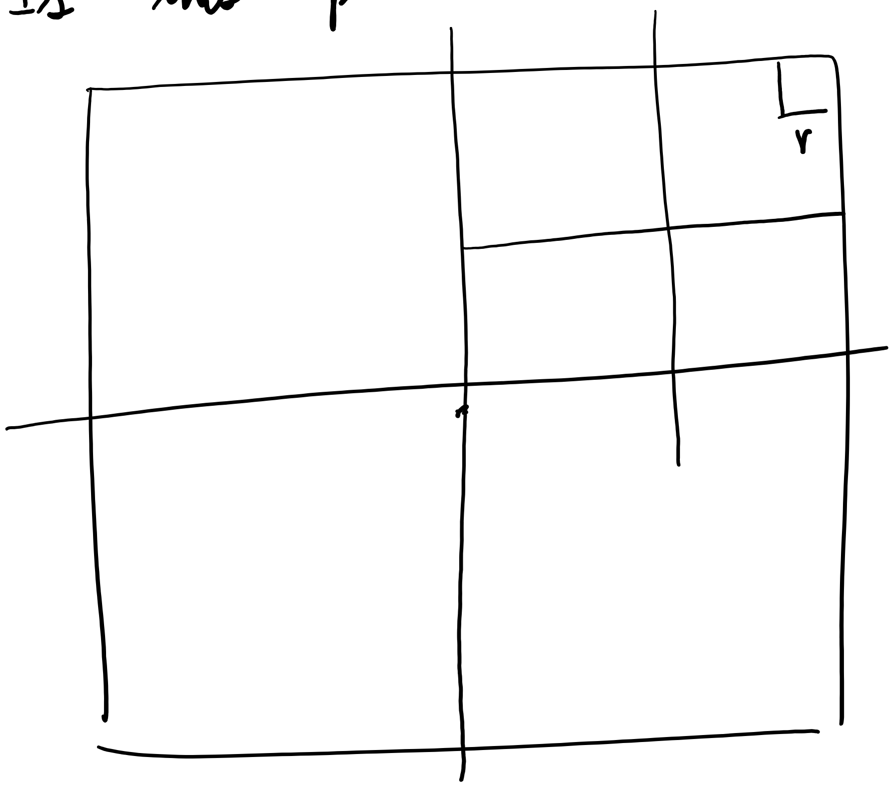
Elimination: Center of Gravity

Wednesday, January 29, 2025 7:32 AM

$x_0 + rB^n \subseteq K \subseteq RB^n$ Ellipsoid takes $O(n^2 \log \frac{R}{r})$ separation oracle queries.

Is this optimal?



R

Each cut eliminates $\leq \frac{1}{2}$

need $\Omega\left(\log_2 \frac{R^n}{r^n}\right)$

$= \Omega\left(n \log \frac{R}{r}\right)$

Cutting Plane Method

E, $x \in E$

$x \in K$? — YES ✓

— No — $H: \{y : \langle v, y-x \rangle \leq 0\}$

$\hat{E} \supseteq E \cap H$

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What family of E?

Start $E^{(0)} = [-R, R]^n$

Set $\hat{E} = E \cap H$ polyhedron

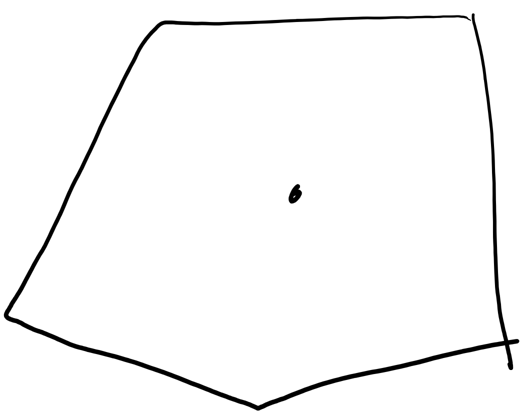
What about x?

$x^{(0)} = 0$. ✓

$\hat{x}$ = "Center"?

$\bar{x}$ = center of gravity of $\hat{E}$

$$= \frac{1}{\mathrm{vol}(\hat{E})} \int_{\hat{E}} x \, dx$$



Q. How to compute $\bar{x}$?

#P-hard!

sample $y^{(1)}, y^{(2)} \ldots y^{(N)} \in E$ uniformly at random.

