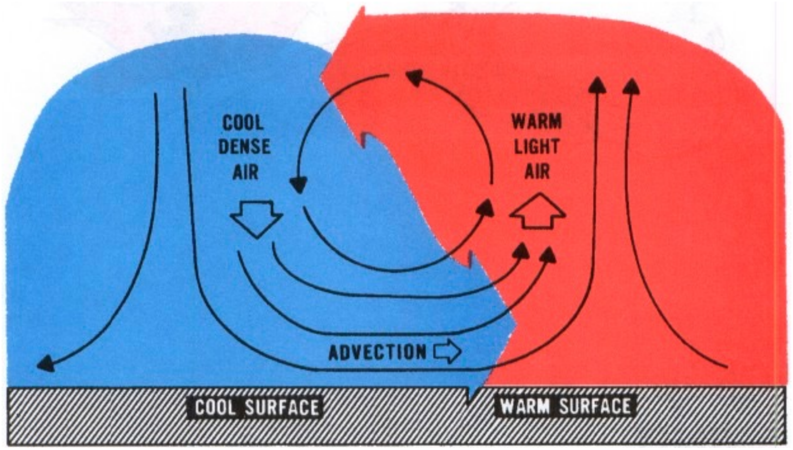
Introduction to Advection

- One of the two fundamental transport mechanisms is advection.
- Advection explains a great number of transport phenomena.
- Advection is considered as a transport mechanism of material or quantity / property through the bulk motion of fluid.
- Advection is not observed in rigid substances as there is no current.

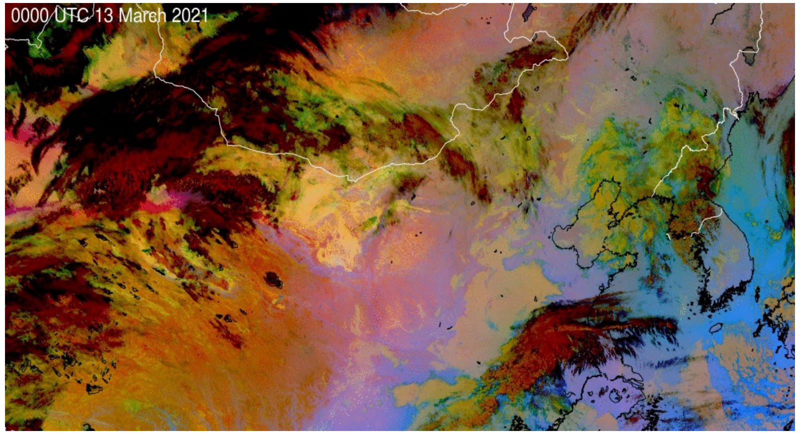
https://www.schamanische-kinesiologie.berlin/sitemap/

https://www.blendspace.com/lessons/InaeSRu5s7e_SA/science-form-3-kssm-chapter-2-respiration-subtopic-2-1

Some Applications of Advection

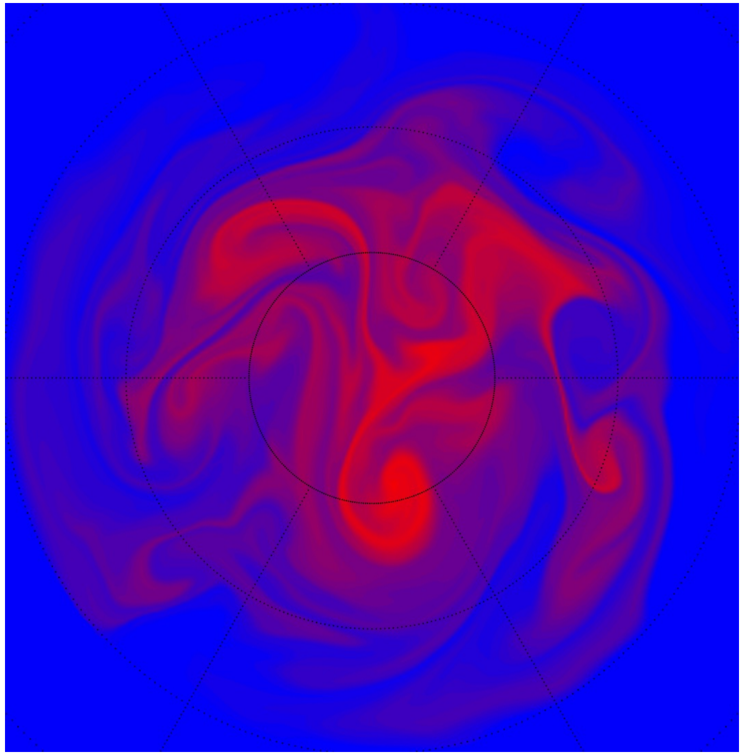


https://www.aviationweather.ws/016_Convection.php



https://cimss.ssec.wisc.edu/

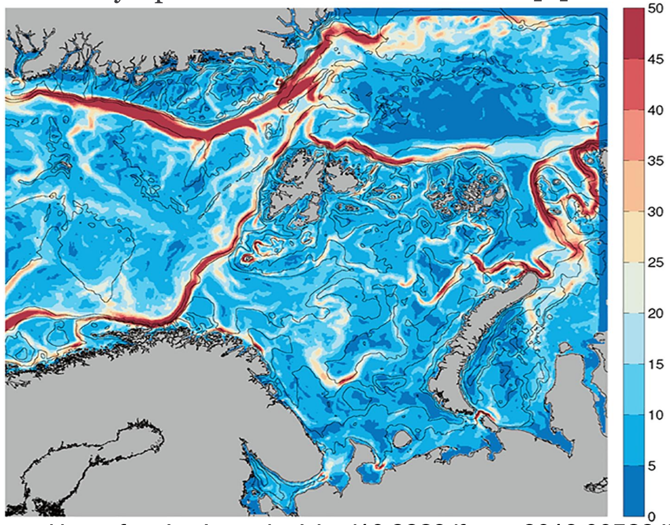
Meteorology / Atmospheric Science



https://en.wikipedia.org/wiki/File:Chaotic_mixing.png

Fluid Dynamics

Phytoplankton Biomass Advection [2]



https://www.frontiersin.org/articles/10.3389/fmars.2019.00583/full

https://www.dreamstime.com/stock-photo-fog-over-cold-current-kara-sea-fog-over-cold-current-kara-sea-arctic-ocean-has-lost-ice-stormy-weather-six-points-image95339677

Oceanography

The Linear Advection Equation

The 1-D linear advection equation is given as,

$$\frac{\partial u(t,x)}{\partial t} + \alpha \frac{\partial u(t,x)}{\partial x} = 0$$

The equation illustrates transport of a quantity or property $\boldsymbol{u(t,x)}$ with a constant characteristic speed α.

