value of y_{r+4}.

TABLE V

x	y (*corrected*)	g (*corrected*)	y (*estd.*)	g (*estd.*)
0	1·00000	1·00000		
0·1	1·11034	1·21034		
0·2	1·24281	1·44281		
0·3	1·39972	1·69972		
0·4	1·58365	1·98365	1·58364	1·98364
0·5	1·79744		1·79743	2·29743

The process is set out in Table V for the equation (4) discussed previously. $y_0,\ldots, y_3$ are found from the Taylor series (5), and $g_1,\ldots, g_3$ are calculated from them. Then (6) with $r = 0$ gives the estimated value $y_4 = 1{\cdot}58364$ with the corresponding $g_4 = 1{\cdot}98364$; using this value in (7) gives the corrected $y_4 = 1{\cdot}58365$, and so on. The fact that the correction is only one in the fifth place indicates that the full power of the method is not being used, and we could either have calculated to more places of decimals or increased the interval h. Clearly the method gives a substantial increase in accuracy over the cruder one discussed earlier without a great increase of labour.

Considering next the second-order equation

$$\frac{d^2y}{dx^2} = g\left(x, y, \frac{dy}{dx}\right), \tag{8}$$

we may either replace the derivatives by differences using § 120 (10) and (13), or we may replace (8) by the pair of simultaneous first-order equations

$$\frac{dy}{dx} = Y, \tag{9}$$

$$\frac{dY}{dx} = g(x, y, Y), \tag{10}$$

which may be solved by an extension of the methods given above. For example, using the Milne–Simpson method, y_0, y_1, y_2, y_3, and $Y_0,\ldots, Y_3$ are found from a Taylor series and $g_0,\ldots, g_3$ are calculated from them. Then, using (6), Y_4 is estimated. Using this result and (7) y_4 is estimated, and then an estimated

g_4 is calculated. Using this and (7) gives the corrected Y_4, and then the corrected y_4. The first two steps of the solution of

$$\frac{d^2y}{dx^2}+4y = 0$$

with $y = 1$, $dy/dx = 0$, when $x = 0$ are given in Table VI.

TABLE VI

x	y	Y	g	y (*estd.*)	Y (*estd.*)	g (*estd.*)
0	1·00000	0	−4·00000			
0·1	0·98007	−0·39734	−3·92028			
0·2	0·92106	−0·77884	−3·68424			
0·3	0·82534	−1·12928	−3·30136			
0·4	0·69671	−1·43472	−2·78684	0·69671	−1·43453	−2·78684
0·5	0·54030	−1·68294		0·54031	−1·68278	−2·16124

One very important point remains to be mentioned. In the earlier chapters the distinction between initial value problems and boundary value problems was stressed. This distinction persists into the numerical methods for their solution. The problem treated above was an initial value problem, and the method used, described as a step-by-step method, was clearly an appropriate one for it. But in a boundary value problem the values of both y and dy/dx when $x = 0$ would not be known, for example dy/dx might not be known. To solve such a problem we would have to calculate solutions for various assumed values of dy/dx until we found one which satisfied the other boundary conditions. This process, though practicable, is laborious, and it will be seen in § 124 that relaxation methods are more suitable for boundary value problems.

122. Partial differential equations

Suppose we are given a partial differential equation in two variables such as those of § 105. It will have to be solved in some region such as the 'open' region of Fig. 92 (a) or the 'closed' region of Fig. 92 (b).

We wish to calculate the values $v_{m,n}$ of the solution v at each point† $x = mh$, $y = nh$, of a rectangular mesh such as that in

† There is no need for the intervals h in x and y to be the same: the modifications to be made if they are not are obvious.

Fig. 92 (*b*). The usual method of solution consists of replacing the partial derivatives by partial differences, and first these latter have to be defined.

