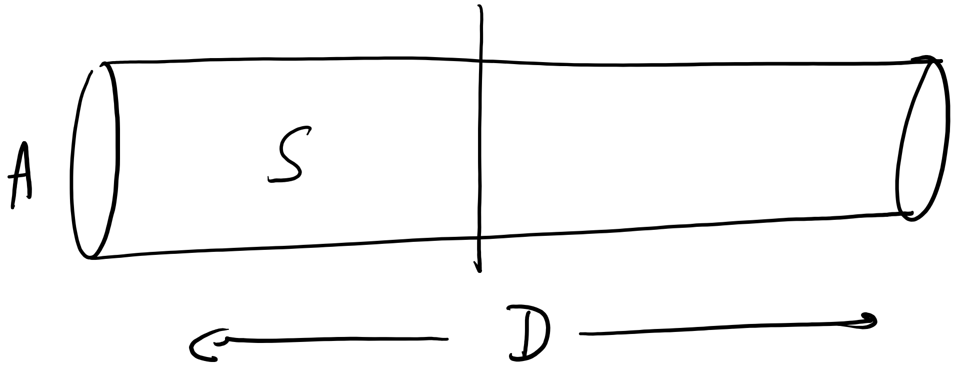
Sampling in Polytime

Tuesday, February 25, 2025 5:47 PM

So now our goal is to bound ϕ, conductance from below.



$$\psi(S) = \frac{A}{A \cdot D/2} = \frac{2}{D} \cdots$$

$$\phi(S) = \Omega\left(\frac{2}{D} \cdot \frac{\delta}{\sqrt{n}}\right) \qquad \frac{1}{\phi^2} = O\left(\frac{n D^2}{\delta^2}\right)$$

δ small enough so that $\ell(u)$ is large $\forall u \in K$.

Assume $\mathrm{vol}(S) \le \mathrm{vol}(K \setminus S)$.

Let $S_1 = \left\{ x \in S : P_x(\bar{S}) < \frac{\ell}{4} \right\}$

$S_2 = \left\{ x \in \bar{S} : P_x(S) < \frac{\ell}{4} \right\}$

$S_3 = K \setminus S_1 \setminus S_2$

$$\int_{S_1} P_u(S_2)\, dQ(u) = \frac{1}{2}\left(\int_{S_1} P_u(S_2)\, dQ(u) + \int_{S_2} P_u(S_1)\, dQ(u) \right)$$

$$\ge \frac{1}{2} \cdot \frac{\ell}{4} \cdot \frac{\mathrm{vol}(S_3)}{\mathrm{vol}(K)}$$

How large is S_3?

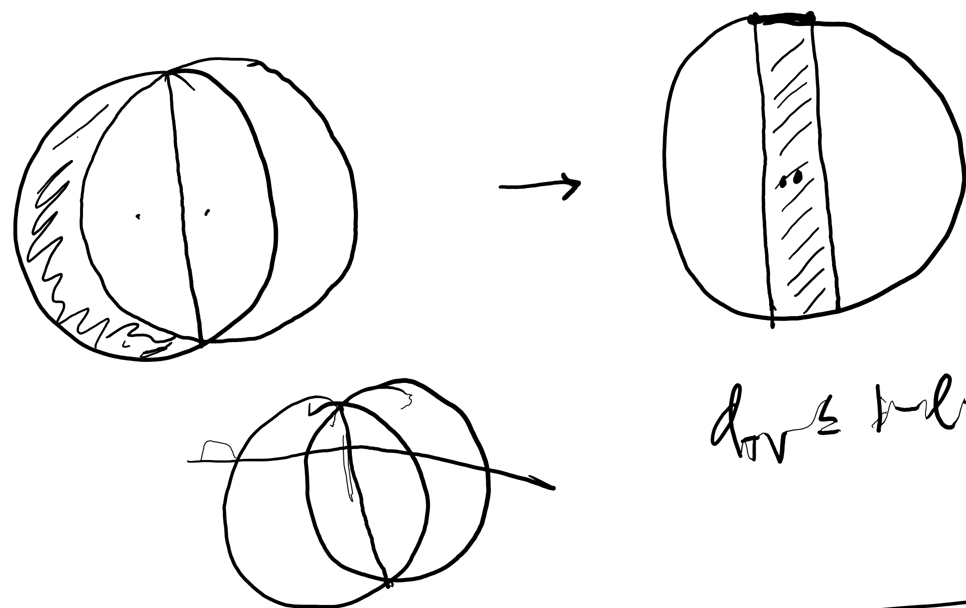
<u>Lema.</u> $u, v \in K$, $\|u-v\|_2 \le \frac{t\delta}{\sqrt{n}} \Rightarrow d_{TV}(P_u, P_v) \le 1 - \ell + t$

$\forall u \in S_1, v \in S_2$, $d_{TV}(P_u, P_v) > 1 - \frac{\ell}{4} - \frac{\ell}{4} = 1 - \frac{\ell}{2}$

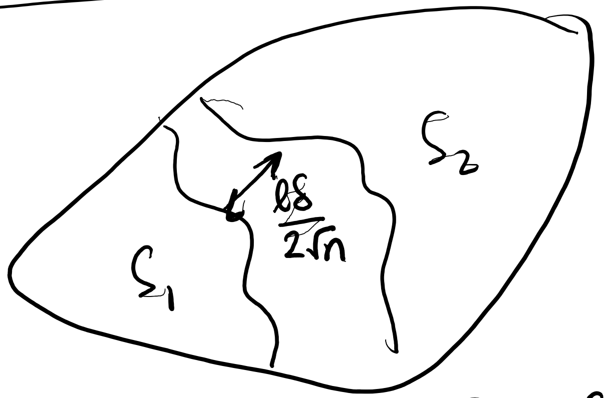
Lema $\Rightarrow$ $\|u-v\|_2 \ge \frac{\ell\delta}{2\sqrt{n}}$.
$(t = \frac{\ell}{2})$

$(t = \frac{\ell}{2})$ $\qquad {}_2 \qquad \frac{\quad}{2\sqrt{n}}$.