In simplified form, we can write the equation as,

$$u_t + \alpha u_x = 0$$

Where,

$$u_t = \frac{\partial u}{\partial t},\ u_x = \frac{\partial u}{\partial x}$$

$$\mathit{The\ domain\ considered\ here\ is}\ \daleth = [-4,8]$$

The Linear Advection Equation

The exact closed form solution of the 1-D advection equation is [1],

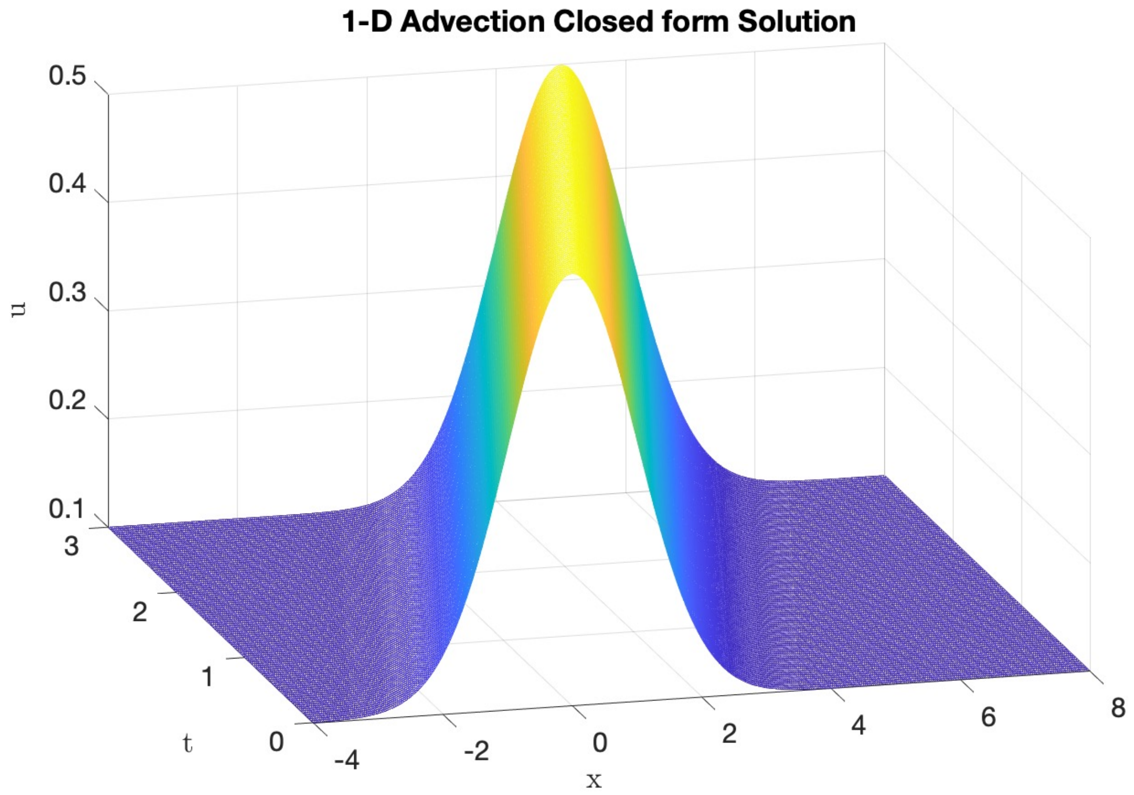
$$u(x,t) = c_o + \frac{1}{\sigma\sqrt{2\pi}} exp\left[\frac{-(x-\alpha t)^2}{2\sigma^2}\right]$$

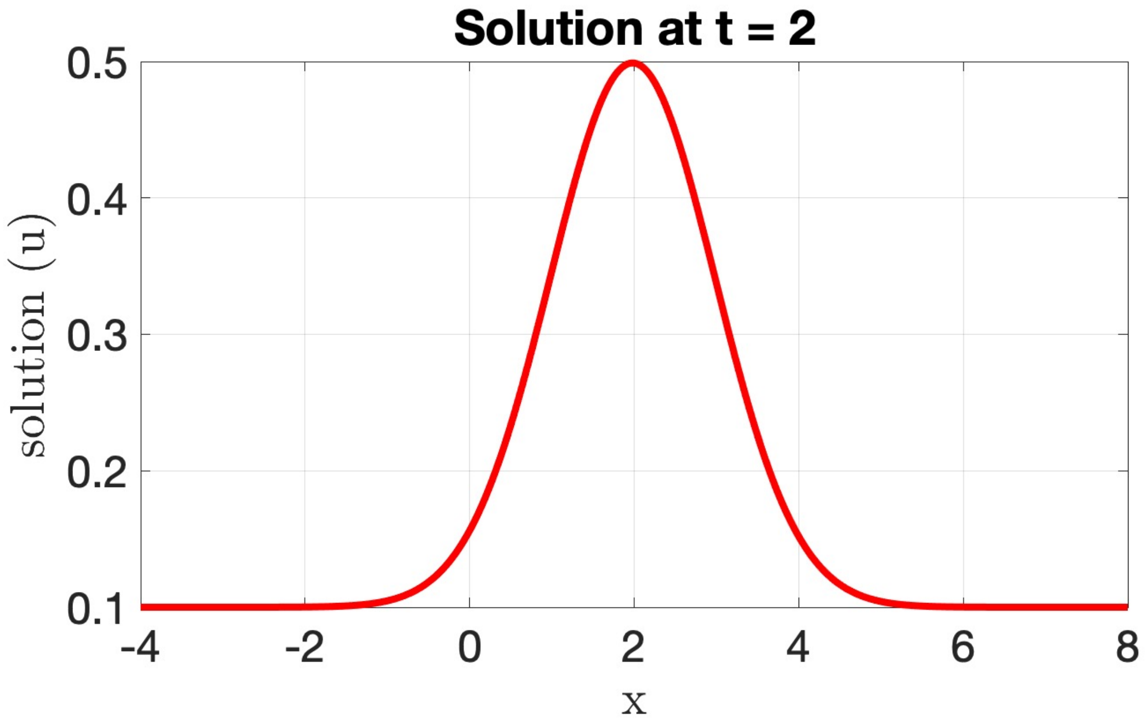
$$\sigma = characteristic\ width\ of\ the\ kernel$$
$$\alpha = characteristic\ wave\ speed$$
$$c_o = simple\ shift\ or\ translation$$