Partial differences with respect to x (that is, y being kept constant) are denoted by a suffix x, so that

$$(\Delta v_{m,n})_x = v_{m+1,n} - v_{m,n}. \tag{1}$$

Similarly, $$(\Delta v_{m,n})_y = v_{m,n+1} - v_{m,n}. \tag{2}$$

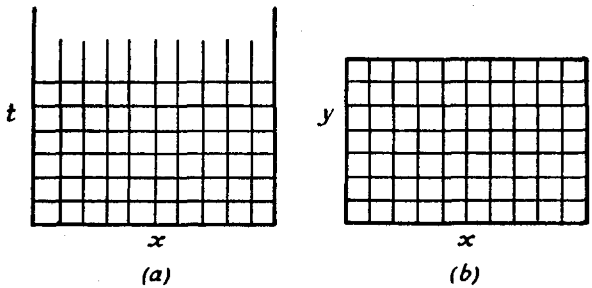


FIG. 92.

Clearly, with these additions, the whole of the results of § 120 can be taken over and, in particular, from § 120 (10) and (13)

$$h\left(\frac{\partial v}{\partial x}\right)_{m,n} = \tfrac{1}{2}(v_{m+1,n} - v_{m-1,n}) - \tfrac{1}{6}(\mu\delta^3 v_{m,n})_x + \dots, \tag{3}$$

$$h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_{m,n} = v_{m+1,n} - 2v_{m,n} + v_{m-1,n} - \tfrac{1}{12}(\delta^4 v_{m,n})_x + \dots, \tag{4}$$

$$h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_{m,n} + h^2\left(\frac{\partial^2 v}{\partial y^2}\right)_{m,n} = v_{m+1,n} + v_{m-1,n} + v_{m,n+1} + v_{m,n-1} - \\ - 4v_{m,n} - \tfrac{1}{12}(\delta^4 v_{m,n})_x - \tfrac{1}{12}(\delta^4 v_{m,n})_y \dots . \tag{5}$$

It follows from (5) that, neglecting fourth differences, Laplace's equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{6}$$

becomes

$$v_{m+1,n} + v_{m-1,n} + v_{m,n+1} + v_{m,n-1} - 4v_{m,n} = 0, \tag{7}$$

that is, the value of v at any point is the arithmetic mean of the values of v at the four points nearest to it.

A distinction similar to that mentioned at the end of § 121 between initial and boundary value problems for ordinary differential equations again appears. Equations of elliptic type such as (6) have to be solved in a closed region such as Fig. 92 (*b*) and relaxation methods are the most suitable for this. Equations of hyperbolic and parabolic types such as § 105 (1) and (2), on the other hand, have to be solved in an open region such as Fig. 92 (*a*) and step-by-step methods have to be used for these.

As an example of the latter we consider the equation of conduction of heat

$$\frac{\partial^2 v}{\partial x^2}-\frac{1}{\kappa}\frac{\partial v}{\partial t}=0 \tag{8}$$

in the region $0 < x < l$, $t > 0$. Using (4) in this, neglecting the fourth differences, and taking for $\partial v/\partial t$ the crudest possible value, namely

$$\tau\left(\frac{\partial v}{\partial t}\right)_{m,n}=(v_{m,n+1}-v_{m,n}), \tag{9}$$

where τ is the interval in t, and the first suffix refers to x and the second to t, gives

$$v_{m+1,n}-2v_{m,n}+v_{m-1,n}-\frac{h^2}{\kappa\tau}(v_{m,n+1}-v_{m,n})=0. \tag{10}$$

If the intervals h in x and τ in t are connected by

$$h^2=2\kappa\tau,$$

(10) takes the specially simple form

$$v_{m,n+1}=\tfrac{1}{2}(v_{m-1,n}+v_{m+1,n}), \tag{11}$$

that is, the temperature at any point at time $t+\tau$ is the arithmetic mean of the temperature at the two neighbouring points at time t. This result is known as Schmidt's method; it is very simple to use either numerically or graphically.

