

Basics of Engineering Mathematics

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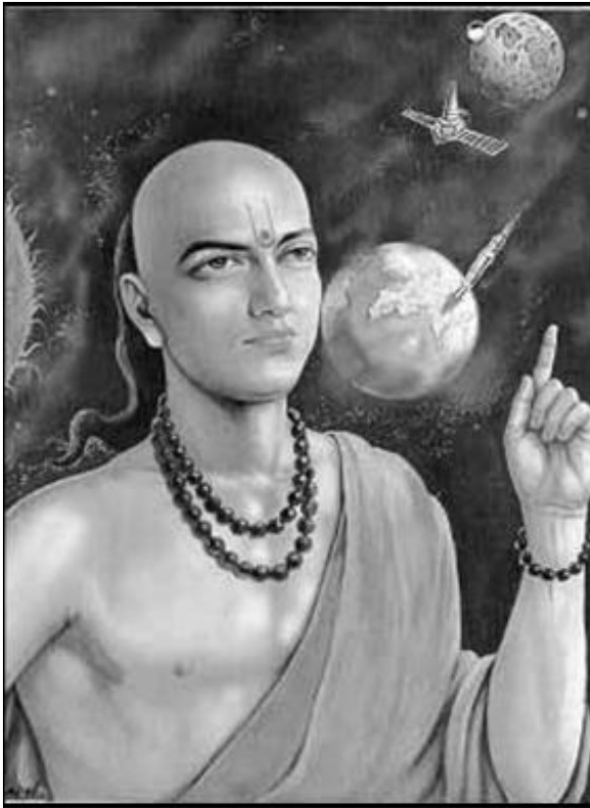


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*“Dedicated to the
mathematicians of
India, who gifted the
world an abundance of
knowledge and a spirit
of enquiry”.*



1 Basic Algebra

$$(a + b)^2 = a^2 + 2ab + b^2 \quad (1)$$

$$(a - b)^2 = a^2 - 2ab + b^2 \quad (2)$$

$$(a + b)(a - b) = a^2 - b^2 \quad (3)$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b) \quad (4)$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b) \quad (5)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad (6)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \quad (7)$$

$$a^4 - b^4 = (a + b)(a - b)(a^2 + b^2) \quad (8)$$

$$a^2 + b^2 = (a + b)^2 - 2ab \quad (9)$$

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac) \quad (10)$$

1.1 Law of Negatives

$$\begin{aligned} -(-a) &= a \\ (-a)(-b) &= ab \\ -ab &= (-a)(b) = a(-b) = -(-a)(-b) \\ a - b &= a + (-b) \end{aligned}$$

1.2 Law of Quotients

$$\begin{aligned} -\frac{a}{b} &= \frac{(-a)}{b} = \frac{a}{(-b)} = -\frac{(-a)}{(-b)} \\ \frac{a}{b} &= \frac{c}{d} \quad \text{if} \quad ad = bc \\ \frac{a}{b} &= \frac{ka}{kb} \end{aligned}$$

2 Polynomial Functions

Zeros of a polynomial: If $f(c) = 0$, then c is called a zero of the polynomial $f(x)$.

Division algorithm for a polynomial: If $f(x)$ and $g(x)$ are two polynomials such that $g(x) \neq 0$ then there exist two unique polynomials $q(x)$ and $r(x)$ such that:

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad (11)$$

Fundamental Theorem of Algebra: Every polynomial with positive degree has at least one complex zero.

Remainder Theorem: If a polynomial $f(x)$ is divided by $x - c$ then the remainder is $f(c)$.

Factor Theorem: Polynomial $f(x)$ has a factor $x - c$ if and only if $f(c) = 0$.

2.1 Corollary to the Fundamental Theorem

1. Every polynomial of positive degree n has a factorization of the form:

$$P(x) = a_n(x - r_1)(x - r_2)\dots\dots(x - r_n) \quad (12)$$

where r_i are not necessarily distinct. If $(x - r_i)$ occurs n times it is said to have a multiplicity of n .

2. It is not always possible to find the factors using exact methods.
3. A polynomial of degree n has at most n complex zeros.
4. Complex zeros of real polynomials with real coefficients occur in complex conjugate pairs.
5. Any polynomial of degree $n > 0$ with real coefficients has complete factorization using linear and quadratic factors, multiplied by the leading coefficient of the polynomial.
6. **Intermediate Value Theorem:** Given a polynomial $f(x)$ with $a < b$ and $f(a) \neq f(b)$ then $f(c)$ takes on every value c in the interval (a, b) .
7. For a polynomial $f(x)$ if $f(a)$ and $f(b)$ have opposite signs then $f(x)$ has at least one zero between a and b .
8. For a polynomial $f(x)$ if $f(a)$ and $f(b)$ have opposite signs then $f(x)$ has at least one zero between a and b .

