

# Basics of Engineering Mathematics

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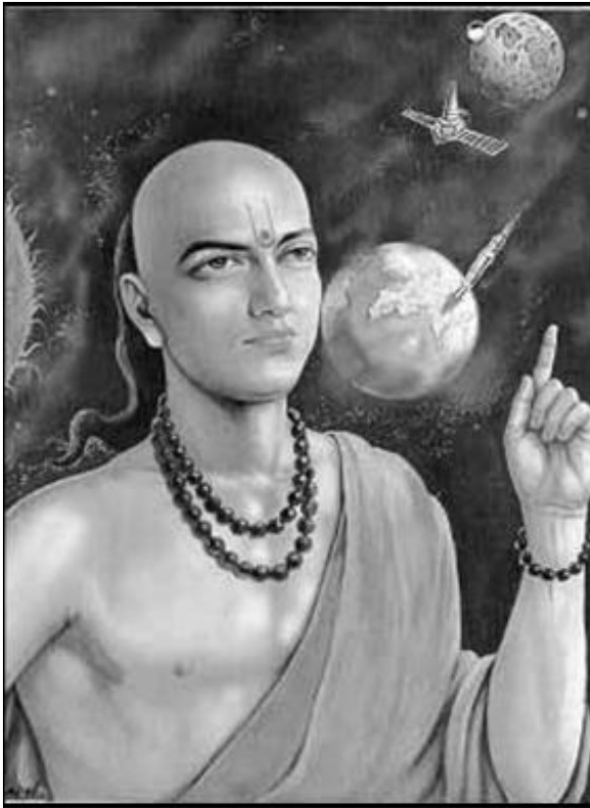


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*“Dedicated to the  
mathematicians of  
India, who gifted the  
world an abundance of  
knowledge and a spirit  
of enquiry”.*



# 1 Basic Algebra

$$(a + b)^2 = a^2 + 2ab + b^2 \quad (1)$$

$$(a - b)^2 = a^2 - 2ab + b^2 \quad (2)$$

$$(a + b)(a - b) = a^2 - b^2 \quad (3)$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b) \quad (4)$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b) \quad (5)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad (6)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \quad (7)$$

$$a^4 - b^4 = (a + b)(a - b)(a^2 + b^2) \quad (8)$$

$$a^2 + b^2 = (a + b)^2 - 2ab \quad (9)$$

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac) \quad (10)$$

## 1.1 Law of Negatives

$$\begin{aligned} -(-a) &= a \\ (-a)(-b) &= ab \\ -ab &= (-a)(b) = a(-b) = -(-a)(-b) \\ a - b &= a + (-b) \end{aligned}$$

## 1.2 Law of Quotients

$$\begin{aligned} -\frac{a}{b} &= \frac{(-a)}{b} = \frac{a}{(-b)} = -\frac{(-a)}{(-b)} \\ \frac{a}{b} &= \frac{c}{d} \quad \text{if} \quad ad = bc \\ \frac{a}{b} &= \frac{ka}{kb} \end{aligned}$$

## 2 Polynomial Functions

**Zeros of a polynomial:** If  $f(c) = 0$ , then  $c$  is called a zero of the polynomial  $f(x)$ .

**Division algorithm for a polynomial:** If  $f(x)$  and  $g(x)$  are two polynomials such that  $g(x) \neq 0$  then there exist two unique polynomials  $q(x)$  and  $r(x)$  such that:

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad (11)$$

**Fundamental Theorem of Algebra:** Every polynomial with positive degree has at least one complex zero.

**Remainder Theorem:** If a polynomial  $f(x)$  is divided by  $x - c$  then the remainder is  $f(c)$ .

**Factor Theorem:** Polynomial  $f(x)$  has a factor  $x - c$  if and only if  $f(c) = 0$ .

### 2.1 Corollary to the Fundamental Theorem

1. Every polynomial of positive degree  $n$  has a factorization of the form:

$$P(x) = a_n(x - r_1)(x - r_2)\dots\dots(x - r_n) \quad (12)$$

where  $r_i$  are not necessarily distinct. If  $(x - r_i)$  occurs  $n$  times it is said to have a multiplicity of  $n$ .