123. Solution of equations

In the course of this book many equations have occurred which have to be solved numerically; these have usually been frequency equations and may be either algebraic or transcendental. Various methods for the numerical solution of such equations are given in text-books on algebra [cf. Whittaker and Robinson or Milne, loc. cit.], but from the present point of view it is simplest to regard the finding of real roots of both types as a problem of inverse interpolation, that is, if we have to solve

$$f(x) = 0, \tag{1}$$

we tabulate $y = f(x)$ at suitably chosen points and have to find the value of x between two of these at which the function takes a given value, namely zero.

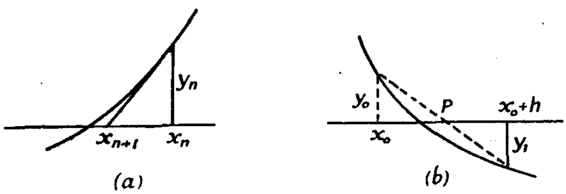


FIG. 93.

The first stage, then, is the approximate location of the roots of the equation from a rough table and graph. The root is then located as precisely as desired by a process of iteration or successive approximation. This may be either controlled, as when an iteration formula is applied successively, or discretionary, in which case the computer uses his judgement.

The simplest example of an iteration formula is Newton's, which states that if x_n is an approximation to a root of the equation (1) and y_n and y'_n are the values of y and its derivative at x_n, then

$$x_{n+1} = x_n - \frac{y_n}{y'_n} \tag{2}$$

is a better approximation. This is obvious from Fig. 93 (*a*). The formula (2) is then applied successively until the desired accuracy is attained. There are many similar simple formulae.

As an example of a simple discretionary method which is particularly useful when dealing with functions which are tabulated, suppose that y_0 and y_1 are the values of y at the points x_0 and x_0+h; then for the case $y_0 > 0$ and $y_1 < 0$ it follows from Fig. 93 (*b*) that if the function ran linearly between these two points it would vanish at P which is $x_0+\alpha h$, where

$$\alpha = \frac{y_0}{y_0+|y_1|}. \tag{3}$$

Two convenient points on either side of $x_0+\alpha h$ at which the function is tabulated are then selected and the process repeated. The solution of $\tan x - 2x = 0$ is entered in Table VII.

TABLE VII

x	$\tan x - 2x$	*Approx. sol.*
1·1	−0·2352	
1·2	0·1722	1·158
1·155	−0·0452	
1·165	−0·0025	1·1656
1·1654	−0·0007	
1·1656	0·0001	1·1656

124. Relaxation methods

These methods were originally developed by Southwell† for the determination of stresses in frames, and subsequently extended to cover ordinary and partial differential equations. The basis of the method is a very flexible iteration procedure for solving a system of n linear algebraic equations in n unknowns.

To illustrate, consider the equations

$$8x_1 - x_2 + x_3 - 200 = 0, \tag{1}$$

$$x_1 - 10x_2 + 2x_3 - 112 = 0, \tag{2}$$

$$-2x_1 - 2x_2 + 6x_3 - 81 = 0. \tag{3}$$

We wish to solve these by a process of successive approximation, that is, to find values of x_1, x_2, x_3 which will make the left-hand sides differ by an arbitrarily small amount from zero. Denote the differences of the left-hand sides from zero for any values of x_1, x_2, x_3 by R_1, R_2, R_3, so that

$$8x_1 - x_2 + x_3 - 200 = R_1, \tag{4}$$

$$x_1 - 10x_2 + 2x_3 - 112 = R_2, \tag{5}$$

$$-2x_1 - 2x_2 + 6x_3 - 81 = R_3. \tag{6}$$

These quantities R_1, R_2, R_3 are called the *residuals*, and the method of solution consists of changing x_1, x_2, x_3 systematically until the residuals are sufficiently near to zero. It is convenient

† *Relaxation Methods in Engineering Science* (Oxford, 1940); *Relaxation Methods in Theoretical Physics* (Oxford, 1946).

for this to construct a table showing the changes δR in the residuals caused by unit changes in x_1, x_2, and x_3. This runs as in Table VIII.

