Basic Engineering Mathematics

DRAFT

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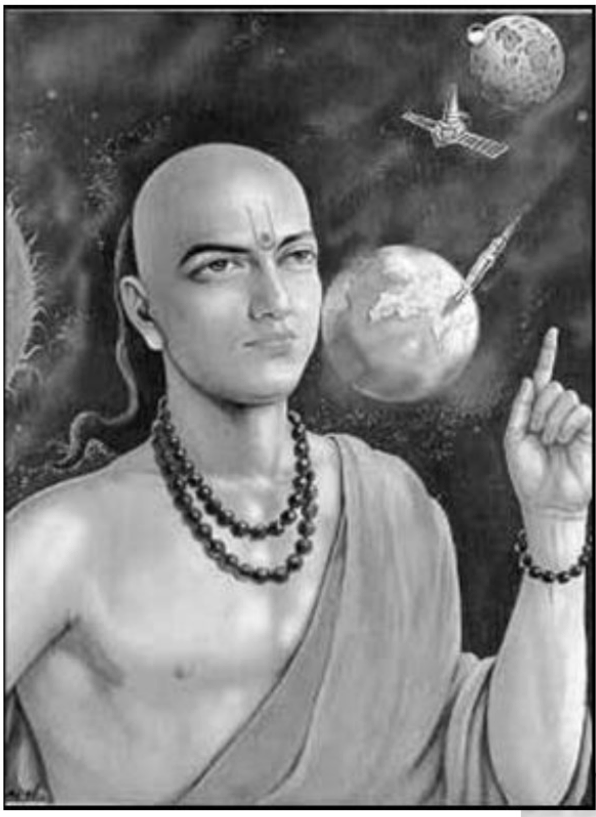

"Dedicated to the mathematicians of India, who gifted the world an abundance of knowledge and a spirit of enquiry".

1 Basic Algebra

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \tag{2}$$

$$(a+b)(a-b) = a^2 - b^2 \tag{3}$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b) \tag{4}$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a-b) \tag{5}$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \tag{6}$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \tag{7}$$

$$a^4 - b^4 = (a+b)(a-b)(a^2 + b^2) \tag{8}$$

$$a^2 + b^2 = (a+b)^2 - 2ab \tag{9}$$

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac) \tag{10}$$

1.1 Law of Negatives

$$\begin{aligned} -(-a) &= a \\ (-a)(-b) &= ab \\ -ab = (-a)(b) = a(-b) &= -(-a)(-b) \\ a - b &= a + (-b) \end{aligned}$$

1.2 Law of Quotients

$$\begin{aligned} -\frac{a}{b} = \frac{(-a)}{b} = \frac{a}{(-b)} &= -\frac{(-a)}{(-b)} \\ \frac{a}{b} = \frac{c}{d} \quad \text{if} \quad ad &= bc \\ \frac{a}{b} &= \frac{ka}{kb} \end{aligned}$$

2 Quadratic Equations

The quadratic equation

$$ax^2 + bx + c = 0 \tag{11}$$

where a, b and c are constants and $a \neq 0$, has two solutions for the variable x:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{12}$$

If the *discriminant* Δ with

$$\Delta = b^2 - 4ac = 0 \tag{13}$$

then the equation (11) has real and equal roots given by:

$$x = \frac{-b}{2a} \tag{14}$$

Sum of the roots:

$$x1 + x2 = \frac{-b}{a} \tag{15}$$

Product of the roots:

$$x1.x2 = \frac{c}{a} \tag{16}$$

3 Logarithmic Functions

For any two variables x and y related as:

$$y = a^x \tag{17}$$

Where a is a constant. We define the logarithmic function as:

$$\log_a y = x \tag{18}$$

3.1 Properties of Logarithmic Functions

$$\log_a mn = \log_a m + \log_a n \tag{19}$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n \tag{20}$$

