

# Duality & Reductions

Monday, January 27, 2020 10:35 PM

(for

ORACLES for a convex set  $K$

$\text{MEM}(x)$  : YES if  $x \in K$

NO O.W.

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$\text{SEP}(x)$  : YES if  $x \in K$

NO O.W., and  $c : c^T y \leq c^T x + y \in K$ .

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$\text{VAL}(c)$  : OUTPUTS  $\max_{x \in K} c^T x$

or "K is EMPTY"

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$\text{OPT}(c)$  :  $x$  s.t.  $c^T x \leq c^T y + x \in K$   
 $\in K$

or "K is EMPTY".

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Cutting Plane Method :  $\text{OPT}_K \rightarrow \text{SEP}_K$ .

$\text{SEP}$  is stronger than  $\text{MEM}$

SEP is stronger than MEM

OPT ————— VAL.

For different problems, different oracles can be more convenient / efficient.

E.g.  $K = \{x : Ax \geq b\}$   $\text{SEP}_K$  is easy - check all constraints  
 $K = \text{Conv Hull } \{a_1, \dots, a_m\}$   $\text{OPT}_K$  is easy -  $\sup_i c^T a_i$

Q. Are these fundamentally equivalent?

ORACLES for convex functions

$\text{EVAL}_f(x) : f(x)$ .

$\text{GRAD}_f(x) : f(x), g$  s.t.  $\forall y \quad f(y) \geq f(x) + g^T(y-x)$ .

( $g$  is a subgradient of  $f$  at  $x$ ).

Recall  $\delta_K(x) = \begin{cases} 0 & x \in K \\ \infty & x \notin K \end{cases}$  convex.

$\text{MEM}_K = \delta_K$ .

A useful (and important) concept.

A useful (and important) ~~concept~~  
 Dual of a convex function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ .

$$f^*(\theta) = \sup_{x \in \mathbb{R}^n} \theta^T x - f(x) \quad \forall \theta \in \mathbb{R}^n.$$

Note:  $f^*$  is max of affine functions (one per  $x$ )  
 so  $f^*$  is convex.

$$- f^*(0) = - \inf_x f(x).$$

$$\begin{aligned} - \delta_K^*(c) &= \sup_x c^T x - \delta_K(x) \\ &= \sup_{x \in K} c^T x \end{aligned} \quad \boxed{\text{EVAL}_{\delta_K^*} = \text{VAL}_K.}$$

Lemma  $\nabla f^*(\theta) = \arg \max_x \theta^T x - f(x)$

Pf.  $x_\theta = \arg \max_x \theta^T x - f(x).$

$$f^*(\theta) = \theta^T x_\theta - f(x_\theta)$$

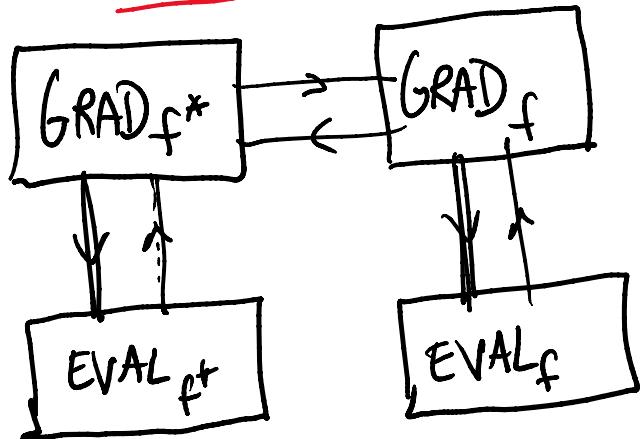
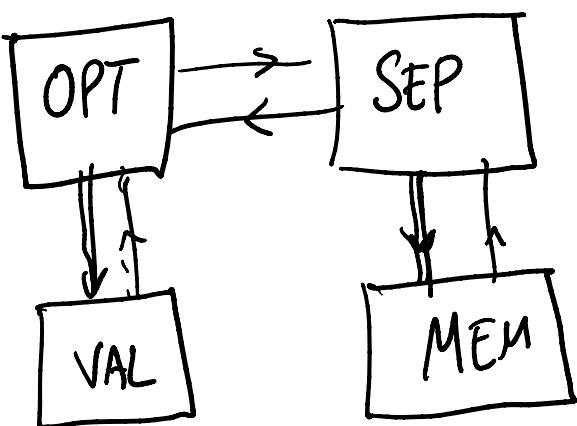
$$\forall \eta \quad f^*(\eta) \geq \eta^T x_\theta - f(x_\theta)$$

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$$\Rightarrow f^*(\eta) - f^*(\theta) \geq x_\theta^T (\eta - \theta)$$

$$\Rightarrow x_\theta \in \text{Subgrad.}(f^*).$$

$$\text{GRAD}_{g_k^*} \equiv \text{OPT}_k.$$



Thm. convex  $f$ ,  $\text{Epi}(f)$  closed, then  $f^{**} = f$ .

Pf.  $\text{Epi}(f) = \{(x, t) : f(x) \leq t\}$  is a convex set.

So it is an intersection of halfspaces  $A$ .

