Yunfeng

The Central Path Method

(P)	(D)
$\min\ c^T x$	$\max\ b^T y$
$Ax = b$	$A^T y + s = c$
$x \geq 0$	$s \geq 0$

$A \in \mathbb{R}^{n \times d}$

Lemma. (Duality gap) x, s feasible $0 \leq c^T x - b^T y \leq x^T s$. P, D nonempty $\Rightarrow$ $\exists x^* \in P, s^* \in D$ $x^{*T} s^* = 0$.

$$c^T x - b^T y = c^T x - x^T A^T y = x^T s.$$

$$\therefore\ c^T x \leq \max_{\substack{A^T y + s = c \\ s \geq 0}} b^T y + x^T s \leq \min_{x \in P} c^T x + \underline{x^T s}$$

$\boxed{\text{Goal: Solve (P)}}$.

How to handle $x \geq 0$?

Consider

(P_μ): $\quad \min\ c^T x - \mu \sum_{i=1}^{n} \ln x_i$

$\mu > 0$. $\quad Ax = b$

Lemma. If P has an interior, then $\mathrm{OPT}(P_\mu)$ is unique and given by

$$xs = \mu$$
$$Ax = b$$
$$A^T y + s = c$$
$$x, s \geq 0.$$

($xs = \mu$ and be a general vector $\Leftrightarrow$ $x_i s_i = \mu_i \ \forall i$.)

Pf.

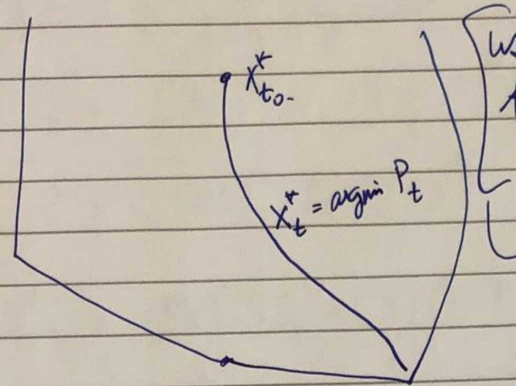
$$\nabla_x \left(c^T x - \mu \sum \ln x_i - y^T (Ax - b) \right) = 0. \quad \nabla_y (\ ) = 0$$

$\Rightarrow Ax = b$

$$c - \frac{\mu}{x} = A^T y \Rightarrow s = \frac{\mu}{x}$$

i.e. $xs = \mu$

Idea. Start with $\mu = t_0$.



Use current $x_{t'}$ as approximate solution for ~~..~~ smaller t. Improve to get closer to x^*_t.

Repeat.

Subproblem: Given approx solution x, s for P_t, how to improve?

$$x \to x+\delta_x \qquad y \to y+\delta_y \qquad s \to s+\delta_s.$$

$$(x+\delta_x)(s+\delta_s) = t \iff x\delta_s + s\delta_x + \delta_x\delta_s = t - xs$$
$$A\delta_x = 0$$
$$A^T\delta_y + \delta_s = 0$$
$$x+\delta_x,\ s+\delta_s > 0.$$

Simplify and linearize by dropping $\delta_x\delta_s$.

$X = \mathrm{Diag}(x)$

$$X\delta_s + S\delta_x = \delta_\mu = t - xs. \qquad (*)$$
$$A\delta_x = 0$$
$$A^T\delta_y = -\delta_s.$$

Lemma. Let $P = XA^T\left(A\frac{X}{S}A^T\right)^+ AS^{-1}$. Then $X\delta_s = P\delta_\mu$, $S\delta_x = (I-P)\delta_\mu$ solves for δ_x, δ_s.

Pf. From the first equation, we have

$$\frac{X}{S}A^T\delta_y + \delta_x = S^{-1}\delta_\mu \Rightarrow A\frac{X}{S}A^T\delta_y = AS^{-1}\delta_\mu$$

$$\therefore \delta_y = \left(A\frac{X}{S}A^T\right)^+ AS^{-1}\delta_\mu \qquad X\delta_s = XA^T\left(A\frac{X}{S}A^T\right)^+AS^{-1}\delta_\mu = P\delta_\mu.$$

$$S\delta_x = \delta_\mu - X\delta_s = (I-P)\delta_\mu.$$

Note that $P^2 = P$ (projection matrix). eigenvalues are 0 or 1 $\boxed{\lambda^2=\lambda}$

But P is not symmetric in general.

Let $P_0 = \left(\frac{X}{S}\right)^{\frac12} A^T \left(A \frac{X}{S} A^T\right)^{+} A \left(\frac{X}{S}\right)^{\frac12} = (XS)^{-\frac12} P (XS)^{\frac12}$

$P = (XS)^{\frac12} P_0 (XS)^{-\frac12}$.

