Basics of Engineering Mathematics

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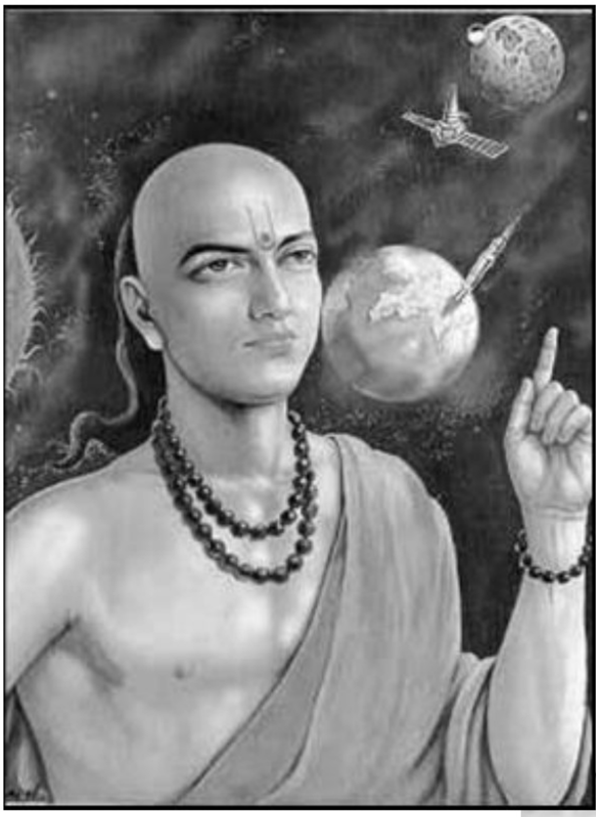

"Dedicated to the mathematicians of India, who gifted the world an abundance of knowledge and a spirit of enquiry".

1 Basic Algebra

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \tag{2}$$

$$(a+b)(a-b) = a^2 - b^2 \tag{3}$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b) \tag{4}$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a-b) \tag{5}$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \tag{6}$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \tag{7}$$

$$a^4 - b^4 = (a+b)(a-b)(a^2 + b^2) \tag{8}$$

$$a^2 + b^2 = (a+b)^2 - 2ab \tag{9}$$

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac) \tag{10}$$

1.1 Law of Negatives

$$\begin{aligned} -(-a) &= a \\ (-a)(-b) &= ab \\ -ab = (-a)(b) = a(-b) &= -(-a)(-b) \\ a - b &= a + (-b) \end{aligned}$$

1.2 Law of Quotients

$$\begin{aligned} -\frac{a}{b} = \frac{(-a)}{b} = \frac{a}{(-b)} &= -\frac{(-a)}{(-b)} \\ \frac{a}{b} = \frac{c}{d} \quad \text{if} \quad ad &= bc \\ \frac{a}{b} &= \frac{ka}{kb} \end{aligned}$$

2 Polynomial Functions

Zeros of a polynomial: If $f(c) = 0$, then c is called a zero of the polynomial $f(x)$.

Division algorithm for a polynomial: If $f(x)$ and $g(x)$ are two polynomials such that $g(x) \neq 0$ then there exist two unique polynomials $q(x)$ and $r(x)$ such that:

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \tag{11}$$

Fundamental Theorem of Algebra: Every polynomial with positive degree has at least one complex zero.

Remainder Theorem: If a polynomial $f(x)$ is divided by $x - c$ then the remainder is $f(c)$.

Factor Theorem: Polynomial $f(x)$ has a factor $x - c$ if and if only $f(c) = 0$.

2.1 Corollary to the Fundamental Theorem

1. Every polynomial of positive degree n has a factorization of the form:

   $$P(x) = a_n(x - r_1)(x - r_2).........(x - r_n) \tag{12}$$

   where r_i are not necessarily distinct. If $(x - r_i)$ occurs n times it is said to have a multiplicity of n.

2. It is not always possible to find the factors using exact methods.

3. A polynomial of degree n has at most n complex zeros.

4. Complex zeros of real polynomials with real coefficients occur in complex conjugate pairs.

5. Any polynomial of degree $n > 0$ with real coefficients has complete factorization using linear and quadratic factors, multiplied by the leading coefficient of the polynomial.

6. **Intermediate Value Theorem**: Given a polynomial $f(x)$ with $a < b$ and $f(a) \neq f(b)$ then $f(c)$ takes on every value c in the interval (a, b).

7. For a polynomial $f(x)$ if $f(a)$ and $f(b)$ have opposite signs then $f(x)$ has at least one zero between a and b.

8. For a polynomial $f(x)$ if $f(a)$ and $f(b)$ have opposite signs then $f(x)$ has at least one zero between a and b.

9. **Descartes rule of signs**: If $f(x)$ is a polynomial with terms arranged in descending order, then the number of positive real roots of $f(x)$ is either equal to the number of sign changes between the successive terms of $f(x)$ or less than this number by an even number. Number of negative real zeros of $f(x)$ is obtained by applying this rule to $f(-x)$.

2.2 Synthetic Division

c is said to be a zero of the polynomial $f(x)$ if $f(c) = 0 \implies x - c$ is a factor of the polynomial $f(x)$. The graph of $f(x)$ has an intercept at c.

1. Arrange the coefficients in descending order in the first row.

2. Third row is formed by bringing down the first coefficient of $f(x)$ then successively multiplying each coefficient in the third row by c, placing the results in second row adding this to the corresponding coefficients in the first row, and placing result in the next position of the third row.