We can assume of the form  $(\theta, b) : \theta^T x \geq b$   $\forall x \in \text{Epi}(f)$ .

(Why? :  $\theta^T x + \alpha t \leq b$

but if  $(x, t) \in \text{Epi}(f)$ , then  $\forall t' \geq t, (x, t') \in \text{Epi}(f)$ .

so take  $\bar{t} = \operatorname{argmax}_t \theta^T x + \alpha t \leq b$   
 $(x, t) \in \text{Epi}(f)$

$$\text{So } \forall t \in \mathbb{R}^+ \quad \exists x_t \in \mathcal{X} \quad \theta^\top x_t \leq b - \alpha t$$

$$f(x) \geq \theta^\top x - b \quad \forall (\theta, b) \in \mathcal{H}$$

$$\text{fix } \theta. \quad b \geq \theta^\top x - f(x) \quad \forall x$$

$$b \geq \sup_x \theta^\top x - f(x) = f^*(\theta)$$

$$\Rightarrow f(x) = \sup_{(\theta, b)} \theta^\top x - b = \sup_\theta \theta^\top x - f^*(\theta) = f^{**}(x) \quad \square$$

Example

$$f(x) = \frac{1}{p} \sum x_i^p$$

$$f^*(x) = \frac{1}{q} \sum x_i^q$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$f(x) = ax - b$$

$$f^*(\theta) = \begin{cases} 0 & \theta = a \\ \infty & \text{otherwise} \end{cases}$$

$$\nabla f \iff \nabla f^*$$

$$h = h^{**}$$

$$\dots n^* \dots L^*(a)$$

$$\begin{aligned}
 \min_x g(x) + h(Ax) &= \min_x \max_{\theta} g(x) + \theta^T Ax - h^*(\theta) \\
 &\Rightarrow = \max_{\theta} \min_x g(x) + (A^T \theta)^T x - h^*(\theta) \\
 &= \max_{\theta} - \max_x (-A^T \theta)^T x - g(x) - h^*(\theta) \\
 &= - \min_{\theta} g^*(-A^T \theta) + h^*(\theta).
 \end{aligned}$$

Sion's minimax Theorem.  $X \subset \mathbb{R}^n$  compact, convex set.

$Y \subset \mathbb{R}^m$  convex.  $f: X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$  st.

$f(x, \cdot)$  is upper semi-continuous and quasi-concave on  $Y$   
 $f(\cdot, y)$  is lower quasi convex on  $X$ .

Then

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y)$$

Example. SDP.

$$C_i v - A_i \cdot x = b_i \quad i=1, 2, \dots, m$$

Primal:  $\max_{X \geq 0} C \cdot X \quad A_i \cdot X = b_i \quad i=1, 2, \dots, m$

Dual:  $\min_y b^T y \quad \sum_{i=1}^m y_i A_i \leq C$

$X$  is  $n \times n$  symmetric.

Primal takes  $O(n^2(Z + n^4))$   $Z$ : total #nnz in  $A_i$ .

Dual takes  $O(m(Z + n^6 + m^2))$

better when  $m < n^2$ , often the case.

But how to recover primal solution from dual?  
(we only solve to some error  $\epsilon$ ).

$$\min_b b^T y = \min_v v^T (\sum_i y_i A_i - C) \quad v \geq 0 \quad \forall v$$

When running the cutting plane method on DUAL,  
...  $v^T (\sum_i y_i A_i - C) \geq 0$

we get planes of the form  $v^T(\sum_i y_i A_i - C)v \geq 0$   
 Let  $S$  be the set of all such  $v$ . At the end,

$$\min_{\substack{v \\ \sum_i y_i A_i - C \\ v \in S}} b^T v \leq \min_{\substack{v \\ v^T(\sum_i y_i A_i - C)v \geq 0 \\ v \in S}} b^T v + \epsilon.$$

Now consider RHS.

$$\begin{aligned} \min_{\substack{v \\ i \\ v \in S}} b^T v &= \min_y \max_{\substack{\lambda_v \geq 0 \\ v \in S}} b^T y - \sum_{v \in S} \lambda_v v^T (\sum_i y_i A_i - C) v \\ &= \max_{\lambda_v \geq 0} \min_y C \cdot \sum_{v \in S} \lambda_v v v^T + b^T y - \sum_i y_i (A_i \cdot \sum_{v \in S} \lambda_v v v^T) \\ &= \max_{\substack{X \\ X = \sum_{v \in S} \lambda_v v v^T, \lambda_v \geq 0}} \min_y C \cdot X + \sum_i y_i (b_i - A_i \cdot X) \\ &= \max_{X = \sum_{v \in S} \lambda_v v v^T, \lambda_v \geq 0} C \cdot X \quad \text{(else the OPT is } -\infty) \end{aligned}$$

$$\lambda = \langle v^*, v^* \rangle$$

$$A_i \cdot X = b_i$$

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This is  $\max_{v \in S} \sum \lambda_v (v^T C v)$

$$\sum_v \lambda_v (v^T A_i v) = b_i, \lambda_v \geq 0$$

an LP!

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