2. It is not always possible to find the factors using exact methods.
3. A polynomial of degree  $n$  has at most  $n$  complex zeros.
4. Complex zeros of real polynomials with real coefficients occur in complex conjugate pairs.
5. Any polynomial of degree  $n > 0$  with real coefficients has complete factorization using linear and quadratic factors, multiplied by the leading coefficient of the polynomial.
6. **Intermediate Value Theorem:** Given a polynomial  $f(x)$  with  $a < b$  and  $f(a) \neq f(b)$  then  $f(c)$  takes on every value  $c$  in the interval  $(a, b)$ .
7. For a polynomial  $f(x)$  if  $f(a)$  and  $f(b)$  have opposite signs then  $f(x)$  has at least one zero between  $a$  and  $b$ .
8. For a polynomial  $f(x)$  if  $f(a)$  and  $f(b)$  have opposite signs then  $f(x)$  has at least one zero between  $a$  and  $b$ .

9. **Descartes rule of signs:** If  $f(x)$  is a polynomial with terms arranged in descending order, then the number of positive real roots of  $f(x)$  is either equal to the number of sign changes between the successive terms of  $f(x)$  or less than this number by an even number. Number of negative real zeros of  $f(x)$  is obtained by applying this rule to  $f(-x)$ .

## 2.2 Synthetic Division

$c$  is said to be a zero of the polynomial  $f(x)$  if  $f(c) = 0 \implies x - c$  is a factor of the polynomial  $f(x)$ . The graph of  $f(x)$  has an intercept at  $c$ .

1. Arrange the coefficients in descending order in the first row.
2. Third row is formed by bringing down the first coefficient of  $f(x)$  then successively multiplying each coefficient in the third row by  $c$ , placing the results in second row adding this to the corresponding coefficients in the first row, and placing result in the next position of the third row.

\*\*\*\*\*



### 3 Quadratic Equations

The quadratic equation

$$ax^2 + bx + c = 0 \quad (13)$$

where  $a, b$  and  $c$  are constants and  $a \neq 0$ , has two solutions for the variable  $x$ :

$$x_{\alpha, \beta} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (14)$$

If the *discriminant*  $\Delta$  with

$$\Delta = b^2 - 4ac = 0 \quad (15)$$

then the equation (13) has real and equal roots given by:

$$x = \frac{-b}{2a} \quad (16)$$

**Sum of the roots:**

$$\alpha + \beta = \frac{-b}{a} \quad (17)$$

**Product of the roots:**

$$\alpha \cdot \beta = \frac{c}{a} \quad (18)$$

#### 3.1 Factorization method

Find the two factors  $x - \alpha$  and  $x - \beta$  which when multiplied produce the quadratic equation  $ax^2 + bx + c$ .

#### 3.2 Completion of squares method

Consider any quadratic equation of the form  $ax^2 + bx + c = 0$ . Dividing throughout by  $a$ .

$$\implies x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$\text{Adding and subtracting } \left(\frac{b}{2a}\right)^2, \quad ,$$

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

$$\therefore ax^2 + bx + c = 0 \implies a(x + d)^2 + e = 0 \quad \text{where,} \quad d = \frac{b}{2a} \quad \text{and} \quad e = \frac{c}{a} - \left(\frac{b}{2a}\right)^2.$$