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3 Quadratic Equations

The quadratic equation

$$ax^2 + bx + c = 0 \tag{13}$$

where a, b and c are constants and $a \neq 0$, has two solutions for the variable x:

$$x_{\alpha,\beta} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{14}$$

If the *discriminant* Δ with

$$\Delta = b^2 - 4ac = 0 \tag{15}$$

then the equation (13) has real and equal roots given by:

$$x = \frac{-b}{2a} \tag{16}$$

Sum of the roots:

$$\alpha + \beta = \frac{-b}{a} \tag{17}$$

Product of the roots:

$$\alpha.\beta = \frac{c}{a} \tag{18}$$

3.1 Factorization method

Find the two factors $x - \alpha$ and $x - \beta$ which when multiplied produce the quadratic equation $ax^2 + bx + c$.

3.2 Completion of squares method

Consider any quadratic equation of the form $ax^2 + bx + c = 0$. Dividing throughout by a.

$$\implies x^2 + \left(\frac{b}{a}\right) x + \frac{c}{a} = 0$$

$$\text{Adding and subtracting } \left(\frac{b}{2a}\right)^2 \qquad ,$$

$$x^2 + \left(\frac{b}{a}\right) x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

$$\therefore ax^2 + bx + c = 0 \implies a(x+d)^2 + e = 0 \quad \text{where,} \quad d = \frac{b}{2a} \quad \text{and} \quad e = \frac{c}{a} - \left(\frac{b}{2a}\right)^2.$$

****

4 Logarithmic Functions

For any two variables x and y related as:

$$y = a^x \tag{19}$$

Where a is a constant. We define the logarithmic function as:

$$\log_a y = x \tag{20}$$

4.1 Properties of Logarithmic Functions

$$\log_a mn = \log_a m + \log_a n \tag{21}$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n \tag{22}$$

$$\log_a m^n = n. \log_a m \tag{23}$$

$$\log_a a = 1 \tag{24}$$

$$\log_a 1 = 0 \tag{25}$$

$$\log_a b = \frac{1}{\log_b a} \tag{26}$$

$$a^x = e^{x. \log a} \tag{27}$$

$$\log_a X = \frac{\ln X}{\ln a} \tag{28}$$

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5 Laws of Exponents

$$x^a.x^b = x^{a+b} \tag{29}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{30}$$

$$(x^a)^b = x^{ab} \tag{31}$$

$$x^a = y^b \implies a = b \tag{32}$$

$$x^{-a} = \frac{1}{x^a} \qquad \text{(for a being any non zero real number)} \tag{33}$$

$$x^0 = 1 \tag{34}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \tag{35}$$

$$\sqrt[n]{xy} = \sqrt[n]{x}.\sqrt[n]{y} \tag{36}$$

$$\left(\frac{x}{y}\right)^{-m} = \left(\frac{y}{x}\right)^m \tag{37}$$

$$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n} \tag{38}$$

5.1 Rational Number Exponents

The principle nth root of x>0 is defined as:

$$x^{1/n} = \begin{cases} \text{is a unique real number y if n is odd or even} \\ 0, if x = 0 \\ \text{not a real number if x<0} \end{cases}$$

$$(x^{1/n})^n = \begin{cases} \text{x,if n is odd/even and x is positive} \\ \text{|x|, if n is even and x is negative} \end{cases}$$

5.2 Simplest Radical Form

No radicand can contain a factor with an exponent greater than or equal to the index of the radical e.g. $\sqrt[3]{16x^3y^5} \implies$ is not in it's standard form. No power of a radicand and index of the radical can have common factor other than 1. e.g. $2xy\sqrt[3]{2y^2} \implies$ not in standard form. No radical appears in denominator. No fractional part appears in radical. e.g. $\sqrt[6]{t^3} = \sqrt{t}$ is in it's standard form.

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6 Complex Numbers

A number of the form $z = x + iy$ consisting of a real part x, denoted by $Re(z)$ and imaginary part y, denoted by $Im(z)$ is known as a complex number.

$$x = r\cos\theta \tag{39}$$
$$y = r\sin\theta \tag{40}$$
$$r = \sqrt{x^2 + y^2} \tag{41}$$
$$\tan\theta = \frac{y}{x} \implies \theta = \tan^{-1}\frac{y}{x} \tag{42}$$
$$e^{i\theta} = \cos\theta + i\sin\theta \tag{43}$$
$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{44}$$

6.1 De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \tag{45}$$

****

7 Permutations and Combinations

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)(n-3).....(n-(r+1))}{r!} \tag{46}$$

Note that:

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad 0! = 1$$

7.1 Binomial Theorem

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \binom{n}{3}x^{n-3}a^3 + \binom{n}{4}x^{n-4}a^4 + ...a^n \tag{47}$$

7.2 Summation of Series

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3...... = \frac{(n)(n+1)}{2} \tag{48}$$

$$\sum_{n=1}^{\infty} n^2 = 1^2 + 2^2 + 3^2...... = \frac{(n)(n+1)(2n+1)}{6} \tag{49}$$

$$\sum_{n=1}^{\infty} n^3 = 1^3 + 2^3 + 3^3...... = \frac{(n)^2(n+1)^2}{4} \tag{50}$$