$$\log_a m^n = n. \log_a m \tag{21}$$

$$\log_a a = 1 \tag{22}$$

$$\log_a 1 = 0 \tag{23}$$

$$\log_a b = \frac{1}{\log_b a} \tag{24}$$

$$a^x = e^{x. \log a} \tag{25}$$

$$\log_a X = \frac{\ln X}{\ln a} \tag{26}$$

4 Exponential Functions

$$a^m.a^n = a^{m+n} \tag{27}$$

$$\frac{a^m}{a^n} = a^{m-n} \tag{28}$$

$$(a^m)^n = a^{mn} \tag{29}$$

$$a^m = a^n \implies m = n \tag{30}$$

$$a^{-n} = \frac{1}{a^n} \qquad \textit{(for x being any non zero real number)} \tag{31}$$

$$a^0 = 1 \tag{32}$$

5 Complex Numbers

A number of the form $z = x + iy$ consisting of a real part x, denoted by $Re(z)$ and imaginary part y, denoted by $Im(z)$ is known as a complex number.

$$x = r\cos\theta \tag{33}$$

$$y = r\sin\theta \tag{34}$$

$$r = \sqrt{x^2 + y^2} \tag{35}$$

$$\tan\theta = \frac{y}{x} \implies \theta = \tan^{-1}\frac{y}{x} \tag{36}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{37}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{38}$$

5.1 De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \tag{39}$$

6 Permutations and Combinations

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)(n-3).....(n-(r+1))}{r!} \tag{40}$$

Note that:

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad 0! = 1$$

6.1 Binomial Theorem

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \binom{n}{3}x^{n-3}a^3 + \binom{n}{4}x^{n-4}a^4 + ...a^n \tag{41}$$

6.2 Summation of Series

$$\sum_{n=1}^{\infty} n = 1+2+3...... = \frac{(n)(n+1)}{2} \tag{42}$$

$$\sum_{n=1}^{\infty} n^2 = 1^2+2^2+3^2...... = \frac{(n)(n+1)(2n+1)}{6} \tag{43}$$

$$\sum_{n=1}^{\infty} n^3 = 1^3+2^3+3^3...... = \frac{(n)^2(n+1)^2}{4} \tag{44}$$

7 Trigonometry

$$\sin\theta = \frac{Perpendicular}{Hypotenuse} \qquad \cos\theta = \frac{Base}{Hypotenuse} \qquad \tan\theta = \frac{Perpendicular}{Base}$$

$$\text{cosec}\theta = \frac{Hypotenuse}{Perpendicular} \qquad \sec\theta = \frac{Hypotenuse}{Base} \qquad \cot\theta = \frac{Base}{Perpendicular}$$

From the above relations it turns out that,

$$\sin\theta.\text{cosec}\theta = 1$$
$$\cos\theta.\sec\theta = 1$$
$$\tan\theta.\cot\theta = 1$$

7.1 Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{45}$$
$$1 + \sec^2\theta = \tan^2\theta \tag{46}$$
$$1 + \text{cosec}^2\theta = \cot^2\theta \tag{47}$$

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\text{cosec}\,\theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Figure 1: Most common trigonometric ratios and values

7.2 Trigonometric ratios of allied angles

- ***For n being an odd multiple of $\pi/2$ the function changes to its composite function.***
- ***For n being an even multiple of $\pi/2$ the function does not change to its composite, but may change sign.***

7.2.1 First Quadrant

$$\sin(2\pi+\theta)=\sin\theta \qquad \cos(2\pi+\theta)=\cos\theta$$
$$\text{cosec}(2\pi+\theta)=\text{cosec}\theta \qquad \sec(2\pi+\theta)=\sec\theta$$
$$\tan(2\pi+\theta)=\tan\theta \qquad \cot(2\pi+\theta)=\cot\theta$$

$$\sin(\pi/2-\theta)=\cos\theta \qquad \cos(\pi/2-\theta)=\sin\theta$$
$$\text{cosec}(\pi/2-\theta)=\sec\theta \qquad \sec(\pi/2-\theta)=\text{cosec}\theta$$
$$\tan(\pi/2-\theta)=\cot\theta \qquad \cot(\pi/2-\theta)=\tan\theta$$