\*\*\*\*\*

## 4 Logarithmic Functions

For any two variables  $x$  and  $y$  related as:

$$y = a^x \quad (19)$$

Where  $a$  is a constant. We define the logarithmic function as:

$$\log_a y = x \quad (20)$$

### 4.1 Properties of Logarithmic Functions

$$\log_a mn = \log_a m + \log_a n \quad (21)$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n \quad (22)$$

$$\log_a m^n = n \cdot \log_a m \quad (23)$$

$$\log_a a = 1 \quad (24)$$

$$\log_a 1 = 0 \quad (25)$$

$$\log_a b = \frac{1}{\log_b a} \quad (26)$$

$$a^x = e^{x \cdot \log a} \quad (27)$$

$$\log_a X = \frac{\ln X}{\ln a} \quad (28)$$

\*\*\*\*\*

## 5 Laws of Exponents

$$x^a . x^b = x^{a+b} \tag{29}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{30}$$

$$(x^a)^b = x^{ab} \tag{31}$$

$$x^a = y^b \implies a = b \tag{32}$$

$$x^{-a} = \frac{1}{x^a} \quad (\text{for } a \text{ being any non zero real number}) \tag{33}$$

$$x^0 = 1 \tag{34}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \tag{35}$$

$$\sqrt[n]{xy} = \sqrt[n]{x} . \sqrt[n]{y} \tag{36}$$

$$\left(\frac{x}{y}\right)^{-m} = \left(\frac{y}{x}\right)^m \tag{37}$$

$$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n} \tag{38}$$

## 5.1 Rational Number Exponents

The principle  $n$ th root of  $x > 0$  is defined as:

$$x^{1/n} = \begin{cases} \text{is a unique real number } y \text{ if } n \text{ is odd or even} \\ 0, \text{ if } x = 0 \\ \text{not a real number if } x < 0 \end{cases}$$

$$(x^{1/n})^n = \begin{cases} x, \text{ if } n \text{ is odd/even and } x \text{ is positive} \\ |x|, \text{ if } n \text{ is even and } x \text{ is negative} \end{cases}$$

## 5.2 Simplest Radical Form

No radicand can contain a factor with an exponent greater than or equal to the index of the radical e.g.  $\sqrt[3]{16x^3y^5} \implies$  is not in it's standard form. No power of a radicand and index of the radical can have common factor other than 1. e.g.  $2xy\sqrt[3]{2y^2} \implies$  not in standard form. No radical appears in denominator. No fractional part appears in radical. e.g.  $\sqrt[6]{t^3} = \sqrt{t}$  is in it's standard form.

\*\*\*\*\*

## 6 Complex Numbers

A number of the form  $z = x + iy$  consisting of a real part  $x$ , denoted by  $Re(z)$  and imaginary part  $y$ , denoted by  $Im(z)$  is known as a complex number.

$$x = r \cos \theta \quad (39)$$

$$y = r \sin \theta \quad (40)$$

$$r = \sqrt{x^2 + y^2} \quad (41)$$

$$\tan \theta = \frac{y}{x} \implies \theta = \tan^{-1} \frac{y}{x} \quad (42)$$

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (43)$$

$$e^{-i\theta} = \cos \theta - i \sin \theta \quad (44)$$

### 6.1 De Moivre's Theorem

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta) \quad (45)$$

\*\*\*\*\*

## 7 Permutations and Combinations

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)(n-3)\dots(n-(r+1))}{r!} \quad (46)$$

Note that:

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad 0! = 1$$

### 7.1 Binomial Theorem

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \binom{n}{3}x^{n-3}a^3 + \binom{n}{4}x^{n-4}a^4 + \dots\dots\dots a^n \quad (47)$$

### 7.2 Summation of Series

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3\dots\dots = \frac{(n)(n+1)}{2} \quad (48)$$

$$\sum_{n=1}^{\infty} n^2 = 1^2 + 2^2 + 3^2\dots\dots = \frac{(n)(n+1)(2n+1)}{6} \quad (49)$$

$$\sum_{n=1}^{\infty} n^3 = 1^3 + 2^3 + 3^3\dots\dots = \frac{(n)^2(n+1)^2}{4} \quad (50)$$

\*\*\*\*\*

## 8 Trigonometry

$$\begin{aligned}\sin \theta &= \frac{\text{Perpendicular}}{\text{Hypotenuse}} & \cos \theta &= \frac{\text{Base}}{\text{Hypotenuse}} & \tan \theta &= \frac{\text{Perpendicular}}{\text{Base}} \\ \operatorname{cosec} \theta &= \frac{\text{Hypotenuse}}{\text{Perpendicular}} & \sec \theta &= \frac{\text{Hypotenuse}}{\text{Base}} & \cot \theta &= \frac{\text{Base}}{\text{Perpendicular}}\end{aligned}$$

