Reduction

Sunday, February 9, 2025 10:08 AM

y/No

$MEM_K(y)$ — YES if $y \in K$ NO otherwise

$SEP_K(y)$ — vector c s.t. $c^T x \le c^T y \;\; \forall x \in K$

$OPT_K(c)$ — vector y s.t. $c^T x \le c^T y \;\; \forall x \in K$ or "K is empty".

$VAL_K(c)$ — $\max_{x \in K} c^T x$ or "K is empty".

Examples

K	Easy Oracle
$\{x : Ax \le b\}$	SEP/MEM
$Conv\{v_1 \dots v_m\}$	OPT
$\{X : X \succeq 0\}$	SEP/MEM
Spanning tree polytope	SEP
Matching polytope	SEP

$\boxed{Q.}$ Are these oracles equivalent?

e.g. Does MEM suffice for OPT?

Function oracles

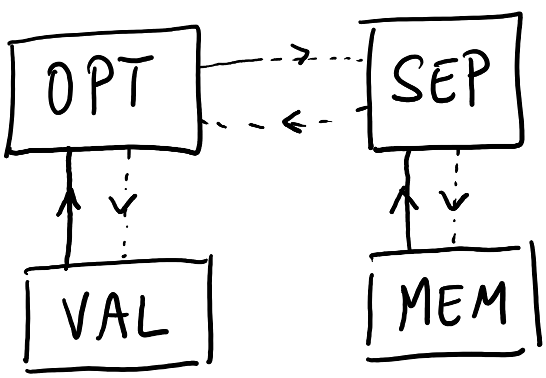
$\mathrm{EVAL}_f(y)$: outputs $f(y)$

$\mathrm{GRAD}_f(y)$: $f(y), g$ s.t. $\forall x$
$$f(x) \geq f(y) + \langle g, x-y \rangle$$

$f = \delta_K$

$$\mathrm{EVAL}_{\delta_K} \equiv \mathrm{MEM}_K$$

$$\mathrm{GRAD}_{\delta_K} \equiv \mathrm{SEP}_K.$$



Dual (Polar) of a convex set K.

$$K^* = \{ \theta \mid \langle \theta, x \rangle \leq 1 \quad \forall x \in K \}$$

$\boxed{\text{Thm. } (K^*)^* = K \text{ iff } 0 \in K}$

$$\mathrm{SEP}_K(y) = \begin{cases} \text{YES} & y \in K \\ \langle \theta, x \rangle \leq \langle \theta, y \rangle \quad \forall x \in K \end{cases}$$

$$\mathrm{SEP}_K(y) = \begin{cases} \\ \theta & : \langle \theta, x\rangle \le \langle \theta, y\rangle \ \forall x \in K \end{cases}$$

i.e. $\exists \theta$ s.t. $\langle \theta, y\rangle \ge 1$

$\langle \theta, x\rangle \le 1 \quad \forall x \in K$ by scaling θ

$\theta \in K^*$.

This is $\mathrm{OPT}_{K^*}(y) = \underset{\theta \in K^*}{\mathrm{argmax}} \langle \theta, y\rangle$

$$\mathrm{SEP}_{K^*}(\theta) = \begin{cases} \text{YES} & \theta \in K^* \\ y \in K & \text{s.t. } \langle \theta, y\rangle > 1. \end{cases}$$

$$\mathrm{OPT}_K(\theta) = \underset{\in K}{y} \text{ s.t. } \langle \theta, x\rangle \le \langle \theta, y\rangle \quad \forall x \in K.$$

$$= \underset{y \in K}{\mathrm{argmax}} \langle \theta, y\rangle$$

What about the dual of a convex function?

Fenchel dual:

$$f^*(\theta) = \sup_x \theta^T x - f(x).$$

<u>Lemma.</u> f^* is convex. $f^*(0) = -\inf_x f(x)$.

Note: $f = \delta_K$ then $f^* = \delta_K^* = \sup_{x \in K} \theta^T x$.