7.2.2 Second Quadrant

$$\sin(\pi/2+\theta)=\cos\theta \qquad \cos(\pi/2+\theta)=-\sin\theta$$
$$\text{cosec}(\pi/2+\theta)=\sec\theta \qquad \sec(\pi/2+\theta)=-\text{cosec}\theta$$
$$\tan(\pi/2+\theta)=-\cot\theta \qquad \cot(\pi/2+\theta)=-\tan\theta$$

$$\sin(\pi-\theta)=\sin\theta \qquad \cos(\pi-\theta)=-\cos\theta$$
$$\text{cosec}(\pi-\theta)=\text{cosec}\theta \qquad \sec(\pi-\theta)=-\sec\theta$$
$$\tan(\pi-\theta)=-\tan\theta \qquad \cot(\pi-\theta)=-\cot\theta$$

7.2.3 Third Quadrant

$$\sin(\pi+\theta)=-\sin\theta \qquad \cos(\pi+\theta)=-\cos\theta$$
$$\text{cosec}(\pi+\theta)=\text{cosec}\theta \qquad \sec(\pi+\theta)=-\sec\theta$$
$$\tan(\pi+\theta)=\tan\theta \qquad \cot(\pi+\theta)=\cot\theta$$

$$\sin(3\pi/2-\theta)=-\cos\theta \qquad \cos(3\pi/2-\theta)=-\sin\theta$$
$$\text{cosec}(3\pi/2-\theta)=-\sec\theta \qquad \sec(3\pi/2-\theta)=-\text{cosec}\theta$$
$$\tan(3\pi/2-\theta)=\cot\theta \qquad \cot(3\pi/2-\theta)=\tan\theta$$

7.2.4 Fourth Quadrant

$$\sin(3\pi/2+\theta)=-\cos\theta \qquad \cos(3\pi/2+\theta)=\sin\theta$$
$$\text{cosec}(3\pi/2+\theta)=-\sec\theta \qquad \sec(3\pi/2+\theta)=\text{cosec}\theta$$
$$\tan(3\pi/2+\theta)=-\cot\theta \qquad \cot(3\pi/2+\theta)=-\tan\theta$$

$$\sin(2\pi-\theta)=-\sin\theta \qquad \cos(2\pi-\theta)=\cos\theta$$
$$\text{cosec}(2\pi-\theta)=-\text{cosec}\theta \qquad \sec(2\pi-\theta)=\sec\theta$$
$$\tan(2\pi-\theta)=-\tan\theta \qquad \cot(2\pi-\theta)=-\cot\theta$$

- ***The values for (2π-θ) and (-θ) are identical.***

7.3 Compound Angle Formulae

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$
$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

7.4 Sum or Difference Formulae

$$\sin(A+B) + \sin(A-B) = 2\sin(A)\cos(B)$$
$$\sin(A-B) - \sin(A-B) = 2\cos(A)\sin(B)$$
$$\cos(A+B) + \cos(A-B) = 2\cos(A)\cos(B)$$
$$\cos(A+B) - \cos(A-B) = -2\sin(A)\sin(B)$$

7.5 Half Angle Formulae

$$\sin(\frac{A}{2}) = \pm\sqrt{\frac{1-\cos(A)}{2}}$$
$$\cos(\frac{A}{2}) = \pm\sqrt{\frac{1+\cos(A)}{2}}$$
$$\tan(\frac{A}{2}) = \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}}$$

