Euclidean Isoperimetry

Sunday, March 2, 2025 8:04 PM

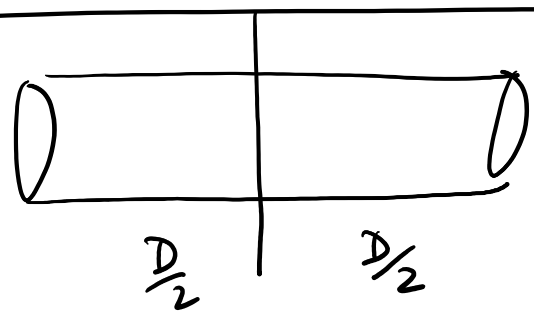
Thm [KLS, DF] $S_1, S_2, S_3 \subseteq K$ Partition of a convex body in $\mathbb{R}^n$.

$$d = d(S_1, S_2) = \min_{u \in S_1, v \in S_2} \|u - v\|_2 \qquad D = \mathrm{Diam}(K).$$

Then
$$\mathrm{vol}(S_3) \geq \frac{2}{D}\, d \, \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}.$$

Equivalently $\forall\, S \subset K$
$$\mathrm{Vol}_{n-1}(\partial S) \geq \frac{2}{D} \min\{\mathrm{vol}(S), \mathrm{vol}(K \setminus S)\}$$

Tight!



Pf. idea we try to find a "minimal" counterexample.

Suppose $\exists\, S_1, S_2, S_3$ s.t.

$$\mathrm{vol}(S_1) > A\, \mathrm{vol}(S_3) \qquad A = \frac{2d}{D}.$$
$$\mathrm{vol}(S_2) > A\, \mathrm{vol}(S_3)$$

$$\int 1_{S_1}(x) - A\, 1_{S_3}(x)\, dx > 0 \qquad \int 1_{S_2}(x) - A\, 1_{S_3}(x)\, dx > 0$$

$$\int \mathbb{1}_{S_1}(x) - A\,\mathbb{1}_{S_3}(x)\,dx > 0\ , \quad \int \mathbb{1}_{S_2}(x) - A\,\mathbb{1}_{S_3}(x)\,dx > 0$$

$$\int_{\mathbb{R}^n} f(x) > 0 \qquad \int_{\mathbb{R}^n} g(x) > 0$$

Suppose f, g are lower semi-continuous with ≥ 2 dim support.

<u>Lem</u> $\exists$ halfspace H through $x \in \text{int}(\text{Supp}(f))$ s.t. $\int_H f = \int_{\mathbb{R}^n \setminus H} f$

<u>Pf</u> define $h(v) = \int_{H(v)} f - \int_{H(-v)} f$. $h(v) = -h(-v)$.
$v \in S^{n-1}$

By continuity $\exists v$ s.t. $h(v) = 0$.

$\therefore \exists H$ s.t. $\int_H f > 0 \quad \int_H g > 0$.

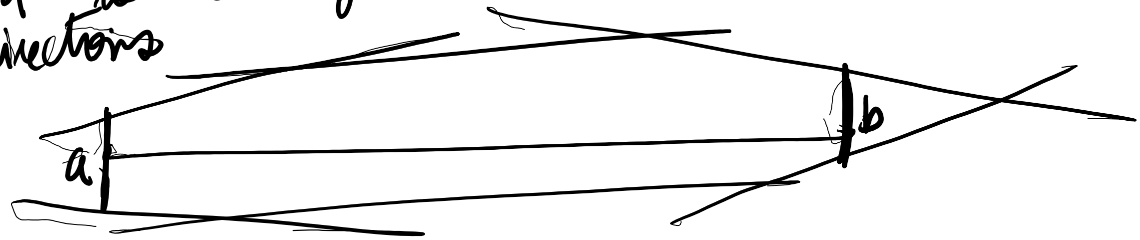
Repeat! Consider $\bar{K} = \bigcap_{\substack{x \text{ rational} \\ \in \text{int}(\text{Supp})}} H_x$. $\bar{K}$ is convex.

<u>Lema</u> $\text{supp}(\bar{K})$ is one-dimesional.

<u>Pf</u> if not, i.e. $\text{Supp}(\bar{K})$ is 2 or higher dim,
$\exists$ rational $x \in \text{int}(\bar{K})$ and we can apply bisection!
... with arbitrarily small widths in

∃ rationals $x \in \text{int}(K)$ ...

So limit is a segment with arbitrarily small widths in other directions



f, g are constant orthogonal to $[a, b]$

Cross-sectional area of a convex body.

So $\exists a, b, h$ s.t.

$$\int f((1-t)a + tb)\, h(t)\, dt > 0 \qquad \int g(\ldots)\, h(t)\, dt > 0$$

$$Z_i = \{t \in [0,1] : (1-t)a + tb \in S_i\}$$

$$\int_{Z_1} h(t) > A \int_{Z_3} h(t), \qquad \int_{Z_2} h(t) > A \int_{Z_3} h(t).$$

h is logconcave!