****

8 Trigonometry

$$\sin\theta = \frac{Perpendicular}{Hypotenuse} \qquad \cos\theta = \frac{Base}{Hypotenuse} \qquad \tan\theta = \frac{Perpendicular}{Base}$$
$$\text{cosec}\theta = \frac{Hypotenuse}{Perpendicular} \qquad \sec\theta = \frac{Hypotenuse}{Base} \qquad \cot\theta = \frac{Base}{Perpendicular}$$

From the above relations it turns out that,

$$\sin\theta.\text{cosec}\theta = 1$$
$$\cos\theta.\sec\theta = 1$$
$$\tan\theta.\cot\theta = 1$$

8.1 Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{51}$$
$$1 + \sec^2\theta = \tan^2\theta \tag{52}$$
$$1 + \text{cosec}^2\theta = \cot^2\theta \tag{53}$$

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\text{cosec}\,\theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Figure 1: Most common trigonometric ratios and values

8.2 Trigonometric ratios of allied angles

- ***For n being an odd multiple of $\pi/2$ the function changes to its composite function.***
- ***For n being an even multiple of $\pi/2$ the function does not change to its composite, but may change sign.***

8.2.1 First Quadrant

$$\begin{aligned} \sin(2\pi+\theta) &= \sin\theta & \cos(2\pi+\theta) &= \cos\theta \\ \text{cosec}(2\pi+\theta) &= \text{cosec}\theta & \sec(2\pi+\theta) &= \sec\theta \\ \tan(2\pi+\theta) &= \tan\theta & \cot(2\pi+\theta) &= \cot\theta \end{aligned}$$

$$\begin{aligned} \sin(\pi/2-\theta) &= \cos\theta & \cos(\pi/2-\theta) &= \sin\theta \\ \text{cosec}(\pi/2-\theta) &= \sec\theta & \sec(\pi/2-\theta) &= \text{cosec}\theta \\ \tan(\pi/2-\theta) &= \cot\theta & \cot(\pi/2-\theta) &= \tan\theta \end{aligned}$$

8.2.2 Second Quadrant

$$\begin{aligned} \sin(\pi/2+\theta) &= \cos\theta & \cos(\pi/2+\theta) &= -\sin\theta \\ \text{cosec}(\pi/2+\theta) &= \sec\theta & \sec(\pi/2+\theta) &= -\text{cosec}\theta \\ \tan(\pi/2+\theta) &= -\cot\theta & \cot(\pi/2+\theta) &= -\tan\theta \end{aligned}$$

$$\begin{aligned} \sin(\pi-\theta) &= \sin\theta & \cos(\pi-\theta) &= -\cos\theta \\ \text{cosec}(\pi-\theta) &= \text{cosec}\theta & \sec(\pi-\theta) &= -\sec\theta \\ \tan(\pi-\theta) &= -\tan\theta & \cot(\pi-\theta) &= -\cot\theta \end{aligned}$$

8.2.3 Third Quadrant

$$\begin{aligned} \sin(\pi+\theta) &= -\sin\theta & \cos(\pi+\theta) &= -\cos\theta \\ \text{cosec}(\pi+\theta) &= \text{cosec}\theta & \sec(\pi+\theta) &= -\sec\theta \\ \tan(\pi+\theta) &= \tan\theta & \cot(\pi+\theta) &= \cot\theta \end{aligned}$$

$$\begin{aligned} \sin(3\pi/2-\theta) &= -\cos\theta & \cos(3\pi/2-\theta) &= -\sin\theta \\ \text{cosec}(3\pi/2-\theta) &= -\sec\theta & \sec(3\pi/2-\theta) &= -\text{cosec}\theta \\ \tan(3\pi/2-\theta) &= \cot\theta & \cot(3\pi/2-\theta) &= \tan\theta \end{aligned}$$

8.2.4 Fourth Quadrant

$$\begin{aligned} \sin(3\pi/2+\theta) &= -\cos\theta & \cos(3\pi/2+\theta) &= \sin\theta \\ \text{cosec}(3\pi/2+\theta) &= -\sec\theta & \sec(3\pi/2+\theta) &= \text{cosec}\theta \\ \tan(3\pi/2+\theta) &= -\cot\theta & \cot(3\pi/2+\theta) &= -\tan\theta \end{aligned}$$

$$\begin{aligned} \sin(2\pi-\theta) &= -\sin\theta & \cos(2\pi-\theta) &= \cos\theta \\ \text{cosec}(2\pi-\theta) &= -\text{cosec}\theta & \sec(2\pi-\theta) &= \sec\theta \\ \tan(2\pi-\theta) &= -\tan\theta & \cot(2\pi-\theta) &= -\cot\theta \end{aligned}$$

- ***The values for (2π-θ) and (-θ) are identical.***

8.3 Compound Angle Formulae

$$
\begin{aligned}
\sin(A+B) &= \sin(A)\cos(B)+\cos(A)\sin(B)\\
\sin(A-B) &= \sin(A)\cos(B)-\cos(A)\sin(B)\\
\cos(A+B) &= \cos(A)\cos(B)-\sin(A)\sin(B)\\
\cos(A-B) &= \cos(A)\cos(B)+\sin(A)\sin(B)\\
\tan(A+B) &= \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}\\
\tan(A-B) &= \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}
\end{aligned}
$$