For our case, we assume,

$$\sigma = 1$$
$$\alpha = 1$$
$$c_o = 0.1$$

The Linear Advection Equation

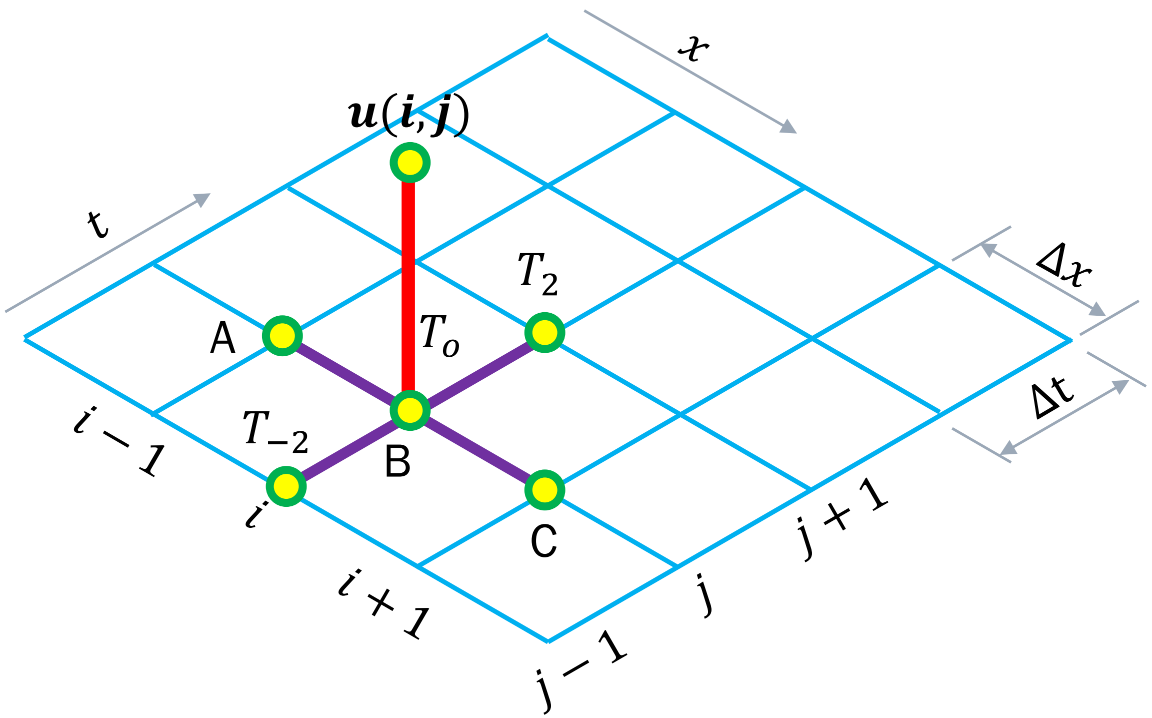


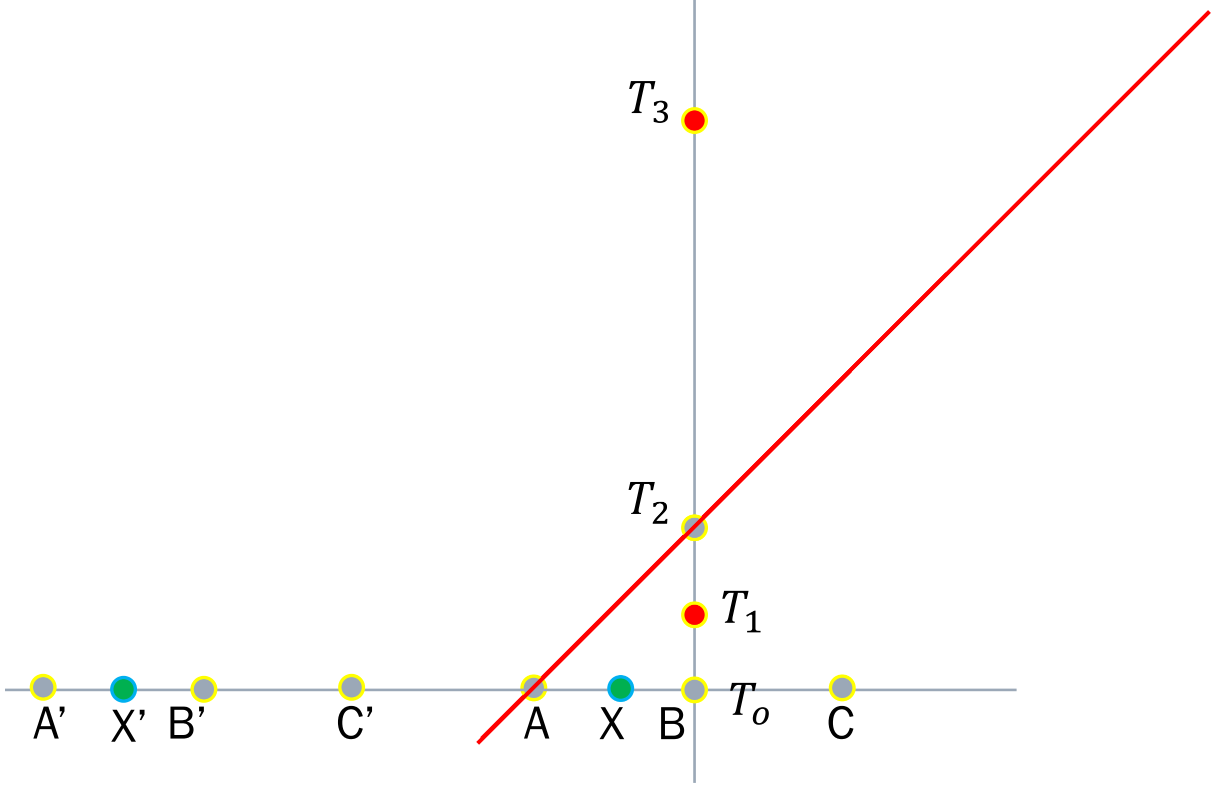


Plots of exact form solution

Finite Difference Scheme

In finite difference schemes, continuous time derivatives are approximated using discrete set of points [4]





*Initial and boundary conditions are obtained from the exact solution available

Finite Difference Scheme

The Courant number for the 1-D advection equation is given as,

$$\boldsymbol{\lambda} = \boldsymbol{\alpha}\frac{\boldsymbol{\Delta t}}{\boldsymbol{\Delta}\mathrm{x}}$$

Where,

$\boldsymbol{\Delta t}$ = step size n time

$\boldsymbol{\Delta x}$ = step size in space

