Hit-and-Run

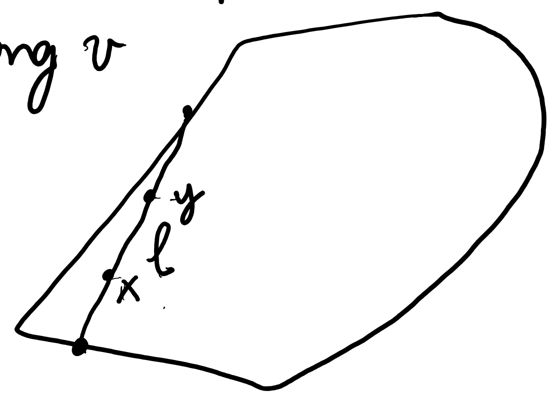
Wednesday, March 5, 2025 9:23 AM

The ball walk leads to polytime sampling and volume computation, but is not rapidly mixing from arbitrary starts for general convex bodies

Hit-and-Run

At $x \in K$,

- pick uniform random unit vector v
- go to y, uniform random point on ℓ chord through x along v



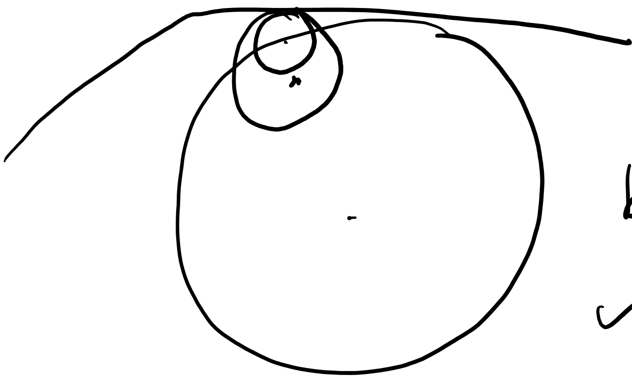
Lemma.

$$P_x(A) = \frac{1}{\mathrm{vol}_{n-1}(S^{n-1})} \int_A \frac{2\,dy}{\|y-x\|^{n-1}\,\ell(x,y)}$$

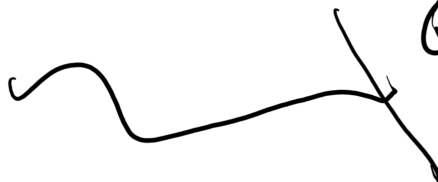
Markov chain is symmetric. Uniform dist is stationary

Lazy $\Rightarrow$ Uniform Q is unique stationary.

Does this potentially fix the issues with the ball walk? small local conductance / step-size near boundary.

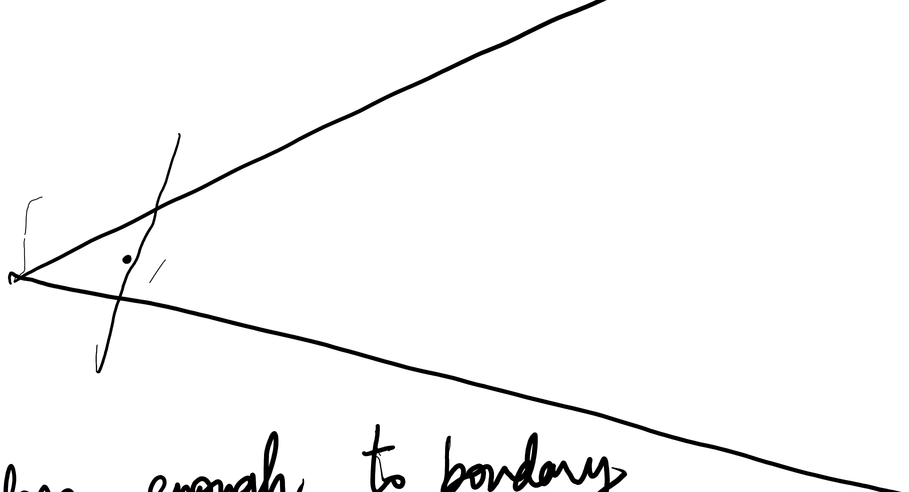


What if we used smaller balls closer to the boundary?