TABLE VIII

	δR_1	δR_2	δR_3
$\delta x_1 = 1$	8	1	-2
$\delta x_2 = 1$	-1	-10	-2
$\delta x_3 = 1$	1	2	6

In its simplest form the method consists of assuming initial values of x_1, x_2, x_3 which may be an obvious approximation if one exists, otherwise they are taken all to be zero. The residuals are then calculated, and the x associated with the largest residual is changed in such a way as to make this residual nearly zero.

TABLE IX

x_1	x_2	x_3	R_1	R_2	R_3
0	0	0	-200	-112	-81
20	0	0	-40	-92	-121
0	0	20	-20	-52	-1
0	-5	0	-15	-2	9
2	0	0	1	0	5
0	0	-1	0	-2	-1
22	-5	19	0	-2	-1
0	0	0	0	-200	-100
0	-20	0	20	0	-60
0	0	10	30	20	0
-4	0	0	-2	16	8
0	2	0	-4	-4	4
0	0	-1	-5	-6	-2
0	-1	0	-4	4	0
-4	-19	9	-4	4	0

If we take all the x to be zero initially, R_1, R_2, R_3, given by (4)–(6), are -200, -112, and -81. These values are entered in Table IX. The largest residual is $R_1 = -200$, and it appears from Table VIII that a change of 20 in x_1 will reduce this to -40 and will increase R_2 by 20 and R_3 by -40. These results are entered in the second row.

Note that there is no point in reducing the residual accurately to zero: the great thing is to choose as simple figures as possible at any stage so that the arithmetic may be performed quickly and accurately.

The largest residual is now $R_3 = -121$ and, by Table VIII, a change of 20 in x_3 reduces this to -1 and adds 20 and 40 to R_1 and R_2 respectively. This gives the third row of Table IX, and the process is continued until in the sixth row the residuals are reduced to one or two units. At this stage it is wise to add the changes in the x to get their final values and to check that these values of x do give the residuals on the right. This is done in the seventh line of Table IX. We notice that this line is equivalent to the statement that, if we write $x_1 = x_1'+22$, $x_2 = x_2'-5$, $x_3 = x_3'+19$, (1) to (3) become

$$8x_1'-x_2'+x_3' = 0, \tag{7}$$

$$x_1'-10x_2'+2x_3'-2 = 0, \tag{8}$$

$$-2x_1'-2x_2'+6x_3'-1 = 0. \tag{9}$$

Since it is always convenient to work with whole numbers only, we form the equations whose roots are 100 times the roots of (7) to (9). These are

$$8x_1''-x_2''+x_3'' = 0, \tag{10}$$

$$x_1''-10x_2''+2x_3''-200 = 0, \tag{11}$$

$$-2x_1''-2x_2''+6x_3''-100 = 0. \tag{12}$$

To solve these we start with $x_1'' = x_2'' = x_3'' = 0$, the residuals being 0, -200, and -100, and proceed as before. Thus, *to get another two places of decimals at any stage we multiply the residuals by* 100. This is done in the second part of Table IX and to this order the solutions are

$$x_1 = 21{\cdot}96, \qquad x_2 = -5{\cdot}19, \qquad x_3 = 19{\cdot}09,$$

the last figures being unreliable. If further accuracy is required we again multiply the residuals by 100, giving -400, 400, 0 and continue the process.

There are many methods by which these processes may be shortened; also it should be mentioned that there are types of equation for which the method in the simple form given above is not suitable (in such cases the residuals will be found to diminish slowly or to oscillate) and special methods have been developed to handle these.

In applying these methods to the solution of ordinary or partial differential equations, the differential equation is replaced by a difference equation, and the set of algebraic equations so obtained is solved by the process given above. Usually the simplest difference equations, neglecting higher differences, are used, so that to get reasonable accuracy the interval h must be taken to be fairly small.

