

Duality & Reductions

Monday, January 27, 2020 10:35 PM

(for

ORACLES for a convex set K

$\text{MEM}(x)$: YES if $x \in K$

NO O.W.

$\text{SEP}(x)$: YES if $x \in K$

NO O.W., and $c : c^T y \leq c^T x + y \in K$.

$\text{VAL}(c)$: OUTPUTS $\max_{x \in K} c^T x$

or "K is EMPTY"

$\text{OPT}(c)$: x s.t. $c^T x \leq c^T y + x \in K$
 $\in K$

or "K is EMPTY".

Cutting Plane Method : $\text{OPT}_K \rightarrow \text{SEP}_K$.

SEP is stronger than MEM

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OPT ————— VAL.

For different problems, different oracles can be more convenient / efficient.

E.g. $K = \{x : Ax \geq b\}$ SEP_K is easy - check all constraints
 $K = \text{Conv Hull } \{a_1, \dots, a_m\}$ OPT_K is easy - $\sup_i c^T a_i$

Q. Are these fundamentally equivalent?

ORACLES for convex functions

$\text{EVAL}_f(x) : f(x)$.

$\text{GRAD}_f(x) : f(x), g$ s.t. $\forall y \quad f(y) \geq f(x) + \bar{g}^T(y-x)$.

(\bar{g} is a subgradient of f at x).

Recall $\delta_K(x) = \begin{cases} 0 & x \in K \\ \infty & x \notin K \end{cases}$ convex.

$\text{MEM}_K = \delta_K$.

A useful (and important) concept.

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 Dual of a convex function $f: \mathbb{R}^n \rightarrow \mathbb{R}$.

$$f^*(\theta) = \sup_{x \in \mathbb{R}^n} \theta^T x - f(x) \quad \forall \theta \in \mathbb{R}^n.$$

Note: f^* is max of affine functions (one per x)
 so f^* is convex.

$$- f^*(0) = - \inf_x f(x).$$

$$\begin{aligned} - \delta_K^*(c) &= \sup_x c^T x - \delta_K(x) \\ &= \sup_{x \in K} c^T x \end{aligned} \quad \boxed{\text{EVAL}_{\delta_K^*} = \text{VAL}_K.}$$

Lemma $\nabla f^*(\theta) = \arg \max_x \theta^T x - f(x)$

Pf. $x_\theta = \arg \max_x \theta^T x - f(x).$

$$f^*(\theta) = \theta^T x_\theta - f(x_\theta)$$

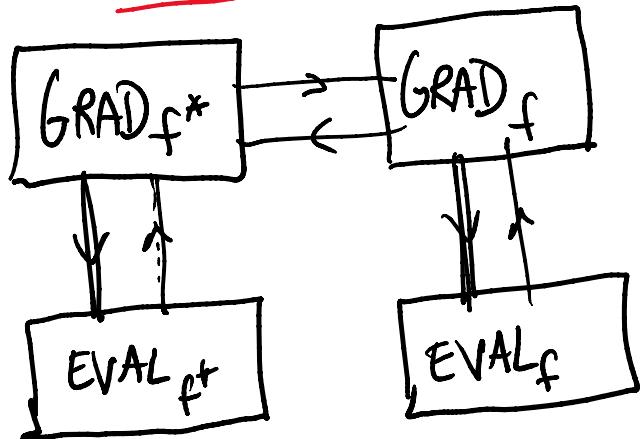
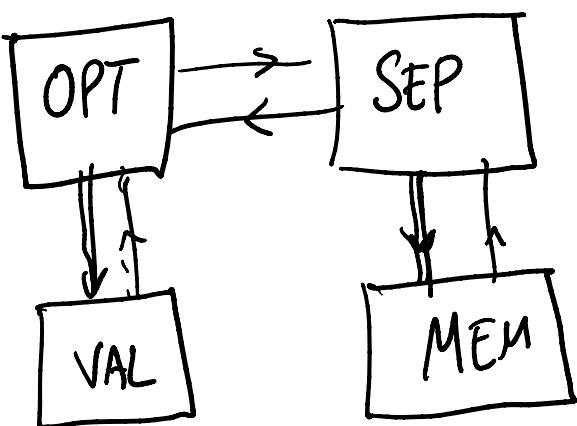
$$\forall \eta \quad f^*(\eta) \geq \eta^T x_\theta - f(x_\theta)$$

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$$\Rightarrow f^*(\eta) - f^*(\theta) \geq x_\theta^T (\eta - \theta)$$

$$\Rightarrow x_\theta \in \text{Subgrad.}(f^*).$$

$$\text{GRAD}_{g_k^*} \equiv \text{OPT}_k.$$



Thm. convex f , $\text{Epi}(f)$ closed, then $f^{**} = f$.

Pf. $\text{Epi}(f) = \{(x, t) : f(x) \leq t\}$ is a convex set.

