VOLUME

Monday, February 3, 2025 8:15 AM

Convex Body K $\quad x_0 + rB^n \subseteq K \subseteq RB^n$

Problem. Compute vol(K). Easy for △ □ ○

#P-hard, even for a polyhedron.

$\varepsilon > 0$ Find V s.t.

$$\mathrm{vol}(K) \le V \le (1+\varepsilon)\,\mathrm{vol}(K).$$

Ans? - Divide & Conquer? too many pieces.

Thm [BF'87] For any algorithm that uses n^a oracle queries and outputs A, B s.t. $A \le \mathrm{vol}(K) \le B$,

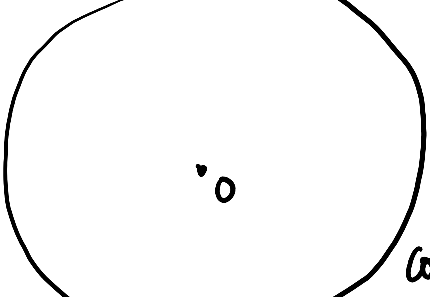
$\exists K$ s.t. $\quad \frac{B}{A} \ge \left(\frac{cn}{a \log n}\right)^{n/2}$

Thm [BF'88] $\quad (1+\varepsilon)^n$ error needs $\left(\frac{1}{\varepsilon}\right)^n$ queries.

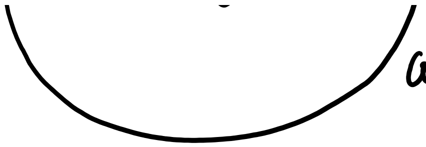
Pf [E'86: m queries $\Rightarrow \frac{2^n}{m}$ error]

$x_i \in B(0,1)$ ✓ YES

$\notin$ — NO



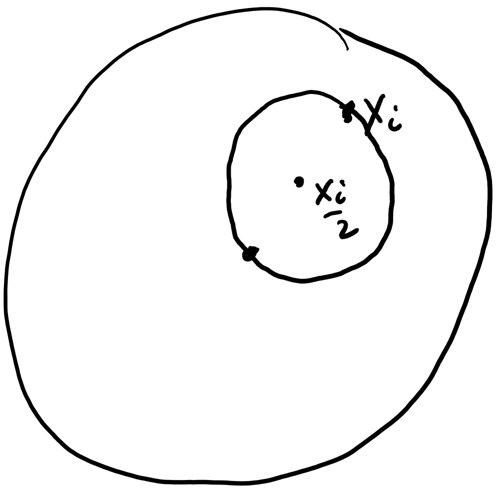
$\mathrm{conv}\{x_1 \ldots x_m\} \subseteq K \subseteq B(0,1)$



$\text{conv}\{x_1 \ldots x_m\} \subseteq K \subseteq B(0,1)$

Lemma. $\dfrac{\text{vol}(\text{conv}\{x_1 \ldots x_m\})}{\text{vol}(B)} \leq \dfrac{m}{2^n}$.

Pf. $B_i = B\left(\dfrac{x_i}{2}, \dfrac{\|x_i\|}{2}\right)$



$B_i \subseteq B$. $\cup B_i \subseteq B$.

$\text{vol}(\cup B_i) \leq m \, \text{vol}(B_i) \leq \dfrac{m}{2^n} \cdot \text{vol}(B)$.

claim. $\text{conv}\{x_1 \ldots x_m\} \subseteq \cup B_i$.

Suppose not. $\exists y \in \text{conv}\{x_1 \ldots x_m\}$ $\forall i$ $y \notin B_i$

$\forall i: \left\|y - \dfrac{x_i}{2}\right\| > \dfrac{\|x_i\|}{2} \Rightarrow \|y\|^2 > \langle y, x_i \rangle$

i.e. the plane $\langle y, x \rangle \leq \|y\|^2$ separates y from all x_i $\Rightarrow y \notin \text{conv}\{x_1 \ldots x_m\}$.

No efficient volume algorithm?

Thm [DFK'89]. $\exists$ Randomized Algorithm that estimates $\text{vol}(K)$ to within $(1+\varepsilon)$ for any $\varepsilon > 0$ w.p. $1-\delta$ using $\text{poly}\left(n, \log \dfrac{R}{r}, \dfrac{1}{\varepsilon}, \log \dfrac{1}{\delta}\right)$ queries and time.

$\text{poly}(n, \log \frac{R}{r}, \frac{1}{\varepsilon}, \log \frac{1}{\delta})$ queries are used.

Today: Volume $\longrightarrow$ centroid.

Thm. Volume can be computed using $O(n \log \frac{Rn}{r})$ centroid computations of convex bodies.

Algo. $r\square \subseteq K \subseteq R\square$.

$z(K) =$ centroid of K. $V = 1$.