9. **Descartes rule of signs:** If $f(x)$ is a polynomial with terms arranged in descending order, then the number of positive real roots of $f(x)$ is either equal to the number of sign changes between the successive terms of $f(x)$ or less than this number by an even number. Number of negative real zeros of $f(x)$ is obtained by applying this rule to $f(-x)$.

2.2 Synthetic Division

c is said to be a zero of the polynomial $f(x)$ if $f(c) = 0 \implies x - c$ is a factor of the polynomial $f(x)$. The graph of $f(x)$ has an intercept at c .

1. Arrange the coefficients in descending order in the first row.
2. Third row is formed by bringing down the first coefficient of $f(x)$ then successively multiplying each coefficient in the third row by c , placing the results in second row adding this to the corresponding coefficients in the first row, and placing result in the next position of the third row.

3 Quadratic Equations

The quadratic equation

$$ax^2 + bx + c = 0 \quad (13)$$

where a, b and c are constants and $a \neq 0$, has two solutions for the variable x :

$$x_{\alpha, \beta} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (14)$$

If the *discriminant* Δ with

$$\Delta = b^2 - 4ac = 0 \quad (15)$$

then the equation (13) has real and equal roots given by:

$$x = \frac{-b}{2a} \quad (16)$$

Sum of the roots:

$$\alpha + \beta = \frac{-b}{a} \quad (17)$$

Product of the roots:

$$\alpha \cdot \beta = \frac{c}{a} \quad (18)$$

3.1 Factorization method

Find the two factors $x - \alpha$ and $x - \beta$ which when multiplied produce the quadratic equation $ax^2 + bx + c$.

3.2 Completion of squares method

Consider any quadratic equation of the form $ax^2 + bx + c = 0$. Dividing throughout by a .

$$\implies x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$\text{Adding and subtracting } \left(\frac{b}{2a}\right)^2, \quad ,$$

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

$$\therefore ax^2 + bx + c = 0 \implies a(x + d)^2 + e = 0 \quad \text{where,} \quad d = \frac{b}{2a} \quad \text{and} \quad e = \frac{c}{a} - \left(\frac{b}{2a}\right)^2.$$

4 Logarithmic Functions

For any two variables x and y related as:

$$y = a^x \tag{19}$$

Where a is a constant. We define the logarithmic function as:

$$\log_a y = x \tag{20}$$

4.1 Properties of Logarithmic Functions

$$\log_a mn = \log_a m + \log_a n \tag{21}$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n \tag{22}$$

$$\log_a m^n = n \cdot \log_a m \tag{23}$$

$$\log_a a = 1 \tag{24}$$

$$\log_a 1 = 0 \tag{25}$$

$$\log_a b = \frac{1}{\log_b a} \tag{26}$$

$$a^x = e^{x \cdot \log a} \tag{27}$$

$$\log_a X = \frac{\ln X}{\ln a} \tag{28}$$

5 Laws of Exponents

$$x^a . x^b = x^{a+b} \tag{29}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{30}$$

$$(x^a)^b = x^{ab} \tag{31}$$

$$x^a = y^b \implies a = b \tag{32}$$

$$x^{-a} = \frac{1}{x^a} \quad (\text{for } a \text{ being any non zero real number}) \tag{33}$$

$$x^0 = 1 \tag{34}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \tag{35}$$

$$\sqrt[n]{xy} = \sqrt[n]{x} . \sqrt[n]{y} \tag{36}$$

$$\left(\frac{x}{y}\right)^{-m} = \left(\frac{y}{x}\right)^m \tag{37}$$

$$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n} \tag{38}$$

5.1 Rational Number Exponents

The principle n th root of $x > 0$ is defined as:

$$x^{1/n} = \begin{cases} \text{is a unique real number } y \text{ if } n \text{ is odd or even} \\ 0, \text{ if } x = 0 \\ \text{not a real number if } x < 0 \end{cases}$$

$$(x^{1/n})^n = \begin{cases} x, \text{ if } n \text{ is odd/even and } x \text{ is positive} \\ |x|, \text{ if } n \text{ is even and } x \text{ is negative} \end{cases}$$

5.2 Simplest Radical Form

No radicand can contain a factor with an exponent greater than or equal to the index of the radical e.g. $\sqrt[3]{16x^3y^5} \implies$ is not in it's standard form. No power of a radicand and index of the radical can have common factor other than 1. e.g. $2xy\sqrt[3]{2y^2} \implies$ not in standard form. No radical appears in denominator. No fractional part appears in radical. e.g. $\sqrt[6]{t^3} = \sqrt{t}$ is in it's standard form.