$\boldsymbol{\alpha}$ = the propagation speed

Courant number is crucial for convergence of the solution the advection equation in finite difference scheme

The stability condition is given as,

$$\boldsymbol{\lambda} = \boldsymbol{\alpha}\frac{\boldsymbol{\Delta t}}{\boldsymbol{\Delta}\mathrm{x}} \leq \mathbf{1}$$

Upwind Finite Difference Scheme

*Also called Forward in Time and Backward in Space (FTBS)

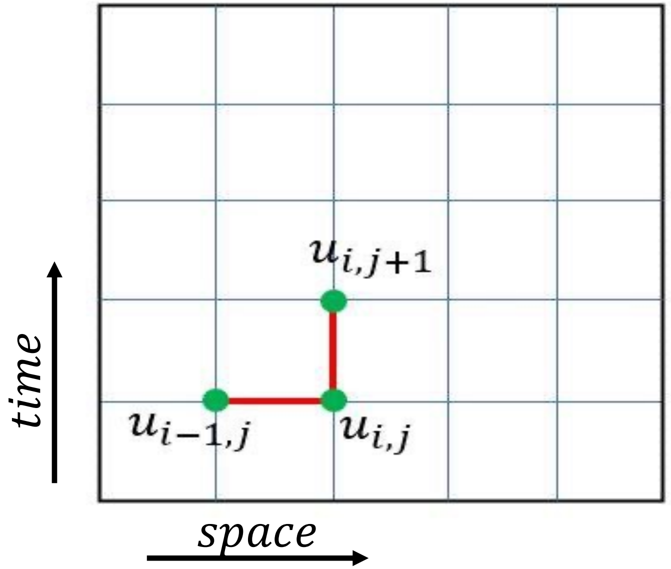
The derivatives are approximated as,

$$u_x = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} \qquad u_t = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

Putting the values in the advection equation,

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} + \alpha \frac{u_{i,j} - u_{i-1,j}}{\Delta x} = 0$$