As an example suppose that we wish to solve Laplace's equation, § 110 (1), in a square $ABCD$, with $v = 100$ on AB, and $v = 0$ on BC, CD, DE. Taking $h = \frac{1}{3}AB$, we know that $v = 100$ at each of the points in AB, and $v = 0$ at each of the points in the other sides. The difference equation is § 122 (7),

$$v_{m+1,n}+v_{m-1,n}+v_{m,n+1}+v_{m,n-1}-4v_{m,n} = 0. \tag{13}$$

This equation has to be satisfied for each internal point of the square. We write

$$R_{m,n} = v_{m+1,n}+v_{m-1,n}+v_{m,n+1}+v_{m,n-1}-4v_{m,n} \tag{14}$$

for the residual corresponding to $v_{m,n}$ and proceed as before to reduce these residuals to zero. First we assume values of v at the points P, Q, R, S, which we may take to be zero; these values are written to the left of the vertical through the point concerned. The corresponding residuals are calculated from (14) and entered to the right of the corresponding vertical. The residuals at P and Q are 100, and those at R and S are zero.

We now proceed as before to reduce the largest residual. By (14), a change of 25 at P reduces the residual there to zero, and adds 25 to the residuals at R and Q. Thus we enter 25 against v at P, and change the residuals at P, Q, R to 0, 125, and 25, respectively. Note that, to save writing, only the changes in v and the new values of the residuals which change are entered. A change of 35 in v at Q makes the residual there -15 and increases the residuals at P and S both to 35. Proceeding in this way we get the result of Fig. 94: at R and S the values of v

are both 12, and at P and Q they are both 37, all the residuals being 1.

As usual, these residuals should be checked. The accurate values, calculated from § 111 (12), are 11·9 at R and S, and 38·2 at P and Q, so that the results above are surprisingly accurate considering the large mesh taken.

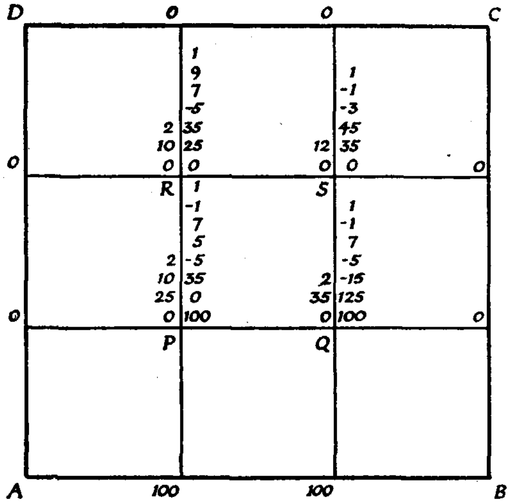


FIG. 94.

As a second example consider the solution of Laplace's equation in the L-shaped corner of Fig. 95, the value of v being 100 on the face ABC, and zero on DEF. For simplicity of demonstration we take only one row of points between the faces as shown. Away from the corner, it is known that v varies linearly between the faces, so we may assume $v = 50$ for the initial values (there is no objection to taking $v = 0$ as before, but if there is an obvious approximation as in this case it shortens the work to use it). The residuals are zero except at the point G midway between B and E where the residual is 100. The solution then proceeds as before, and is shown in Fig. 95. By increasing the number of points the accuracy can be improved and values of v found over the whole of the interior of the corner: there is more arithmetic but the process remains simple.

In the past few years Southwell and his co-workers (loc. cit.) have applied the method to a wide variety of partial differential equations and boundary conditions. It has the advantage that it applies to 'mixed' boundary conditions, that is, to problems in which boundary conditions of different types occur on the boundary. Also it can be extended to cover bounding curves of any shape.