6 Complex Numbers

A number of the form $z = x + iy$ consisting of a real part x , denoted by $Re(z)$ and imaginary part y , denoted by $Im(z)$ is known as a complex number.

$$x = r \cos \theta \tag{39}$$

$$y = r \sin \theta \tag{40}$$

$$r = \sqrt{x^2 + y^2} \tag{41}$$

$$\tan \theta = \frac{y}{x} \implies \theta = \tan^{-1} \frac{y}{x} \tag{42}$$

$$e^{i\theta} = \cos \theta + i \sin \theta \tag{43}$$

$$e^{-i\theta} = \cos \theta - i \sin \theta \tag{44}$$

6.1 De Moivre's Theorem

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta) \tag{45}$$

7 Permutations and Combinations

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)(n-3)\dots(n-(r+1))}{r!} \quad (46)$$

Note that:

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad 0! = 1$$

7.1 Binomial Theorem

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \binom{n}{3}x^{n-3}a^3 + \binom{n}{4}x^{n-4}a^4 + \dots\dots\dots a^n \quad (47)$$

7.2 Summation of Series

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3\dots\dots = \frac{(n)(n+1)}{2} \quad (48)$$

$$\sum_{n=1}^{\infty} n^2 = 1^2 + 2^2 + 3^2\dots\dots = \frac{(n)(n+1)(2n+1)}{6} \quad (49)$$

$$\sum_{n=1}^{\infty} n^3 = 1^3 + 2^3 + 3^3\dots\dots = \frac{(n)^2(n+1)^2}{4} \quad (50)$$

8 Trigonometry

$$\begin{aligned}\sin \theta &= \frac{\text{Perpendicular}}{\text{Hypotenuse}} & \cos \theta &= \frac{\text{Base}}{\text{Hypotenuse}} & \tan \theta &= \frac{\text{Perpendicular}}{\text{Base}} \\ \operatorname{cosec} \theta &= \frac{\text{Hypotenuse}}{\text{Perpendicular}} & \sec \theta &= \frac{\text{Hypotenuse}}{\text{Base}} & \cot \theta &= \frac{\text{Base}}{\text{Perpendicular}}\end{aligned}$$

From the above relations it turns out that,

$$\sin \theta \cdot \operatorname{cosec} \theta = 1$$

$$\cos \theta \cdot \sec \theta = 1$$

$$\tan \theta \cdot \cot \theta = 1$$

8.1 Trigonometric Identities

$$\sin^2 \theta + \cos^2 \theta = 1 \tag{51}$$

$$1 + \sec^2 \theta = \tan^2 \theta \tag{52}$$

$$1 + \operatorname{cosec}^2 \theta = \cot^2 \theta \tag{53}$$

θ	0°	30°	45°	60°	90°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot \theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec \theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\operatorname{cosec} \theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Figure 1: Most common trigonometric ratios and values

8.2 Trigonometric ratios of allied angles

- *For n being an odd multiple of $\pi/2$ the function changes to its composite function.*
- *For n being an even multiple of $\pi/2$ the function does not change to its composite, but may change sign.*

8.2.1 First Quadrant

$$\begin{array}{ll}\sin(2\pi + \theta) = \sin \theta & \cos(2\pi + \theta) = \cos \theta \\ \operatorname{cosec}(2\pi + \theta) = \operatorname{cosec} \theta & \sec(2\pi + \theta) = \sec \theta \\ \tan(2\pi + \theta) = \tan \theta & \cot(2\pi + \theta) = \cot \theta\end{array}$$

$$\begin{array}{ll}\sin(\pi/2 - \theta) = \cos \theta & \cos(\pi/2 - \theta) = \sin \theta \\ \operatorname{cosec}(\pi/2 - \theta) = \sec \theta & \sec(\pi/2 - \theta) = \operatorname{cosec} \theta \\ \tan(\pi/2 - \theta) = \cot \theta & \cot(\pi/2 - \theta) = \tan \theta\end{array}$$

8.2.2 Second Quadrant

$$\begin{array}{ll}\sin(\pi/2 + \theta) = \cos \theta & \cos(\pi/2 + \theta) = -\sin \theta \\ \operatorname{cosec}(\pi/2 + \theta) = \sec \theta & \sec(\pi/2 + \theta) = -\operatorname{cosec} \theta \\ \tan(\pi/2 + \theta) = -\cot \theta & \cot(\pi/2 + \theta) = -\tan \theta\end{array}$$

$$\begin{array}{ll}\sin(\pi - \theta) = \sin \theta & \cos(\pi - \theta) = -\cos \theta \\ \operatorname{cosec}(\pi - \theta) = \operatorname{cosec} \theta & \sec(\pi - \theta) = -\sec \theta \\ \tan(\pi - \theta) = -\tan \theta & \cot(\pi - \theta) = -\cot \theta\end{array}$$