$$\Delta t = time - steps,$$
$$\Delta x = space - steps$$

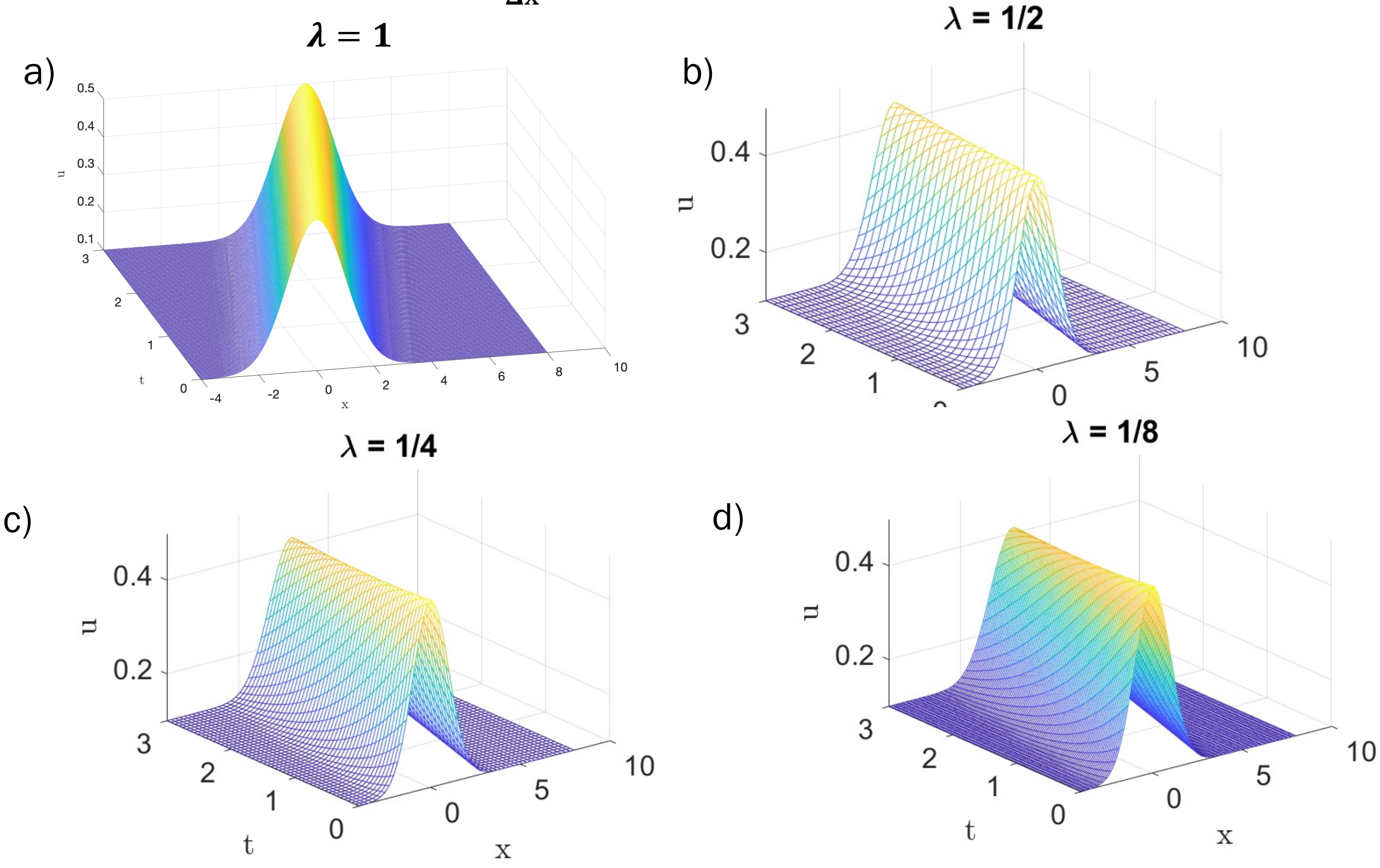


The explicit upwind scheme is given by,

$$u_{i,j+1} = u_{i,j}\left(1 - \frac{\alpha\, \Delta t}{\Delta x}\right) + \frac{\alpha\, \Delta t}{\Delta x} u_{i-1,j}$$

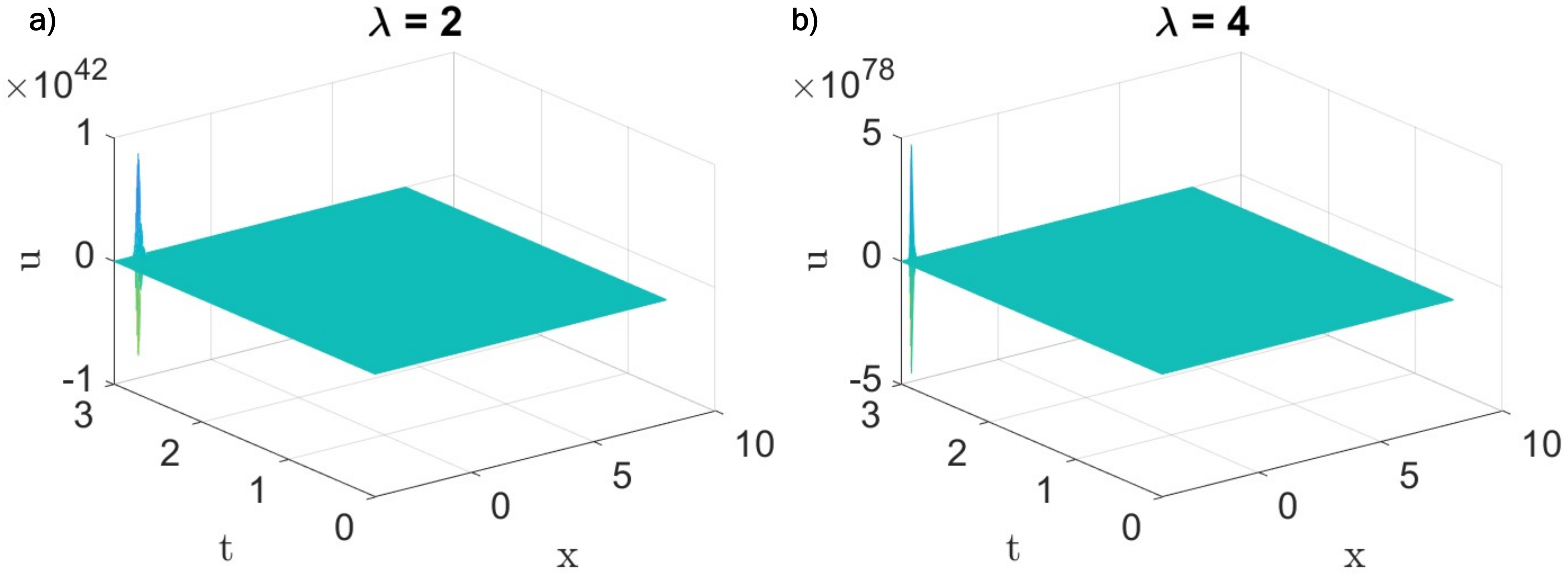
Upwind Finite Difference Scheme

Upwind scheme Courant Number, $\boldsymbol{\lambda = \alpha \frac{\Delta t}{\Delta x} \leq 1}$