8.4 Sum or Difference Formulae

$$
\begin{aligned}
\sin(A+B)+\sin(A-B) &= 2\sin(A)\cos(B)\\
\sin(A-B)-\sin(A-B) &= 2\cos(A)\sin(B)\\
\cos(A+B)+\cos(A-B) &= 2\cos(A)\cos(B)\\
\cos(A+B)-\cos(A-B) &= -2\sin(A)\sin(B)
\end{aligned}
$$

8.5 Half Angle Formulae

$$
\begin{aligned}
\sin(\frac{A}{2}) &= \pm\sqrt{\frac{1-\cos(A)}{2}}\\
\cos(\frac{A}{2}) &= \pm\sqrt{\frac{1+\cos(A)}{2}}\\
\tan(\frac{A}{2}) &= \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}}
\end{aligned}
$$

8.6 Sub Multiple Angle Formulae

$$
\begin{aligned}
\sin(2A) &= 2\sin(A)\cos(A)\\
\cos(2A) &= cos^2(A)-sin^2(A) = 2\cos^2(A)-1 = 1-2\sin^2(A)\\
\tan(2A) &= \frac{2\tan(A)}{1-\tan^2(A)}\\
\sin(2A) &= \frac{2\tan(A)}{1+\tan^2(A)}\\
\cos(2A) &= \frac{1-\tan^2(A)}{1+\tan^2(A)}\\
\cos^2(A) &= \frac{1+\cos(2A)}{2}\\
\sin^2(A) &= \frac{1-\cos(2A)}{2}\\
\sin(A) &= 2\sin(A/2)\cos(A/2)\\
\cos(A) &= cos^2(A/2)-sin^2(A/2) = 2\cos^2(A/2)-1 = 1-2\sin^2(A/2)
\end{aligned}
$$

8.7 Triple Angle Formulae

$$
\begin{aligned}
\sin(3x) &= 3\sin(x) - 4\sin^3(x) \\
\cos(3x) &= 4\cos^3(x) - 3\cos(x) \\
\tan(3x) &= \frac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)} \\
\sin(4x) &= 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x) \\
\tan(4x) &= \frac{4\tan(x) - 4\tan^3(x)}{1 - 6\tan^2(x) + \tan^4(x)} \\
\cos(4x) &= 8\cos^4(x) - 8\cos^2(x) + 1
\end{aligned}
$$

8.8 Product Formulae

If $A + B = C$ and $A - B = D$
Then $A = (C + D)/2$ and $B = (C - D)/2$, the product formulae are defined as

$$
\begin{aligned}
\sin(C) + \sin(D) &= 2\sin\frac{(C+D)}{2}\cos\frac{(C-D)}{2} \\
\sin(C) - \sin(D) &= 2\cos\frac{(C+D)}{2}\sin\frac{(C-D)}{2} \\
\cos(C) + \cos(D) &= 2\cos\frac{(C+D)}{2}\cos\frac{(C-D)}{2} \\
\cos(C) - \cos(D) &= -2\sin\frac{(C+D)}{2}\sin\frac{(C-D)}{2}
\end{aligned}
$$

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9 Inverse Trigonometric Functions

If $x = \sin\theta$ then we define the inverse trigonometric function as $\sin^{-1}(x) = \theta$, the same holds true for all other trigonometric functions.

9.1 Properties of Inverse Trigonometric Functions

$$
\begin{aligned}
\sin^{-1}(\sin\theta) &= \theta & \qquad \mathrm{cosec}^{-1}(\mathrm{cosec}\theta) &= \theta \\
\cos^{-1}(\cos\theta) &= \theta & \qquad \sec^{-1}(\sec\theta) &= \theta \\
\tan^{-1}(\tan\theta) &= \theta & \qquad \cot^{-1}(\cot\theta) &= \theta
\end{aligned}
$$

$$
\begin{aligned}
\sin(\sin^{-1}(x)) &= x & \qquad \mathrm{cosec}(\mathrm{cosec}^{-1}(x)) &= x \\
\cos(\cos^{-1}(x)) &= x & \qquad \sec(\sec^{-1}(x)) &= x \\
\tan(\tan^{-1}(x)) &= x & \qquad \cot(\cot^{-1}(x)) &= x
\end{aligned}
$$

Functions	Domain	Range
$\sin^{-1}x$	[-1,1]	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	[-1,1]	$[0, \pi]$
$\tan^{-1}x$	R	$[-\pi/2, \pi/2]$
$\cot^{-1}x$	R	$[0, \pi]$
$\sec^{-1}x$	R-(-1,1)	$[0, \pi] - [\pi/2]$
$\mathrm{cosec}^{-1}x$	R-(-1,1)	$[-\pi/2, \pi/2]-\{0\}$

Figure 2: Domain and range of inverse trigonometric functions

$$
\begin{aligned}
\sin^{-1}(x) &= \cos^{-1}\sqrt{1-x^2} &= \tan^{-1}\frac{x}{\sqrt{1-x^2}} \\
\cos^{-1}(x) &= \sin^{-1}\sqrt{1-x^2} &= \tan^{-1}\frac{\sqrt{1-x^2}}{x} \\
\tan^{-1}(x) &= \sin^{-1}\frac{x}{\sqrt{1+x^2}} &= \cos^{-1}\frac{1}{\sqrt{1+x^2}}
\end{aligned}
$$