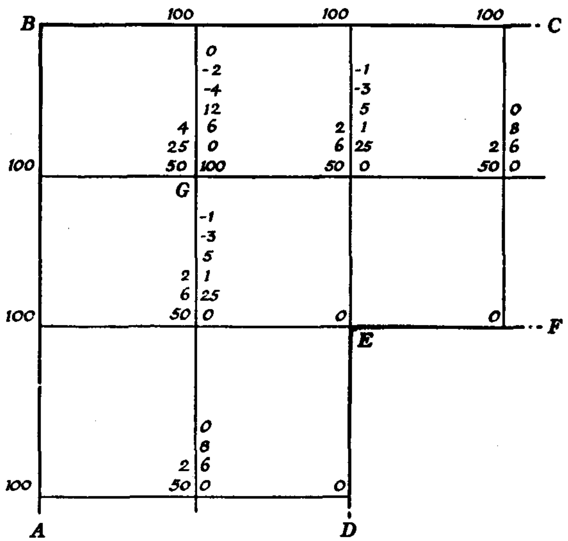


FIG. 95.

PROBLEMS ON CHAPTER XIV

1. Form a difference table from the following

x	$\cos x$	x	$\cos x$
5·0	0·28366	5·5	0·70867
5·1	0·37798	5·6	0·77557
5·2	0·46852	5·7	0·83471
5·3	0·55437	5·8	0·88552
5·4	0·63469	5·9	0·92748
		6·0	0·96017

Estimate values of cos 5·075, cos 5·25, and cos 4·97. [0·35472, 0·51208, 0·25477.]

2. Show that the differences of the set of numbers ...0, 0, 1, 0, 0,... are ...0, 0, 1, -1, 0, 0,..., ...0, 1, -2, 1, 0,..., etc., the numbers in the nth difference being

$$(-1)^r\,{}^nC_r.$$

Discuss the effect of rounding-off errors on the successive differences.

3. The most common errors in numerical work are: (i) interchanging two figures, e.g. writing 9478 for 9748, and (ii) doubling the wrong figure, e.g. writing 9447 for 9477. Discuss the effect of these errors on the differences and show how they can be traced (in the first case the errors in all differences are multiples of nine).

Show *that in the following* set of values of a function

1·0000, 0·9406, 0·8824, 0·8245, 0·7696, 0·7150, 0·6616, 0·6094,...

the fourth should probably read 0·8254.

4. Show that
$$\Delta^r y_n = \sum_{s=0}^{r} (-1)^s \binom{r}{s} y_{n+r-s},$$

and
$$y_{n+r} = \sum_{s=0}^{r} \binom{r}{s} \Delta^{r-s} y_n.$$

5. Show that
$$[x]^{n+1} = (x-n)[x]^n.$$

Deduce that if $S_r^{(n)}$ is the coefficient of x^{n-r} in $[x]^n$,
$$S_r^{(n+1)} = S_r^{(n)} - nS_{r-1}^{(n)}.$$

Using this result, build up the first few $S_r^{(n)}$, starting from $S_0^{(1)} = 1$. Show how to use these results to express a polynomial as a sum of factorial polynomials, and prove that
$$x^4 - 4x^3 + 5x^2 + x = [x]^4 + 2[x]^3 + 3[x].$$

6. From the following entries

x	Ei(x)	δ^2	δ^4
5·0	40·18527	0·23761	0·00176
5·1	43·27571	0·25871	0·00192

find the value of Ei(5·073). [42·41671.]

7. If $f(a-\theta h)$ is the polynomial which takes the values y_{-r} at $a-rh$, $r = 0, 1,..., n$, show that
$$f(a-\theta h) = y_0 - \theta\Delta y_{-1} + \binom{\theta}{2}\Delta^2 y_{-2} + ... + (-1)^n \binom{\theta}{n}\Delta^n y_{-n}.$$

This is the 'Gregory–Newton backwards' formula. It can be deduced from the Gregory–Newton formula for the function which has the values y_{-r} at the points $a+rh$ by comparing the difference tables for the two functions. Alternatively the argument of § 119 (10) to (15) may be used, assuming $f(a+\theta h) = \alpha_0 + \alpha_1[\theta]^1 + \alpha_2[\theta+1]^2 + ... + \alpha_n[\theta+n-1]^n$.