$= \{ 0 \quad x \in K$

$-$ EVAL (δ_K^*).

$$= \begin{cases} 0 & x \in K \\ \infty & x \notin K \end{cases}$$

$$VAL_K \equiv EVAL(\delta_K^*).$$

<u>Lema.</u> f continuos.

$$\nabla f^*(\theta) = \operatorname*{argmax}_x \; \theta^T x - f(x)$$

<u>Pf.</u> Need to show <u>sup</u> is achieved.

$$x_\theta = \operatorname*{argmax}_x \; \theta^T x - f(x)$$

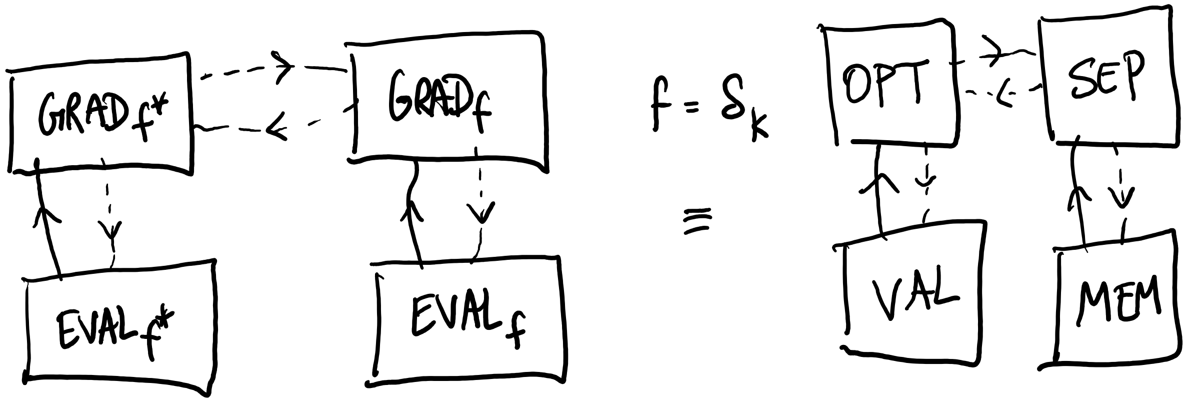
$$f^*(\theta) = \theta^T x_\theta - f(x_\theta)$$

$$\forall \eta \quad f^*(\eta) \geq \eta^T x_\theta - f(x_\theta)$$

$$\Rightarrow f^*(\eta) - f^*(\theta) \geq \langle x_\theta, \eta - \theta \rangle$$

$$\therefore \; x_\theta \in \nabla f^*(\theta).$$

$$GRAD\,\delta_K^* = OPT_K$$



Example 1. $\quad f(x) = a^T x - b$

Exampe 1. $f(x) = a^T x - b$

$$f^*(\theta) = \sup_x \theta^T x - (a^T x - b)$$

$$= \max_x \langle (\theta - a), x \rangle + b = \begin{cases} b & \text{if } \theta = a \\ \infty & \text{o.w.} \end{cases}$$

$$(f^*)^*(x) = \sup_\theta \theta^T x - f^*(\theta) = a^T x - b.$$

<u>Ex. 2.</u> $f(x) = \frac{1}{p} \|x\|_p^p = \frac{1}{p} \sum_i x_i^p$. $p \geq 1$.

$$f^*(\theta) = \max_x \theta^T x - \frac{1}{p} \|x\|_p^p$$

Setting $\nabla = 0$ $\theta = \begin{pmatrix} x_i^{p-1} \\ \\ \\ \end{pmatrix}$ $x_i = \theta_i^{\frac{1}{p-1}}$

$$= \sum_i \theta_i^{1+\frac{1}{p-1}} - \frac{1}{p} \sum_i \theta_i^{\frac{p}{p-1}}$$

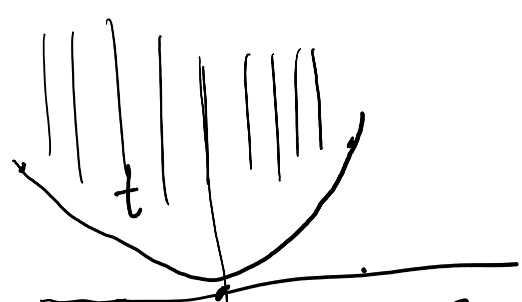
$$= \frac{p-1}{p} \cdot \sum_i \theta_i^{\frac{p}{p-1}}$$

$\frac{1}{p} + \frac{1}{q} = 1$ $\frac{1}{q} = 1 - \frac{1}{p}$

$$= \frac{1}{q} \|\theta\|_q^q$$

$$\mathrm{Epi}(f) = \{ (x, t) : f(x) \leq t \}$$

$f(x) = x^2$



<u>Thm</u>. $\mathrm{Epi}(f)$ is closed

$\Rightarrow (f^*)^* = f$.