7.6 Sub Multiple Angle Formulae

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = cos^2(A) - sin^2(A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A)$$
$$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$
$$\sin(2A) = \frac{2\tan(A)}{1+\tan^2(A)}$$
$$\cos(2A) = \frac{1-\tan^2(A)}{1+\tan^2(A)}$$
$$\cos^2(A) = \frac{1+\cos(2A)}{2}$$
$$\sin^2(A) = \frac{1-\cos(2A)}{2}$$
$$\sin(A) = 2\sin(A/2)\cos(A/2)$$
$$\cos(A) = cos^2(A/2) - sin^2(A/2) = 2\cos^2(A/2) - 1 = 1 - 2\sin^2(A/2)$$

7.7 Triple Angle Formulae

$$\sin(3x) = 3\sin(x) - 4\sin^3(x)$$
$$\cos(3x) = 4\cos^3(x) - 3\cos(x)$$
$$\tan(3x) = \frac{3\tan(x) - \tan^3(x)}{1 - 3\tan^2(x)}$$
$$\sin(4x) = 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x)$$
$$\tan(4x) = \frac{4\tan(x) - 4\tan^3(x)}{1 - 6\tan^2(x) + \tan^4(x)}$$
$$\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$$

7.8 Product Formulae

If $A + B = C$ and $A - B = D$
Then $A = (C + D)/2$ and $B = (C - D)/2$, the product formulae are defined as

$$\sin(C) + \sin(D) = 2\sin\frac{(C+D)}{2}\cos\frac{(C-D)}{2}$$
$$\sin(C) - \sin(D) = 2\cos\frac{(C+D)}{2}\sin\frac{(C-D)}{2}$$
$$\cos(C) + \cos(D) = 2\cos\frac{(C+D)}{2}\cos\frac{(C-D)}{2}$$
$$\cos(C) - \cos(D) = -2\sin\frac{(C+D)}{2}\sin\frac{(C-D)}{2}$$

8 Inverse Trigonometric Functions

If $x = \sin\theta$ then we define the inverse trigonometric function as $\sin^{-1}(x) = \theta$, the same holds true for all other trigonometric functions.

8.1 Properties of Inverse Trigonometric Functions

$$\sin^{-1}(\sin\theta) = \theta \qquad \mathrm{cosec}^{-1}(\mathrm{cosec}\theta) = \theta$$
$$\cos^{-1}(\cos\theta) = \theta \qquad \sec^{-1}(\sec\theta) = \theta$$
$$\tan^{-1}(\tan\theta) = \theta \qquad \cot^{-1}(\cot\theta) = \theta$$

$$\sin(\sin^{-1}(x)) = x \qquad \mathrm{cosec}(\mathrm{cosec}^{-1}(x)) = x$$
$$\cos(\cos^{-1}(x)) = x \qquad \sec(\sec^{-1}(x)) = x$$
$$\tan(\tan^{-1}(x)) = x \qquad \cot(\cot^{-1}(x)) = x$$

Functions	Domain	Range
$\sin^{-1}x$	[-1,1]	[-π /2,π/2]
$\cos^{-1}x$	[-1,1]	[0, π]
$\tan^{-1}x$	R	[-π /2,π/2]
$\cot^{-1}x$	R	[0, π]
$\sec^{-1}x$	R-(-1,1)	[0, π] – [π/2]
$\mathrm{cosec}^{-1}x$	R-(-1,1)	[-π /2,π/2]-{0}