From the above relations it turns out that,

$$\sin \theta \cdot \operatorname{cosec} \theta = 1$$

$$\cos \theta \cdot \sec \theta = 1$$

$$\tan \theta \cdot \cot \theta = 1$$

### 8.1 Trigonometric Identities

$$\sin^2 \theta + \cos^2 \theta = 1 \tag{51}$$

$$1 + \sec^2 \theta = \tan^2 \theta \tag{52}$$

$$1 + \operatorname{cosec}^2 \theta = \cot^2 \theta \tag{53}$$

$\theta$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot \theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec \theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\operatorname{cosec} \theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Figure 1: Most common trigonometric ratios and values

### 8.2 Trigonometric ratios of allied angles

- *For  $n$  being an odd multiple of  $\pi/2$  the function changes to its composite function.*
- *For  $n$  being an even multiple of  $\pi/2$  the function does not change to its composite, but may change sign.*

### 8.2.1 First Quadrant

$$\begin{array}{ll}\sin(2\pi + \theta) = \sin \theta & \cos(2\pi + \theta) = \cos \theta \\ \operatorname{cosec}(2\pi + \theta) = \operatorname{cosec} \theta & \sec(2\pi + \theta) = \sec \theta \\ \tan(2\pi + \theta) = \tan \theta & \cot(2\pi + \theta) = \cot \theta\end{array}$$

$$\begin{array}{ll}\sin(\pi/2 - \theta) = \cos \theta & \cos(\pi/2 - \theta) = \sin \theta \\ \operatorname{cosec}(\pi/2 - \theta) = \sec \theta & \sec(\pi/2 - \theta) = \operatorname{cosec} \theta \\ \tan(\pi/2 - \theta) = \cot \theta & \cot(\pi/2 - \theta) = \tan \theta\end{array}$$

### 8.2.2 Second Quadrant

$$\begin{array}{ll}\sin(\pi/2 + \theta) = \cos \theta & \cos(\pi/2 + \theta) = -\sin \theta \\ \operatorname{cosec}(\pi/2 + \theta) = \sec \theta & \sec(\pi/2 + \theta) = -\operatorname{cosec} \theta \\ \tan(\pi/2 + \theta) = -\cot \theta & \cot(\pi/2 + \theta) = -\tan \theta\end{array}$$

$$\begin{array}{ll}\sin(\pi - \theta) = \sin \theta & \cos(\pi - \theta) = -\cos \theta \\ \operatorname{cosec}(\pi - \theta) = \operatorname{cosec} \theta & \sec(\pi - \theta) = -\sec \theta \\ \tan(\pi - \theta) = -\tan \theta & \cot(\pi - \theta) = -\cot \theta\end{array}$$

### 8.2.3 Third Quadrant

$$\begin{array}{ll}\sin(\pi + \theta) = -\sin \theta & \cos(\pi + \theta) = -\cos \theta \\ \operatorname{cosec}(\pi + \theta) = \operatorname{cosec} \theta & \sec(\pi + \theta) = -\sec \theta \\ \tan(\pi + \theta) = \tan \theta & \cot(\pi + \theta) = \cot \theta\end{array}$$

$$\begin{array}{ll}\sin(3\pi/2 - \theta) = -\cos \theta & \cos(3\pi/2 - \theta) = -\sin \theta \\ \operatorname{cosec}(3\pi/2 - \theta) = -\sec \theta & \sec(3\pi/2 - \theta) = -\operatorname{cosec} \theta \\ \tan(3\pi/2 - \theta) = \cot \theta & \cot(3\pi/2 - \theta) = \tan \theta\end{array}$$

