Fourier Learning

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Motivated by learning DNF.
Still an open problem to PAC learn DNF or decision trees (lists we can learn).

So we consider special distributions

- uniform
- product
- Gaussian etc

and allow for membership queries, i.e., "what is the label of x?".

Assume $x \in \{-1, 1\}^n$ $\quad f: \{-1, 1\}^n \to \{-1, 1\}$

Note that any such $f \in \{-1, 1\}^{2^n}$ (as a table)

Defining inner product of f and g with respect to distribution D as

$$\langle f, g \rangle_D = \sum D(x) f(x) g(x)$$

$$\langle f, g\rangle_D = \sum_x D(x) f(x) g(x)$$

$$\langle f, f\rangle_D = \|f\|_D^2 = 1 \quad (\text{since } f(x)^2 = 1)$$

Viewing f as a vector, the standard basis is $e_1, e_2, \ldots, e_{2^n}$.

But we can use any basis and write

$f(x) = \sum_v \langle f, v\rangle v$ since $\{v\}$ is an orthonormal basis.

What's an interesting basis?

The set of parity functions!

$\forall S \subseteq [n]$, $\chi_S(x) = \prod_{i \in S} x_i$ $\quad 2^n$ functions.

$$\langle \chi_S, \chi_S\rangle_D = 1$$

$$\langle \chi_S, \chi_T\rangle_D = \mathbb{E}_D\left(\prod_{i\in S} x_i \prod_{j \in T} x_j\right) = 0$$

for a product distribution D.

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Hence $\{\chi_S\}$ is an orthonormal basis!

So any f can be written as

$$f(x) = \sum_S \hat{f}_S \chi_S(x) \quad \text{where } \hat{f}_S = \langle f, \chi_S \rangle_D .$$

Thm 1. (Parseval) $\langle f, f \rangle_D = \langle \hat{f}, \hat{f} \rangle$

Thm 2. (Plancherel) $\langle f, g \rangle_D = \langle \hat{f}, \hat{g} \rangle$.

Pf. $$\sum_x D(x) \sum_S \hat{f}_S \chi_S(x) \sum_T \hat{g}_T \chi_T(x)$$

$$= \sum_{S,T} \hat{f}_S \hat{g}_T \mathbb{E}_D (\chi_S(x) \chi_T(x))$$

$$= \sum_S \hat{f}_S \hat{g}_S = \langle \hat{f}, \hat{g} \rangle .$$

A decision tree is a Boolean function f.

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of its significant Fourier coefficients $\hat{f}_S$.

Our approx is g.

$$\Pr_D(g(x) \neq f(x)) \leq \mathbb{E}_D\left((f(x)-g(x))^2\right) = \sum_S (\hat{f}_S - \hat{g}_S)^2.$$

We will learn all $\hat{f}_S$ for which $\hat{f}_S \geq \tau$.

Note. $\sum_S \hat{f}_S^2 = 1 \Rightarrow |\hat{f}_S| \leq 1$

Lemma. If a decision tree has m leaves

then $\|f\|_1 = \sum |\hat{f}_S| \leq 2m+1$.

Thm. If we learn all $\hat{f}_S \geq \dfrac{\varepsilon}{\|f\|_1}$

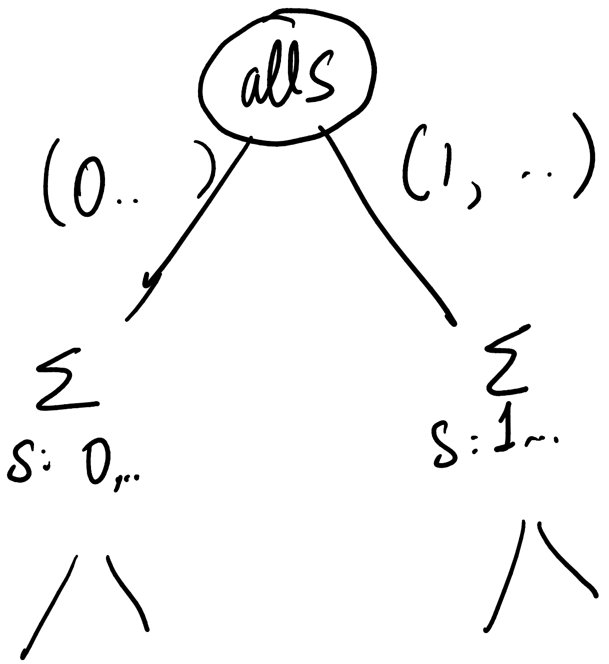
then $\|\hat{f}_S - \hat{g}_S\|^2 \leq \varepsilon$.

Pf. $$\|\hat{f}_S - \hat{g}_S\|^2 = \sum_{S : |\hat{f}_S| \leq \frac{\varepsilon}{\|\hat{f}\|}} \hat{f}_S^2 \leq \sum_S |\hat{f}_S| \cdot \frac{\varepsilon}{\|\hat{f}\|_1}$$

$$s.t.\ |\hat{f}_S| \ge \frac{\varepsilon}{\|\hat{f}\|_1} \qquad \dots$$

$$= \varepsilon.$$

How to learn all large Fourier coefficients.



At each node estimate whether

$$\sum_{S:\ \text{prefix}\ \alpha} \hat{f}_S^2 \geq \tau.$$

width $\le \frac{1}{\tau}$, depth $\le n$

$\#$ nodes $\le \frac{n}{\tau}$.

How to estimate

$$\sum_{S \in S_\alpha} \hat{f}_S^2 \ ?$$

Suppose $\alpha = (\underbrace{0, \dots 0}_{k})$.

Claim $\sum \hat{f}^2 \quad \mathbb{E}\ \big(f(yx)\, f(zx)\big)$

<u>Claim</u>. $\sum_{S_\alpha} \hat{f}_S^2 = \mathbb{E}_{x \sim \{0,1\}^{n-k},\, y, z \sim \{0,1\}^k} \left(f(yx) f(zx) \right)$

Suppose f is a parity function

if f agrees with α, then $f(yx) = f(zx)$ so we get 1.

else $\Pr(f(yx) = f(zx)) = \frac{1}{2} \Rightarrow$ we get 0.

Any f can be written as a weighted sum of parities. So,

$$f = \sum_U \hat{f}_U \chi_U$$

$$\mathbb{E}(f(yx) f(zx)) = \mathbb{E}\left(\sum_U \hat{f}_U \chi_U(yx) \sum_V \hat{f}_V \chi_V(zx) \right)$$

$$= \sum_{U,V} \hat{f}_U \hat{f}_V \underbrace{\mathbb{E}\left(\chi_U(yx) \chi_V(zx)\right)}_{=0 \text{ if } U \neq V}$$

$$= \sum_U \hat{f}_U^2 \underbrace{\mathbb{E}\left(\chi_U(yx) \chi_U(zx)\right)}_{=0 \text{ if } U \text{ does not}}$$

$$= \sum_{U \in \mathcal{U}_\alpha} \hat{f}_U^2 .$$

$= 0$ if U does not agree with $\alpha = (0, \dots, 0)$

What about general α?

<u>Lemma.</u> $$\sum_{S \in S_\alpha} \hat{f}_S^2 = \mathop{\mathbb{E}}_{\substack{x \sim \{0,1\}^{n-k} \\ y, z \sim \{0,1\}^k}} \left(f(yx) f(zx) \chi_\alpha(y) \chi_\alpha(z) \right).$$