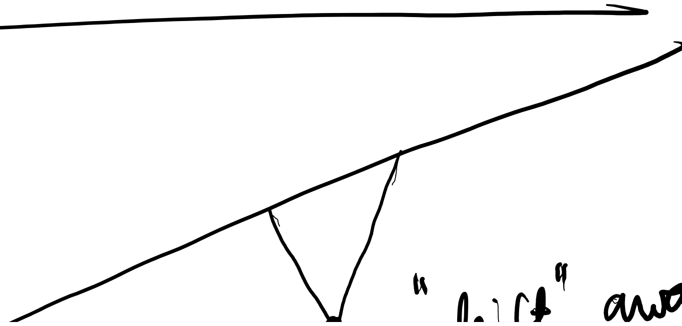
$Q_t \longrightarrow \partial K$.

What about H-&-R?

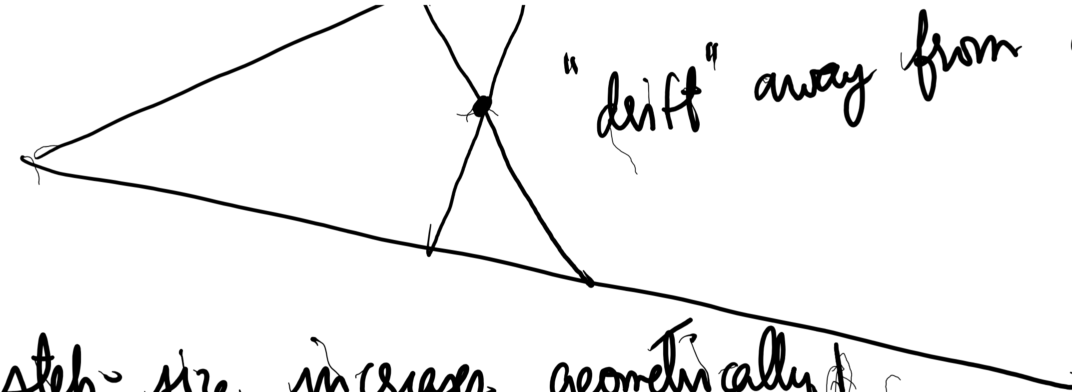


close enough to boundary
most steps will be arbitrarily small!

But,



"lift" away from corner!



"drift" away from corner!

step-size increases geometrically!

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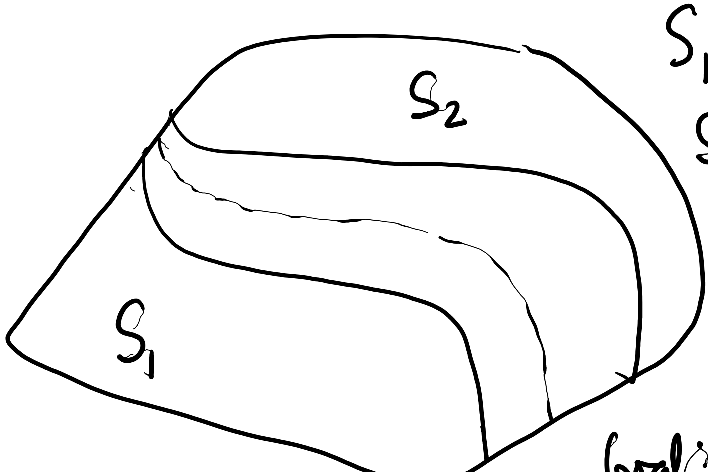
local conductance is not an issue, formally.

How to prove global conductance?