So it is an intersection of halfspaces A .

We can assume of the form $(\theta, b) : \theta^T x \geq b$ $\forall x \in \text{Epi}(f)$.

(Why?: $\theta^T x + \alpha t \leq b$

but if $(x, t) \in \text{Epi}(f)$, then $\forall t' \geq t, (x, t') \in \text{Epi}(f)$.

so take $\bar{t} = \underset{t}{\operatorname{argmax}} \theta^T x + \alpha t \leq b$
 $(x, t) \in \text{Epi}(f)$

$$\text{So } \forall t \in \mathbb{R}^+ \quad \exists x_t \in \mathcal{X} \quad \theta^\top x_t \leq b - \alpha t$$

$$f(x) \geq \theta^\top x - b \quad \forall (\theta, b) \in \mathcal{H}$$

$$\text{fix } \theta. \quad b \geq \theta^\top x - f(x) \quad \forall x$$

$$b \geq \sup_x \theta^\top x - f(x) = f^*(\theta)$$

$$\Rightarrow f(x) = \sup_{(\theta, b)} \theta^\top x - b = \sup_\theta \theta^\top x - f^*(\theta) = f^{**}(x) \quad \square$$

Example

$$f(x) = \frac{1}{p} \sum x_i^p$$

$$f^*(x) = \frac{1}{q} \sum x_i^q$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$f(x) = ax - b$$

$$f^*(\theta) = \begin{cases} 0 & \theta = a \\ \infty & \text{otherwise} \end{cases}$$

$$\nabla f \iff \nabla f^*$$

$$h = h^{**}$$

$$\dots n^* \dots L^*(a)$$

$$\begin{aligned}
 \min_x g(x) + h(Ax) &= \min_x \max_{\theta} g(x) + \theta^T Ax - h^*(\theta) \\
 &\Rightarrow = \max_{\theta} \min_x g(x) + (A^T \theta)^T x - h^*(\theta) \\
 &= \max_{\theta} - \max_x (-A^T \theta)^T x - g(x) - h^*(\theta) \\
 &= - \min_{\theta} g^*(-A^T \theta) + h^*(\theta).
 \end{aligned}$$

Sion's minimax Theorem. $X \subset \mathbb{R}^n$ compact, convex set.

$Y \subset \mathbb{R}^m$ convex. $f: X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ st.

$f(x, \cdot)$ is upper semi-continuous and quasi-concave on Y
 $f(\cdot, y)$ is lower quasi convex on X .

Then

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y)$$

Example. SDP.

$$\begin{aligned}
 P \cdot v &= b \quad i=1, 2, \dots, m
 \end{aligned}$$

Primal: $\max_{X \geq 0} C \cdot X \quad A_i \cdot X = b_i \quad i=1, 2, \dots, m$

Dual: $\min_y b^T y \quad \sum_{i=1}^m y_i A_i \leq C$

X is $n \times n$ symmetric.

Primal takes $O(n^2(Z + n^4))$ Z : total #nnz in A_i .

Dual takes $O(m(Z + n^6 + m^2))$

better when $m < n^2$, often the case.

But how to recover primal solution from dual?
(we only solve to some error ϵ).

$$\min_b b^T y = \min_v v^T (\sum_i y_i A_i - C) \quad v \geq 0 \quad \forall v$$

When running the cutting plane method on DUAL,
... $v^T (\sum_i y_i A_i - C) \geq 0$

we get planes of the form $v^T(\sum_i y_i A_i - C)v \geq 0$
 Let S be the set of all such v . At the end,

$$\min_{\sum_i y_i A_i \leq C} b^T y \geq \min_{\substack{v \in S \\ v^T(\sum_i y_i A_i - C)v \geq 0}} b^T y$$

Now consider RHS.

$$\begin{aligned} \min_{\substack{i \\ v \in S}} b^T y &= \min_y \max_{\substack{\lambda_v \geq 0 \\ v \in S}} b^T y - \sum \lambda_v v^T (\sum_i y_i A_i - C) v \\ &= \max_{\lambda_v \geq 0} \min_y C \cdot \sum \lambda_v v v^T + b^T y - \sum_i y_i (A_i \cdot \sum_{v \in S} \lambda_v v v^T) \\ &= \max_{\substack{X \\ X = \sum \lambda_v v v^T, \lambda_v \geq 0}} C \cdot X + \sum_i y_i (b_i - A_i \cdot X) \\ &= \max_{X = \sum \lambda_v v v^T, \lambda_v \geq 0} C \cdot X \quad \text{(else the OPT is } -\infty) \end{aligned}$$

$$\lambda = \langle v^*, v^* \rangle$$

$$A_i \cdot X = b_i$$

This is $\max_{v \in S} \sum \lambda_v (v^T C v)$

$$\sum_v \lambda_v (v^T A_i v) = b_i, \lambda_v \geq 0$$

an LP!