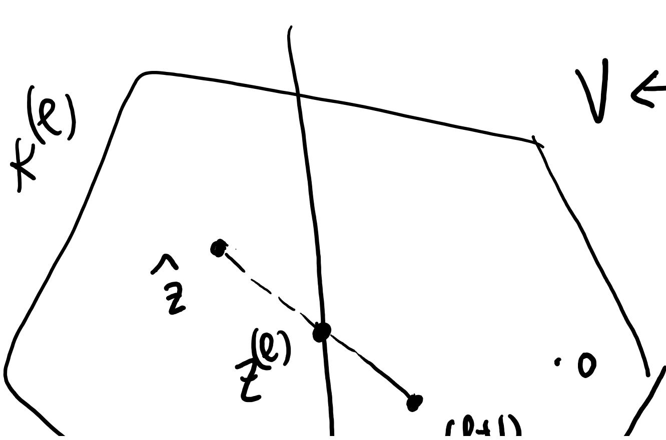
While $\exists i$ width(K) along $e_i > r$:

set $v = e_i$ or $-e_i$ s.t. $H = \{x : v^T x \leq v^T z^{(\ell)}\}$ contains 0.

$$K^{(\ell+1)} = K^{(\ell)} \cap H$$

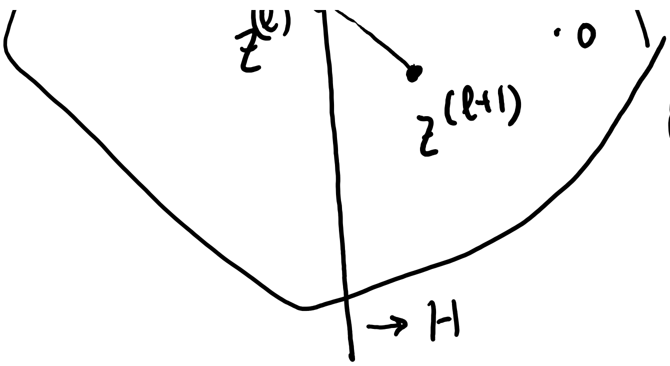
$$z^{(\ell+1)} = \text{centroid}(K^{(\ell+1)})$$

$$\hat{z} = \text{centroid}(K^{(\ell)} \setminus K^{(\ell+1)})$$



$$V \leftarrow V \cdot \frac{\|\hat{z} - z^{(\ell+1)}\|}{\|\hat{z} - z^{(\ell)}\|}$$

at last: $V \cdot \prod^{n}$ width$(K^{(\ell)})$.

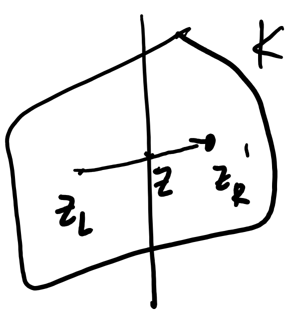


Output: $V \cdot \prod_{i=1}^{n} \text{width}(K^{(\ell)})$.

Lemma. width along e_i goes from $\leq R$ to $\geq \frac{r}{n+1}$

Lemma. K s.t. support along e_1 is $[a, b]$
$0 = \text{centroid}(K)$.
Then $|a| \geq \frac{b}{n}$.

Lemma.



$$\frac{\text{vol}(K_R)}{\text{vol}(K)} = \frac{\|z_L - z\|}{\|z_L - z_R\|}.$$

Pf. (Thm). Each cut reduces width along some e_i by $\left(\frac{n}{n+1}\right)$. $\Longrightarrow O\left(n^2 \log \frac{Rn}{r}\right)$ iterations?

Volume? drops by $\left(1 - \frac{1}{e}\right)$ each iteration

$\therefore$ # iterations $= \log \left(\frac{R^n}{(r\ldots)^n}\right)$

$$\therefore \quad \#\text{iterations} = \log_{\frac{e}{e-1}} \left(\frac{K}{\left(\frac{r}{n+1}\right)^n} \right)$$

$$= O\left(n \log \frac{Rn}{r}\right).$$

$$\begin{tikzcd} \boxed{\text{VOL}} \arrow[r] & \boxed{\text{CENTROID}} \arrow[r] & \boxed{\text{SAMPLING}} \\ \boxed{\text{OPT}} \arrow[ur] \arrow[urr, bend right, "?"'] & & \end{tikzcd}$$

$$\min_{x \in K} c^T x.$$

Sample $x \sim e^{-\alpha c^T x} \mathbb{1}_K$ for α large enough!