Recall Proof Strategy

- Overlap of one-step distributions
  ("nearby" points have overlapping P_u, P_v)
- Isoperimetry (large subsets have large boundaries)



$S_1 = \{x \in S \mid P_x(\bar{S}) < 0.001\}$

$S_2 = \{x \in \bar{S} \mid P_x(S) < 0.001\}$

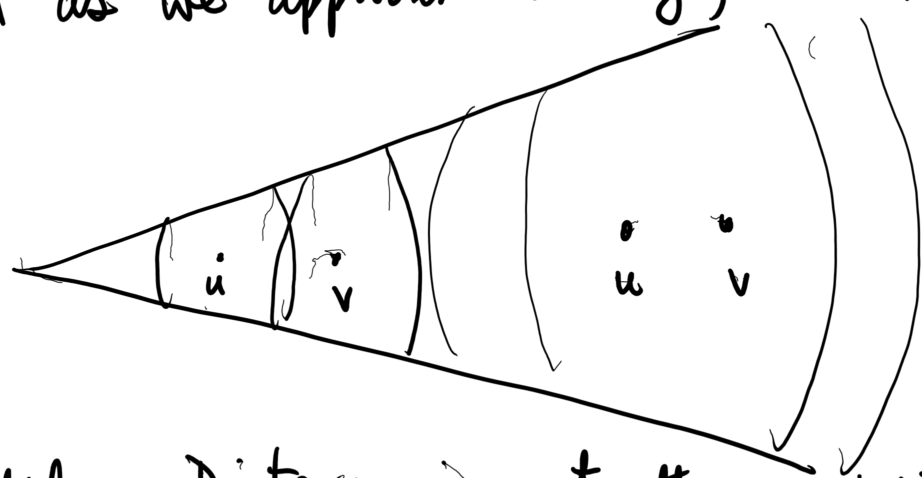
$S_3 = K \setminus S_1 \setminus S_2$

Goal: show S_3 is large

- show S_1, S_2 are far
- use isoperimetry.

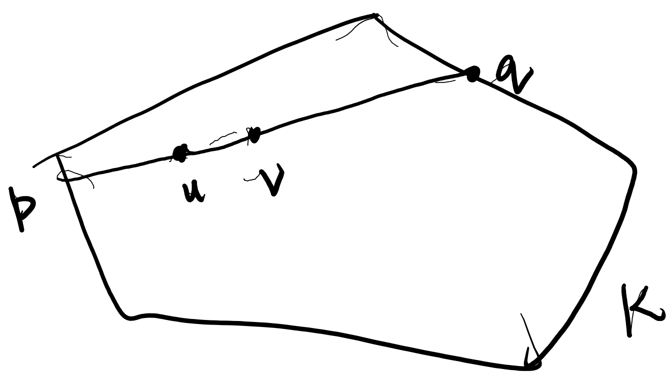
— use isoperimetry.

But as we approach boundary, overlap of nearby points decreases.



Euclidean Distance is not the right notion.

$$d_K(u,v) = \frac{|u-v||p-q|}{|p-u||v-q|} = (p:v:u:q)$$ "cross-ratio" distance



not a metric.

But $d_H(u,v) = \ln(1 + d_K(u,v))$ is a metric ("Hilbert Distance").

Next we define a "median" step-size.

$u \in K$ $F(u)$ is s.t. $P_{x \sim P_u}\left(\|u-x\|_2 \leq F(u)\right) = \frac{1}{8}$.

With prob. $\frac{7}{8}$, random step from u is farther than $F(u)$.

Lemma. $u, v \in K$, $d_K(u,v) \leq \frac{1}{8}$, $\|u-v\|_2 \leq \frac{2}{\sqrt{n}} \max\{F(u), F(v)\}$

Lemma. $u, v \in K$, $d_K(u,v) \le \frac{1}{8}$, $\|u-v\|_2 \le \frac{2}{\sqrt{n}} \max\{F(u), F(v)\}$

$$d_{TV}(P_u, P_v) < 1 - \frac{1}{500}.$$

$\Rightarrow$ for $u \in S_1$, $v \in S_2$, $d_{TV}(P_u, P_v) > 1 - \frac{1}{500}$ $\Rightarrow$

either $d_K(u,v) > \frac{1}{8}$ or $\|u-v\|_2 > \frac{2}{\sqrt{n}} \max\{F(u), F(v)\}$.

