Intro & Overview.

Monday, January 6, 2020 5:42 PM

Yellow

Algorithms are discrete-time procedures to solve problems.

e.g. SORT

Does f have a SATISFYING assignment?

What should my next move be?

Translate "Vágyok boldog" to English

$\boxed{P1}$ Optimization: $\min_{x \in \Omega} f(x)$.

very general. E.g. $\exists x : f(x) = 1$

$\iff \min |f(x) - 1|$

<u>also intractable</u>. $f(x) = \begin{cases} 1 & x \neq x^* \\ 0 & \text{o.w.} \end{cases}$ $\cdot x^*$

$\min f(x)$

takes an infinite # evaluations of f.

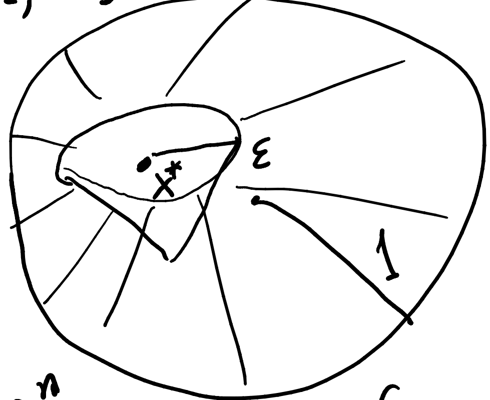
But what if f is Lipshitz, i.e. $\|\nabla f\| \leq 1$.

$L(f) = \sup_{X,Y} \frac{|f(x) - f(y)|}{\|X - Y\|}$. and with bounded domain.

$$L(f) = \sup_{X,Y} \frac{\|f(x) - f(y)\|}{\|X-Y\|}.$$

$$f(x) = \min\{\|x - x^*\|_2, \varepsilon\} \qquad x \in B_2^n$$

is 1-Lipshitz.



$$\mathrm{vol}(\{x : \|x - x^*\|_2 \le \varepsilon\}) \le \varepsilon^n \cdot \mathrm{vol}(B_2^n)$$

Finding x takes $\Omega\left(\frac{1}{\varepsilon}\right)^n$ calls to f.

<u>Exercise</u>. Show that $O\left(\frac{1}{\varepsilon}\right)^n$ calls suffice.

$\boxed{P2}$ Sampling. $f : \mathbb{R}^n \to \mathbb{R}$, sample x with density

$$\nu(x) \propto e^{-f(x)}.$$

e.g. $f(x) = \frac{\|x\|^2}{2}$, $\quad f(x) = \begin{cases} 0 & x \in K \\ \infty & \text{o.w.} \end{cases}$

OPT $\to$ Sampling (binary search)

$\therefore$ <u>intractable</u>!

<u>Examples in practice</u>: flows, matching, LP

regression, sampling Gaussian, convex body.

Suppose we assume f is convex.

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$$f(\lambda x+(1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

then all kinds of wonderful things happen!

① Any local minimum $\nabla f(x^*)=0 \Rightarrow$ global min $f(x^*) \le f(x)\ \forall x$.

Why?.. Lemma. $f(y) \geq f(x) + \nabla f(x)^T (y-x)$

Pf. $g(\lambda) = f((1-\lambda)x + \lambda y)$ $\quad g(0) = x \quad g(1) = y$

by convexity $\quad g(\lambda) \le (1-\lambda) g(0) + \lambda g(1)$.

$$g(1) \geq g(0) + \frac{g(\lambda) - g(0)}{\lambda}$$

$\lambda \to 0$

$$g(1) \geq g(0) + g'(0)$$

$$g'(0) = \nabla f(x)^T (y-x)$$

$$f(y) \geq f(x) + \nabla f(x)^T (y-x).$$

② Seperation. K is closed convex, i.e. $x, y \in K \Rightarrow [x,y] \subseteq K$.

$y \notin K \Rightarrow \exists a : \quad a^T y > \max_{x \in K} a^T x$.

This lets us do binary search!

Examples of convex functions.

Gradient Descent.

while $\|\nabla f(x)\| > \varepsilon$:

$x \leftarrow x - h \nabla f(x)$

Goal: Find x: $\|\nabla f(x)\| \le \varepsilon$.

Q. how many iterations of GD? What h to use?

Why this goal: x is near-minimum in its neighborhood.

But even this is not true if ∇f can change quickly.

Assume. $\|\nabla^2 f\|_{op} \le L$ i.e. ∇f is L-Lipshitz.

Thm. $\nabla^2 f$ is L-Lip, $\forall \varepsilon > 0$, starting at x^0, we reach x with $\|\nabla f(x)\| \le \varepsilon$ in at most

$$2\frac{L}{\varepsilon^2}\left(f(x_0) - f(x^*)\right) \text{ steps}$$

Lem 1.
$$f\left(x - \frac{1}{L}\nabla f(x)\right) \le f(x) - \frac{1}{2L}\|\nabla f(x)\|^2$$

Pf.

Thm [Taylor]. $\forall$ $k+1$ times differentiable $g: \mathbb{R} \to \mathbb{R}$, $\forall x, y$

$$g(y) = g(x) + \sum_{i=1}^{k} \frac{(y-x)^i}{i!} g^{(i)}(x) + \frac{(y-x)^{k+1} g^{(k+1)}(\zeta)}{(k+1)!} \qquad \zeta \in [x, y]$$

$$g(y) = g(x) + \sum_{i=1} \frac{(y-x)^i g^{(i)}(\cdot)}{i!} \cdots \frac{\cdots}{(k+1)!} \qquad \xi \in [x,y]$$

$$g(1) = g(0) + g'(0) + \frac{1}{2} g''(z) \quad z \in [0,1].$$

$$g(t) = f((1-\lambda)x + \lambda y)$$

$$f(y) = f(x) + \nabla f(x)^T (y-x) + \frac{1}{2}(y-x)^T \nabla^2 f(z)(y-x)$$

$y = x - \frac{1}{L}\nabla f(x)$ gives

$$f\left(x - \frac{1}{L}\nabla f(x)\right) \leq f(x) - \frac{1}{L}\|\nabla f(x)\|^2 + \frac{1}{2} L \cdot \frac{\|\nabla f(x)\|^2}{L^2}$$

$$\leq f(x) - \frac{1}{2L} \cdot \|\nabla f(x)\|^2.$$

Start at $f(x^0)$, min is $f(x^*)$. each step decrease is $\geq \frac{\varepsilon^2}{2L}$.

$$\Rightarrow \# \text{steps} \leq \frac{2L}{\varepsilon^2} \left(f(x^0) - f(x^*)\right). \qquad \blacksquare.$$

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f convex

$f(x) = x^2$

$$\mathrm{Epi}(f) = \{(x,t) : f(x) \leq t\}$$

<u>Exercise:</u> $\mathrm{Epi}(f)$ is convex.

$$\min f(x) = \underset{t}{\arg\min}\ \mathrm{Epi}(f).$$

min convex function $\iff$ minimizing Linear function over convex sets.

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Next time: GD for convex f.