### 8.2.4 Fourth Quadrant

$$\begin{array}{ll}\sin(3\pi/2 + \theta) = -\cos \theta & \cos(3\pi/2 + \theta) = \sin \theta \\ \operatorname{cosec}(3\pi/2 + \theta) = -\sec \theta & \sec(3\pi/2 + \theta) = \operatorname{cosec} \theta \\ \tan(3\pi/2 + \theta) = -\cot \theta & \cot(3\pi/2 + \theta) = -\tan \theta\end{array}$$

$$\begin{array}{ll}\sin(2\pi - \theta) = -\sin \theta & \cos(2\pi - \theta) = \cos \theta \\ \operatorname{cosec}(2\pi - \theta) = -\operatorname{cosec} \theta & \sec(2\pi - \theta) = \sec \theta \\ \tan(2\pi - \theta) = -\tan \theta & \cot(2\pi - \theta) = -\cot \theta\end{array}$$

- *The values for  $(2\pi - \theta)$  and  $(-\theta)$  are identical.*



### 8.3 Compound Angle Formulae

$$\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$$

$$\sin(A - B) = \sin(A) \cos(B) - \cos(A) \sin(B)$$

$$\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$$

$$\cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B)$$

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A) \tan(B)}$$

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A) \tan(B)}$$

### 8.4 Sum or Difference Formulae

$$\sin(A + B) + \sin(A - B) = 2 \sin(A) \cos(B)$$

$$\sin(A - B) - \sin(A - B) = 2 \cos(A) \sin(B)$$

$$\cos(A + B) + \cos(A - B) = 2 \cos(A) \cos(B)$$

$$\cos(A + B) - \cos(A - B) = -2 \sin(A) \sin(B)$$

### 8.5 Half Angle Formulae

$$\sin\left(\frac{A}{2}\right) = \pm \sqrt{\frac{1 - \cos(A)}{2}}$$

$$\cos\left(\frac{A}{2}\right) = \pm \sqrt{\frac{1 + \cos(A)}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \pm \sqrt{\frac{1 - \cos(A)}{1 + \cos(A)}}$$

### 8.6 Sub Multiple Angle Formulae

$$\sin(2A) = 2 \sin(A) \cos(A)$$

$$\cos(2A) = \cos^2(A) - \sin^2(A) = 2 \cos^2(A) - 1 = 1 - 2 \sin^2(A)$$

$$\tan(2A) = \frac{2 \tan(A)}{1 - \tan^2(A)}$$

$$\sin(2A) = \frac{2 \tan(A)}{1 + \tan^2(A)}$$

$$\cos(2A) = \frac{1 - \tan^2(A)}{1 + \tan^2(A)}$$

$$\cos^2(A) = \frac{1 + \cos(2A)}{2}$$

$$\sin^2(A) = \frac{1 - \cos(2A)}{2}$$

$$\sin(A) = 2 \sin(A/2) \cos(A/2)$$

$$\cos(A) = \cos^2(A/2) - \sin^2(A/2) = 2 \cos^2(A/2) - 1 = 1 - 2 \sin^2(A/2)$$

## 8.7 Triple Angle Formulae

$$\sin(3x) = 3\sin(x) - 4\sin^3(x)$$

$$\cos(3x) = 4\cos^3(x) - 3\cos(x)$$

$$\tan(3x) = \frac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)}$$

$$\sin(4x) = 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x)$$

$$\tan(4x) = \frac{4\tan(x) - 4\tan^3(x)}{1 - 6\tan^2(x) + \tan^4(x)}$$

$$\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$$

## 8.8 Product Formulae

If  $A + B = C$  and  $A - B = D$

Then  $A = (C + D)/2$  and  $B = (C - D)/2$ , the product formulae are defined as

$$\sin(C) + \sin(D) = 2\sin\frac{(C + D)}{2}\cos\frac{(C - D)}{2}$$

$$\sin(C) - \sin(D) = 2\cos\frac{(C + D)}{2}\sin\frac{(C - D)}{2}$$

$$\cos(C) + \cos(D) = 2\cos\frac{(C + D)}{2}\cos\frac{(C - D)}{2}$$

$$\cos(C) - \cos(D) = -2\sin\frac{(C + D)}{2}\sin\frac{(C - D)}{2}$$

\*\*\*\*\*

## 9 Inverse Trigonometric Functions

If  $x = \sin \theta$  then we define the inverse trigonometric function as  $\sin^{-1}(x) = \theta$ , the same holds true for all other trigonometric functions.

