Volume

Wednesday, April 9, 2025 9:47 AM

[illegible]

$B^n \subseteq K \subseteq R\,B^n \qquad \varepsilon > 0 \;, \; \delta > 0$

Estimate $\mathrm{vol}(K)$ to within $(1+\varepsilon)$. | with prob. $1-\delta$.

$$(1-\varepsilon)\mathrm{vol}(K) \le A \le (1+\varepsilon)\mathrm{vol}(K)$$

$$\mathrm{Vol}(K) = \mathrm{Vol}(B = K_1)\,\frac{\mathrm{Vol}(K_2)}{\mathrm{Vol}(K_1)} \cdots\cdots \frac{\mathrm{Vol}(K = K_m)}{\mathrm{Vol}(K_{m-1})}$$

$$K_i = K \cap 2^{i/n} B \qquad m = \lceil n \log_2 R \rceil.$$

Lem. $\mathrm{Vol}(K_{i+1}) \le 2\,\mathrm{Vol}(K_i)$

Algo. Sample $X^{(1)} \ldots X^{(N)} \sim K_{i+1}$

$$Y_i = \frac{|\{X^{(j)} \in K_i\}|}{N}$$

Output $\dfrac{\mathrm{Vol}(B)}{\prod_{i=1}^{m} Y_i}$.

Thm With $N = O\left(\frac{m}{\varepsilon^2}\right)$ samples in each phase, algo finds a $(1+\varepsilon)$ approximation with prob. $\ge \frac{3}{4}$.

a $(1+\varepsilon)$ approximation with prob. $\geq \frac{3}{4}$.

(Can be boosted by taking median of $O(\log \frac{1}{\delta})$ ind. trials)

Pf. $\mathbb{E}(Y_i) = r_i = \frac{Vol(K_i)}{Vol(K_{i+1})}$ $Vol(K) = Vol(B) \cdot \frac{1}{\prod_i r_i}$.

$\frac{1}{2} \leq r_i \leq 1$.

$$Var(Y_i) = \frac{1}{N} r_i (1-r_i) \leq \frac{1}{4N}.$$

$$Pr\left(|Y_i - r_i| > \frac{t}{2\sqrt{N}} \right) \leq \frac{1}{t^2}$$

We will have m Y_i's. So set $t = \frac{1}{2\sqrt{m}}$

Then overall failure prob $\leq \frac{1}{4m} \cdot m = \frac{1}{4}$.

$Error_i \leq \frac{1}{4}\sqrt{\frac{m}{N}}$ error in $\prod_i Y_i$ could be

$\leq \frac{1}{2}\sqrt{\frac{m}{N}}\, r_i$

$$\left(1 + \frac{1}{2}\sqrt{\frac{m}{N}}\right)^m \qquad N \sim \frac{m^3}{\varepsilon^2}$$

$$\longrightarrow \left(1 + \frac{\varepsilon}{2m}\right)^m \leq (1+\varepsilon).$$

Hmm ....

$Y = \prod_i Y_i$ we need $\frac{Var(Y)}{\mathbb{E}(Y)^2}$ to be small.

$Var(Y) = \mathbb{E}(Y^2) \quad -1 \quad = \quad \prod \mathbb{E}(Y_i^2) \quad -1$

$$\frac{Var(Y)}{\mathbb{E}(Y)^2} = \frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} - 1 = \prod_i \frac{\mathbb{E}(Y_i^2)}{\mathbb{E}(Y_i)^2} - 1$$

