Integration and Rounding

Sunday, April 13, 2025 5:09 PM

Yunlong

We used annealing for volume computation.

$\int_K dx$. What about general logconcave integration? Let $\int f$ be the goal.

We can still use

$$\int f = \int f_0 \cdot \frac{\int f_1}{\int f_0} \cdots \frac{\int f_m}{\int f_{m-1}}$$

$f_m = f$ $\quad$ f_0 is "easy".

We need some bounds on f similar to $rB_n \subseteq K \subseteq RB_n$.

(1) $\mathbb{E}(\|x - x_0\|^2) \leq R^2$

(2) $L_f(\frac{1}{8})$ contains a ball of radius r.

(1) is weaker than $K \subseteq RB_n$

supp(f) can be unbounded.

supp(f) can be unbounded.

(2) is similar to $rB_n \subseteq K$.

But is it reasonable?

E.g. is it true for isotropic logconcave f?

$f \leftrightarrow \pi_f$ $\quad \mathbb{E}_{\pi_f}(X) = 0 \quad \mathbb{E}_{\pi_f}(XX^T) = I$

$\mathbb{E}(\|X\|^2) = n$.

We say that f or π_f is <u>a-rounded</u> if any level set of measure p contains a ball of radius ap.

<u>Lemma 1.</u> Isotropic logconcave f is $\frac{1}{e}$-rounded.

<u>Thm 1.</u> Hit-and-run mixes in $\tilde{O}\left(n^2 \frac{R^2}{r^2}\right)$ steps for (r, R)-rounded f from χ^2-warm start.

<u>Thm 2.</u> Ball walk mixes in $\tilde{O}\left(n^2 \frac{R^2}{r^2}\right)$ from $O(1)$-warm start.

$(Q_0 \ldots \quad \chi^2(Q, Q) \quad \mathbb{E}\left(\frac{dQ_0}{\ldots} - 1\right)^2$

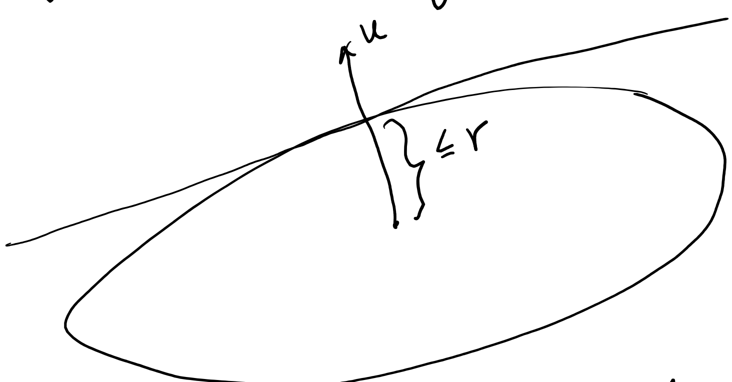
$$\left(\text{Recall} \quad \chi^2(Q_0, Q) = \mathbb{E}_Q\left(\frac{dQ_0}{dQ} - 1\right)^2\right)$$

M-warm start: $\dfrac{dQ_0}{dQ} \le M$

<u>Pf (Lem 1)</u>. f isotropic density. L: level set of f of measure p.

$$h(x) = \begin{cases} f(x) & x \in L \\ 0 & \text{o.w.} \end{cases}$$ restriction of f to L.

Suppose h does not contain a ball of radius r centered at mean of h. Then $\exists u$ s.t. a supporting plane of L along u is at distance $\le r$.



Consider projection of h along u.

$$\int_{z_u \le u \cdot x \le r + z_u} h(x) \le 1 \cdot r$$

$\uparrow$ max isotropic f.

But $$\int_{x \ge z_u} h(x) \ge \frac{1}{e} \int h(x) \Rightarrow r \ge \frac{p}{e}.$$

$x \geq z_u$

Rounding: Given oracle access to logconcave f, find affine transformation to put f in near-isotropic position. $\left[\frac{1}{C} \leq \mathbb{E}((u^T x)^2) \leq C \right]$

Thm. $X^{(1)} \dots X^{(N)} \sim f$ logconcave in $\mathbb{R}^n$.

$$Z = \frac{1}{N} \sum_{j=1}^{N} X^{(j)} \qquad A = \frac{1}{N} \sum (X^{(j)} - Z)(X^{(j)} - Z)^T.$$

$N = O\left(\frac{n}{\varepsilon^2} \ln \frac{1}{\delta}\right)$, with prob. $\geq 1-\delta$

$$(1-\varepsilon)\, \mathrm{Cov}\, f \preceq A \preceq (1+\varepsilon)\, \mathrm{Cov}(f).$$

$$\begin{tikzcd} & \text{Integration} \arrow[dr] & \\ \text{ROUNDING} \arrow[rr, bend left] & & \text{SAMPLING} \arrow[ll, bend left] \end{tikzcd}$$

We can use the following annealing to integrate f:
- Round f (near-isotropic)
- Restrict f to ball of radius $c\sqrt{n}$. $K = \mathrm{supp}(f) \cap B(0, c\sqrt{n})$
- start with f_0 = uniform in K.

