Consider a thin rod of uniform cross-section and homogeneous material placed along the x-axis with ends at x = 0 and x = L. with L = 1 m, and $\alpha = 0.1$.

Heat conduction in the rod is described by the heat equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad x \in [0, L], t \geq 0.$$

Assume one end is kept at constant temperature and one end is insulated. The boundary conditions are

$$u(0, t) = 0,$$

$$\frac{\partial u}{\partial x}(L, t) = 0, t \geq 0$$

The initial condition is:

$$u(x, 0) = 5\sin(3\pi x/2),$$

The closed-form solution is,

$$\boxed{u(x,t) = 5 \sin\left(\frac{3\pi}{2} x\right) e^{-\left(\frac{3}{2}\alpha\right)^2 t}}$$

Implementing the boundary conditions in finite difference method:

Given, $u(0, t) = 0$

we will make,

$u_1^n = 0$ ($j=1$, n - each time).

also, $\frac{\partial u}{\partial x}(L, t) = 0$, $t \geq 0$

$\Rightarrow u(L, t) = c$ (c = arbitrary constant) $\longrightarrow$ 5

Now, Given initial condition,

$u(x,0) = 5\sin(3\pi x/2)$

$\therefore$ using 5,

$u(L,0) = 5 \sin(3\pi . L/2) = c$

$L = 1$

$\therefore$ we implement the 2nd boundary condition

as, $u_j^n = 5\sin\left(\frac{3\pi}{2}\right)$ (where j = no. of steps of space)

Implementing initial conditions:

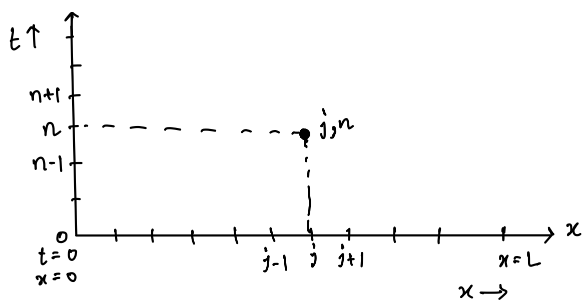
Given,

$$u(x, 0) = 5\sin(3\pi x/2)$$

This is implemented as,

$$u_j^1 = 5\sin(3\pi x_j/2) \quad (n=1 \text{ for each } j)$$

FTCS (Forward in Time & Central difference in Space method:



We assume,

$$u(x_j, t_n) = u_j^n$$

$$\therefore \frac{\partial u(x_j, t_n)}{\partial t} = \frac{u_j^{n+1} - u_j^n}{\Delta t} \quad \text{(forward in time)}$$

(Δt is step size of time)

$$\therefore \frac{\partial^2 u(x_j, t_n)}{\partial x^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

(Δx is the step size of space)

$\therefore$ the given heat equation becomes,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$\Rightarrow \boxed{u_j^{n+1} = u_j^n + \frac{\alpha^2 \Delta t}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)}$$

Crank Nicolson Method :

let, $u(x_j, t_n) = u_j^n$

In this method, we take the average between time steps $(n-1)$ and n. (Boundary conditions are addressed in the same way explained in FTCS method)

$$\frac{u_j^n - u_j^{n-1}}{\Delta t} = \frac{1}{2}\cdot\alpha^2\left[\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right) + \left(\frac{u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}}{\Delta x^2}\right)\right]$$

$$\Rightarrow u_j^n = u_j^{n-1} + r\left[u_{j+1}^n - 2u_j^n + u_{j-1}^n + u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}\right]$$

$$\Rightarrow u_j^n - r u_{j+1}^n + 2r u_j^n - r u_{j-1}^n = u_j^{n-1} + r u_{j+1}^{n-1} - 2r u_j^{n-1} + r u_{j-1}^{n-1}$$

$$\Rightarrow -r u_{j+1}^n + (1+2r)u_j^n - r u_{j-1}^n = r u_{j+1}^{n-1} + (1-2r)u_j^{n-1} + r u_{j-1}^{n-1}$$

$$\Rightarrow \underbrace{\begin{bmatrix} -r & (1+2r) & r & 0 & \cdots \\ 0 & -r & (1+2r) & 0 & \cdots \\ \cdots & --- & & - & - \\ \cdot & --- & - & - & -- \end{bmatrix}}_{A} \begin{bmatrix} u_{j+1}^n \\ u_j^n \\ u_{j-1}^n \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} r u_{j+1}^{n-1} + (1-2r)u_j^{n-1} + r u_{j-1}^{n-1} \\ \vdots \\ \vdots \end{bmatrix}}_{B}$$

$$\therefore u(x_j, t_n) = A^{-1}B.$$