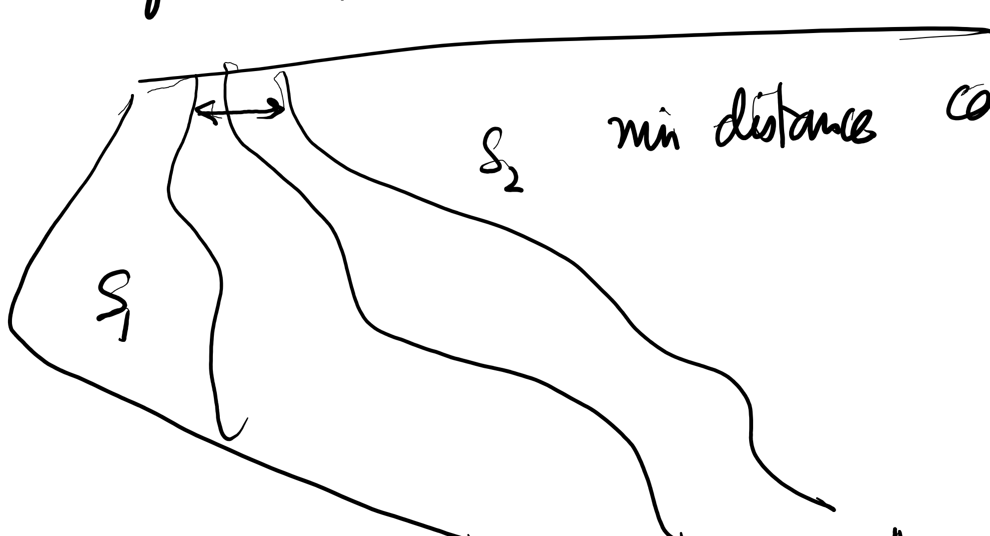
What about Isoperimetry?

Thm For any partition S_1, S_2, S_3 of convex body K,

$$\mathrm{vol}(S_3) \ge d_K(S_1, S_2) \frac{\mathrm{vol}(S_1)\,\mathrm{vol}(S_2)}{\mathrm{vol}(K)}.$$

(i.e. $Q(S_3) \ge d_K(S_1, S_2)\, Q(S_1)\, Q(S_2)$)

Is this enough? No!



min distance can be tiny!

"warm" start,

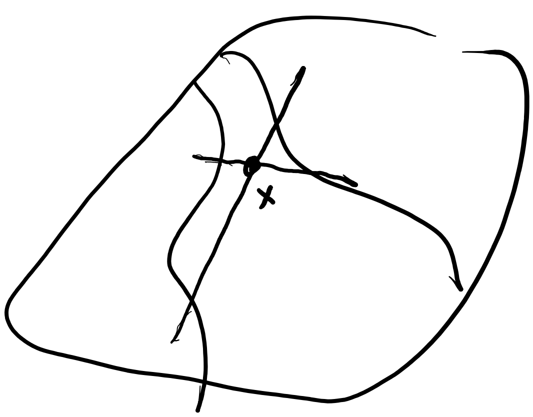
(allows proving mixing from a "warm" start, similar to the ball walk)

We need to somehow <u>average</u> the distance between S_1 and S_2. $\mathbb{E}_{\substack{u \in S_1 \\ v \in S_2}} (d(u,v))$ doesn't make sense as it can be <u>high</u> for any partition.

Neither does $\mathbb{E}_{\substack{u \in \partial S_1 \\ v \in \partial S_2}} d(S_1, S_2)$.

Instead we will weight every point in K.

$$\left[\begin{array}{ll} h : K \to \mathbb{R}_+ & \forall u,v \quad u \in S_1 \quad v \in S_2 \\ & x \in [u,v] \quad h(x) \le \frac{1}{3} \min\{1, d_K(u,v)\} \end{array}\right] (*)$$



<u>Thm</u> [Average distance Isoperimetry] Let $h(x)$ satisfy (*). S_1, S_2, S_3 partition of K.

Thm [Average distance Isoperimetry] ...