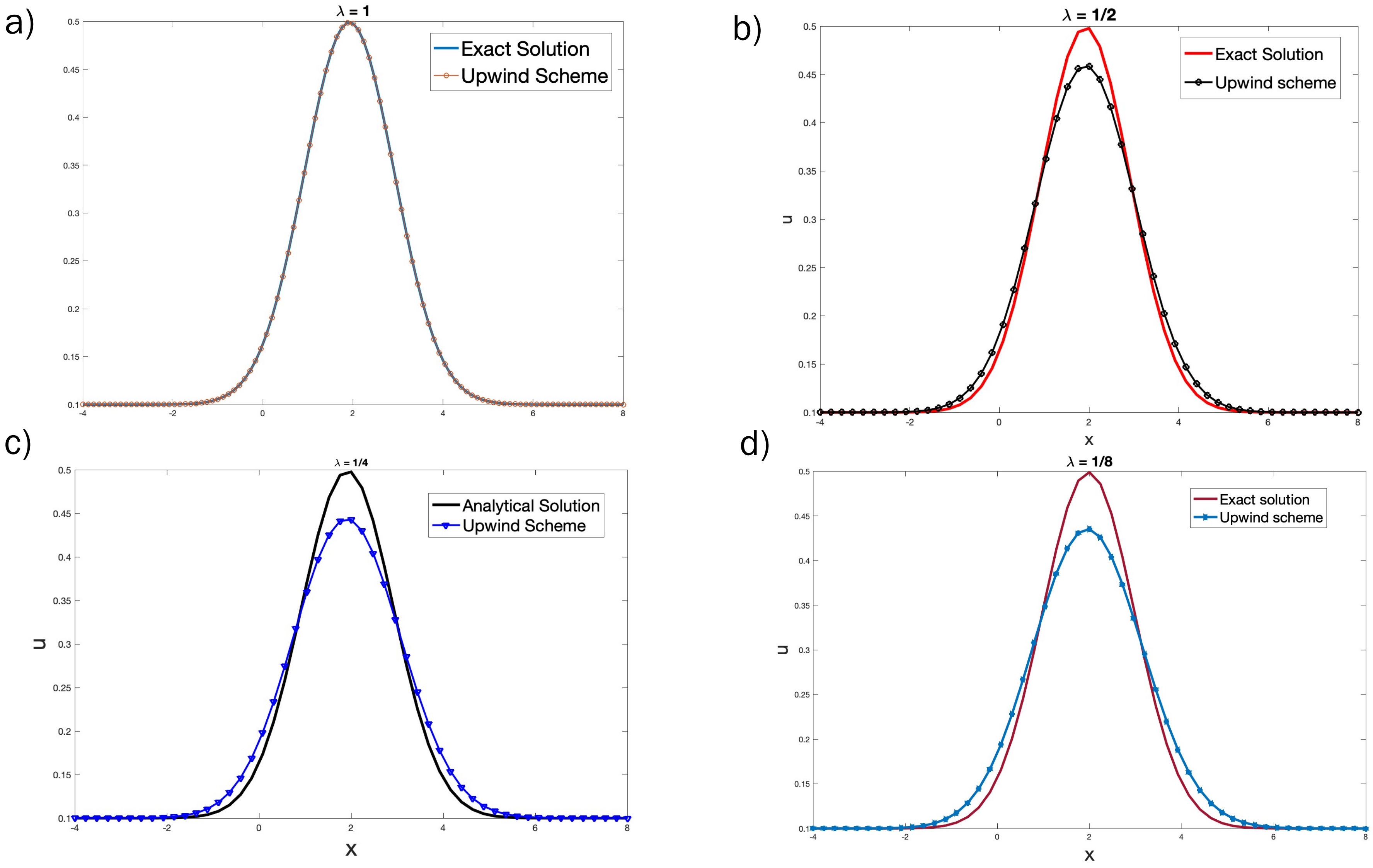


Upwind Finite Difference Scheme

Upwind scheme Courant Number, $\boldsymbol{\lambda = \alpha \frac{\Delta t}{\Delta x} > 1}$



Upwind Finite Difference Scheme

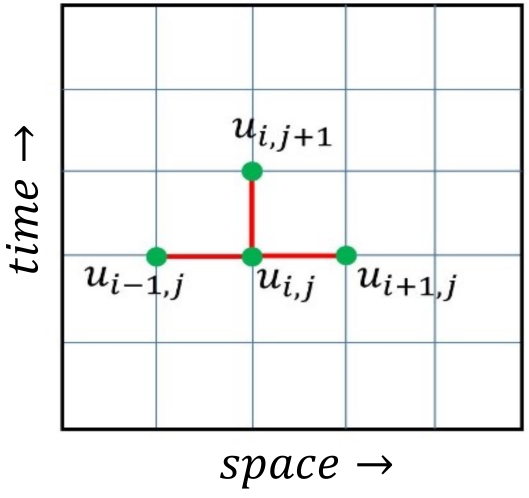


Comparison of Upwind Scheme with Exact solution at t=2 sec

FTCS Scheme with small diffusion term

We consider small diffusion term in the 1-D advection equation as below,

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = f \frac{\partial^2 u}{\partial x^2}$$



The derivatives are approximated as,

$$u_x = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} \qquad u_t = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} \qquad u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$$

The explicit scheme is,

$$u_{i,j+1} = \Delta t \left(u_{i,j} \left(\frac{1}{\Delta t} - \frac{\alpha}{\Delta x} + \frac{2f}{\Delta x^2} \right) + u_{i-1,j} \left(\frac{\alpha}{\Delta x} - \frac{f}{\Delta x^2} \right) + u_{i+1,j} \left(\frac{f}{\Delta x^2} \right) \right)$$

$If\ we\ consider\ equal\ step\ sizes\ for\ time\ and\ space, i.e.\ \Delta t = \Delta x = h\ \ the\ explicit\ scheme\ becomes,$

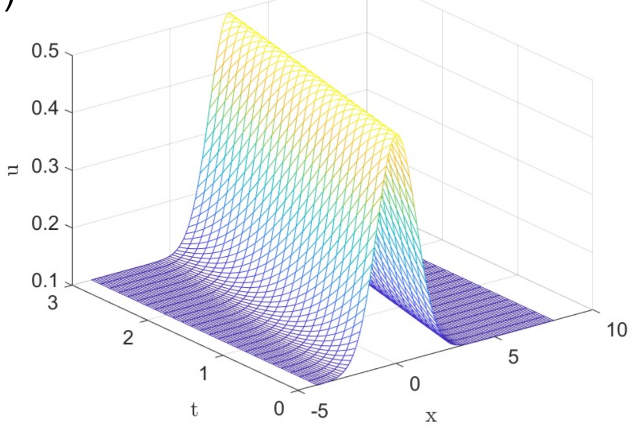
$$u_{i,j+1} = \left[u_{i-1,j} \frac{\alpha h - f}{h} + u_{i,j} \frac{h(1-\alpha) + 2f}{h^2} + u_{i+1,j} \frac{f}{h} \right]$$

