Convergence

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$Q, \quad Q_t \qquad\qquad Q_t \to Q.$

Lazy : stay with prob. $\frac{1}{2}$.

Thm. Q is stationary, Markov chain is lazy
$\Rightarrow$ Q is unique stationary.

Rate

$d_{TV}(Q_t, Q)$ is not monotonic.

$$h_t(x) = \sup_{g, A} \int_{y \in A} g(y)\, dQ_t(y) - Q(A)$$

$\int g\, dQ = x$

$g : \Omega \to [0,1]$

Lemma. (1) If Q has no atoms,

$$h_t(x) = \sup_A Q_t(A) - Q(A)$$

$t \cdot \cdot \quad \cdot$

A

(2) $h_t(x)$ is concave in x

(3) $\forall t \geq 1 \qquad y = \min\{x, 1-x\}$

$$h_t(x) \leq \frac{1}{2} h_{t-1}(x - 2\phi y) + \frac{1}{2} h_{t-1}(x + 2\phi y)$$

<u>Thm</u>. $h_0(x) \leq C_0 + C_1 \min\{\sqrt{x}, \sqrt{1-x}\}$

$$\Rightarrow h_t(x) \leq C_0 + C_1 \min\{\sqrt{x}, \sqrt{1-x}\} \cdot \left(1 - \frac{\phi^2}{2}\right)^t.$$

Pf: Lemma (i) pick y with $\frac{dQ_t(y)}{dQ(y)}$ as large as possible till budget of x is exhausted.

(ii) $h_t(\lambda x + (1-\lambda) y) \geq \lambda h_t(x) + (1-\lambda) h_t(y)$

$$\qquad\qquad\qquad g_\lambda \qquad\qquad g_{1-\lambda}$$

take $g = \lambda g_\lambda + (1-\lambda) g_{1-\lambda} : \Omega \to [0,1]$

(iii) Let A be s.t. $Q(A) = x$

$$g_1(u) = \begin{cases} 2P_u(A) - 1 & u \in A \\ 0 & u \notin A \end{cases} \qquad g_2(u) = \begin{cases} 1 & u \in A \\ 2P_u(A) & u \notin A \end{cases}$$

Then $g_i(u) \in [0, 1]$ since $P_u(\{u\}) \geq \frac{1}{2}$

and $\frac{1}{2}(g_1(u) + g_2(u)) = P_u(A)$.

$\int g_1(u) dQ(u) = x_1$ and $\int g_2(u) dQ(u) = x_2$.

$$\frac{1}{2}(x_1 + x_2) = \frac{1}{2}\int (g_1(u) + g_2(u)) dQ(u) = \int P_u(A) dQ(u) = Q(A) = x.$$

Fix A to be the subset that achieves $h_t(x)$.

$$\begin{aligned} h_t(x) &= Q_t(A) - Q(A) \\ &= \int P_u(A) dQ_{t-1}(u) - x \\ &= \frac{1}{2}\int g_1(u)\, dQ_{t-1}(u) + \frac{1}{2}\int g_2(u) dQ_{t-1}(u) - \frac{1}{2}(x_1 + x_2) \\ &= \frac{1}{2}\left[\int g_1(u) dQ_{t-1}(u) - x_1 + \int g_2(u) dQ_{t-1}(u) - x_2\right] \end{aligned}$$

$$\leq \frac{1}{2} h_{t-1}(x_1) + \frac{1}{2} h_{t-1}(x_2).$$

Note

$$x_1 = \int g_1(u)\, dQ(u)$$

$$= \int_A (2P_u(A)-1)\, dQ(u)$$

$$= 2\int_A P_u(A)\, dQ(u) - Q(A)$$

$$= 2\int_A (1 - P_u(\Omega \setminus A))\, dQ(u) - Q(A)$$

$$= Q(A) - 2\int_A P_u(\Omega\setminus A)\, dQ(u)$$

$$= x - 2\phi(A)x \leq x(1-2\phi) \quad (\text{assuming } x \leq \tfrac{1}{2})$$

$$x_1 \leq x(1-2\phi) \leq x \leq x(1+2\phi) \leq x_2$$

$$\Rightarrow \quad h_t(x) \leq \frac{1}{2} h_{t-1}(x(1-2\phi)) + \frac{1}{2} h_{t-1}(x(1+2\phi)).$$

Pf (Thm). By induction. Assume $x \leq \frac{1}{2}$

$$h_t(x) \leq C + \frac{1}{2} C_1 \left(\sqrt{x(1-2\phi)} + \sqrt{x(1+2\phi)}\right)\left(1-\frac{\phi^2}{2}\right)^{t-1}$$

$$h_t(x) \le C_0 + \frac{1}{2} C_1 \left(\sqrt{x(1-2\phi)} + \sqrt{x(1+2\phi)}\right)\left(1-\frac{\phi^2}{2}\right)^{t-1}$$

$$\le C_0 + \frac{1}{2} C_1 \sqrt{x}\left(\sqrt{1-2\phi} + \sqrt{1+2\phi}\right)\left(1-\frac{\phi^2}{2}\right)$$

$$\le C_0 + \frac{1}{2} C_1 \sqrt{x} \cdot 2 \cdot \left(1-\frac{\phi^2}{2}\right) \cdot \left(1-\frac{\phi^2}{2}\right)^{t-1}$$

$$= C_0 + C_1 \sqrt{x}\left(1-\frac{\phi^2}{2}\right)^t .$$

$$\boxed{\sqrt{1-2a} + \sqrt{1+2a} \le 2\left(1-\frac{a^2}{2}\right).}$$

Cor. $X \sim Q_0$ s.t. $\forall A \quad \dfrac{Q_0(A)}{Q(A)} \le M.$

$$h_0(x) \le \min\{1, Mx\}$$

$$\le \sqrt{Mx}. \qquad C_1 = \sqrt{M}.$$

$$\Rightarrow h_t(x) \le \sqrt{Mx} \cdot \left(1-\frac{\phi^2}{2}\right)^t .$$

$$d_{TV}(Q_t, Q) \le \sqrt{M}\left(1-\frac{\phi^2}{2}\right)^t .$$