### 9.1 Properties of Inverse Trigonometric Functions

$$\begin{array}{ll} \sin^{-1}(\sin \theta) = \theta & \operatorname{cosec}^{-1}(\operatorname{cosec} \theta) = \theta \\ \cos^{-1}(\cos \theta) = \theta & \sec^{-1}(\sec \theta) = \theta \\ \tan^{-1}(\tan \theta) = \theta & \cot^{-1}(\cot \theta) = \theta \end{array}$$

$$\begin{array}{ll} \sin(\sin^{-1}(x)) = x & \operatorname{cosec}(\operatorname{cosec}^{-1}(x)) = x \\ \cos(\cos^{-1}(x)) = x & \sec(\sec^{-1}(x)) = x \\ \tan(\tan^{-1}(x)) = x & \cot(\cot^{-1}(x)) = x \end{array}$$

Functions	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$\mathbb{R}$	$[-\pi/2, \pi/2]$
$\cot^{-1}x$	$\mathbb{R}$	$[0, \pi]$
$\sec^{-1}x$	$\mathbb{R} - (-1, 1)$	$[0, \pi] - [\pi/2]$
$\operatorname{cosec}^{-1}x$	$\mathbb{R} - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$

Figure 2: Domain and range of inverse trigonometric functions

$$\begin{array}{llll} \sin^{-1}(x) & = & \cos^{-1} \sqrt{1-x^2} & = & \tan^{-1} \frac{x}{\sqrt{1-x^2}} \\ \cos^{-1}(x) & = & \sin^{-1} \sqrt{1-x^2} & = & \tan^{-1} \frac{\sqrt{1-x^2}}{x} \\ \tan^{-1}(x) & = & \sin^{-1} \frac{x}{\sqrt{1+x^2}} & = & \cos^{-1} \frac{1}{\sqrt{1+x^2}} \end{array}$$

$$\begin{aligned}
\sin^{-1}(-x) &= -\sin^{-1}(x) & x \in [-1, 1] \\
\cos^{-1}(-x) &= \pi - \cos^{-1}(x) & x \in [-1, 1] \\
\tan^{-1}(-x) &= -\tan^{-1}(x) & x \in \mathbb{R} \\
\operatorname{cosec}^{-1}(-x) &= -\operatorname{cosec}^{-1}(x) & x \in (-\infty, -1] \cup [1, \infty) \\
\sec^{-1}(-x) &= \pi - \sec^{-1}(x) & x \in (-\infty, -1] \cup [1, \infty) \\
\cot^{-1}(-x) &= \pi - \cot^{-1}(x) & x \in \mathbb{R}
\end{aligned}$$

$$\begin{aligned}
\sin^{-1}\left(\frac{1}{x}\right) &= \operatorname{cosec}^{-1}(x) & x \in (-\infty, -1] \cup [1, \infty) \\
\cos^{-1}\left(\frac{1}{x}\right) &= \sec^{-1}(x) & x \in (-\infty, -1] \cup [1, \infty)
\end{aligned}$$

$$\tan^{-1}(1/x) = \begin{cases} \cot^{-1}(x), & x > 0 \\ \cot^{-1}(x) - \pi, & x < 0 \end{cases}$$

$$\begin{aligned}
\sin^{-1}(x) + \cos^{-1}(x) &= (\pi/2) \\
\tan^{-1}(x) + \cot^{-1}(x) &= (\pi/2) \\
\sec^{-1}(x) + \operatorname{cosec}^{-1}(x) &= (\pi/2)
\end{aligned}$$

$$\begin{aligned}
\tan^{-1}(x) + \tan^{-1}(y) &= \tan^{-1} \frac{x+y}{1-xy} \\
\tan^{-1}(x) - \tan^{-1}(y) &= \tan^{-1} \frac{x-y}{1+xy}
\end{aligned}$$

$$2 \tan^{-1}(x) = \begin{cases} 2 \sin^{-1} \frac{2x}{1+x^2}, & x > 0 \\ \cos^{-1} \frac{1-x^2}{1+x^2}, & x < 0 \end{cases}$$