Set $x = \frac{1}{N} \sum_{i=1}^{N} y^{(i)}$

$$\mathbb{E}(x) = \mathbb{E}(y^{(i)}) = \int x \cdot \frac{dx}{\mathrm{vol}(E)}$$

$\mathbb{E}(x) = \mathbb{E}(\ldots) \quad \int \ldots \, \mathrm{vol}(\ldots)$

$= \bar{x}$.

Lemma 1. [Grünbaum] Convex body K, H contains $z(K)$ center of gravity.

Then $\dfrac{\mathrm{vol}(K \cap H)}{\mathrm{vol}(K)} \geq \left(\dfrac{n}{n+1}\right)^n > \dfrac{1}{e}$.

Def: $f : \mathbb{R}^n \to \mathbb{R}_+$ is logconcave if $\log f$ is concave.

$\forall x, y \in \mathbb{R}^n, \ \forall t \in [0,1]$

$$f(tx + (1-t)y) \geq f(x)^t f(y)^{1-t}.$$

E.g. $f(x) = e^{-\frac{\|x\|^2}{2}}$, $e^{-g(x)}$ g is convex

$\mathbb{1}_K(x)$ K is convex

Thm. [Prekopa-Leindler; Dinghas]. f, g logconcave

fg, $\min\{f, g\}$, $f \circledast g$ logconcave

any marginal is logconcave.

If f is a density ($\int f = 1$) then

the dist. function $P_f(t) = P_f(x \leq t)$ is logconcave.

the dist. functn $P_f(t) = \Pr_f(x \le t)$ is ...

Lemma 2 For logconcave density p with mean 0 $\mathbb{E}_p(x)=0$

$$\Pr_p(x \ge 0) \ge \frac{1}{e}$$

if $\mathrm{Var}_p(x) = 1$

$$\forall t \ge 0 \quad \Pr_p(x \ge t) \ge \frac{1}{e} - t.$$

Lemma 3 $z = \frac{1}{N}\sum_{i=1}^{N} y^{(i)}$ $\quad y^{(i)} \sim \mathrm{unif}(K)$

$$\mathbb{E}(\|z - \bar{z}\|^2) \le \frac{1}{N}\mathbb{E}(\|y - \bar{y}\|^2) = \frac{1}{N}\mathrm{Tr}(\mathrm{cov}_K(y))$$

Thm! $z = \frac{1}{N}\sum_{i=1}^{N} y^{(i)}$ $\quad y^{(i)} \sim \mathrm{UNIF}(K)$

H : halfspace containing $z(K)$

$$\mathbb{E}(\mathrm{Vol}(K \cap H)) \ge \left(\frac{1}{e} - \sqrt{\frac{n}{N}}\right)\mathrm{vol}(K).$$

Cor! CPM with approx. center-of-gravity terminates in $O(n \log \frac{R}{r})$ expected queries.

Pf (1.1) First: symmetrize

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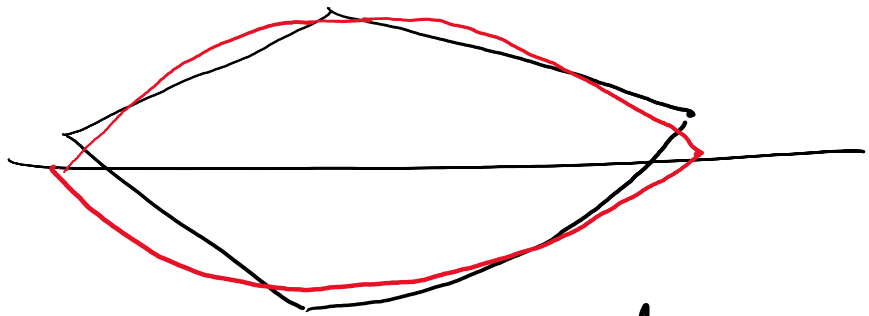
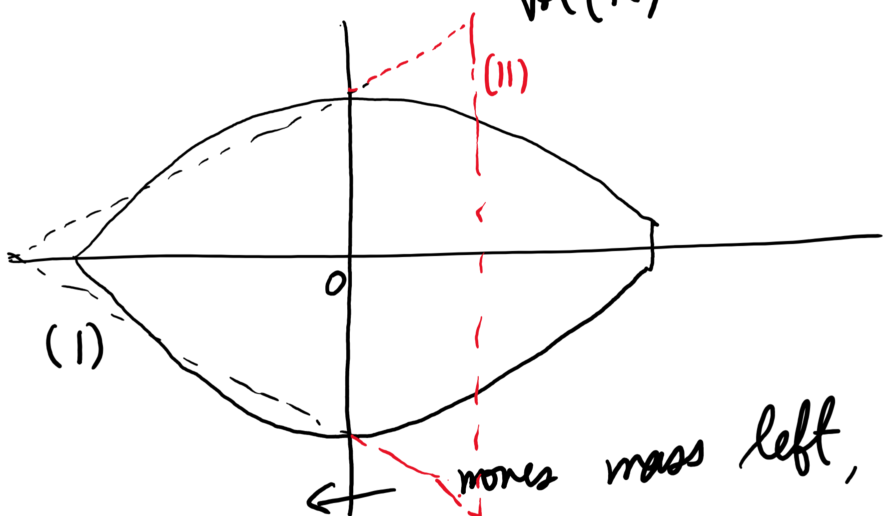