8.2.3 Third Quadrant

$$\begin{array}{ll}\sin(\pi + \theta) = -\sin \theta & \cos(\pi + \theta) = -\cos \theta \\ \operatorname{cosec}(\pi + \theta) = \operatorname{cosec} \theta & \sec(\pi + \theta) = -\sec \theta \\ \tan(\pi + \theta) = \tan \theta & \cot(\pi + \theta) = \cot \theta\end{array}$$

$$\begin{array}{ll}\sin(3\pi/2 - \theta) = -\cos \theta & \cos(3\pi/2 - \theta) = -\sin \theta \\ \operatorname{cosec}(3\pi/2 - \theta) = -\sec \theta & \sec(3\pi/2 - \theta) = -\operatorname{cosec} \theta \\ \tan(3\pi/2 - \theta) = \cot \theta & \cot(3\pi/2 - \theta) = \tan \theta\end{array}$$

8.2.4 Fourth Quadrant

$$\begin{array}{ll}\sin(3\pi/2 + \theta) = -\cos \theta & \cos(3\pi/2 + \theta) = \sin \theta \\ \operatorname{cosec}(3\pi/2 + \theta) = -\sec \theta & \sec(3\pi/2 + \theta) = \operatorname{cosec} \theta \\ \tan(3\pi/2 + \theta) = -\cot \theta & \cot(3\pi/2 + \theta) = -\tan \theta\end{array}$$

$$\begin{array}{ll}\sin(2\pi - \theta) = -\sin \theta & \cos(2\pi - \theta) = \cos \theta \\ \operatorname{cosec}(2\pi - \theta) = -\operatorname{cosec} \theta & \sec(2\pi - \theta) = \sec \theta \\ \tan(2\pi - \theta) = -\tan \theta & \cot(2\pi - \theta) = -\cot \theta\end{array}$$

- *The values for $(2\pi - \theta)$ and $(-\theta)$ are identical.*

8.3 Compound Angle Formulae

$$\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$$

$$\sin(A - B) = \sin(A) \cos(B) - \cos(A) \sin(B)$$

$$\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$$

$$\cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B)$$

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A) \tan(B)}$$

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A) \tan(B)}$$

8.4 Sum or Difference Formulae

$$\sin(A + B) + \sin(A - B) = 2 \sin(A) \cos(B)$$

$$\sin(A - B) - \sin(A - B) = 2 \cos(A) \sin(B)$$

$$\cos(A + B) + \cos(A - B) = 2 \cos(A) \cos(B)$$

$$\cos(A + B) - \cos(A - B) = -2 \sin(A) \sin(B)$$

8.5 Half Angle Formulae

$$\sin\left(\frac{A}{2}\right) = \pm \sqrt{\frac{1 - \cos(A)}{2}}$$

$$\cos\left(\frac{A}{2}\right) = \pm \sqrt{\frac{1 + \cos(A)}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \pm \sqrt{\frac{1 - \cos(A)}{1 + \cos(A)}}$$

8.6 Sub Multiple Angle Formulae

$$\sin(2A) = 2 \sin(A) \cos(A)$$

$$\cos(2A) = \cos^2(A) - \sin^2(A) = 2 \cos^2(A) - 1 = 1 - 2 \sin^2(A)$$

$$\tan(2A) = \frac{2 \tan(A)}{1 - \tan^2(A)}$$

$$\sin(2A) = \frac{2 \tan(A)}{1 + \tan^2(A)}$$

$$\cos(2A) = \frac{1 - \tan^2(A)}{1 + \tan^2(A)}$$

$$\cos^2(A) = \frac{1 + \cos(2A)}{2}$$

$$\sin^2(A) = \frac{1 - \cos(2A)}{2}$$

$$\sin(A) = 2 \sin(A/2) \cos(A/2)$$

$$\cos(A) = \cos^2(A/2) - \sin^2(A/2) = 2 \cos^2(A/2) - 1 = 1 - 2 \sin^2(A/2)$$

8.7 Triple Angle Formulae

$$\sin(3x) = 3\sin(x) - 4\sin^3(x)$$

$$\cos(3x) = 4\cos^3(x) - 3\cos(x)$$

$$\tan(3x) = \frac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)}$$

$$\sin(4x) = 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x)$$

$$\tan(4x) = \frac{4\tan(x) - 4\tan^3(x)}{1 - 6\tan^2(x) + \tan^4(x)}$$

$$\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$$

8.8 Product Formulae

If $A + B = C$ and $A - B = D$

Then $A = (C + D)/2$ and $B = (C - D)/2$, the product formulae are defined as

$$\sin(C) + \sin(D) = 2\sin\frac{(C + D)}{2}\cos\frac{(C - D)}{2}$$

$$\sin(C) - \sin(D) = 2\cos\frac{(C + D)}{2}\sin\frac{(C - D)}{2}$$

$$\cos(C) + \cos(D) = 2\cos\frac{(C + D)}{2}\cos\frac{(C - D)}{2}$$

$$\cos(C) - \cos(D) = -2\sin\frac{(C + D)}{2}\sin\frac{(C - D)}{2}$$

9 Inverse Trigonometric Functions

If $x = \sin \theta$ then we define the inverse trigonometric function as $\sin^{-1}(x) = \theta$, the same holds true for all other trigonometric functions.