$P_0 = P_0^T$, $P_0^2 = P_0$. | Symmetric projection matrix. $\Longleftrightarrow$ orthogonal projection.

So $X^{-1}\delta_x = (SX)^{-1} S\delta_x = (SX)^{-1}(I-P)\delta_\mu$

$= (SX)^{-\frac12}(I - P_0)(SX)^{-\frac12}\delta_\mu$.

Lemma 1. $\|xs - t\|_2 \le \varepsilon t \Rightarrow x + \delta_x > 0$ for $\varepsilon < \frac12$.

Pf. $\|X^{-1}\delta_x\|_2 \le \|(XS)^{-\frac12}\|_2 \, \|(I-P_0)(XS)^{-\frac12}\delta_\mu\|_2$

$\le \frac{1}{\sqrt{(1-\varepsilon)t}} \|(XS)^{-\frac12}\delta_\mu\|_2 \le \frac{1}{(1-\varepsilon)t}\|\delta_\mu\|_2 = \frac{\|xs-t\|_2}{(1-\varepsilon)t} \le \frac{\varepsilon}{(1-\varepsilon)}$

$\varepsilon < \frac12 \Rightarrow \ < 1$.

Lemma 2. $x \to \hat{x}$, $s \to \hat{s}$. solving (*).

$\|xs - t\|_2 \le \varepsilon t \Rightarrow \|\hat{x}\hat{s} - t\|^2 \le (\varepsilon^4 + 16\varepsilon^5)t^2$.

Pf. $\|\hat{x}\hat{s} - t\|_2^2 = \sum_i \left(x_i s_i + x_i \delta_{s_i} + s_i \delta_{x_i} + \delta_{x_i}\delta_{s_i} - t\right)^2$

$= \sum_i (\delta_{x_i}\delta_{s_i})^2 \le ((1+\varepsilon)t)^2 \sum_i \left(\frac{\delta_{x_i}}{x_i}\right)^2 \left(\frac{\delta_{s_i}}{s_i}\right)^2$

$x_i s_i \le (1+\varepsilon)t$

(C-S) $\le (1+\varepsilon)^2 t^2 \left\|\frac{\delta_x}{x}\right\|_4^2 \left\|\frac{\delta_s}{s}\right\|_4^2$

$\le (1+\varepsilon)^2 t^2 \left(\frac{\varepsilon}{1-\varepsilon}\right)^4 \le (\varepsilon^4 + 16\varepsilon^5)t^2$.

Thm. x_0, s_0 feasible with $\|x_0 s_0 - t_0\|_2 \le \varepsilon t_0$. $\varepsilon < \frac{1}{4}$.

Repeat $\left[\begin{array}{l} \text{Compute } x+\delta_x,\ s+\delta_s \\ t \leftarrow \alpha t. \end{array}\right.$ with $\alpha = 1 - \frac{1}{10\sqrt{n}}$.

CPM converges to duality gap εn in $O\left(\sqrt{n}\log\left(\frac{t_0}{\varepsilon}\right)\right)$ iterations.

Pf. $\phi(t) = \|xs - t\|_2^2$. We maintain by induction that $\phi(t) \le \frac{t^2}{16}$.

After one step $\phi(t)(\hat{x}, \hat{s}) \le \frac{t^2}{50}$.

Setting $t' = \left(1 - \frac{1}{10\sqrt{n}}\right)t$ $\qquad h = 1-\alpha = \frac{1}{10\sqrt{n}}$.

$$\phi(t') = \sum_i \left(\hat{x}_i \hat{s}_i - t(1-h)\right)^2 \le 2\sum_i \left(\hat{x}_i\hat{s}_i - t\right)^2 + 2\sum_i t^2 h^2$$

$$\le 2 \cdot \frac{t^2}{50} + 2 \cdot \cancel{n} \cdot t^2 \cdot \frac{1}{100\cancel{n}} = \frac{3t^2}{50}$$

$$< \frac{(1-h)^2 t^2}{16}.$$ ∎

Q1 How to get initial feasible x_0, s_0?

Q2. Is $\sqrt{n}$ the correct bound? Why?!