<u>Pf.</u> By convexity $\mathrm{Epi}(f) = \bigcap \{ (x, t) : t \geq \theta^T x - b \}$

$$\mathcal{H} = \{ (\theta, b) : f(x) \geq \theta^T x - b \}$$

$$f(x) = \sup \theta^T x - b$$

$\mathcal{H}$: set of supporting halfspaces

$f(x) = \sup \quad \theta x - b$

$\mathcal{H} = \{(\theta, b) : \ \dots\}$ $\quad$ $\mathcal{H}$: set of supporting halfspaces of $\text{Epi}(f)$.

Fix θ. Then $\quad b \geq \theta^T x - f(x) \quad \forall x$

$$b^*_\theta = \sup_x \ \theta^T x - f(x) = f^*(\theta)$$

$$\therefore \quad f(x) = \sup_\theta \ \theta^T x - f^*(\theta) = (f^*)^*(x).$$

<u>Ex 3.</u>

$$f(x) = \sum_i e^{x_i}$$

$$f^*(\theta) = \max_x \ \theta^T x - \sum_i e^{x_i}$$

$$= \begin{cases} 0 & \text{if } \theta_i = 0 \ \forall i \\ \infty & \text{if } \exists i \ \ \theta_i < 0 \\ \sum_i \theta_i \ln \theta_i - \theta_i & \text{o.w.} \end{cases}$$

Duality helps in many ways.

Efficiency.

(P) $\quad \max \ C \cdot X$	$\min \ b^T y \quad$ (D)
$A_i \cdot X = b_i \quad i = 1..m$	$\sum y_i A_i \succeq C$

$A_i \cdot X = b_i \quad i = 1..m$ $X \succeq 0$	$\sum y_i A_i \succeq C$
$n^2 (Z + n^4) \qquad \#\text{var} = n^2$	$m(Z + n^\omega + m^2) \qquad \#\text{var} = m$

if $m < n^2$, faster to solve dual.

But how to recover primal solution from dual?

We use duality!

Suppose we solve (D) using CPM

and obtain y:

$$b^T y \le \underbrace{\min_{\substack{\sum y_i A_i \succeq C \\ y_i \ge 0}} b^T y}_{} + \varepsilon$$

$$= \min_{v^T \left(\sum_i y_i A_i - C\right) v \ge 0 \ \forall v, \|v\| = 1} b^T y$$

By CPM, with cutting planes S

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$$\min_{\substack{v^T(\sum y_i A_i - C)v \geq 0 \\ v \in S}} b^T y \leq OPT + \varepsilon$$

Now take dual of LHS.

$$\min_{\substack{v^T(\sum y_i A_i - C)v \geq 0 \\ v \in S}} b^T y = \min_{y} \max_{\substack{\lambda_v \geq 0 \\ v \in S}} b^T y - \sum_v \lambda_v v^T \left(\sum_i y_i A_i - C\right) v$$

$$= \max_{\lambda_v \geq 0} \min_{y} b^T y + C \cdot \sum_v \lambda_v v v^T - \sum_i y_i \sum_v \lambda_v v^T A_i v$$

$$= \max_{\substack{\lambda_v \geq 0 \\ X = \sum \lambda_v v v^T}} \min_{y} C \cdot X + \sum_i y_i (A_i \cdot X - b_i)$$

$$= \underbrace{\max_{\substack{X = \sum_{v \in S} \lambda_v v v^T \\ A_i \cdot X = b_i \\ \lambda_v \geq 0}} C \cdot X} \geq OPT - \varepsilon.$$

LP!

$$\max \sum_{v} \lambda_v \, v^T C v$$

$$\sum_{v} \lambda_v \left(v^T A_i v \right) = b_i \quad i = 1 \ldots m$$

$$\lambda_v \geq 0$$