$$= \prod_i \left(\frac{Var(Y_i)}{\mathbb{E}(Y_i)^2} + 1 \right) - 1$$

For us, $\frac{Var(Y_i)}{\mathbb{E}(Y_i)^2} = \frac{r_i(1-r_i)}{N \cdot r_i^2}$

$$\leq \frac{1}{N}.$$

$$\leq \left(1 + \frac{1}{N}\right)^m - 1$$

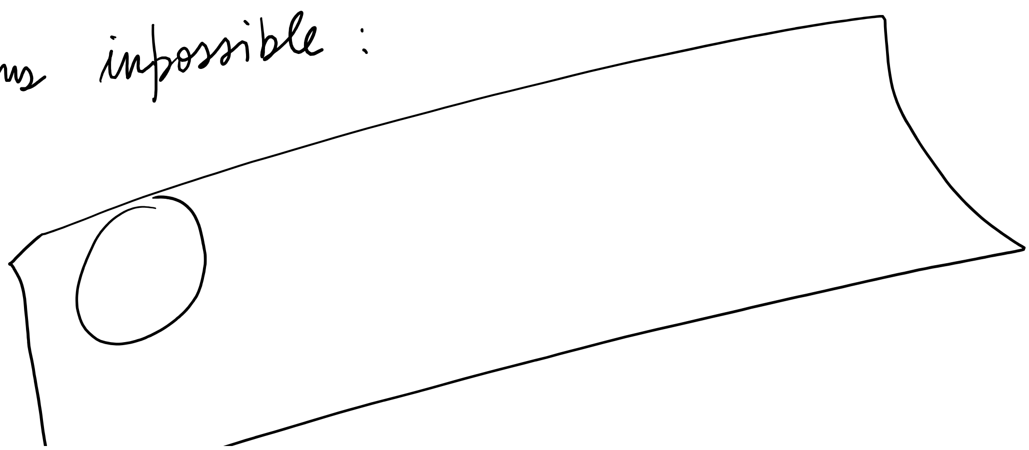
$$\boxed{N = \frac{8m}{\varepsilon^2}} \qquad \leq \frac{\varepsilon^2}{4}.$$

$$\therefore \Pr\left(\left|Y - \prod_i r_i\right| > \varepsilon \right) < \frac{1}{4}.$$

Total #samples $= m \cdot \frac{8m}{\varepsilon^2} = O\left(\frac{n^2}{\varepsilon^2} (\log_2 R)^2 \right)$.

Can we do better?

Seems impossible:



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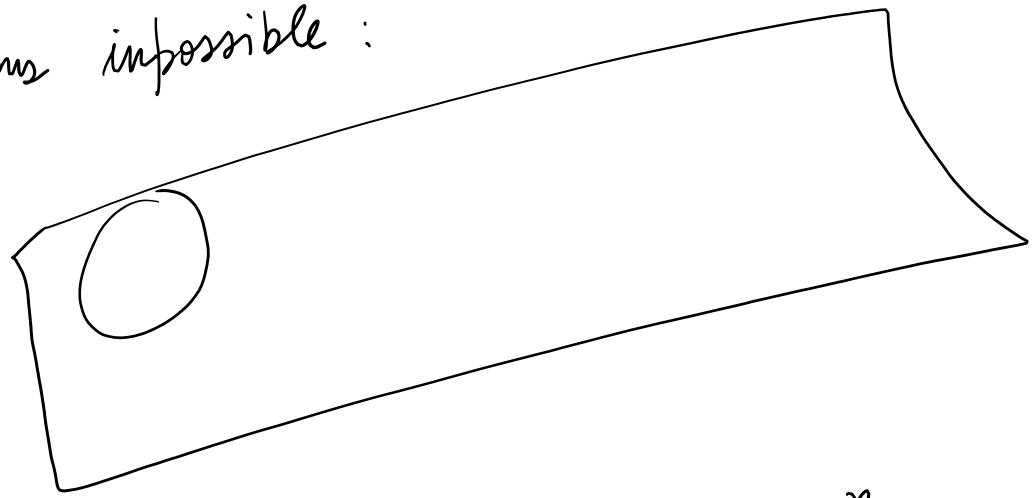

$$\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B)} \text{ can be } \sim n^n$$

if we have $m = o(n)$ phases, then some ratio $r_i \gg \mathrm{poly}(n)$

and then we need a huge # samples ...

$$(n^n)^{\frac{1}{m}} \cdot m \quad \text{minimized at } m = n \log n$$

Hmm... but do we need r_i's to be small?

e.g. suppose $\mathbb{E}(f(x))$ is large.

But $\dfrac{\mathrm{Var}(f(x))}{\mathbb{E}(f(x))^2}$ is small. Then we still need few samples.