9.1 Properties of Inverse Trigonometric Functions

$$\begin{array}{ll} \sin^{-1}(\sin \theta) = \theta & \operatorname{cosec}^{-1}(\operatorname{cosec} \theta) = \theta \\ \cos^{-1}(\cos \theta) = \theta & \sec^{-1}(\sec \theta) = \theta \\ \tan^{-1}(\tan \theta) = \theta & \cot^{-1}(\cot \theta) = \theta \end{array}$$

$$\begin{array}{ll} \sin(\sin^{-1}(x)) = x & \operatorname{cosec}(\operatorname{cosec}^{-1}(x)) = x \\ \cos(\cos^{-1}(x)) = x & \sec(\sec^{-1}(x)) = x \\ \tan(\tan^{-1}(x)) = x & \cot(\cot^{-1}(x)) = x \end{array}$$

Functions	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	\mathbb{R}	$[-\pi/2, \pi/2]$
$\cot^{-1}x$	\mathbb{R}	$[0, \pi]$
$\sec^{-1}x$	$\mathbb{R} - (-1, 1)$	$[0, \pi] - [\pi/2]$
$\operatorname{cosec}^{-1}x$	$\mathbb{R} - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$

Figure 2: Domain and range of inverse trigonometric functions

$$\begin{array}{llll} \sin^{-1}(x) & = & \cos^{-1} \sqrt{1-x^2} & = & \tan^{-1} \frac{x}{\sqrt{1-x^2}} \\ \cos^{-1}(x) & = & \sin^{-1} \sqrt{1-x^2} & = & \tan^{-1} \frac{\sqrt{1-x^2}}{x} \\ \tan^{-1}(x) & = & \sin^{-1} \frac{x}{\sqrt{1+x^2}} & = & \cos^{-1} \frac{1}{\sqrt{1+x^2}} \end{array}$$

$$\begin{aligned}
\sin^{-1}(-x) &= -\sin^{-1}(x) & x \in [-1, 1] \\
\cos^{-1}(-x) &= \pi - \cos^{-1}(x) & x \in [-1, 1] \\
\tan^{-1}(-x) &= -\tan^{-1}(x) & x \in \mathbb{R} \\
\operatorname{cosec}^{-1}(-x) &= -\operatorname{cosec}^{-1}(x) & x \in (-\infty, -1] \cup [1, \infty) \\
\sec^{-1}(-x) &= \pi - \sec^{-1}(x) & x \in (-\infty, -1] \cup [1, \infty) \\
\cot^{-1}(-x) &= \pi - \cot^{-1}(x) & x \in \mathbb{R}
\end{aligned}$$

$$\begin{aligned}
\sin^{-1}\left(\frac{1}{x}\right) &= \operatorname{cosec}^{-1}(x) & x \in (-\infty, -1] \cup [1, \infty) \\
\cos^{-1}\left(\frac{1}{x}\right) &= \sec^{-1}(x) & x \in (-\infty, -1] \cup [1, \infty)
\end{aligned}$$

$$\tan^{-1}(1/x) = \begin{cases} \cot^{-1}(x), & x > 0 \\ \cot^{-1}(x) - \pi, & x < 0 \end{cases}$$

$$\begin{aligned}
\sin^{-1}(x) + \cos^{-1}(x) &= (\pi/2) \\
\tan^{-1}(x) + \cot^{-1}(x) &= (\pi/2) \\
\sec^{-1}(x) + \operatorname{cosec}^{-1}(x) &= (\pi/2)
\end{aligned}$$

$$\begin{aligned}
\tan^{-1}(x) + \tan^{-1}(y) &= \tan^{-1} \frac{x+y}{1-xy} \\
\tan^{-1}(x) - \tan^{-1}(y) &= \tan^{-1} \frac{x-y}{1+xy}
\end{aligned}$$

$$2 \tan^{-1}(x) = \begin{cases} 2 \sin^{-1} \frac{2x}{1+x^2}, & x > 0 \\ \cos^{-1} \frac{1-x^2}{1+x^2}, & x < 0 \end{cases}$$

9.2 Expressions and suggested substitutions

Expression	Substitution
$a^2 + x^2$	$x = a \tan \theta \quad \text{or} \quad x = a \cot \theta$
$a^2 - x^2$	$x = a \sin \theta \quad \text{or} \quad x = a \cos \theta$
$x^2 - a^2$	$x = a \operatorname{cosec} \theta \quad \text{or} \quad x = a \sec \theta$
$\sqrt{\frac{a-x}{a+x}} \quad \text{or} \quad \sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}} \quad \text{or} \quad \sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x = a^2 \cos 2\theta$