<u>Lemma</u>. For any logconcave $h : [0,1] \to \mathbb{R}_+$, and any partition of Z_1, Z_2, Z_3 of $[0,1]$,

$$\int_{Z_3} h \geq 2\, d(Z_1, Z_2) \min \int_{Z_1} h, \int_{Z_2} h$$

For us, $d(Z_1, Z_2) \geq \underline{d(S_1, S_2)} \geq \underline{d(S_1, S_2)}$.

For us, $d(Z_1, Z_2) \geq \dfrac{d(S_1, S_2)}{|a-b|} \geq \dfrac{d(S_1, S_2)}{D}$.

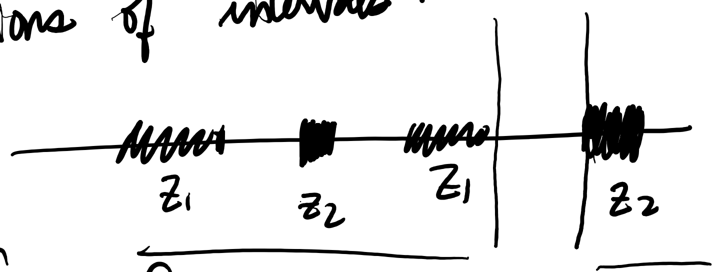
So $\displaystyle\int_{Z_3} h \geq A \min \int_{Z_1} h, \int_{Z_2} h$

contradiction!

<u>Pf (Lemma)</u>. Suffices to prove when Z_1, Z_2, Z_3 are single intervals.

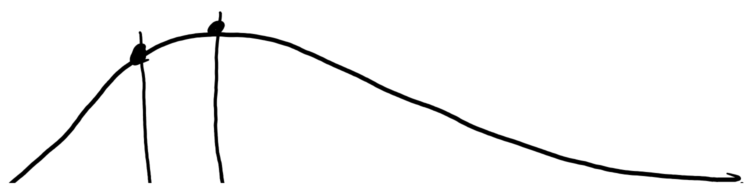
Why? Suppose Z_i are unions of intervals.

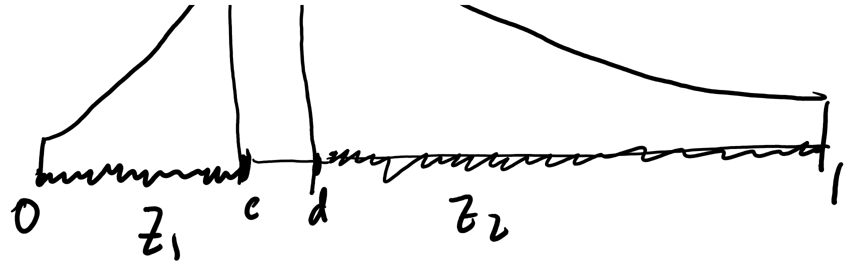
$\forall\ I \in Z_1, \quad J \in Z_2$



$$\int_{Z_3} h \geq 2\, d(I, J) \min \int_{\text{"Left"}} h, \int_{\text{"Right"}} h.$$

If all of Z_1 or all of Z_2 is accounted for, done!
Else $\exists\ I \in Z_1, \quad J \in Z_2$ but then we can account for at least one of them.





Suppose $h(c) \le h(d)$. Then by unimodality of h (weaker than logconcavity)
$h(c) \geq h(t) \quad \forall t \in Z_1$.

Then $\int_{Z_1} h \le c \; h(c)$

and $\int_{Z_3} h \geq |c-d| \, h(c) \geq \frac{|c-d|}{c} \int_{Z_1} h$

$\geq d(Z_1, Z_2) \int_{Z_1} h$.

To get the optimal $2\,d(Z_1,Z_2)$, we show that truncated exponential is the worst distribution.

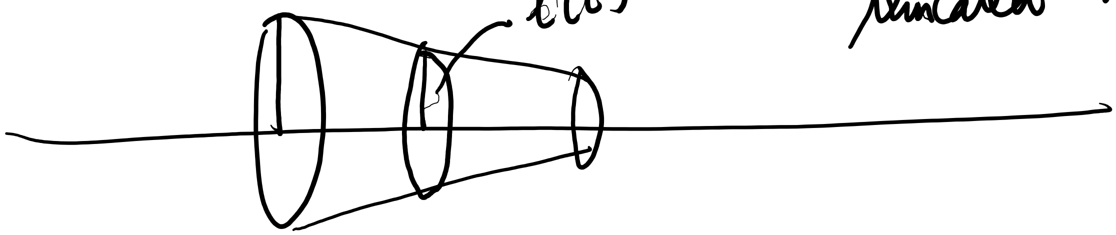
<u>Lemma</u> [Localization] $f, g : \mathbb{R}^n \to \mathbb{R}$ lower semi-continuous

$\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g > 0$. Then $\exists\, a, b \in \mathbb{R}^n$ and

linear function $l : [0,1] \to \mathbb{R}_+$, s.t.

linear function $\ell : [0,1] \to \mathbb{R}_+$ s.t.

$$\int_0^1 f((1-t)a + tb)\, \ell(t)^{n-1}\, dt \,, \quad \int_0^1 g((1-t)a + tb)\, \ell(t)^{n-1}\, dt > 0 .$$



truncated needle

We will see other applications!