v: normal to H.

Replace each cross-section with a (n-1)-dim ball of same volume. $\frac{vol(K\cap H)}{vol(K)}$ is unchanged.



moves mass left, so $z(K) \leftarrow$.

Why? Because K is convex!

ie. $r(t)$ is a concave function.

Why?

Thm [Brunn-Minkowski] A, B measurable subsets of $\mathbb{R}^n$.

$$vol(A+B)^{\frac{1}{n}} \geq vol(A)^{\frac{1}{n}} + vol(B)^{\frac{1}{n}}$$

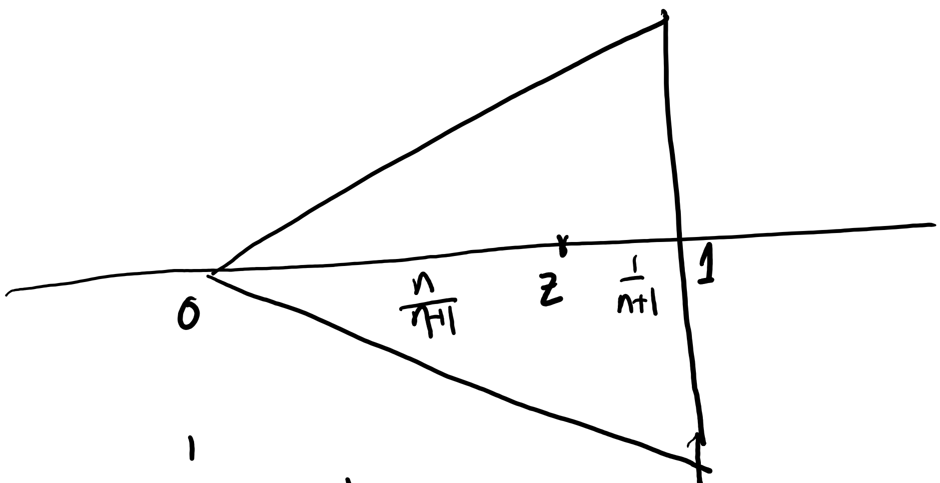
$$vol(tA+(1-t)B)^{\frac{1}{n}} \geq t\, vol(A)^{\frac{1}{n}} + (1-t) vol(B)^{\frac{1}{n}}$$

$$\mathrm{vol}\,(tA+(1-t)B)^{\frac{1}{n}} \geq t\,\mathrm{vol}(A)^{\frac{1}{n}}+(1-t)\,\mathrm{vol}(B)^{\frac{1}{n}}$$

$$\left(\Longleftrightarrow \ \forall\, t\in[0,1]\quad \mathrm{vol}\,(tA+(1-t)B)^{\frac{1}{n}} \geq t\,\mathrm{vol}(A)^{\frac{1}{n}}+(1-t)\,\mathrm{vol}(B)^{\frac{1}{n}}\right)$$

$$r(t)=\left(\frac{\mathrm{vol}(A(t))}{\mathrm{vol}(B^n)}\right)^{\frac{1}{n}}.$$

So we can assume K is a pointed rotational cone.