From the figures given in Ex. 1, estimate the value of cos 5·95. [0·94501.]

8. Replacing θ by $-\theta$ in the result of Ex. 7, derive the extrapolation formula
$$f(a+\theta h) = y_0 + \theta\Delta y_{-1} + \binom{\theta+1}{2}\Delta^2 y_{-2} + ... + \binom{\theta+n-1}{n}\Delta^n y_{-n}.$$

From the figures given in Ex. 1 estimate the value of $\cos 6{\cdot}03$. [0·96812.]

Show that

$$\int_0^1 f(a+\theta h)\,d\theta = y_0+\tfrac{1}{2}\Delta y_{-1}+\tfrac{5}{12}\Delta^2 y_{-2}+\tfrac{3}{8}\Delta^3 y_{-3}+\tfrac{251}{720}\Delta^4 y_{-4}+\dots.$$

9. Calculate the derivative of $\cos x$ from the values in Ex. 1 for the following values of x: 5·0, 5·05, 5·4, 6·0. [0·9589, 0·9436, 0·7728, 0·2794.]

10. y_0, y_1, y_2, y_3 are the values of a function at four equally spaced points. Show that its derivatives at the first two of these are

$$\frac{1}{6h}(-11y_0+18y_1-9y_2+2y_3)-\frac{1}{4h}\Delta^4 y_0,$$

$$\frac{1}{6h}(-2y_0-3y_1+6y_2-y_3)+\frac{1}{12h}\Delta^4 y_0.$$

11. Deduce the results of Ex. 10 without the difference correction by assuming a polynomial of the third degree to pass through the four points. Deduce § 120 (14) and (15) in the same way by assuming an appropriate quadratic.

12. If y_r, y'_r, y''_r,... are the values of a function and its successive differential coefficients at the point $a+rh$, show by Taylor's theorem that

$$\mu\delta y_0 = hy'_0+\frac{h^3}{3!}y'''_0+\frac{h^5}{5!}y_0^{(\mathrm{v})}+\dots,$$

$$\delta^2 y_0 = h^2 y''_0+\tfrac{1}{12}h^4 y_0^{(\mathrm{iv})}+\tfrac{1}{360}h^6 y_0^{(\mathrm{vi})}+\dots,$$

$$\delta^4 y_0 = h^4 y_0^{(\mathrm{iv})}+\tfrac{1}{6}h^6 y_0^{(\mathrm{vi})}+\dots.$$

13. Show as in § 120 that

$$\int_a^{a+h} y\,dx = h\{y_0+\tfrac{1}{2}\Delta y_0-\tfrac{1}{12}\Delta^2 y_{-1}-\tfrac{1}{24}\Delta^3 y_{-1}+\tfrac{11}{720}\Delta^4 y_{-2}+\dots\}$$

$$= \frac{h}{24}\{-y_{-1}+13y_0+13y_1-y_2\}+\frac{11h}{720}\Delta^4 y_{-2}+\dots.$$

If y_{-1}, y_0, y_1,..., y_{n+1} are the values of y at $x = rh$, $r = -1, 0,\dots, n+1$, show by adding results of the above type and neglecting the fourth difference correction that

$$\int_0^{nh} y\,dx = h\{\tfrac{1}{2}y_0+y_1+y_2+\dots+y_{n-1}+\tfrac{1}{2}y_n\}+\frac{h}{24}\{y_1-y_{-1}+y_{n-1}-y_{n+1}\}.$$

The first term of this is the trapezoidal rule, and the second a correcting term which improves its accuracy considerably.

14. Evaluate
$$\int_{5\cdot 1}^{5\cdot 9} \cos x\, dx$$
from the values given in Ex. 1 using (i) Simpson's rule § 120 (14); (ii) Milne's formula, § 120 (15); (iii) the result of Ex. 13, neglecting fourth differences in all cases. Compare the results with the accurate value, which is 0·55194.