This is not the case for $f(x) = \mathbb{1}_{K_i}$

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Consider a more general "annealing".

Start at easy distribution

f_0 (e.g. Gaussian or $e^{-\|x\|}$) $\qquad\qquad$ $f_m = \mathbb{1}_K$.

f_1

$\ldots$

$$\int f_m = \int f_0 \cdot \frac{\int f_1}{\int f_0} \cdot \frac{\int f_2}{\int f_1} \cdot \;\cdots\; \frac{\int f_m}{\int f_{m-1}}$$

How to estimate $\dfrac{\int f_{i+1}}{\int f_i}$?

- Sample $x \sim f_i$
- $Y = \dfrac{f_{i+1}(x)}{f_i(x)}$

Lemma. $\mathbb{E}(Y) = \displaystyle\int \frac{f_{i+1}(x)}{f_i(x)} \cdot \frac{f_i(x)}{\int f_i(x)}\, dx$

$$f_i(x) \qquad \int f_i(x)$$

$$= \frac{\int f_{i+1}(x)}{\int f_i(x)}.$$

What about variance? How should we choose f_i?

$f_i(x) = f(a_i x)$ or $f(x)^{a_i}$ $\qquad f_0(x) = e^{-\frac{n\|x\|^2}{2}}$

or $f_0(x) = e^{-\sqrt{n}\|x\|}$

$a_0 = 2n$

$\downarrow$

$a_m = \underbrace{\varepsilon}_{2R}$

$$\int_K f(a_0 x) \geq \int_{B^n} f(a_0 x) \geq \left(1 - \frac{1}{2^n}\right) \int f(a_0 x).$$

$$= \left(1 - \frac{1}{2^n}\right) \cdot C_0.$$

e.g. $\displaystyle\int_{\mathbb{R}^n} e^{-\frac{n\|x\|^2}{2}} dx = \left(\sqrt{\frac{2\pi}{n}}\right)^n = C_0.$

$$\int_K f(a_m x) \geq \left(1 - \frac{\varepsilon}{2}\right) \int_K dx = \left(1 - \frac{\varepsilon}{2}\right) \mathrm{vol}(K).$$

How to set a_i?

How to set a_i ?

Let's estimate variance.

$$Y = \frac{f_{i+1}(x)}{f_i(x)} = \frac{f(x)^{a_{i+1}}}{f(x)^{a_i}}$$

$$\mathbb{E}(Y) = \frac{\int_K f(x)^{a_{i+1}}}{\int_K f(x)^{a_i}} = \frac{F(a_{i+1})}{F(a_i)}$$

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} = \frac{\int_K \frac{f(x)^{2a_{i+1}}}{f(x)^{2a_i}} \cdot \frac{f(x)^{a_i}}{F(a_i)} dx}{F(a_{i+1})^2 / F(a_i)^2} = \frac{F(2a_{i+1} - a_i)\, F(a_i)}{F(a_{i+1})^2} .$$

Suppose F were logconcave.

Then RHS ≤ 1 since $a_{i+1} = \frac{(2a_{i+1} - a_i) + a_i}{2}$

But this is impossible.

Lemma 1. (a) $\forall$ convex K, logconcave f : $a^n \int_K f(ax)\,dx$ is logconcave for $a > 0$.

(b) $a^n \int f(x)^a$

(b) $\forall$ logconcave integrable f : $a^n \int f(x)^a$ logconcave + ____________.

So $a^n F(a)$ is logconcave.

$$(a_{i+1}^n)^2 F(a_{i+1})^2 \geq (2a_{i+1}-a_i) F(2a_{i+1}-a_i)\, a_i F(a_i)$$

or
$$\frac{F(2a_{i+1}-a_i)F(a_i)}{F(a_{i+1})^2} \leq \left(\frac{a_{i+1}^2}{a_i(2a_{i+1}-a_i)}\right)^n$$

Lemma 2 For $a_{i+1} = a_i\left(1-\frac{1}{\sqrt{n}}\right)$, $\frac{E(Y^2)}{E(Y)^2} \leq 5$ for $n \geq 8$.