$$
\begin{aligned}
\sin^{-1}(-x) &= -\sin^{-1}(x) \quad x \in [-1,1] \\
\cos^{-1}(-x) &= \pi - \cos^{-1}(x) \quad x \in [-1,1] \\
\tan^{-1}(-x) &= -\tan^{-1}(x) \quad x \in \mathrm{R} \\
\mathrm{cosec}^{-1}(-x) &= -\mathrm{cosec}^{-1}(x) \quad x \in (-\infty,-1] \cup [1,\infty) \\
\sec^{-1}(-x) &= \pi - \sec^{-1}(x) \quad x \in (-\infty,-1] \cup [1,\infty) \\
\cot^{-1}(-x) &= \pi - \cot^{-1}(x) \quad x \in \mathrm{R}
\end{aligned}
$$

$$
\begin{aligned}
\sin^{-1}(\frac{1}{x}) &= \mathrm{cosec}^{-1}(x) \quad x \in (-\infty,-1] \cup [1,\infty) \\
\cos^{-1}(\frac{1}{x}) &= \sec^{-1}(x) \quad x \in (-\infty,-1] \cup [1,\infty) \\
\tan^{-1}(1/x) &= \begin{cases} \cot^{-1}(x), & x > 0 \\ \cot^{-1}(x) - \pi, & x < 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\sin^{-1}(x) + \cos^{-1}(x) &= (\pi/2) \\
\tan^{-1}(x) + \cot^{-1}(x) &= (\pi/2) \\
\sec^{-1}(x) + \mathrm{cosec}^{-1}(x) &= (\pi/2)
\end{aligned}
$$

$$
\begin{aligned}
\tan^{-1}(x) + \tan^{-1}(y) &= \tan^{-1}\frac{x+y}{1-xy} \\
\tan^{-1}(x) - \tan^{-1}(y) &= \tan^{-1}\frac{x-y}{1+xy} \\
2\tan^{-1}(x) &= \begin{cases} 2\sin^{-1}\frac{2x}{1+x^2}, & x > 0 \\ \cos^{-1}\frac{1-x^2}{1+x^2}, & x < 0 \end{cases}
\end{aligned}
$$

9.2 Expressions and suggested substitutions

Expression	Substitution
$a^2 + x^2$	$x = a\tan\theta$ *or* $x = a\cot\theta$
$a^2 - x^2$	$x = a\sin\theta$ *or* $x = a\cos\theta$
$x^2 - a^2$	$x = a\text{cosec}\theta$ *or* $x = a\sec\theta$
$\sqrt{\dfrac{a-x}{a+x}}$ *or* $\sqrt{\dfrac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$ *or* $\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}$	$x = a^2\cos 2\theta$

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10 Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \text{cosech(x)} = \frac{2}{e^x - e^{-x}}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{sech(x)} = \frac{2}{e^x + e^{-x}}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

10.1 Properties of Hyperbolic Functions

$$\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= 1 \\
\cosh^2(x) + \sinh^2(x) &= \cosh(2x) \\
1 - \tanh^2(x) &= \mathrm{sech}^2(x) \\
\coth^2(x) - 1 &= \mathrm{cosech}^2(x) \\
\sinh(2x) &= 2\sinh(x)\cosh(x) \\
\sin(ix) &= \frac{e^{ix} - e^{-ix}}{2} \\
\cos(ix) &= \frac{e^{ix} + e^{-ix}}{2} \\
\sin(ix) &= i\sinh(x) \\
\cos(ix) &= \cosh(x) \\
\tan(ix) &= i\tanh(x)
\end{aligned}$$

****

11 Linear Inequalities

Inequalities of the form:

$$ax + b < 0$$

$$ax + b > 0$$

$$ax + b \leq 0$$

$$ax + b \geq 0$$

are known as linear inequalities.

1. Add and subtract any number k without change in the inequality on both sides.
2. When multiplying or dividing by constant k, reverse the sign of the inequality only when k is negative.
3. Linear inequalities have infinite solution sets, these can be obtained by isolating the variable and solving them in a manner similar to solving equations.

Inequality	Representation	Graph
$a < x < b$	(a,b)	
$a \leq x \leq b$	$[a, b]$	
$a < x \leq b$	$(a, b]$	
$a \leq x < b$	$[a, b)$	
$x > a$	(a, ∞)	
$x \geq a$	$[a, \infty)$	
$x < b$	$(-\infty, b)$	
$x \leq b$	$(-\infty, b]$	

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12 Limits and Differentiation

12.1 Fundamental principles of limits

1. The limit may exist at a point even if the function is undefined.
2. If a function $f(x)$ is defined at a point a i.e. $f(a)$ exists, it is not necessary that the limit at a must exist. Moreover, even if the limit exists it need not be equal to $f(a)$.
3. **Indeterminate forms:** Any function assuming either of these forms is said to be indeterminate - $\frac{0}{0}, 0.\infty, \frac{\infty}{\infty}, \infty - \infty, 0^0, 1^\infty, \infty^0$.

