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Data-driven Identification and Output Regulation using Partially Observed Actuated Trajectories : A Koopman Bilinear Approach

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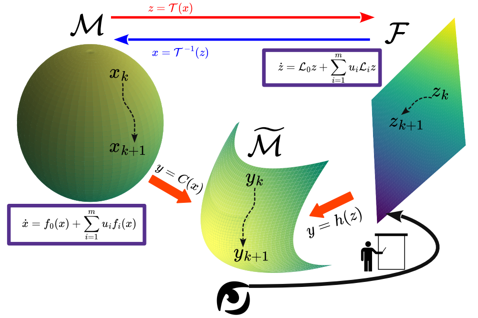
Motivations

- Control-affine systems exhibit a bilinear structure in their Koopman generators, which can be leveraged to develop finite-dimensional approximations of the dynamics.
- A key challenge in constructing Koopman generators for actuated systems lies in selecting appropriate basis functions or observables, as no unified framework exists to guide this choice.
- In many real-world applications, the full system state remains unobserved, requiring observables that capture how outputs depend on the supplied input sequences.
- Recent advancements in Koopman generator-based bilinear surrogate modeling with linear reconstruction have shown promise for actuated systems without requiring explicit dictionaries, even in the presence of noise and partial trajectory observations. However, further investigation is necessary to evaluate the method's efficacy for complex nonlinear systems with actuation and its sensitivity to initialization.
- The integration of a nonlinear decoder with a Koopman-bilinear model can yield superior surrogate model order reduction and enhanced performance.

Koopman Generator-based Bilinear Realization



$x \in \mathcal{M}$: State of the dynamical system
$f_0(\cdot)$: Drift vector field
$f_i(\cdot)$: Control vector field
$u \in \mathbb{R}^m$: Control inputs
$z \in \mathcal{F}$: Collection of Koopman observables (lifted-space)
$\mathcal{T} : \mathcal{M} \to \mathcal{F}$: Mapping from the state-space to the lifted-space
$\mathcal{T}^{-1} : \mathcal{F} \to \mathcal{M}$: Mapping from the lifted-space to the state-space
$\mathcal{L}_0, \mathcal{L}_i$: Koopman generators
$y \in \widetilde{\mathcal{M}}$: Observations
$C : \mathcal{M} \to \widetilde{\mathcal{M}}$: Observation map
$h : \mathcal{F} \to \widetilde{\mathcal{M}}$: Observation map for the lifted-space

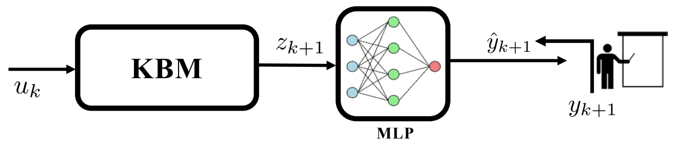
Koopman Generator-based Bilinear Model with Nonlinear Reconstruction (KBM-NL)

$$\dot z = \mathcal{L}_0 z + \sum_{i=1}^m u_i \mathcal{L}_i z \quad \approx \quad \begin{bmatrix} 1 \\ z_{k+1} \end{bmatrix} = \mathcal{U}_k^T \begin{bmatrix} 1 \\ z_k \end{bmatrix}$$

KBM

$$\mathcal{U}_k := \begin{bmatrix} 1 & b_k^T \\ 0 & A_k^T \end{bmatrix} = \left(I + \Delta t \textstyle\sum_{i=0}^m u_{i_k} V_i\right)$$

Matrix Approximation of Koopman Generator

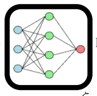


KBM-NL as a Hidden Markov Model (HMM)

Δt : fixed sampling time,
$\hat w_k \sim \mathcal{N}(0, \Sigma_w)$: process noise,
$\hat v_k \sim \mathcal{N}(0, \Sigma_v)$: measurement noise + modeling error,
$z_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$: initial condition.

$$\begin{bmatrix} 1 \\ z_{k+1} \end{bmatrix} = \mathcal{U}_k^T \begin{bmatrix} 1 \\ z_k \end{bmatrix} + \begin{bmatrix} 0 \\ \hat w_k \end{bmatrix}$$

$h(\cdot)$



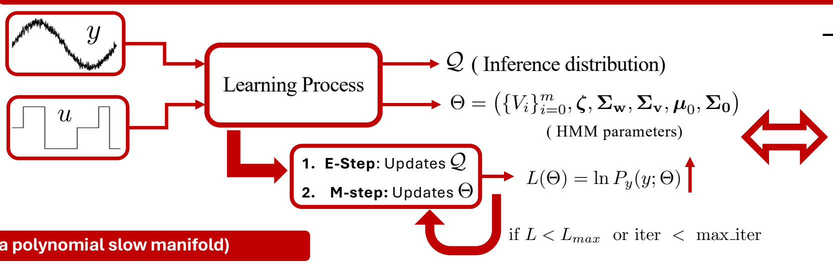
ReLU

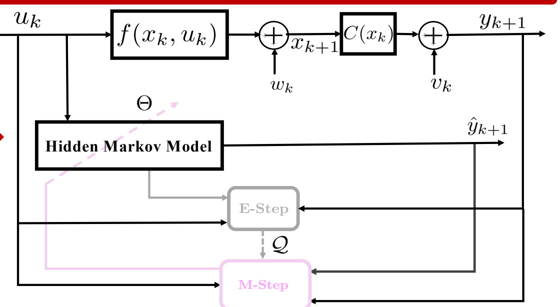
Parameters ζ

$$z_{k+1} = A_k z_k + b_k + \hat w_k, \qquad y_k = h(z_k) + \hat v_k$$

Hidden Markov Model

The Learning Process





Model Predictive Control (MPC) for Output Regulation using HMM

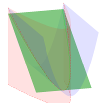
The Optimal Control Problem at any time instant k