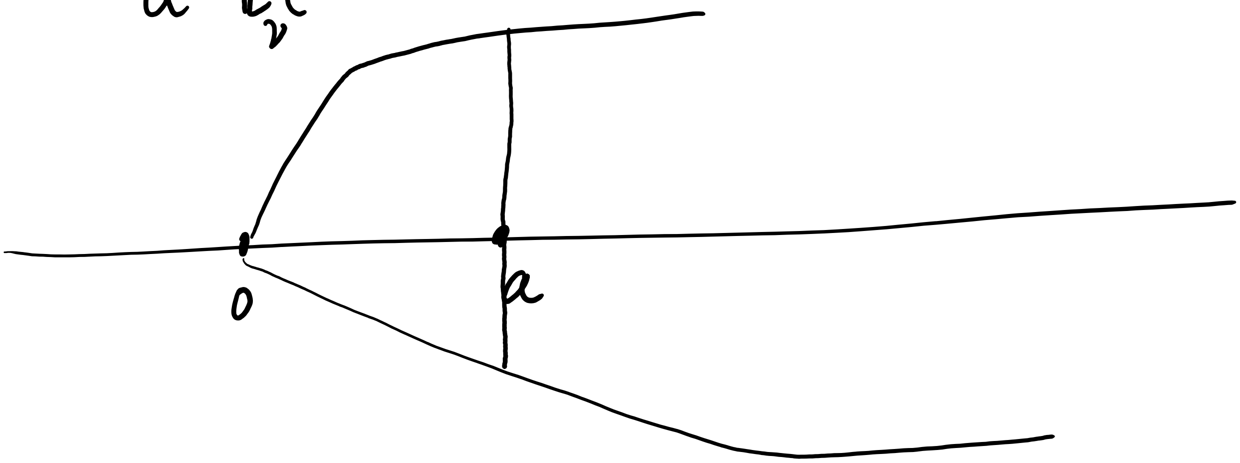
<u>Lemma.</u> $$\mathbb{E}_{x \sim e^{-\alpha c^T x} \mathbb{1}_K} \left(c^T x\right) \leq \min_K c^T x + \frac{n}{\alpha}.$$

Setting $\alpha = \frac{n}{\varepsilon} \Rightarrow \quad OPT + \varepsilon \quad \checkmark$.

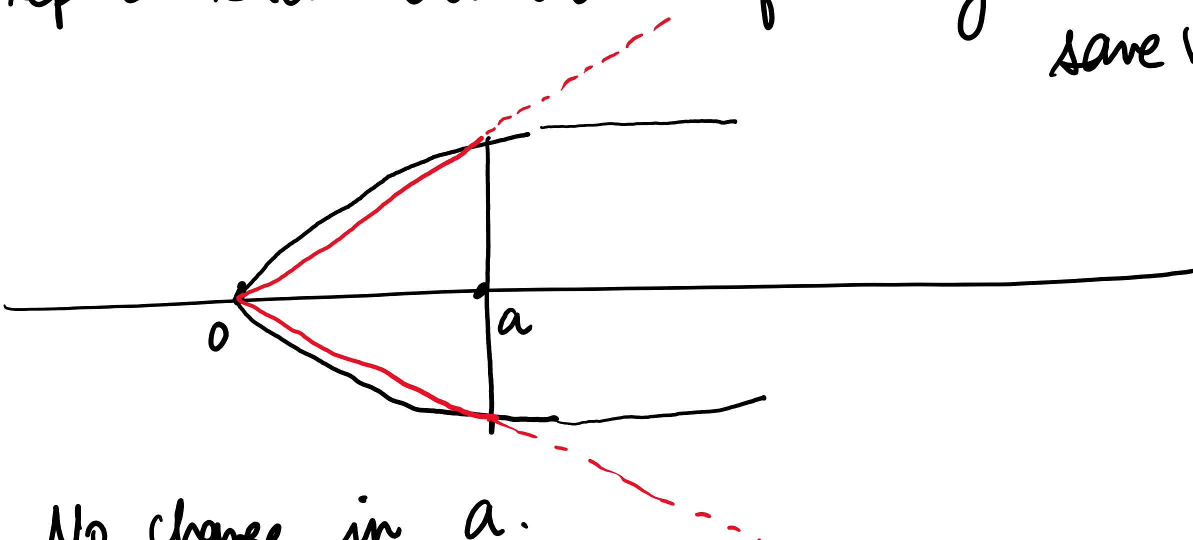
<u>Pf.</u> Assume $c = e_1$. $\quad 0 = \operatorname{argmin}_K e^T x = x_1$.

Pf. Assume $c = e_1$. $0 = \arg\min_K$

$a = \mathbb{E}_\nu(c^T x)$



Replace each cross-section of K by ball of same volume.



No change in a.

Replace "left" half of K with cone.

$\mathbb{E}(c^T x)$ moves right $\longrightarrow$ worse gap.

Replace "Right" half with infinite cone extension.

Again $\mathbb{E}(c^T x) \longrightarrow$.

Now. $\int^{\infty} e^{-\alpha t}\, t^{n-1}\, dt$

Now.

$$\mathbb{E}(c^T x) = \frac{\int_0^\infty t\, e^{-\alpha t}\, t^{n-1}\, dt}{\int_0^\infty e^{-\alpha t}\, t^{n-1}\, dt}$$

$$\int_0^\infty e^{-\alpha t} t^n \, dt = \left[-\frac{1}{\alpha} e^{-\alpha t} t^n \right]_0^\infty + \frac{n}{\alpha} \int_0^\infty e^{-\alpha t} t^{n-1} dt$$

$$= \frac{n!}{\alpha^n}.$$

$$= \frac{n!}{\alpha^n \cdot \frac{(n-1)!}{\alpha^{n-1}}} = \frac{n}{\alpha}.$$

Q. How to Sample Efficiently??!