12.2 Properties

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x).$$
$$\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x). \lim_{x\to a} g(x).$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{provided } \lim_{x\to a} g(x) \neq 0.$$

12.3 Standard Limits

$$\lim_{x\to 0} \sin x = 0 \tag{54}$$
$$\lim_{x\to 0} \tan x = 0 \tag{55}$$
$$\lim_{x\to 0} \cos x = 1 \tag{56}$$
$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{57}$$
$$\lim_{x\to 0} \frac{\tan x}{x} = 1 \tag{58}$$
$$\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 \tag{59}$$
$$\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1 \tag{60}$$
$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a \quad \text{where } \ a > 0 \tag{61}$$
$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \tag{62}$$
$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e \quad \text{where } \ a > 0 \ \text{ and } \ a \neq 0 \tag{63}$$

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1 \tag{64}$$
$$\lim_{x\to 0} \frac{\log x}{x^m} = 0 \quad \text{where } \ m > 0 \tag{65}$$
$$\lim_{x\to 0} \frac{(1+x)^m - 1}{x} = m \tag{66}$$
$$\lim_{x\to 0} \frac{x^n - a^n}{x - a} = n.a^{n-1} \tag{67}$$

12.4 Dealing with indeterminate forms

If $f(x)$ and $g(x)$ are two functions such that $f(x) \to 0$ as $x \to a$ and $g(x) \to \infty$ as $x \to a$ such that $[1 + f(x)]^{g(x)}$ assumes the form 1^∞ then:

$$\lim_{x\to a} [1 + f(x)]^{g(x)} = e^{\lim_{x\to a} f(x).g(x)}$$

12.5 L'Hospital's Rule

L'Hospital's rule states that for functions $f(x)$ and $g(x)$ which are differentiable on an open interval I except possibly at a point c contained in I, if:

$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$ and $g^{'}(x) \neq 0$ for all x in I with $x \neq c$ and $\lim_{x\to a} \frac{f^{'}(x)}{g^{'}(x)}$ exists then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^{'}(x)}{g^{'}(x)}$$

12.6 Differentiation Formulae

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$$

$$\frac{d}{dx} f(x).g(x) = \frac{d}{dx} f(x). \frac{d}{dx} g(x)$$

$$\frac{dc}{dx} = 0 \quad \text{where c is a real constant}$$

$$\frac{dx^n}{dx} = nx^{n-1} \quad \text{where n is any real number}$$

$$\frac{de^x}{dx} = e^x$$

$$\frac{da^x}{dx} = a^x \log_e a$$

$$\frac{d}{dx} \log_a x = \frac{1}{x. \log a}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

12.7 Differentiation of Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\mathrm{cosec} x = -\mathrm{cosec} x.\cot x$$

$$\frac{d}{dx}\sec x = \sec x.\tan x$$

$$\frac{d}{dx}\cot x = -\mathrm{cosec}^2 x$$

12.8 Differentiation of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\mathrm{cosec}^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

12.9 Differentiation of Hyperbolic Functions

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = \mathrm{sech}^2 x$$

$$\frac{d}{dx}\mathrm{cosech} x = -\mathrm{cosech} x.\coth x$$

$$\frac{d}{dx}\mathrm{sech} x = -\mathrm{sech} x.\tanh x$$

$$\frac{d}{dx}\coth x = -\mathrm{cosech}^2 x$$

12.10 Product Rule

$$\frac{d}{dx}.uv = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

12.11 Quotient Rule

$$\frac{d}{dx}.\frac{u}{v} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

12.12 Derivative of Inverse of a function

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

12.13 Chain Rule of Differentiation (Derivative of composite function)

If $u = \phi(x)$ and $y = \phi(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

12.14 Differentiation of Parametric Functions

Parametric Function: Consider two variables x and y which can be expressed in terms of another variable t, this t is termed as the parameter. Hence both the variables can be expressed in terms of a third variable and the relation between them is known as a parametric function. Therefore if $x = f(t)$ and $y = g(t)$ then:

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$\text{But} \quad \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

.

Therefore,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

12.15 Logarithmic Differentiation

If $y = u^v$ where both u and v are functions of variable x.

Taking log on both sides,and differentiating w.r.t x,

$\log y = v \log u$

$$\frac{1}{y}.\frac{dy}{dx} = v.\frac{d}{dx}\log u + \log u.\frac{dv}{dx}$$

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13 Indefinite Integrals

Evaluation of indefinite integrals can be accomplished using one of the possible methods:

- Decomposition of given integral into sum of integrals which can be reduced using standard formulas.
- Integration by substitution
- Integration by parts
- Integration by successive reduction

13.1 Integration by substitution

If $f(x)$ is a function such that x can be substituted as $x = \phi(t)$, then:

$$\int f(x).dx = \int f(\phi(t)).\phi'(t).dt \quad \text{where } x = \phi(t)$$

Proof:

$$v = \int f(x).dx = \int f(\phi(t)) \text{ substituting } x = \phi(t)$$
$$\frac{dv}{dx} = f(x)$$
$$\frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt}$$
$$\Rightarrow \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = f(x).\frac{dx}{dt}$$

Integrating both sides w.r.t we get t

$$v = \int f(x).\frac{dx}{dt} = \int f(\phi(t)).\phi'(t).dt for\ x = \phi(t)$$

13.1.1 $\int \frac{\mathbf{f'(x)}}{\mathbf{f(x)}}.\mathbf{dx} = \log \mathbf{f(x)}$

Proof:

Let $f(x) = t$

$$f'(x) = \frac{dt}{dx}$$
$$f'(x).dx = dt$$
$$\therefore \int \frac{f'(x)}{f(x)}dx = \int \frac{dt}{t} = \log t = \log f(x)$$

Integral of a fraction whose numerator is the derivative of the denominator is equal to the logarithm of the denominator.

