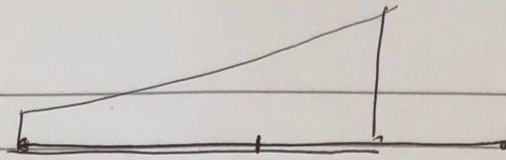
10 Mar 2024



Lemma f, g lower semi-continuous $\mathbb{R}^n \to \mathbb{R}$ integrable | T: bounded open convex set.

$$\int_{\mathbb{R}^n} f > 0 \qquad \int_{\mathbb{R}^n} g > 0 \qquad (\text{or}) \qquad \int_T f = 0 \qquad \int_T g > 0$$

$$\Rightarrow \exists\, a, b \in \mathbb{R}^n \quad \ell : [0,1] \to \mathbb{R}_+$$

$$\int_0^1 f((1-t)a+tb)\, \ell(t)^{n-1}\, dt > 0 \qquad \Big| \qquad \Rightarrow \int_0^1 f((1-t)a+tb)\, \ell(t)^{n-1} = 0$$

$$\int g(\ .\ )\ell(\ )^{n-1}\, dt > 0 \qquad \Big| \qquad \int g(\quad)\ell(\ )^{n-1} > 0.$$

Sometimes more convenient to have a product form.

Lemma. $f_1, f_2, f_3, f_4 : \mathbb{R}^n \to \mathbb{R}_+$ integrable; f_1, f_2 upper semi-continuous, f_3, f_4 lower semi-continuous.

Then the following are equivalent

(a) $$\int_K f_1 \int_K f_2 \le \int_K f_3 \int_K f_4 \quad \forall \text{ convex body } K$$

(b) $$\int_N f_1 \int_N f_2 \le \int_N f_3 \int_N f_4 \quad \forall \text{ needles } N.$$

$$\boxed{N: \quad a, b \in \mathbb{R}^n,\ \ell : [0,1] \to \mathbb{R}_+. \qquad \int_N f = \int_0^1 f((1-t)a+tb)\, \ell(t)^{n-1}\, dt.}$$

Isoperimetric theorems for convex bodies can typically be generalized to logconcave functions.

Thm: Π: logconcave distribution with support of diameter D.

$\forall S_1, S_2, S_3$ partition of $\mathbb{R}^n$.

$$\Pi(S_3) \geq \frac{2}{D}\, d(S_1, S_2) \min\{\Pi(S_1), \Pi(S_2)\}.$$

Thm π ~~f~~ logconcave ~~density~~ distribution on R^n with support K. S_1, S_2, S_3 partition of K into measurable sets.

$$\pi(S_3) \geq d_K(S_1, S_2)\, \pi(S_1)\, \pi(S_2).$$

Thm π in R^n. S_1, S_2, S_3. $\mathbb{E}_{\pi}(\|X - \bar{x}\|^2) \leq R^2$.

$$\pi(S_3) \geq \frac{\ln 2}{R}\, d(S_1, S_2) \min \pi(S_1), \pi(S_2).$$

(unbounded support is OK).

Recall our proof of Euclidean Isoperimetry

convex body in $\mathbb{R}^n \longrightarrow$ logconcave function in $\mathbb{R}$. (needle)

If we apply the same methods to logconcave f

logconcave f in $\mathbb{R}^n \longrightarrow$ logconcave function in $\mathbb{R}$.

Lemma. $f_1\, f_2,\, f_3\, f_4$ $\mathbb{R}^n \to \mathbb{R}_+$ continuous. Then the following are equivalent:

(a) $\forall$ logconcave $F: \mathbb{R}^n \to \mathbb{R}_+$ with compact support

$$\int_{\mathbb{R}^n} F f_1 \int F f_2 \leq \int F f_3 \int F f_4$$

(b) $\forall$ exponential needle E

$$\int_E f_1 \int_E f_2 \leq \int_E f_3 \int_E f_4.$$

$$\boxed{E: a, b \in \mathbb{R}^n,\ \gamma \in \mathbb{R}. \quad \int_E f = \int_0^1 f((1-t)a + tb)\, e^{\gamma t}\, dt}$$

Let us prove the d_K isoperimetry for general logconcave f. (3)

f logconcave $\longleftrightarrow$ π_f distribution. support K.

we want to show

~~Suppose~~ $\forall S_1, S_2, S_3$ s.t. $\pi_f(S_3) \geq d_K(S_1,S_2)\, \pi_f(S_1)\, \pi_f(S_2)$.

$$f_i = \mathbb{1}_{S_i} \quad f_4 = \mathbb{1}_K$$

$$\int f f_1 \int f f_2 \le d_K(S_1,S_2) \int f f_3 \int f f_4 .$$

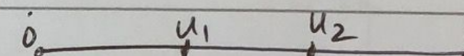
By LL, $\Longleftarrow$ $\forall E$

$$\int_E f_1 \int_E f_2 \le d_K(S_1,S_2) \int_E f_3 \int_E f_4 .$$

in 1-d, it suffices to consider the case when S_1, S_2 are single intervals.

for needle E, a, b, γ:

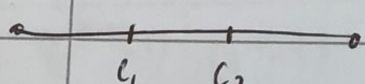
$$Z_i = \{ t \in [0,1] : a(1-t) + tb \in S_i \}$$



WLOG $\quad Z_1 = [0, u_1] \quad Z_2 = [u_2, 1] \quad Z_3 = [u_1, u_2]$.

$$c_1 = (1-u_1)a + u_1 b \qquad c_2 = (1-u_2)a + u_2 b$$

$$d_K(c_1, c_2) = \frac{|c_1 - c_2|\,|a-b|}{|a-c_1|\,|c_2-b|} = \frac{|u_1-u_2|}{u_1(1-u_2)}$$



we need to show that

$$\frac{\int_{u_1}^{u_2} e^{\gamma t} \int_0^1 e^{\gamma t}}{\int_0^{u_1} e^{\gamma t} \int_{u_2}^1 e^{\gamma t}} \le \frac{|u_1-u_2|}{u_1(1-u_2)} .$$

Lem. $\quad 0 < u < v < w$. $\quad \dfrac{(e^v - e^u)(e^w - 1)}{(e^u - 1)(e^w - e^v)} \geq \dfrac{(v-u)w}{u(w-v)}$.

Pf. $g(t) = \dfrac{e^t - 1}{t}$ is logconvex.

$$\left(\ln\left(\frac{e^t-1}{t}\right)\right)' = \frac{t}{e^t-1} \cdot \frac{te^t - (e^t-1)}{t^2} = \frac{1 - e^t + te^t}{t(e^t-1)}$$

$$\left(\quad\right)'' = \frac{e^t + e^{-t} - 2 - t^2}{(e^t-1)^2 t^2 e^{-t}} \geq 0.$$

$\therefore \forall\ 0 \leq a < b, c < d \quad a+d = b+c.$

$$\frac{e^a-1}{a} \cdot \frac{e^d-1}{d} \geq \frac{(e^b-1)}{b} \cdot \frac{(e^c-1)}{c}$$

$a = v-u \quad b = v$

$d = w \quad c = w-u$

$$\frac{(e^{v-u}-1)}{v-u} \cdot \frac{e^w-1}{w} \geq \frac{e^v-1}{v} \, \frac{e^{w-u}-1}{w-u}$$

$$\frac{(e^v-1)(e^{w-u}-1)}{(e^{v-u}-1)(e^w-1)} \leq \frac{v(w-u)}{w(v-u)}.$$

$$\frac{(e^v-1)(e^{w-u}-1) - (e^{v-u}-1)(e^w-1)}{(e^{v-u}-1)(e^w-1)} \leq \frac{vw - vu - wv + wu}{w(v-u)}$$

$$= \frac{(e^v - e^{v-u})(e^{w-v}-1)}{(e^{v-u}-1)(e^w-1)} \leq \frac{u(w-v)}{w(v-u)} \quad \blacksquare$$

Back to OPT.

$$\min\ c^T x \quad \geq \quad \max\ b^T y$$

$$Ax = b \qquad\qquad A^T y + s = c$$

$$x \geq 0. \qquad\qquad s \geq 0.$$

$$(P) \qquad\qquad (D).$$

interior $x > 0$ $\qquad\qquad s > 0$.

Thm 1. $x \in P$, $s \in D$ OPTIMAL iff $x^T s = 0$.
If P, D are nonempty $\exists\, x^* \in P$, $s^* \in D$ s.t. $x^{*T} s = 0$
$x^* + s^* > 0$.

Lemma. $\forall x \in P$, $s \in D$
$$c^T x - b^T y = x^T s.$$

Pf. $c^T x - b^T y = c^T x - x^T (A^T) y = x^T s.$

$$c^T x = b^T y + x^T \cancel{s} \leq \max_{\substack{A^T y + s = c \\ s \geq 0}} b^T y + x^T s \leq \min_{x \in P} c^T x + x^T s.$$

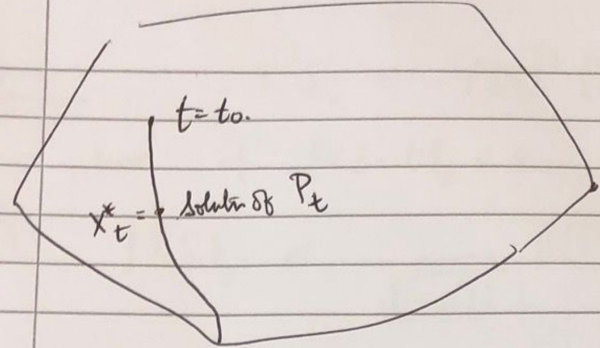
The trouble with solving P is the $x \geq 0$ constraint.
Let's smoothen this out? $\quad \mu > 0$.

$$(P_\mu): \quad \min\ c^T x - \mu \sum_{i=1}^n \ln x_i$$
$$Ax = b.$$

Lemma. If P has an ~~in~~ interior, then $OPT(P_\mu)$ is unique and $\mu > 0$. given by

$$xs = \mu$$
$$Ax = b$$
$$A^T y + s = c$$
$$x, s \geq 0.$$

($xs = t$ could be a general vector, means $x_i s_i = t_i$).