Then, $\mathrm{vol}(S_3) \geq \mathbb{E}(h(x)) \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}$.

What is a suitable h?

$$s(x) = \sup\left\{\lambda : \frac{\mathrm{vol}(x + \lambda B_n \cap K)}{\mathrm{vol}(\lambda B_n)} \geq \gamma\right\}$$

"step-size"

Lemma (1) $s(x)$ is a concave function

(2) $F(x) \geq \frac{s(x)}{32}$ for $\gamma \geq \frac{63}{64}$

(3) $B_n \subseteq K \Rightarrow \mathbb{E}_K(s(x)) \geq \frac{1-\gamma}{\sqrt{n}}$.

Thm Conductance of Hit-and-Run is $\Omega\left(\frac{1}{nD}\right)$

Pf Consider arbitrary $S \subset K$.

S_1, S_2, S_3 as before.

$u \in S_1, v \in S_2$, either $d_K(u,v) > \frac{1}{8}$ or $\|u-v\|_2 > \frac{2}{\sqrt{n}} \max\{F(u), F(v)\}$

Set $h(x) = \frac{s(x)}{48\sqrt{n}D}$ Then $h(x) \leq \frac{1}{24}$. $\forall x$.

$$\frac{\quad}{48\sqrt{n}D}$$

If $d_K(u,v) > \frac{1}{8}$ then $h(x) \le \frac{1}{3}\min\{1, d_K(u,v)\}$.

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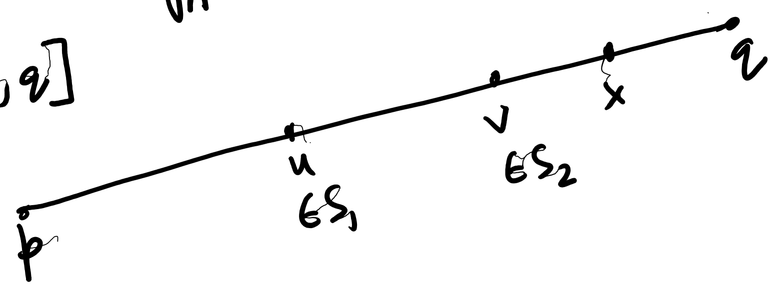
else if $\|u-v\|_2 > \frac{2}{\sqrt{n}} \max\{F(u), F(v)\}$,

Suppose $x \in [u, q]$



Since $s(x)$ is concave

$$s(x) \le \frac{\|p-x\|}{\|p-u\|} s(u) \le 32 \frac{\|p-q\|}{\|p-u\|} F(u)$$

$$\le 16\sqrt{n} \frac{\|p-q\| \, \|u-v\|}{\|p-u\| \, \|v-q\|} D$$

$$\le 16\sqrt{n} D \, d_K(u,v)$$

$$h(x) = \frac{s(x)}{48\sqrt{n}D} \le \frac{d_K(u,v)}{3}.$$

Applying isoperimetry $\mathrm{vol}(S_3) \ge \mathbb{E}(h(x)) \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}$

WLOG $\mathrm{vol}(S_1) \ge \frac{1}{2}\mathrm{vol}(S)$, $\mathrm{vol}(S_2) \ge \frac{1}{2}\mathrm{vol}(\bar{S})$

$\{\mathrm{vol}(S_1)\ \mathrm{vol}(S_2)\}$

WLOG $\mathrm{vol}(S_1) \leq \frac{1}{2}$

$$\mathrm{vol}(S_3) \geq \frac{1}{48\sqrt{n}D} \cdot \frac{1}{64\sqrt{n}} \cdot \frac{1}{2} \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}$$

$$= \frac{1}{CnD} \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}$$

$$\int_S P_x(\bar{S})\, dQ(x) \geq \frac{1}{2} \cdot \frac{1}{1000} \frac{\mathrm{vol}(S_3)}{\mathrm{vol}(K)} \geq \frac{1}{C'nD} \min\{Q(S_1), Q(S_2)\}.$$

$$\phi \geq \frac{1}{C'nD}.$$