## 9.2 Expressions and suggested substitutions

Expression	Substitution
$a^2 + x^2$	$x = a \tan \theta$ or $x = a \cot \theta$
$a^2 - x^2$	$x = a \sin \theta$ or $x = a \cos \theta$
$x^2 - a^2$	$x = a \operatorname{cosec} \theta$ or $x = a \sec \theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x = a^2 \cos 2\theta$

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## 10 Hyperbolic Functions

$$\begin{aligned}\sinh(x) &= \frac{e^x - e^{-x}}{2} & \operatorname{cosech}(x) &= \frac{2}{e^x - e^{-x}} \\ \cosh(x) &= \frac{e^x + e^{-x}}{2} & \operatorname{sech}(x) &= \frac{2}{e^x + e^{-x}} \\ \tanh(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} & \coth(x) &= \frac{e^x + e^{-x}}{e^x - e^{-x}}\end{aligned}$$

### 10.1 Properties of Hyperbolic Functions

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= 1 \\ \cosh^2(x) + \sinh^2(x) &= \cosh(2x) \\ 1 - \tanh^2(x) &= \operatorname{sech}^2(x) \\ \coth^2(x) - 1 &= \operatorname{cosech}^2(x) \\ \sinh(2x) &= 2 \sinh(x) \cosh(x) \\ \sin(ix) &= \frac{e^{ix} - e^{-ix}}{2} \\ \cos(ix) &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin(ix) &= i \sinh(x) \\ \cos(ix) &= \cosh(x) \\ \tan(ix) &= i \tanh(x)\end{aligned}$$

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# 11 Linear Inequalities

Inequalities of the form:

$$ax + b < 0$$

$$ax + b > 0$$

$$ax + b \leq 0$$

$$ax + b \geq 0$$

are known as linear inequalities.

1. Add and subtract any number  $k$  without change in the inequality on both sides.
2. When multiplying or dividing by constant  $k$ , reverse the sign of the inequality only when  $k$  is negative.
3. Linear inequalities have infinite solution sets, these can be obtained by isolating the variable and solving them in a manner similar to solving equations.

Inequality	Representation	Graph
$a < x < b$	$(a, b)$	
$a \leq x \leq b$	$[a, b]$	
$a < x \leq b$	$(a, b]$	
$a \leq x < b$	$[a, b)$	
$x > a$	$(a, \infty)$	
$x \geq a$	$[a, \infty)$	
$x < b$	$(-\infty, b)$	
$x \leq b$	$(-\infty, b]$	

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## 12 Limits and Differentiation

### 12.1 Fundamental principles of limits

1. The limit may exist at a point even if the function is undefined.
2. If a function  $f(x)$  is defined at a point  $a$  i.e.  $f(a)$  exists, it is not necessary that the limit at  $a$  must exist. Moreover, even if the limit exists it need not be equal to  $f(a)$ .
3. **Indeterminate forms:** Any function assuming either of these forms is said to be indeterminate -  $\frac{0}{0}, 0.\infty, \frac{\infty}{\infty}, \infty - \infty, 0^0, 1^\infty, \infty^0$ .