Pf.
$$\left(\frac{\left(1-\frac{1}{\sqrt{n}}\right)^2}{\left(2-\frac{2}{\sqrt{n}}-1\right)}\right)^n = \left(\frac{1-\frac{2}{\sqrt{n}}+\frac{1}{n}}{1-\frac{2}{\sqrt{n}}}\right)^n = \left(1+\frac{1}{n\left(1-\frac{2}{\sqrt{n}}\right)}\right)^n$$
$$\leq e^{\frac{n}{n-2\sqrt{n}}} < 5.$$

Pf (L1). (a) $g(x,t) = g(x)\cdot \mathbb{1}_{\frac{x}{t}\in K}$

is logconcave if g is logconcave, K is convex.

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So, by (P-L) $\int_x g(x,t)$ is logconcave.

$$= \int_{x \in tK} g(x)\, dx \qquad y = \frac{x}{t} \qquad dy = \frac{dx}{t^n}$$

$$= t^n \int_{y \in K} g(ty)\, dy \quad \text{is logconcave!}$$

(b) $g(x,t) = g\left(\frac{x}{t}\right)^t$ is logconcave

$$g\left(\frac{x+y}{2}, \frac{t+s}{2}\right) = g\left(\frac{x+y}{t+s}\right)^{\frac{t+s}{2}}$$

$$= g\left(\frac{x}{t}\cdot\frac{t}{t+s} + \frac{y}{s}\cdot\frac{s}{t+s}\right)^{\frac{t+s}{2}}$$

$$\geq \left(g\left(\frac{x}{t}\right)^{\frac{t}{t+s}} \cdot g\left(\frac{y}{s}\right)^{\frac{s}{t+s}}\right)^{\frac{t+s}{2}}$$

$$= \sqrt{g\left(\frac{x}{t}\right)^t g\left(\frac{y}{s}\right)^s}.$$

Hence $\int g(x,t)$ is logconcave

Hence $\int_X g(x,t)$ is logconcave

$$= \int_X g\left(\frac{x}{t}\right)^t dx \qquad y = \frac{x}{t}$$

$$dy = \frac{dx}{t^n}$$

$$= t^n \int_y g(y)^t dy$$

So need only $m = O\left(\sqrt{n} \log \frac{a_0}{a_m}\right) = O\left(\sqrt{n} \log nR\right)$

and therefore $N = O\left(\frac{m}{\varepsilon^2}\right)$ samples per phase

and $O\left(\frac{m^2}{\varepsilon^2}\right) = O\left(\frac{n}{\varepsilon^2}(\log nR)^2\right)$ samples

total!

Moreover, each phase provides a "warm" start to the next phase.

$$\mathbb{E}_{f_i}\left(\frac{f_i/F_i}{f_{i+1}/F_{i+1}}\right) = \int \frac{f_i(x)}{f_{i+1}(x)} \cdot \frac{F_{i+1}}{F_i} \cdot \frac{f_i(x)}{F_i} \, dx$$

$$= \frac{F(2a_i - a_{i+1})\, F(a_{i+1})}{F(a_i)^2}$$

$$\le \left(\frac{a_i^2}{a_{i+1}(2a_i - a_{i+1})} \right)^n = \left(\frac{1}{(1-\frac{1}{\sqrt{n}})(1+\frac{1}{\sqrt{n}})} \right)^n$$

$$= \left(\frac{1}{(1-\frac{1}{n})} \right)^n$$

$$= \left(1 + \frac{1}{n-1}\right)^n < 4$$

for $n \ge 2$.

L_2 warm start

or χ^2

Time. $\tilde{O}(n)$ samples. time per sample.

amortized $\tilde{O}(n^3)$ per sample in general.

Thm. Complexity of $(1+\varepsilon)$ volume estimation is $\tilde{O}(n^4)$.

Thm __________ $(1+\varepsilon)$ logconcave integration is $\tilde{O}(n^4)$.

Some technical issues :

— samples are not perfectly independent

— distribution is not exactly target.

⋮

— need to do rounding to ensure sampling time depends polynomially in $\log R$ and not R.