15. Show that if $y_0, y_1, y_2, \ldots$ are the values of y at the points $0, h, 2h, \ldots$
$$\int_0^h \frac{y\, dx}{x^{\frac{1}{2}}} = \tfrac{2}{15}h^{\frac{1}{2}}(9y_0+7y_1-y_2)+\tfrac{22}{315}h^{\frac{1}{2}}\Delta^3 y_0.$$
Derive the corresponding result for
$$\int_0^h x^{-n}y\, dx \quad (0 < n < 1).$$

16. y_{r+2}, y_{r+1}, y_r are consecutive values of y in the solution of
$$\frac{dy}{dx}+y = 0$$
by the method of § 121 (3). Show that
$$y_{r+2}+2hy_{r+1}-y_r = 0.$$
Proceeding as in § 85 Ex., show that this is satisfied by
$$y_r = A\{-h-(h^2+1)^{\frac{1}{2}}\}^r+B\{-h+(h^2+1)^{\frac{1}{2}}\}^r,$$
where A and B are constants determined by y_0 and y_1.

Discuss the errors involved in this method of solution.

17. Solve the differential equation
$$\frac{dy}{dx} = x-\tfrac{1}{2}y^2$$
at intervals of 0·1 in x, with $y = 1$ when $x = 0$. Show that $y = 0{\cdot}9725$ when $x = 0{\cdot}8$.

18. Solve the differential equation
$$\frac{d^2y}{dx^2}-xy = 0$$
at intervals of 0·1 in x with $y = 1$, $dy/dx = 0$, when $x = 0$. Show that $y = 1{\cdot}0868$ when $x = 0{\cdot}8$.

19. y satisfies the differential equation
$$\frac{dy}{dx} = g(x,y).$$
$y_0, y_1, y_2, \ldots$ are the values of y at the points $x = rh$, $r = 0, 1, 2, \ldots$, and $g_0, g_1, g_2, \ldots$ are the values of $g(x,y)$ at these points. Using the result of Ex. 8, show that
$$y_{r+1}-y_r = h\{g_r+\tfrac{1}{2}\Delta g_{r-1}+\tfrac{5}{12}\Delta^2 g_{r-2}+\tfrac{3}{8}\Delta^3 g_{r-3}+\ldots\}.$$

Solve the differential equation § 121 (4) with $h = 0{\cdot}1$ using the values of y_0, y_1, y_2, y_3 given in § 121, Table V. Find the values of g_0, g_1, g_2, g_3 and difference them, then calculate y_4 from the above formula and repeat the process. This is the Adams–Bashforth method.

20. Show that the simplest finite difference equivalent of Laplace's equation in three dimensions is the statement that the value of the function at any point is the arithmetic mean of its values at the six nearest points of a rectangular lattice.

21. Show that the first four roots of

$$x \tan x = 0{\cdot}5$$

are 0·6533, 3·2923, 6·3616, 9·4775.

22. Show that the smallest root of

$$\cosh x \cos x = 1$$

is 4·7300.

23. Show that the roots of

$$x^4 - 2x^3 - 6x^2 + 25x - 28 = 0,$$

to three places of decimals, are $-3{\cdot}193$, $2{\cdot}193$, $1{\cdot}500 \pm 1{\cdot}323i$.

24. Find an approximate solution of the equations

$$7x_1 + 2x_2 + x_3 = 750,$$
$$2x_1 + 12x_2 + 3x_3 = 1850,$$
$$x_1 - 3x_2 + 6x_3 = 260.$$

[$x_1 = 59$, $x_2 = 121$, $x_3 = 94$.]

25. Taking $h = AB/4$, show that the solution of Laplace's equation in a square $ABCD$ which has the values 100 on AB and zero on the other sides has the value 25, approximately, at the centre of the square.

INDEX

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