### 12.2 Properties

$$\begin{aligned}\lim_{x \rightarrow a} [f(x) \pm g(x)] &= \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x). \\ \lim_{x \rightarrow a} [f(x).g(x)] &= \lim_{x \rightarrow a} f(x). \lim_{x \rightarrow a} g(x). \\ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{provided } \lim_{x \rightarrow a} g(x) \neq 0.\end{aligned}$$

### 12.3 Standard Limits

$$\lim_{x \rightarrow 0} \sin x = 0 \quad (54)$$

$$\lim_{x \rightarrow 0} \tan x = 0 \quad (55)$$

$$\lim_{x \rightarrow 0} \cos x = 1 \quad (56)$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (57)$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \quad (58)$$

$$\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1 \quad (59)$$

$$\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1 \quad (60)$$

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a \quad \text{where } a > 0 \quad (61)$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \quad (62)$$

$$\lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} = \log_a e \quad \text{where } a > 0 \text{ and } a \neq 1 \quad (63)$$

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \quad (64)$$

$$\lim_{x \rightarrow 0} \frac{\log x}{x^m} = 0 \quad \text{where } m > 0 \quad (65)$$

$$\lim_{x \rightarrow 0} \frac{(1+x)^m - 1}{x} = m \quad (66)$$

$$\lim_{x \rightarrow 0} \frac{x^n - a^n}{x - a} = n.a^{n-1} \quad (67)$$



## 12.4 Dealing with indeterminate forms

If  $f(x)$  and  $g(x)$  are two functions such that  $f(x) \rightarrow 0$  as  $x \rightarrow a$  and  $g(x) \rightarrow \infty$  as  $x \rightarrow a$  such that  $[1 + f(x)]^{g(x)}$  assumes the form  $1^\infty$  then:

$$\lim_{x \rightarrow a} [1 + f(x)]^{g(x)} = e^{\lim_{x \rightarrow a} f(x) \cdot g(x)}$$

## 12.5 L'Hospital's Rule

L'Hospital's rule states that for functions  $f(x)$  and  $g(x)$  which are differentiable on an open interval  $I$  except possibly at a point  $c$  contained in  $I$ , if:

$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$  or  $\pm\infty$  and  $g'(x) \neq 0$  for all  $x$  in  $I$  with  $x \neq c$  and  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  exists then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

## 12.6 Differentiation Formulae

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$$

$$\frac{d}{dx} f(x) \cdot g(x) = \frac{d}{dx} f(x) \cdot \frac{d}{dx} g(x)$$

$$\frac{dc}{dx} = 0 \quad \text{where } c \text{ is a real constant}$$

$$\frac{dx^n}{dx} = nx^{n-1} \quad \text{where } n \text{ is any real number}$$

$$\frac{de^x}{dx} = e^x$$

$$\frac{da^x}{dx} = a^x \log_e a$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \cdot \log a}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

## 12.7 Differentiation of Trigonometric Functions

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x$$

$$\frac{d}{dx} \sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$$

## 12.8 Differentiation of Inverse Trigonometric Functions

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \operatorname{cosec}^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$$

## 12.9 Differentiation of Hyperbolic Functions

$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$$

$$\frac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \cdot \coth x$$

$$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x$$

$$\frac{d}{dx} \coth x = -\operatorname{cosech}^2 x$$

## 12.10 Product Rule

$$\frac{d}{dx}.uv = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

## 12.11 Quotient Rule

$$\frac{d}{dx} \cdot \frac{u}{v} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

## 12.12 Derivative of Inverse of a function

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

## 12.13 Chain Rule of Differentiation (Derivative of composite function)

If  $u = \phi(x)$  and  $y = \phi(u)$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

## 12.14 Differentiation of Parametric Functions

**Parametric Function:** Consider two variables  $x$  and  $y$  which can be expressed in terms of another variable  $t$ , this  $t$  is termed as the parameter. Hence both the variables can be expressed in terms of a third variable and the relation between them is known as a parametric function. Therefore if  $x = f(t)$  and  $y = g(t)$  then:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\text{But } \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

Therefore,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

## 12.15 Logarithmic Differentiation

If  $y = u^v$  where both  $u$  and  $v$  are functions of variable  $x$ .

Taking log on both sides, and differentiating w.r.t  $x$ ,

$\log y = v \log u$

$$\frac{1}{y} \cdot \frac{dy}{dx} = v \cdot \frac{d}{dx} \log u + \log u \cdot \frac{dv}{dx}$$

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