FTCS Scheme with small diffusion term

FTCS Scheme Courant Number, $\lambda = \alpha \frac{\Delta t}{\Delta x} \leq 1$

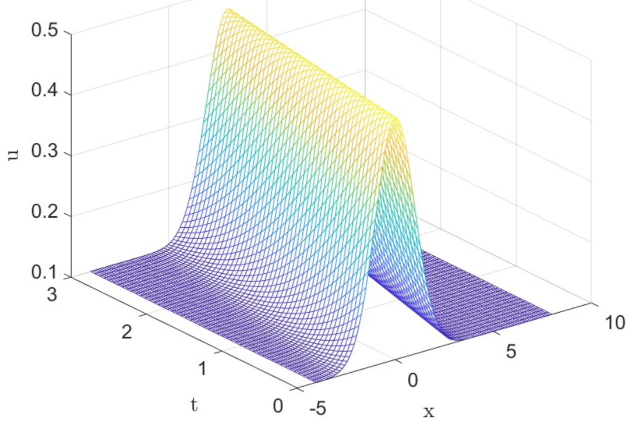
$\lambda = 1$

a)



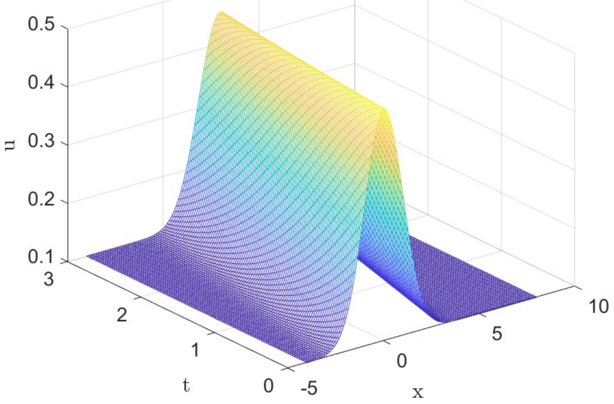
$\lambda = 1/2$

b)



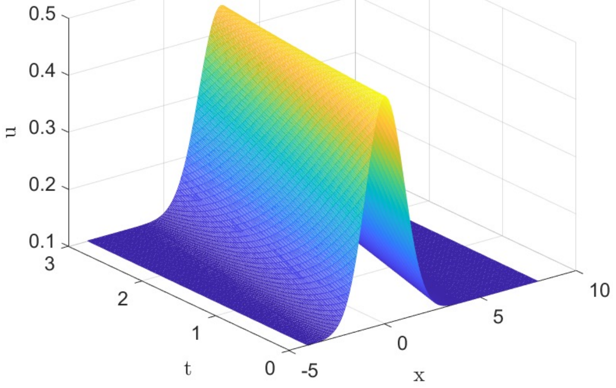
$\lambda = 1/4$

c)



$\lambda = 1/8$

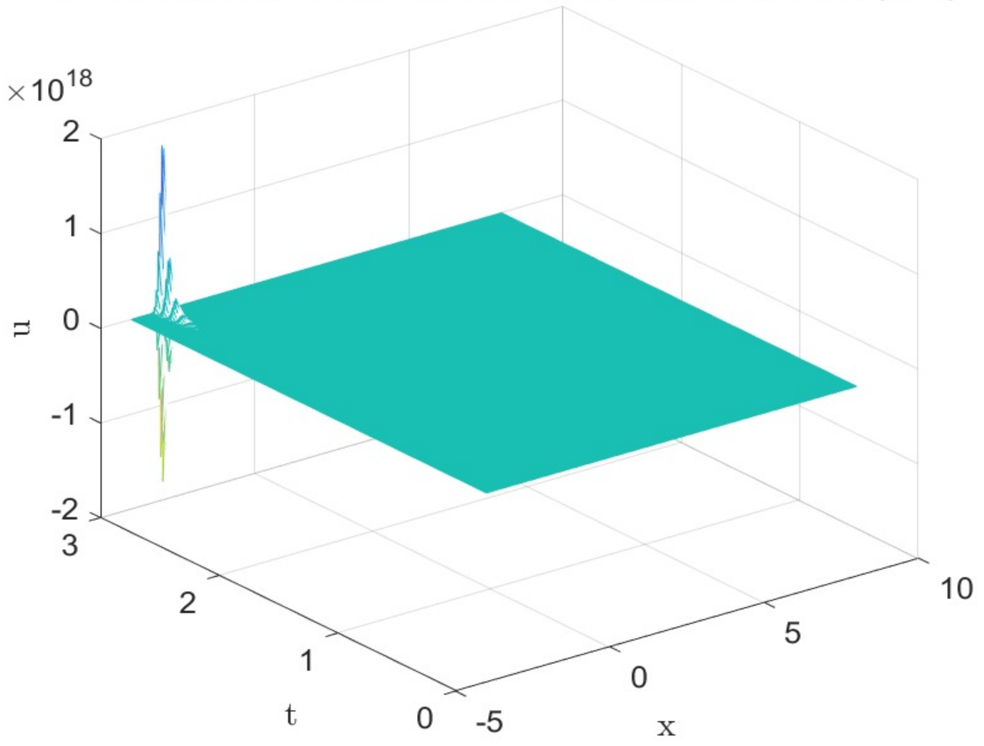
d)