10 Hyperbolic Functions

$$\begin{aligned}\sinh(x) &= \frac{e^x - e^{-x}}{2} & \operatorname{cosech}(x) &= \frac{2}{e^x - e^{-x}} \\ \cosh(x) &= \frac{e^x + e^{-x}}{2} & \operatorname{sech}(x) &= \frac{2}{e^x + e^{-x}} \\ \tanh(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} & \operatorname{coth}(x) &= \frac{e^x + e^{-x}}{e^x - e^{-x}}\end{aligned}$$

10.1 Properties of Hyperbolic Functions

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= 1 \\ \cosh^2(x) + \sinh^2(x) &= \cosh(2x) \\ 1 - \tanh^2(x) &= \operatorname{sech}^2(x) \\ \operatorname{coth}^2(x) - 1 &= \operatorname{cosech}^2(x) \\ \sinh(2x) &= 2 \sinh(x) \cosh(x) \\ \sin(ix) &= \frac{e^{ix} - e^{-ix}}{2} \\ \cos(ix) &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin(ix) &= i \sinh(x) \\ \cos(ix) &= \cosh(x) \\ \tan(ix) &= i \tanh(x)\end{aligned}$$

11 Linear Inequalities

Inequalities of the form:

$$ax + b < 0$$

$$ax + b > 0$$

$$ax + b \leq 0$$

$$ax + b \geq 0$$

are known as linear inequalities.

1. Add and subtract any number k without change in the inequality on both sides.
2. When multiplying or dividing by constant k , reverse the sign of the inequality only when k is negative.
3. Linear inequalities have infinite solution sets, these can be obtained by isolating the variable and solving them in a manner similar to solving equations.

Inequality	Representation	Graph
$a < x < b$	(a, b)	
$a \leq x \leq b$	$[a, b]$	
$a < x \leq b$	$(a, b]$	
$a \leq x < b$	$[a, b)$	
$x > a$	(a, ∞)	
$x \geq a$	$[a, \infty)$	
$x < b$	$(-\infty, b)$	
$x \leq b$	$(-\infty, b]$	

12 Limits and Differentiation

12.1 Fundamental principles of limits

1. The limit may exist at a point even if the function is undefined.
2. If a function $f(x)$ is defined at a point a i.e. $f(a)$ exists, it is not necessary that the limit at a must exist. Moreover, even if the limit exists it need not be equal to $f(a)$.
3. **Indeterminate forms:** Any function assuming either of these forms is said to be indeterminate - $\frac{0}{0}, 0.\infty, \frac{\infty}{\infty}, \infty - \infty, 0^0, 1^\infty, \infty^0$.

12.2 Properties

$$\begin{aligned}\lim_{x \rightarrow a} [f(x) \pm g(x)] &= \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x). \\ \lim_{x \rightarrow a} [f(x).g(x)] &= \lim_{x \rightarrow a} f(x). \lim_{x \rightarrow a} g(x). \\ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{provided } \lim_{x \rightarrow a} g(x) \neq 0.\end{aligned}$$

12.3 Standard Limits

$$\lim_{x \rightarrow 0} \sin x = 0 \quad (54)$$

$$\lim_{x \rightarrow 0} \tan x = 0 \quad (55)$$

$$\lim_{x \rightarrow 0} \cos x = 1 \quad (56)$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (57)$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \quad (58)$$

$$\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1 \quad (59)$$

$$\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1 \quad (60)$$

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a \quad \text{where } a > 0 \quad (61)$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \quad (62)$$

$$\lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} = \log_a e \quad \text{where } a > 0 \text{ and } a \neq 0 \quad (63)$$

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \quad (64)$$

$$\lim_{x \rightarrow 0} \frac{\log x}{x^m} = 0 \quad \text{where } m > 0 \quad (65)$$

$$\lim_{x \rightarrow 0} \frac{(1+x)^m - 1}{x} = m \quad (66)$$

$$\lim_{x \rightarrow 0} \frac{x^n - a^n}{x - a} = n.a^{n-1} \quad (67)$$

12.4 Dealing with indeterminate forms

If $f(x)$ and $g(x)$ are two functions such that $f(x) \rightarrow 0$ as $x \rightarrow a$ and $g(x) \rightarrow \infty$ as $x \rightarrow a$ such that $[1 + f(x)]^{g(x)}$ assumes the form 1^∞ then:

$$\lim_{x \rightarrow a} [1 + f(x)]^{g(x)} = e^{\lim_{x \rightarrow a} f(x) \cdot g(x)}$$

