PFA : Markov Chains

Wednesday, September 25, 2019 5:48 AM

. GAME

Last Time we introduced PFAs.

- states Q
- Transition matrix $P \geq 0$, row sums $= 1$.
- starting distribution $\pi^{(0)}$.
- End states.

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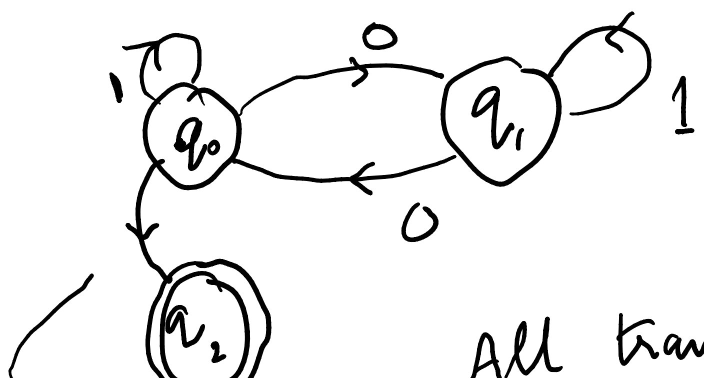
e.g. PFA that only outputs strings with an even # 1's:



no output.

All transitions have > 0 probability.

PFA might not have an end state.

If P is primitive, or P is irreducible + aperiodic, ... $\Rightarrow$ directed cycles in

If P is primitive, or P is irreducible + aperiodic,

[Aperiodic : GCD of all lengths of directed cycles in P is 1]

then $\pi^{(t)} \longrightarrow \pi$ unique.

We consider a simple and natural setting today. $G = (Q, E)$ be the support of the PFA.

Let the degree of Q_i be d_i

Let $P_{ij} = \frac{1}{d_i}$. $\left(P_{ji} = \frac{1}{d_j}\right)$

i.e. we pick a transition at random.

$$P := \begin{pmatrix} \frac{1}{d_1} & \frac{1}{d_1} & \cdots & \\ & & \frac{1}{d_j} & \\ & & & \end{pmatrix}$$

Note $P\mathbb{1} = \mathbb{1}$

$$\left(P^T \pi^{(t)}\right)_i = \sum_j P_{ji} \pi_j^{(t)} = \sum_{j : (j,i) \in E} \frac{\pi_j^{(t)}}{d_j}$$

<u>Lemma</u>. $\pi = \frac{1}{\sum d_i} \begin{pmatrix} d_1 \\ \vdots \\ \vdots \end{pmatrix}$ is stationary for P.

Lemma. $\pi = \frac{1}{\sum d_i}\begin{pmatrix} \vdots \\ d_n \end{pmatrix}$ is stationary ...

Pf. $(P^T\pi)_i = \sum_{j:(j,i)\in E} \frac{\pi_j}{d_j} = \frac{d_i}{\sum d_i} = \pi_i$.

Lemma. If G is connected and non-bipartite then π is the unique stationary distribution and $\pi^{(t)} \longrightarrow \pi$.

Q. What is the probability of going from i to j in the steady state π?

$$\pi_i \cdot P_{ij} = \frac{d_i}{\sum d_i} \cdot \frac{1}{d_i} = \frac{1}{\sum d_i} = \frac{1}{2m}.$$

"each transition is equally likely".

What we have is a simple random walk on a graph. This is an object of much study and has many applications.

Parameters: Access time $H(i,j) = E$(# steps to go ...

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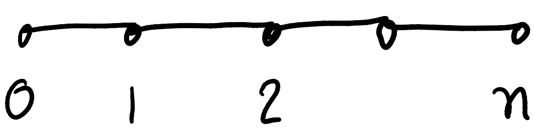
Cover time $C(i) = E(\#\text{ steps to visit all vertices starting at } i)$

Mixing time $\mu = \sup_{t \to \infty} \max_{i,j} \left| p_{ij}^{(t)} - \pi_j \right|^{1/t}$.