13.1.2 $\int [\mathbf{f(x)}]^{\mathbf{n}}\mathbf{f'(x).dx} = \frac{[\mathbf{f(x)}]^{\mathbf{n+1}}}{\mathbf{n+1}}$, such that n $\neq$ 0

Proof:

Let $f(x) = t$

$f'(x)dx = dt$

$$\therefore \int [f(x)]^n.f'(x)dx = \int t^n.dt = \frac{t^{n+1}}{n+1} = \frac{[f(x)]^{n+1}}{n+1}$$

for n $\neq -1$

13.1.3 $\int \mathbf{f'(ax+b)dx} = \frac{\mathbf{f(ax+b)}}{\mathbf{a}}$

Proof:

Let $ax + b = t$

$adx = dt \implies$

$$\therefore \int [f(x)]^n.f'(x)dx = \int t^n.dt = \frac{t^{n+1}}{n+1} = \frac{[f(x)]^{n+1}}{n+1}$$

for n $\neq -1$

Integral of a function f(ax+b) is of the same form as that of f(x), divided by the coefficient of x.

Summary

$$\int \frac{f'(x)}{f(x)}.dx = \log f(x)$$
$$\int [f(x)]^n f'(x).dx = \frac{[f(x)]^{n+1}}{n+1}$$
$$\int f'(ax+b)dx = \frac{1}{a}.f(ax+b)$$

13.2 Integration of Trigonometric Functions

$$\int \sin x.dx = -\cos x + c$$

$$\int \cos x.dx = \sin x + c$$

$$\int \tan x.dx = -\log|\cos x| + c = \log|\sec x| + c$$

$$\int \cot x.dx = \log|\sin x| + c = -\log|\text{cosec} x| + c$$

$$\int \sec x.dx = \log|\sec x + \tan x| + c = \log\tan\left[\frac{\pi}{4} + \frac{x}{2}\right]$$

$$\int \text{cosec} x.dx = \log|\text{cosec} x - \cot x| + c = \log\tan\frac{x}{2}$$

13.3 Standard Integrals

$$\int \frac{1}{a^2+x^2}.dx = \frac{1}{a}.\tan^{-1}\frac{x}{a} + c = -\frac{1}{a}.\cot^{-1}\frac{x}{a} + c$$

$$\int \frac{1}{x^2-a^2}.dx = \frac{1}{2a}.\log\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{1}{a^2-x^2}.dx = \frac{1}{2a}.\log\left|\frac{a+x}{a-x}\right| + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}.dx = \sin^{-1}\frac{x}{a} + c = -\cos^{-1}\frac{x}{a}$$

$$\int \frac{1}{\sqrt{a^2+x^2}}.dx = \sinh^{-1}\frac{x}{a} + c = \log\frac{x+\sqrt{x^2+a^2}}{a}$$

$$\int \frac{1}{\sqrt{x^2-a^2}}.dx = \cosh^{-1}\frac{x}{a} + c = \log\frac{x+\sqrt{x^2-a^2}}{a}$$

$$\int \sqrt{a^2-x^2}.dx = \frac{1}{2}x.\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$\int \sqrt{a^2+x^2}.dx = \frac{1}{2}x.\sqrt{a^2+x^2} + \frac{a^2}{2}\log\frac{x+\sqrt{x^2+a^2}}{a} + c$$

$$\int \sqrt{x^2-a^2}.dx = \frac{1}{2}x.\sqrt{x^2-a^2} - \frac{1}{2}.a^2.\log\frac{x+\sqrt{x^2-a^2}}{a}$$

Corollary:

$$\sinh^{-1}\frac{x}{a} = \log\frac{x+\sqrt{x^2+a^2}}{a}$$

$$\cosh^{-1}\frac{x}{a} = \log\frac{x+\sqrt{x^2-a^2}}{a}$$

13.4 Integration by Parts

If u and v are two functions of x then:

$$\int u.v.dx = u.\int v.dx - \int \frac{du}{dx}.\left[\int v.dx\right].dx$$

If no second function is available unity is taken as the second function.

Integral of Product of two functions = First Function x Integral of Second Function - Integral of [Differential of First function x Integral of Second Function]

Criteria for choosing first and second function : ***ILATE***

- *I:Inverse Trigonometric Function*
- *L:Logarithmic Function*
- *A:Algebraic Function*
- *T:Trigonometric Function*
- *E:Exponential Function*

13.5 Bernoulli's Theorem

$\displaystyle\int u.v.dx \rightarrow u$ is an algebraic function which becomes zero after differentiating for finite steps then:

$$\int u.v.dx = u \int v.dx - u' \int \int v.dx + u'' \int \int \int v.dx............$$

Example:

$$\int x^2 \cos x.dx = x^2 \int cosx - 2.x \int \int \cos x + 2 \int \int \int cosx$$
$$x^2.sinx + 2x \cos x - 2 \sin x$$

13.6 Evaluation of Integrals using e^x

13.6.1 $\int \mathbf{e^x[f(x) + f'(x)].dx}$

Integrating by parts,

$$\int e^x f(x).dx = e^x f(x) - \int e^x f'(x)dx$$

Now $\displaystyle\int e^x[f(x) + f'(x)].dx = \int e^x f(x).dx + \int e^x f'(x)dx$

We get

$$\int \mathbf{e^x[f(x) + f'(x)].dx = e^x.f(x)}$$

13.6.2 $\int \mathbf{e^{ax}} \cos(\mathbf{bx + c})$

$$\int e^{ax} \cos(bx + c).dx = \frac{e^{ax}}{a^2+b^2}.(a\cos(bx + c) + b\sin(bx + c)) = \frac{e^{ax}\cos\left(bx+c-\tan^{-1}\frac{b}{a}\right)}{\sqrt{a^2+b^2}}$$