- start with f_0 = uniform in K.

$$f_i(x) = f(x)^{a_i} \qquad a_i = \frac{1}{B}\left(1+\frac{1}{\sqrt{n}}\right)^i$$

$$B = \ln\left(\frac{\max_K f}{\min_K f}\right)$$

$$a_m = 1.$$

- \# phases $= \sqrt{n} \ln B$

\# samples per phase $= O\left(\frac{\sqrt{n}}{\varepsilon^2} \ln B\right)$.

How to round?

$M_f = \max f$

$$t_k = \frac{M_f}{2^{(1+\frac{1}{n})^k}}$$

1. Put $L = \{x : f(x) \geq f(x_0)/2\}$ in near-isotropic position.

2. For $k = 1, 2, \ldots$ $C n \log n$

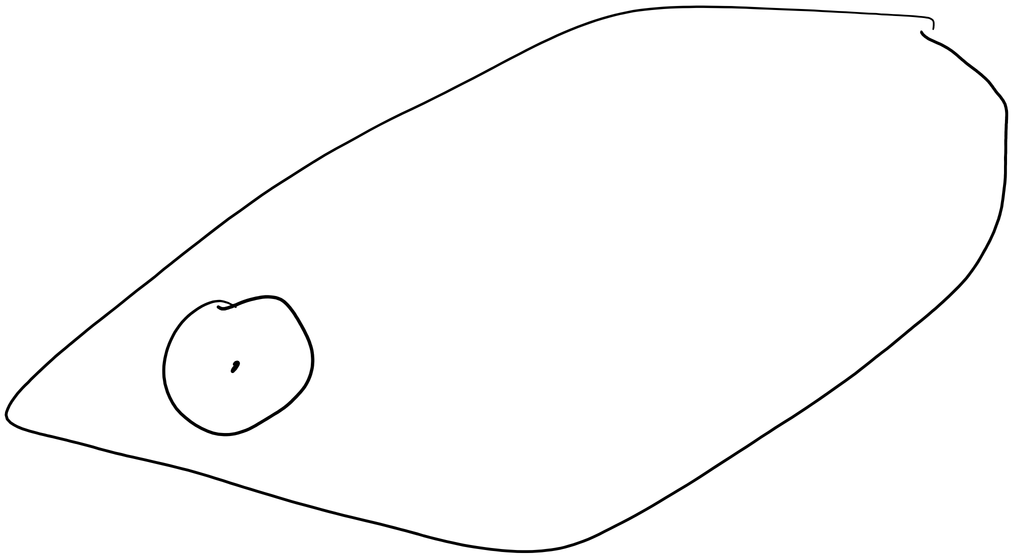
Put f restricted to $\{x : f(x) \geq t_k\}$

near-isotropic position

- Sample from $L(t_k)$

- Sample from $L(t_k)$
- Estimate covariance C_k. Apply $C_k^{-1/2}$.

How to do (1)? Make K isotropic.



$$K_i = K \cap 2^{i/n} B.$$

<u>Lemma 2</u>. If K_i is (2-) isotropic, then K_{i+1} is (2×)2-isotropic

Need to show $\forall u$

$$\mathbb{E}_{K_{i+1}}\left((u^T x)^2\right) = \frac{\int_{K_{i+1}} (u^T x)^2}{\mathrm{Vol}(K_{i+1})} \geq \frac{\int_{K_i} (u^T x)^2}{2\,\mathrm{Vol}(K_i)} \geq \frac{\frac{1}{2}}{2} \geq \frac{1}{4}.$$

$$\mathbb{E}_{K_{i+1}}\left((u^T x)^2\right) = \frac{\int_{K_{i+1}} (u^T x)^2}{\cdots} \leq \frac{\int_{2^{1/n} K_i} (u^T x)^2 \, dx}{\cdots} \leq 2^{1+\frac{2}{n}} \frac{\int_{K_i} (u^T y)^2}{\mathrm{Vol}(K_i)} \cdots$$

$$\frac{K_{i+1}}{Vol(K_{i+1})} = \frac{2^{1/n} K_i}{Vol(K_i)} \qquad \frac{\cdot\cdot}{Vol(K_i)}$$

$y = 2^{-1/n} x$

$\leq 4.$

Lemma 3. $0 < s < t \leq M_f$

f_t: restriction of f to $\mathcal{L}_f(t)$.

f_t is isotropic $\Rightarrow$ f_s is 6-isotropic.

Rounding: $\tilde{O}(n)$ phases

$O(n)$ samples per phase

$\tilde{O}(n^2)$ per sample since f_i is isotropic and has $O(1)$-warm start.

$= \tilde{O}(n^4)$.

Integration $O(\sqrt{n})$ phases

$O(\sqrt{n})$ samples per phase

$O(n^3)$ per sample. f_i is "well-rounded" $\frac{R^2}{r^2} = O(n)$.

$= O^*(n^4)$

$= O^*(n^4).$

$\overline{r^2}$