Idea: reduce $t \to 0$. $\qquad t \approx xs$

x^*_t is approximate solution for $t' < t^*$

improve | $\tilde{x}_{t'} \to x^*_{t'}$

Repeat |

$x \to x+\delta_x \qquad y \to y+\delta_y \qquad s \to s+\delta_s$.

$$(x+\delta_x)(s+\delta_s) = t - xs.$$
$$A\delta_x = 0$$
$$A^T\delta_y + \delta_s = 0.$$
$$x+\delta_x,\ s+\delta_s > 0.$$

$$(*) \qquad X\delta_s + S\delta_x = \delta_\mu := t - xs.$$
$$A\delta_x = 0$$
$$A^T\delta_y = -\delta_s.$$

Lemma: Let $P = X A^T \left(A \frac{X}{S} A^T\right)^{+} A S^{-1}$

Then $X\delta_s = P\delta_\mu \qquad S\delta_x = (I-P)\delta_\mu$.

$P^2 = P$. projection matrix. (But is not symmetric)

$$P_0 = \left(\frac{X}{S}\right)^{\frac{1}{2}} A^T \left(A \frac{X}{S} A^T\right)^{+} A \left(\frac{X}{S}\right)^{\frac{1}{2}} = (XS)^{-\frac{1}{2}} P \, (XS)^{\frac{1}{2}}$$

$$P = (XS)^{\frac{1}{2}} P_0 (XS)^{-\frac{1}{2}}.$$

$P_0 = P_0^T$

$P_0^2 = P_0$.

Lem. $$X^{-1}\delta_x = (SX)^{-1/2}(I - P_0)(SX)^{-1/2}\delta_\mu$$

and if $\|XS - t\|_2 \le \varepsilon$ then $X + \delta_x > 0$ for $\varepsilon < \frac{1}{2}$.

Pf. $$\|X^{-1}\delta_x\|_2 \le \frac{1}{\sqrt{(1-\varepsilon)t}} \|(I - P_0)(SX)^{-1/2}\delta_\mu\|_2$$

$\delta_\mu = t - XS$

$$\le \frac{1}{\sqrt{(1-\varepsilon)t}} \|(SX)^{-1/2}\delta_\mu\|_2 \le \frac{1}{(1-\varepsilon)t}\|\delta_\mu\|_2 \le \frac{\varepsilon t}{(1-\varepsilon)t} = \frac{\varepsilon}{1-\varepsilon}$$

< 1.

Lem. $X \to \hat{X}$ $\quad S \to \hat{S}$

$$\|XS - t\|_2 \le \varepsilon t \Rightarrow \|\hat{X}\hat{S} - t\|^2 \le (\varepsilon^4 + 16\varepsilon^5)t^2$$

Pf. $$\|\hat{X}\hat{S} - t\|_2^2 = \sum_i (x_i s_i + \underline{x_i \delta_{s_i} + s_i \delta x_i} + \delta_{x_i}\delta_{s_i} - t)^2$$

$$= \sum_i (\delta_{x_i}\delta_{s_i})^2 \le (1+\varepsilon)^2 t^2 \sum_i \left(\frac{\delta_{x_i}}{x_i}\right)^2\left(\frac{\delta_{s_i}}{s_i}\right)^2$$

(Cauchy - Schwarz) $\qquad |x_i s_i| \le (1+\varepsilon)t$

$$\le (1+\varepsilon)^2 t^2 \left\|\frac{\delta_x}{x}\right\|_4^2 \left\|\frac{\delta_s}{s}\right\|_4^2$$

$$\le (1+\varepsilon)^2 t^2 \left(\frac{\varepsilon}{1-\varepsilon}\right)^4 \le (\varepsilon^4 + 16\varepsilon^5)t^2.$$

Thm. x_0, s_0 feasible. $\|XS - t\| \le \varepsilon t$ $\quad \varepsilon < \frac{1}{4}$.

$$\left[\begin{array}{l} \text{compute } x + \delta_x,\ s + \delta_s \\ t \leftarrow \alpha t \end{array}\right.$$

with $\alpha = 1 - \frac{1}{10\sqrt{n}}$

converges in $O\left(\sqrt{n}\log\left(\frac{t_0}{\varepsilon}\right)\right)$ iterations.

Pf $\phi(t) = \|xs - t\|_2^2$ By induction $\phi(t) \le \frac{t^2}{16}$.

after one step $\phi(\hat{x}, \hat{s}) \le \frac{t^2}{50}$. $h = \frac{1}{10\sqrt{n}}$.

Set $t' = t(1-h)$

$$\phi(t(1-h)) = \sum_i \left(x_i s_i - t(1-h)\right)^2 \le 2\sum(x_i s_i - t)^2 + 2\sum_i t^2 h^2$$

$$\le 2\frac{t^2}{50} + 2\cancel{n}\frac{t^2 \cdot 1}{100\cancel{n}} = \frac{3t^2}{50}.$$

$$< \frac{t^2(1-h)^2}{16}.$$