Figure 2: Domain and range of inverse trigonometric functions

$$\begin{aligned}
\sin^{-1}(x) &= \cos^{-1}\sqrt{1-x^2} &&= \tan^{-1}\frac{x}{\sqrt{1-x^2}} \\
\cos^{-1}(x) &= \sin^{-1}\sqrt{1-x^2} &&= \tan^{-1}\frac{\sqrt{1-x^2}}{x} \\
\tan^{-1}(x) &= \sin^{-1}\frac{x}{\sqrt{1+x^2}} &&= \cos^{-1}\frac{1}{\sqrt{1+x^2}}
\end{aligned}$$

$$
\begin{aligned}
\sin^{-1}(-x) &= -\sin^{-1}(x) \quad x \in [-1,1] \\
\cos^{-1}(-x) &= \pi - \cos^{-1}(x) \quad x \in [-1,1] \\
\tan^{-1}(-x) &= -\tan^{-1}(x) \quad x \in \mathrm{R} \\
\mathrm{cosec}^{-1}(-x) &= -\mathrm{cosec}^{-1}(x) \quad x \in (-\infty,-1] \cup [1,\infty) \\
\sec^{-1}(-x) &= \pi - \sec^{-1}(x) \quad x \in (-\infty,-1] \cup [1,\infty) \\
\cot^{-1}(-x) &= \pi - \cot^{-1}(x) \quad x \in \mathrm{R}
\end{aligned}
$$

$$
\begin{aligned}
\sin^{-1}(\frac{1}{x}) &= \mathrm{cosec}^{-1}(x) \quad x \in (-\infty,-1] \cup [1,\infty) \\
\cos^{-1}(\frac{1}{x}) &= \sec^{-1}(x) \quad x \in (-\infty,-1] \cup [1,\infty) \\
\tan^{-1}(1/x) &= \begin{cases} \cot^{-1}(x), & x > 0 \\ \cot^{-1}(x) - \pi, & x < 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\sin^{-1}(x) + \cos^{-1}(x) &= (\pi/2) \\
\tan^{-1}(x) + \cot^{-1}(x) &= (\pi/2) \\
\sec^{-1}(x) + \mathrm{cosec}^{-1}(x) &= (\pi/2)
\end{aligned}
$$

$$
\begin{aligned}
\tan^{-1}(x) + \tan^{-1}(y) &= \tan^{-1}\frac{x+y}{1-xy} \\
\tan^{-1}(x) - \tan^{-1}(y) &= \tan^{-1}\frac{x-y}{1+xy} \\
2\tan^{-1}(x) &= \begin{cases} 2\sin^{-1}\frac{2x}{1+x^2}, & x > 0 \\ \cos^{-1}\frac{1-x^2}{1+x^2}, & x < 0 \end{cases}
\end{aligned}
$$

8.2 Expressions and suggested substitutions

Expression	**Substitution**
$a^2 + x^2$	$x = a\tan\theta$ *or* $x = a\cot\theta$
$a^2 - x^2$	$x = a\sin\theta$ *or* $x = a\cos\theta$
$x^2 - a^2$	$x = a\mathrm{cosec}\theta$ *or* $x = a\sec\theta$
$\sqrt{\frac{a-x}{a+x}}$ *or* $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ *or* $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x = a^2\cos 2\theta$

9 Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^- x}{2} \qquad \text{cosech(x)} = \frac{2}{e^x - e^- x}$$
$$\cosh(x) = \frac{e^x + e^- x}{2} \qquad \text{sech(x)} = \frac{2}{e^x + e^- x}$$
$$\tanh(x) = \frac{e^x - e^- x}{e^x + e^- x} \qquad \coth(x) = \frac{e^x + e^- x}{e^x - e^- x}$$

9.1 Properties of Hyperbolic Functions

$$\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= 1 \\
\cosh^2(x) + \sinh^2(x) &= \cosh(2x) \\
1 - \tanh^2(x) &= \text{sech}^2(x) \\
\coth^2(x) - 1 &= \text{cosech}^2(x) \\
\sinh(2x) &= 2\sinh(x)\cosh(x) \\
\sin(ix) &= \frac{e^{ix} - e^{-ix}}{2} \\
\cos(ix) &= \frac{e^{ix} + e^{-ix}}{2} \\
\sin(ix) &= i\sinh(x) \\
\cos(ix) &= \cosh(x) \\
\tan(ix) &= i\tanh(x)
\end{aligned}$$