$$\min_u \sum_{j=0}^{N_H - 1} (y_j - y_j^{\text{ref}})^T Q (y_j - y_j^{\text{ref}}) + u_j^T R u_j$$

$$\text{s.t.} \quad \begin{bmatrix} 1 \\ z_{j+1} \end{bmatrix} = \left(I + \Delta t \textstyle\sum_{i=0}^m u_{i_j} V_i\right)^T \begin{bmatrix} 1 \\ z_j \end{bmatrix}, \quad y_j = h(z_j), \quad z_0 = \mu_k, \quad \underline{u} \le u \le \bar u$$

N_H : # of divisions of the MPC horizon
$Q,\ R$: cost function weightings
μ_k : estimated state based on previous history of inputs and outputs.

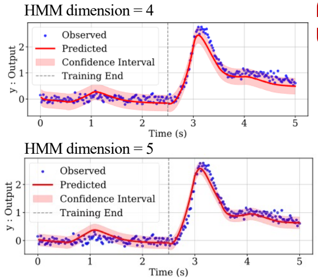
Results : Example 1 (Actuated system with a polynomial slow manifold)

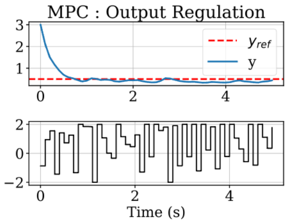


$$\dot x_1 = -\alpha x_1 + u, \quad \dot x_2 = \beta(x_1^3 - x_2), \quad y = x_2, \quad \alpha = 1, \beta = 5$$

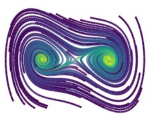
$$\frac{d}{dt}\begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 & \beta \\ 0 & 0 & 0 & -2\alpha & 0 \\ 0 & 0 & 0 & 0 & -3\alpha \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1^3 \end{bmatrix} + u \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1^3 \end{bmatrix}$$

Sampling time 0.02 sec, 25 independent trajectories with i.i.d. piecewise constant control [-5,5], zero process noise, and i.i.d. measurement noise with variance 0.01. The initial conditions are i.i.d. [-2,2]. MLP : 1 hidden layer with dimension = HMM dimension.



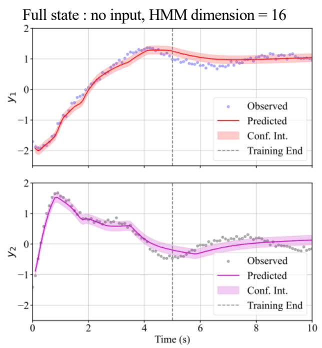


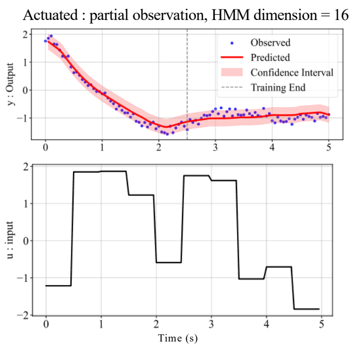
Results : Example 2 (Duffing Equation)



$$\dot x_1 = x_2, \quad \dot x_2 = -\delta x_2 - x_1(\beta + \alpha x_1^2).$$
$y = [x_1, x_2]$ if full-state observation,
$y = x_1$ if partial observation,
with $\delta = 0.5$, $\beta = -1$, $\alpha = 1$.

Sampling time 0.1 sec, 100 independent trajectories with i.i.d. piecewise constant control [-2,2] (if actuated), zero process noise, and i.i.d. measurement noise with variance 0.01. The initial conditions are i.i.d. [-2,2]. MLP : 1 hidden layer with dimension = HMM dimension.

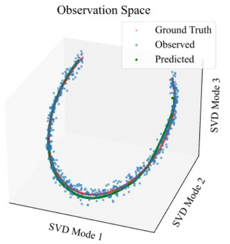


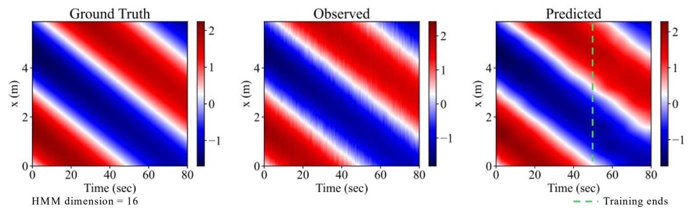


Results : Example 3 (Kuramoto–Sivashinsky (KS) equation)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0, \quad x \in [0, 2\pi], \quad t > 0.$$
$$u(x,0) = a\sin(x), \quad u(x + 2\pi, t) = u(x,t)$$

Sampling time 0.1 sec, 20 independent trajectories, 16 spatial grid points, zero process noise, and i.i.d. measurement noise with variance 0.01. The initial conditions are drawn as a=1 with i.i.d. noise [-0.1,0.1]. MLP : 1 hidden layer with dimension = HMM dimension.





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Future Scope

- Hyperparameter tuning with validation set and exploring model-order reduction.
- To explore efficient initialization methods for the KBM generator matrices.
- Robust MPC with stability guarantees using this framework.
- Implementation in more practical systems.