12.5 L'Hospital's Rule

L'Hospital's rule states that for functions $f(x)$ and $g(x)$ which are differentiable on an open interval I except possibly at a point c contained in I , if:

$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$ or $\pm\infty$ and $g'(x) \neq 0$ for all x in I with $x \neq c$ and $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

12.6 Differentiation Formulae

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$$

$$\frac{d}{dx} f(x) \cdot g(x) = \frac{d}{dx} f(x) \cdot \frac{d}{dx} g(x)$$

$$\frac{dc}{dx} = 0 \quad \text{where } c \text{ is a real constant}$$

$$\frac{dx^n}{dx} = nx^{n-1} \quad \text{where } n \text{ is any real number}$$

$$\frac{de^x}{dx} = e^x$$

$$\frac{da^x}{dx} = a^x \log_e a$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \cdot \log a}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

12.7 Differentiation of Trigonometric Functions

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x$$

$$\frac{d}{dx} \sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$$

12.8 Differentiation of Inverse Trigonometric Functions

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \operatorname{cosec}^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$$

12.9 Differentiation of Hyperbolic Functions

$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$$

$$\frac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \cdot \coth x$$

$$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x$$

$$\frac{d}{dx} \coth x = -\operatorname{cosech}^2 x$$

12.10 Product Rule

$$\frac{d}{dx}.uv = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

12.11 Quotient Rule

$$\frac{d}{dx} \cdot \frac{u}{v} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

12.12 Derivative of Inverse of a function

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

12.13 Chain Rule of Differentiation (Derivative of composite function)

If $u = \phi(x)$ and $y = \phi(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

12.14 Differentiation of Parametric Functions

Parametric Function: Consider two variables x and y which can be expressed in terms of another variable t , this t is termed as the parameter. Hence both the variables can be expressed in terms of a third variable and the relation between them is known as a parametric function. Therefore if $x = f(t)$ and $y = g(t)$ then:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\text{But } \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

Therefore,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

12.15 Logarithmic Differentiation

If $y = u^v$ where both u and v are functions of variable x .

Taking log on both sides, and differentiating w.r.t x ,

$$\log y = v \log u$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = v \cdot \frac{d}{dx} \log u + \log u \cdot \frac{dv}{dx}$$

13 Indefinite Integrals

Evaluation of indefinite integrals can be accomplished using one of the possible methods:

- Decomposition of given integral into sum of integrals which can be reduced using standard formulas.
- Integration by substitution
- Integration by parts
- Integration by successive reduction

13.1 Integration by substitution

If $f(x)$ is a function such that x can be substituted as $x = \phi(t)$, then:

$$\int f(x).dx = \int f(\phi(t)).\phi'(t).dt \quad \text{where } x = \phi(t)$$

Proof:

$$v = \int f(x).dx = \int f(\phi(t)) \text{ substituting } x = \phi(t)$$

$$\frac{dv}{dx} = f(x)$$

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = f(x) \cdot \frac{dx}{dt}$$

Integrating both sides w.r.t we get t

$$v = \int f(x) \cdot \frac{dx}{dt} = \int f(\phi(t)).\phi'(t).dt \text{ for } x = \phi(t)$$

13.1.1 $\int \frac{f'(x)}{f(x)}.dx = \log f(x)$

Proof:

$$\text{Let } f(x) = t$$

$$f'(x) = \frac{dt}{dx}$$

$$f'(x).dx = dt$$

$$\therefore \int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t} = \log t = \log f(x)$$

Integral of a fraction whose numerator is the derivative of the denominator is equal to the logarithm of the denominator.

$$\mathbf{13.1.2} \quad \int [\mathbf{f(x)}]^{\mathbf{n}} \mathbf{f'(x).dx} = \frac{[\mathbf{f(x)}]^{\mathbf{n+1}}}{\mathbf{n+1}} , \text{ such that } \mathbf{n \neq 0}$$

Proof:

$$\text{Let } f(x) = t$$

$$f'(x)dx = dt$$

$$\therefore \int [f(x)]^n \cdot f'(x)dx = \int t^n \cdot dt = \frac{t^{n+1}}{n+1} = \frac{[f(x)]^{n+1}}{n+1}$$

$$\text{for } n \neq -1$$

$$\mathbf{13.1.3} \quad \int \mathbf{f'(ax + b)dx} = \frac{\mathbf{f(ax+b)}}{\mathbf{a}}$$