13.6.3 $\int \mathbf{e^{ax}} \sin(\mathbf{bx + c})$

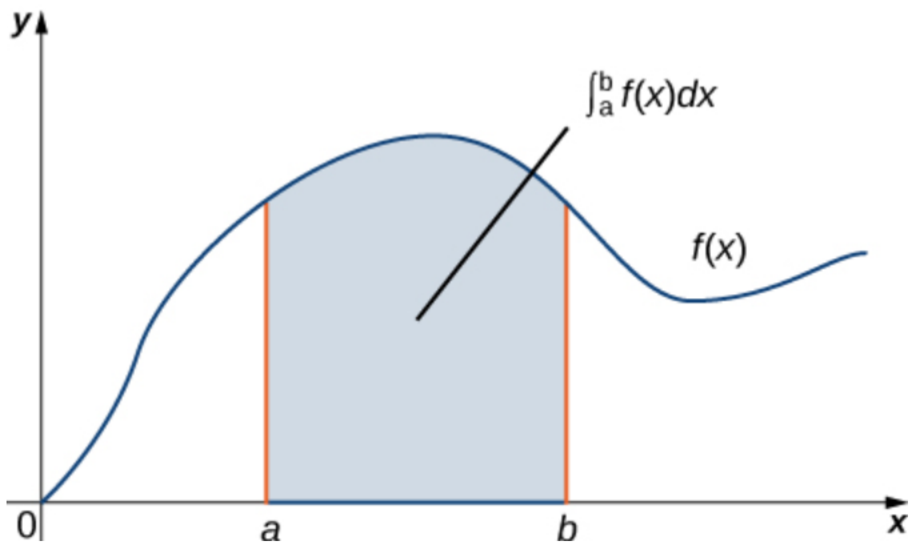
$$\int e^{ax} \sin(bx + c).dx = \frac{e^{ax}}{a^2+b^2}.(a\sin(bx + c) - b\cos(bx + c)) = \frac{e^{ax}\sin\left(bx+c-\tan^{-1}\frac{b}{a}\right)}{\sqrt{a^2+b^2}}$$

14 Definite Integrals

Property I : First Fundamental Theorem of Calculus

Let f be a continuous function on an interval $[a, b]$ and $A(x)$ be the area function.

$$\text{Area Function A(x)} = \int_a^b f(x).dx$$



Property II : Second Fundamental Theorem of Calculus

$$\int_a^b f(x).dx = [F(x)]_a^b = F(b) - F(a)$$

$$\int_a^b f(x).dx = -\int_b^a f(x).dx \tag{68}$$

$$\int_a^b f(x).dx = \int_a^c f(x).dx + \int_c^b f(x).dx \quad \text{where } a < c < b \tag{69}$$

$$\int_0^a f(x).dx = \int_0^a f(a-x).dx \tag{70}$$

$$\int_{-a}^a f(x).dx = \begin{cases} 2.\int_0^a f(x).dx \quad \text{if } f(x) = f(-x) \\ \\ 0 \quad if\ f(x) = -f(-x) \end{cases} \tag{71}$$

$$\int_0^{2a} f(x).dx = \begin{cases} 2.\int_0^a f(x).dx \quad if\ f(2a-x) = f(x) \\ \\ 0 \quad if\ f(2a-x) = -f(x) \end{cases} \tag{72}$$

Mean value of a function in an interval (a, b) is given by:

$$\frac{1}{b-a}\int_a^b f(x).dx$$

14.1 Evaluation of Infinite/Improper Integrals

Consider a definite integral $\int_a^b f(x).dx$ - the limits $[a, b]$ are finite and $f(x)$ exists at every point $c \in [a, b]$. The indefinite integral is defined as:

Type I

$$\int_a^{\infty} f(x).dx \text{ or } \int_{-\infty}^b f(x).dx$$

Evaluate $\lim\limits_{t\to\infty}\int_a^t f(x).dx$ if it exists and is finite then:

$$\int_a^{\infty} f(x).dx = \lim_{t\to\infty}\int_a^t f(x).dx$$

Evaluate $\lim\limits_{t\to-\infty}\int_t^b f(x).dx$ if it exists and is finite then:

$$\int_{-\infty}^b f(x).dx = \lim_{t\to-\infty}\int_t^b f(x).dx$$

Type II $f(x) \to \infty$ as $x \to a$ at no other point except at a.

Evaluate $\lim\limits_{h\to0}\int_{a+h}^b f(x).dx$ for $f(x) \to \infty$ as $x \to a$ if it exists and is finite then:

$$\int_a^b f(x).dx = \lim_{h\to0}\int_{a+h}^b f(x).dx$$

Evaluate $\lim_{h \to 0} \int_a^{b-h} f(x).dx$ for $f(x) \to \infty$ as $x \to b$ if it exists and is finite then:

$$\int_a^b f(x).dx = \lim_{h \to 0} \int_a^{b-h} f(x).dx$$

If $f(x) \to \infty$ at some point $c \in [a, b]$ then:

$$\int_a^b f(x).dx = \int_a^c f(x).dx + \int_c^b f(x).dx$$

$$\int_{-\infty}^{\infty} f(x).dx = \int_{-\infty}^{a} f(x).dx + \int_a^{\infty} f(x).dx$$

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