Consider the continuous algorithm: It maintains/solves this ODE:

$$S_t \frac{d x_t}{dt} + X_t \frac{d}{dt} s_t = \mathbb{1}.$$

$$A \frac{d x_t}{dt} = 0 \qquad A^T \frac{d y_t}{dt} + \frac{d s_t}{dt} = 0.$$

$$\Rightarrow S_t \frac{d x_t}{dt} = (I - P_t)\mathbb{1}, \quad X_t \frac{d s_t}{dt} = P_t \mathbb{1}. \quad P_t = X_t A_t^T \left(A \frac{X_t}{S_t} A^T\right)^{-1} A S_t^{-1}.$$

Q2: $S_t^{-1} \frac{ds_t}{dt} = S_t^{-1} X_t^{-1} P_t \mathbb{1}$
$$= \frac{1}{t} P_t \mathbb{1}.$$

$$\frac{d \ln s_t}{d \ln t} = P_t \mathbb{1}. \quad \text{and } \| \text{ly} \quad \frac{d \ln x_t}{d \ln t} = (I - P_t)\mathbb{1}.$$

So rate of change of s_t, x_t in relative terms depends on $\|P_t \mathbb{1}\|_\infty$.

$$\|P_t \mathbb{1}\|_\infty \leq \|P_t \mathbb{1}\|_2 = \sqrt{n}.$$

Q1: Lift & Relax the LP

$$\bar{A} = \begin{pmatrix} A & -A & 0 \\ \mathbb{1} & 0 & 1 \end{pmatrix} \quad \bar{c} = (c, \tilde{c}) \quad \bar{b} = (b, \tilde{b})$$

Easy starting point:

$$x_c^+ = \frac{t}{c + t/\bar{R}}, \quad x_c^- = x_c^+ - A^+(AA^+)^+ b$$

$(\bar{R} \geq 10R)$

$t \geq 8L\bar{R}$

$$\bar{x} = (x_c^+, x_c^-, \bar{R}).$$
$$\bar{s} = \frac{x}{t}.$$

$$\tilde{c} = \frac{t}{x_c^-} \quad \tilde{b} = \sum_i x_{c,i}^+ + \bar{R}.$$

$\bar{P}$

$$A(x^+ - x^-) = b \qquad x^+, x^-, x^\theta \geq 0.$$
$$\sum_i x_i^+ + x^\theta = \tilde{b}$$
$$\text{Min} \quad c^T x^+ + \tilde{c}^T x^-$$

$\bar{D}$

$$A^T y + \lambda \mathbb{1} + s^+ = c$$
$$-A^T y + s^- = \tilde{c}$$
$$\lambda + s^\theta = 0.$$
$$s^+, s^-, s^\theta \geq 0.$$
$$\max \quad b^T y + \tilde{b} \lambda$$

Thm: Setting $t_{end} = \delta \frac{LR}{2n}$

we get $x = x^+ - x^- \quad s = s^+ - s^\theta > 0$

s.t. $Ax = b \quad A^T y + s = c$

$x, s \geq 0. \quad x^T s \leq \delta L R.$

(1)

CPM. Summary. (P) (D)

I. Find x, s on central path of (P) by solving auxiliary $\bar{P}, \bar{D}$.

$$\min \ \bar{c}^T x^+ + \bar{c}^T x^-$$
$$A(x^+ - x^-) = b$$
$$\sum_i x_i^+ + x^\theta = \tilde{b}$$
$$x^+, x^-, x^\theta \geq 0$$

$$\max \ b^T y + \tilde{b}\lambda$$
$$A^T y + \lambda \mathbb{1} + s^+ = c$$
$$-A^T y + s^- = \tilde{c}$$
$$\lambda + s^\theta = 0.$$
$$s^+, s^-, s^\theta \geq 0$$

$\bar{P}, \bar{D}$ is easy to initialize: on central path.

$$x_c^+ = \frac{t}{c + t/\bar{R}}$$
$$x_c^- = x_c^+ - A^T(AA^T)^{-1} b$$
$$x^\theta = \bar{R}$$
$$\tilde{b} = \sum_i x_{c,i}^+ + \bar{R}.$$

$$s^+ = \frac{t}{x^+} \qquad s^- = \frac{t}{x^-} = \tilde{c}$$
$$s^\theta = \frac{t}{x^\theta}. \qquad y = 0.$$

$$\boxed{\begin{array}{c} \bar{R} = 10R \\ t_{init} = Cn^2 \frac{R}{r} \cdot LR. \end{array}}$$

Follow CP of $(\bar{P}), (\bar{D})$

Apply iteration to reach $t \approx LR$.

then $x = x^+$, $s = s^+$ is on CPM of (P), (D).

II. Follow CP of (P), (D) from this till t end.

$Ax = b, x \geq 0 \subseteq B(0, R)$. $\|C\|_2 \leq L$.

$\exists x, \ x_i \geq r, \ \forall i, \ Ax = b.$

Thm. #iterations $= O\left(\sqrt{n} \log\left(\frac{nRL}{r\varepsilon}\right)\right)$ to find $x \geq 0$, $Ax = b$

$c^T x \leq OPT + \varepsilon.$