<u>Pf</u> (Lemma). First take $\ell = 1$ $\qquad t\delta/\sqrt{n} \qquad d_{TV} \le t$.



$d_{TV} \le 1 - \ell + t$



<u>Thm</u> [Euclidean Isoperimetry] S_1, S_2, S_3 partition of K, K of diameter D.

$$\mathrm{Vol}(S_3) \ge \frac{2}{D}\, d(S_1, S_2) \min\{\mathrm{Vol}(S_1), \mathrm{Vol}(S_2)\}.$$

$$\therefore \quad \mathrm{Vol}(S_3) \ge \frac{2}{2R} \frac{\ell\delta}{2\sqrt{n}} \min\{\mathrm{Vol}(S_1), \mathrm{Vol}(S_2)\}$$

$\therefore \quad vol(S_3) \geq \frac{\ell}{2R} \cdot \frac{\delta}{2\sqrt{n}} \cdot 2$

We can assume $vol(S_1) \geq \frac{1}{2} vol(S)$, $vol(S_2) \geq \frac{1}{2} vol(\bar{S})$

else $\phi(S) \geq \frac{1}{2} \cdot \frac{\ell}{4} \cdot \frac{vol(S)/vol(K)}{Q(S)} = \frac{\ell}{8}$.

$vol(S_3) \geq \frac{\ell \delta}{4R\sqrt{n}} \min\{vol(S), vol(\bar{S})\}$

So $\phi(S) \geq \frac{\ell}{8} \cdot \frac{\ell \delta}{4R\sqrt{n}} = \frac{\ell^2 \delta}{32 R\sqrt{n}}$

Thm $K \subseteq B(0,R)$, $\ell(u) \geq \ell \;\; \forall u \in K$. $\phi \geq \frac{\ell^2 \delta}{32 R\sqrt{n}}$

Cor. Mixing rate $= O\left(\frac{nR^2}{\ell^4 \delta^2}\right)$

$$d_{TV}(Q_t, Q) \leq \sqrt{M} \left(1 - \frac{\phi^2}{2}\right)^t.$$

$\boxed{\text{Is this polytime ?!}}$

$O(n^4 R^2) \leftarrow \delta = \frac{1}{n^{3/2}}$, $B \subseteq K$.

(2) R can be exponentially large!

(1) And M? How to start?

Q_0: uniform in $B(0,1)$.

Then $M \le \dfrac{Vol(K)}{Vol(B(0,1))} \le R^n$

So after $O\left(n^4 R^2 \log \frac{M}{\varepsilon}\right) = O\left(n^5 R^2 \log \frac{R}{\varepsilon}\right)$ steps,

$$d_{TV}(Q_t, Q) \le \varepsilon.$$

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(2). Affine transformation to make R small.

$\forall K$, $\exists$ affine transformation T s.t. TK is isotropic

$$B \subseteq TK \subseteq nB$$

But hard to find for any deterministic algorithm in the oracle model!

What is T? $\quad AK + b \quad$ $b = \mathbb{E}_K(x)$

$$A = \mathbb{E}_K\left((x-b)(x-b)^T\right)^{-1/2}$$

just need samples! :)

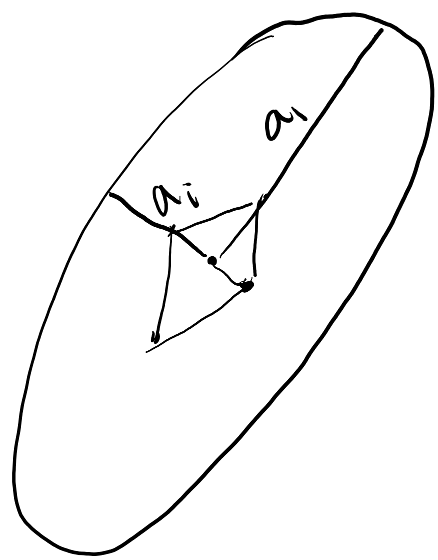
Fast Sampling needs rounding

Fast Sampling $\underset{\text{needs}}{\overset{\text{needs}}{\rightleftarrows}}$ rounding hmm.....

Bootstrap!

Deterministic rounding via Ellipsoid algorithm.

At some point $E_i = E(z, A_i)$ $K \subseteq E_i$



Check if $z \pm \frac{a_i}{n} \in K$.

if all belong to K

then $z + \operatorname{conv}\{\pm \frac{a_i}{n}\} \in K$

and $\frac{1}{n^{3/2}} E_i \subseteq K \subseteq E_i$

$\Rightarrow R \le n^{3/2}$.

if $\exists i$ s.t. $z + a_i \notin K$ (or $z - a_i$)

continue with Ellipsoid algorithm!

Thm. Deterministic polytime algorithm to find E

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Thm Deterministic polytime algorithm to ...
st $E \subseteq K \subseteq n^{3/2} E$.

Polytime Sampling! Polytime Optimization!