$$z=\frac{\int_0^1 t\cdot t^{n-1}\,dt}{\int_0^1 t^{n-1}\,dt}=\frac{\frac{1}{n+1}}{\frac{1}{n}}=\frac{n}{n+1}$$

$$\frac{\mathrm{vol}(K\cap H)}{\mathrm{vol}(K)}=\frac{\int_0^{\frac{n}{n+1}} t^{n-1}}{\int_0^1 t^{n-1}}=\frac{\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n}}{\frac{1}{n}}=\left(\frac{n}{n+1}\right)^n.$$

$\blacksquare$

Pf. (L2).

$$P(x)=\int_{-\infty}^{x} p(x)\,dx$$

$$\int^{M} x\,p(x) = 0 \qquad \text{(we will send } M\to\infty\text{)}$$

$$\int_{-M}^{M} x\,p(x) = 0 \qquad \text{(we will send } M \to \infty)$$

integration by parts

$$\Big[x\,p(x)\Big]_{-M}^{M} - \int_{-M}^{M} p(x) \quad \longrightarrow \quad \int_{-M}^{M} p(x) = M$$

$p(x)$ is logconcave, monotone increasing. $\quad p(x) \le p(0)\, e^{cx} \qquad c = \dfrac{p'(0)}{p(0)}$

$$\left(-\ln p(x) \geq -\ln(p(0)) - \frac{p'(0)}{p(0)} \cdot x \right)$$

$$M = \int_{-M}^{M} p(x) \le p(0) \int_{-\infty}^{\frac{1}{c}} e^{cx}\,dx + \int_{\frac{1}{c}}^{M} 1\,dx$$

$$M \le \frac{p(0)}{c}\, e + M - \frac{1}{c}$$

$$p(0) \geq \frac{1}{e}$$

A distribution D in $\mathbb{R}^n$ is <u>ISOTROPIC</u> if $\mathbb{E}_D(x) = 0$ and $\mathbb{E}_D(x x^T) = I$.

Fact: Any distribution with bounded second moments

Fact. Any distribution with bounded second moments (above and below) can be made isotropic by an affine transformation.

Pf. Suppose $\mathbb{E}_D(x) = a$ $\quad \mathbb{E}_D((x-a)(x-a)^T) = C$

covariance.

Then $y = x - a$ satisfies $\mathbb{E}_D(y) = 0$ $\quad \mathbb{E}_D(yy^T) = C$

Now $C \succ 0$, let $z = C^{-\frac{1}{2}} y$

Then $\mathbb{E}(z) = 0$ and $\mathbb{E}(zz^T) = \mathbb{E}(C^{-\frac{1}{2}} y y^T C^{-\frac{1}{2}})$

$= C^{-\frac{1}{2}} \mathbb{E}(yy^T) C^{-\frac{1}{2}} = C^{-\frac{1}{2}} C C^{-\frac{1}{2}}$

$= I.$

Lemma. For an isotropic 1-dim logconcave density p

$\max_x p(x) \le 1.$

Pf (L2 contd..) $t \geq 0$ $\quad \Pr(X \geq t) \geq \Pr(X \geq 0) - \Pr(0 \le X \le t)$

$\geq \frac{1}{e} - t \cdot \max p(x)$

$\geq \frac{1}{e} - t.$

Pf. (L3) $\mathbb{E}(\|z-\bar{z}\|^2) = \sum_{i=1}^{n} \mathrm{Var}(z_i)$

$$= \frac{1}{N}\sum_{i=1}^{n} \mathrm{Var}(y_i) = \frac{1}{N}\mathbb{E}(\|y-\bar{y}\|^2)$$

$$\mathbb{E}(\|y-\bar{y}\|^2) = \sum_{i=1}^{n} \mathrm{Cov}(y)_{ii} = \mathrm{Tr}(\mathrm{Cov}(y)).$$

Pf. (T). $\mathbb{E}\left(\frac{\mathrm{Vol}(K\cap H)}{\mathrm{Vol}(K)}\right) \geq \mathbb{E}\left(\frac{1}{e} - t\right) = \frac{1}{e} - \mathbb{E}(t).$

(apply isotropic transformation to K.)

$$\mathbb{E}(t^2) \leq \mathbb{E}(\|z-\bar{z}\|^2) \leq \frac{\mathrm{Tr}(\mathrm{Cov}(y))}{N} = \frac{n}{N}$$

$$\mathbb{E}(t) \leq \sqrt{\frac{n}{N}}. \qquad \blacksquare$$

Set $N = 10n$. $\Rightarrow$ # iteration $= O(n \log \frac{R}{r})$

WHP.