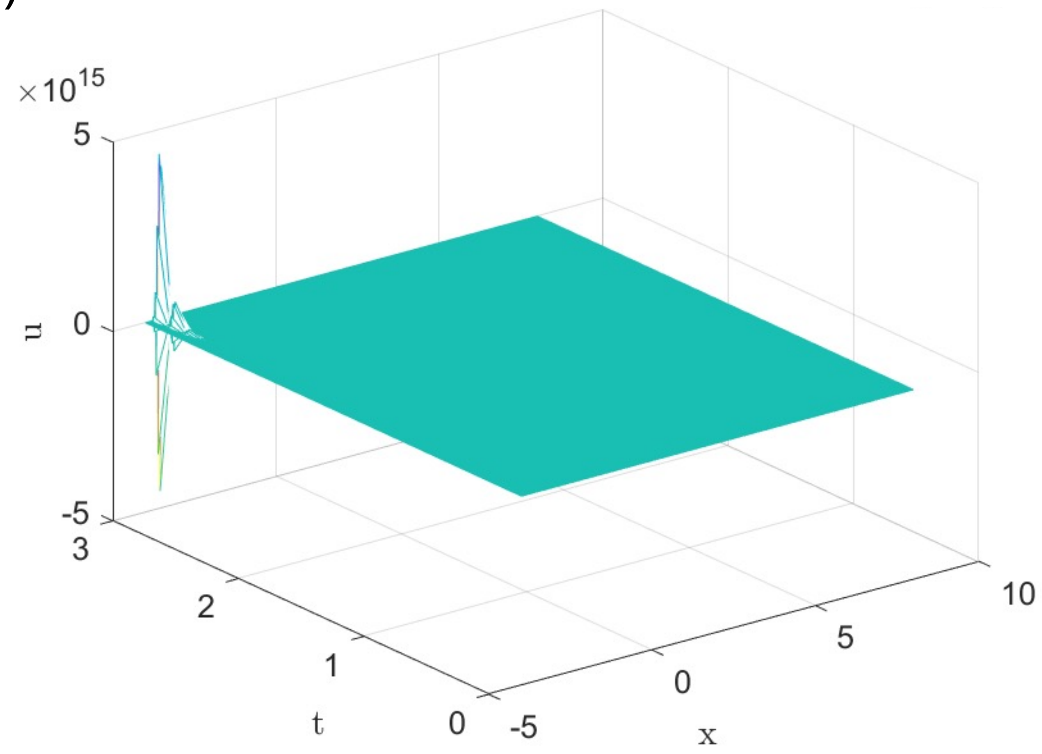
FTCS Scheme with small diffusion term

FTCS Scheme Courant Number, $\boldsymbol{\lambda = \alpha \frac{\Delta t}{\Delta x} > 1}$

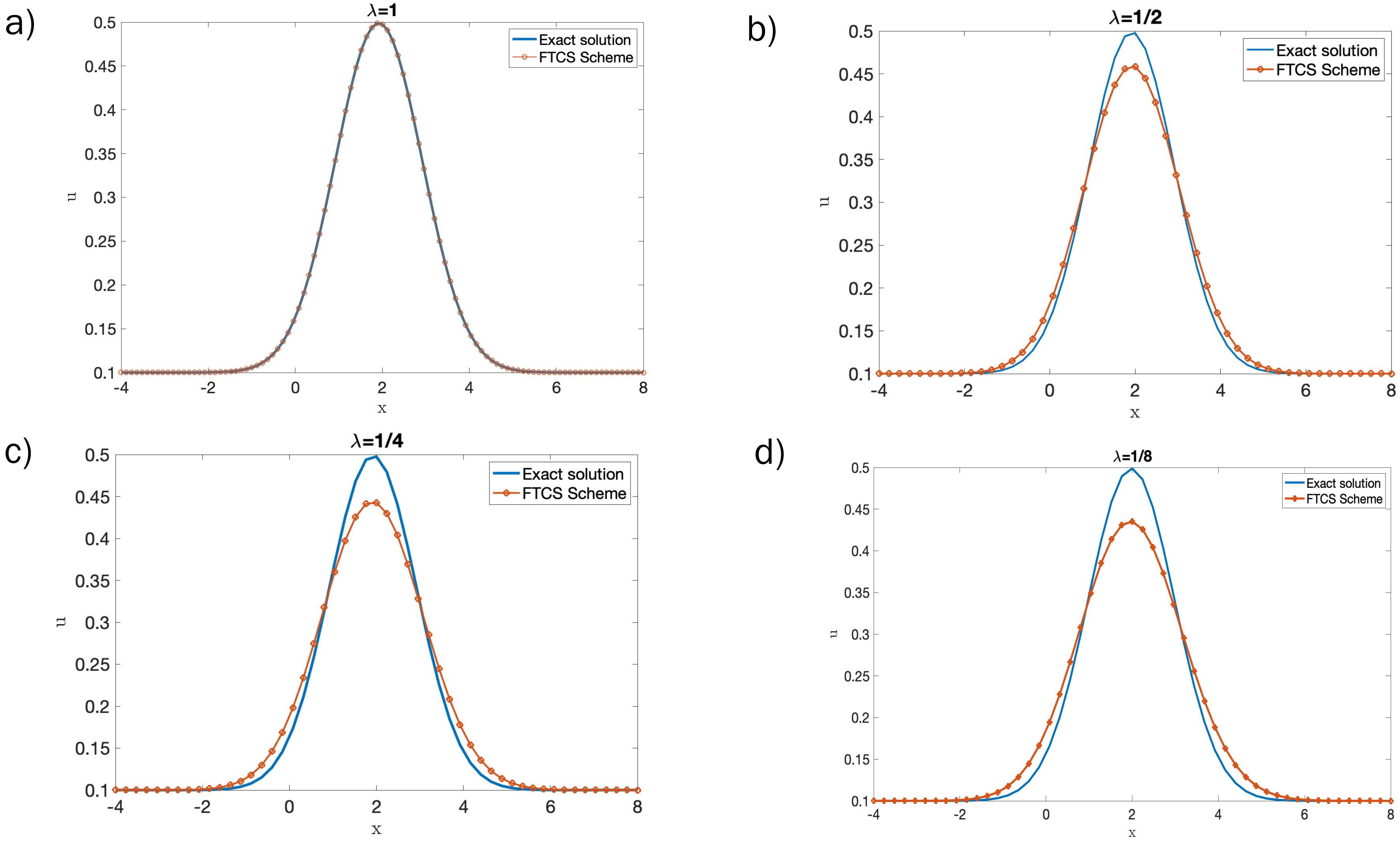
a) $\boldsymbol{\lambda = 2}$



b) $\boldsymbol{\lambda = 4}$



FTCS Scheme with small diffusion term



Comparison of FTCS Scheme with Exact solution at t=2 sec