Proof:

$$\text{Let } ax + b = t$$

$$adx = dt \implies$$

$$\therefore \int [f(x)]^n \cdot f'(x)dx = \int t^n \cdot dt = \frac{t^{n+1}}{n+1} = \frac{[f(x)]^{n+1}}{n+1}$$

$$\text{for } n \neq -1$$

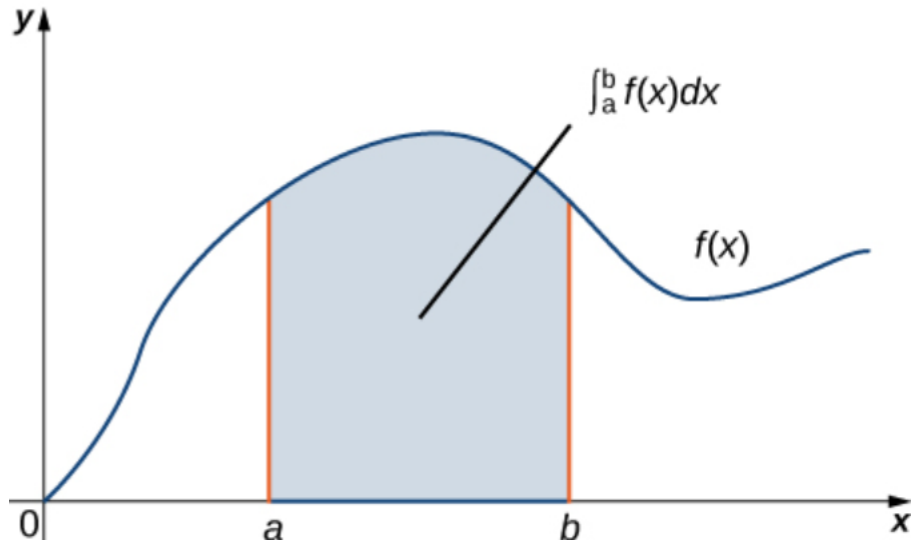
Integral of a function $f(ax+b)$ is of the same form as that of $f(x)$, divided by the coefficient of x .

14 Definite Integrals

Property I : First Fundamental Theorem of Calculus

Let f be a continuous function on an interval $[a, b]$ and $A(x)$ be the area function.

$$\text{Area Function } A(x) = \int_a^b f(x).dx$$



Property II : Second Fundamental Theorem of Calculus

$$\int_a^b f(x).dx = [F(x)]_a^b = F(b) - F(a)$$

$$\int_a^b f(x).dx = - \int_b^a f(x).dx \quad (68)$$

$$\int_a^b f(x).dx = \int_a^c f(x).dx + \int_c^b f(x).dx \quad \text{where } a < c < b \quad (69)$$

$$\int_0^a f(x).dx = \int_0^a f(a-x).dx \quad (70)$$

$$\int_{-a}^a f(x).dx = \begin{cases} 2. \int_0^a f(x).dx & \text{if } f(x) = f(-x) \\ 0 & \text{if } f(x) = -f(-x) \end{cases} \quad (71)$$

$$\int_0^{2a} f(x).dx = \begin{cases} 2. \int_0^a f(x).dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases} \quad (72)$$

Mean value of a function in an interval (a, b) is given by:

$$\frac{1}{b-a} \int_a^b f(x).dx$$

14.1 Evaluation of Infinite/Improper Integrals

Consider a definite integral $\int_a^b f(x).dx$ - the limits $[a, b]$ are finite and $f(x)$ exists at every point $c \in [a, b]$. The indefinite integral is defined as:

Type I

$$\int_a^\infty f(x).dx \text{ or } \int_{-\infty}^b f(x).dx$$

Evaluate $\lim_{t \rightarrow \infty} \int_a^t f(x).dx$ if it exists and is finite then:

$$\int_a^\infty f(x).dx = \lim_{t \rightarrow \infty} \int_a^t f(x).dx$$

Evaluate $\lim_{t \rightarrow -\infty} \int_t^b f(x).dx$ if it exists and is finite then:

$$\int_{-\infty}^b f(x).dx = \lim_{t \rightarrow -\infty} \int_t^b f(x).dx$$

Type II $f(x) \rightarrow \infty$ as $x \rightarrow a$ at no other point except at a .

Evaluate $\lim_{h \rightarrow 0} \int_{a+h}^b f(x).dx$ for $f(x) \rightarrow \infty$ as $x \rightarrow a$ if it exists and is finite then:

$$\int_a^b f(x).dx = \lim_{h \rightarrow 0} \int_{a+h}^b f(x).dx$$

Evaluate $\lim_{h \rightarrow 0} \int_a^{b-h} f(x).dx$ for $f(x) \rightarrow \infty$ as $x \rightarrow b$ if it exists and is finite then:

$$\int_a^b f(x).dx = \lim_{h \rightarrow 0} \int_a^{b-h} f(x).dx$$

If $f(x) \rightarrow \infty$ at some point $c \in [a, b]$ then:

$$\int_a^b f(x).dx = \int_a^c f(x).dx + \int_c^b f(x).dx$$

$$\int_{-\infty}^{\infty} f(x).dx = \int_{-\infty}^a f(x).dx + \int_a^{\infty} f(x).dx$$