"rate at which $\pi^{(t)} \longrightarrow \pi$".

Examples.

1. Path



$H(0,n)$?

$\pi(0) = \frac{1}{2n}$ $\quad \pi(n) = \frac{1}{2n}$

$\pi(i) = \frac{2}{2n} = \frac{1}{n}$ $\quad i = 1, 2, \ldots n-1$

— break ——

$H(i-1, i) = 2i - 1$.

$H(i,j) = H(i, j-1) + 2j - 1$

$H(i,n) = 2n-1 + 2n-3 + 2n-5 \ldots . 2i+1$

$\sum^{n} 2j-1 \quad - \quad \sum^{i} 2j-1$

$$H(i,n) = \sum_{j=1}^{n} 2j-1 - \sum_{j=1}^{i} 2j-1$$

$$= n^2 - i^2$$

$$\boxed{H(0,n) = n^2}$$

2. Complete graph.

$$P_{ij} = \frac{1}{n-1} \qquad \pi(i) = \frac{1}{n}.$$

$$H(i,j) = n-1.$$

What about Cover time?

— thinking break —

$$0 = t_1 \le t_2 \ldots t_i \le t_n$$

first time visiting i vertices

$$\Pr(\text{new vertex after } t_i) = \frac{n-i}{n-1}$$

$$\mathbb{E}(t_{i+1} - t_i) = n-1$$

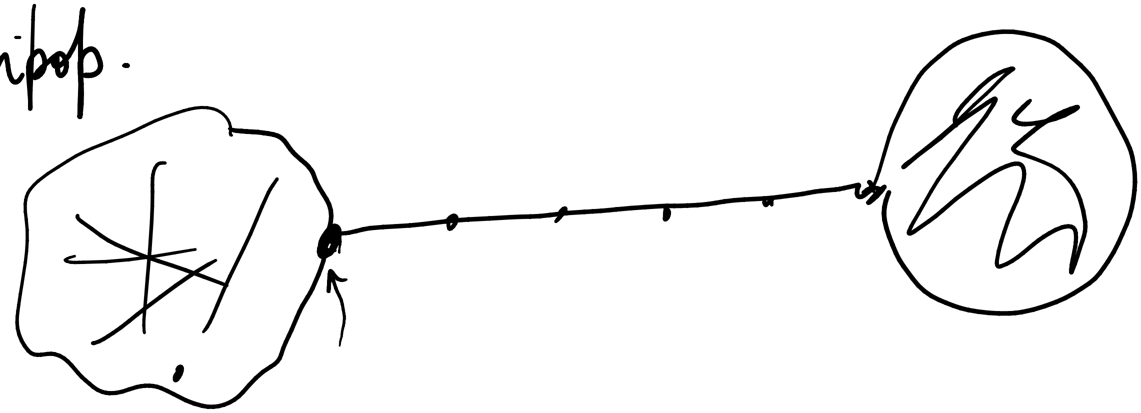
...

$$\mathbb{E}(t_{i+1} - t_i) = \frac{n-1}{n-i}$$

$$\mathbb{E}(t_n) = \sum_{i=0}^{n-1} \mathbb{E}(t_{i+1} - t_i) = \sum_{i=0}^{n-1} \frac{n-1}{n-i} = (n-1) \sum_{i=1}^{n} \frac{1}{i}$$

$$\simeq n \ln n.$$

3. Lollipop.



$\Theta(n^3)$.

<u>Thm</u>. The cover time for any connected undirected graph is $O(n^3)$.

<u>Mixing time</u>. $P^T \pi = \pi$ eigenvalue 1

$\forall v \neq \pi \quad P^T v = \lambda v \quad \lambda < 1$

If P is primitive, $|\lambda| < 1$.

$$\chi^2(\pi^{(t+1)}, \pi) \le \lambda \cdot \chi^2(\pi^{(t)}, \pi).$$

So distance drops by factor λ in each step.
