# IPM

Tuesday, February 25, 2020 9:24 AM

LP. $\min c^T x - t\sum_i \ln x_i$ $\quad t \downarrow 0$

$Ax = b$

$K$: $x \geq 0$ $\quad x_i > 0$

$K$: $Ax \geq b$ $\quad -\sum_i \ln((Ax)_i - b_i)$

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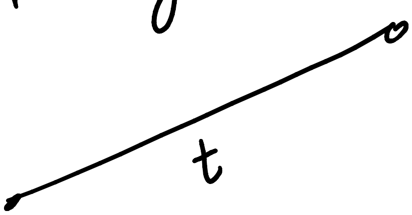
$f$: convex $\mathbb{R}^n \to \mathbb{R}$

$$\|v\|_x^2 = v^T \nabla^2 f(x)\, v$$

<u>self-concordance</u> $\forall h \quad \forall x \quad f(x) < \infty$

$$\boxed{D^3 f(x)[h,h,h]} \leq 2\|h\|_x^3 = 2\left(D^2 f(x)[h,h]\right)^{3/2}$$

On every line $\quad \boxed{g'''(t) \leq 2\, g''(t)^{3/2}}$




<u>Example</u> $g(x) = -\ln x$ $\quad g'(x) = \frac{-1}{x}$

$\frac{1}{x^2}$

$-\frac{2}{x^3}$

$-2 \leq 2 \cdot (1 \quad )^{3/2}$

$$-\frac{2}{x^3} \le 2 \cdot \left(\frac{1}{x^2}\right)^{3/2} \qquad x^3$$

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$$f(X) = -\ln\det X \qquad X \succ 0$$

$$Df[H] = -\operatorname{tr}(X^{-1}H)$$

$$D^2 f[H,H] = \operatorname{tr}(X^{-1}HX^{-1}H) = \operatorname{tr}(A^2)$$

$$D^3 f(H,H,H) = -2\operatorname{tr}(X^{-1}HX^{-1}HX^{-1}H) = -2\operatorname{tr}(A^3)$$

$$A = X^{-\frac{1}{2}} H X^{-\frac{1}{2}}$$

$$-2\operatorname{tr}(A^3) \le 2\operatorname{tr}(A^2)^{3/2} \qquad \checkmark$$

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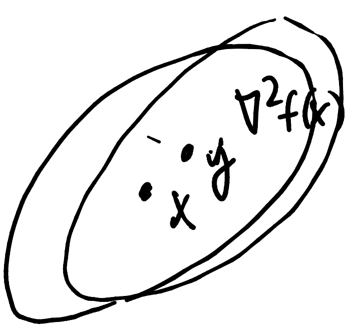
<u>Lema</u>. S.C. $\Rightarrow$ $D^3 f[h_1, h_2, h_3] \le 2\|h_1\|_x \|h_2\|_x \|h_3\|_x$

<u>Lema</u>. $\forall x, y$: $\|y-x\|_x < 1$ then

$$(1-\|y-x\|_x)^2 \nabla^2 f(x) \preccurlyeq \nabla^2 f(y) \preccurlyeq \frac{1}{(1-\|y-x\|_x)^2} \nabla^2 f(x)$$




$$\overbrace{u^T \nabla^2 f(x + t(y-x))\, u}^{D^2 f[u,u]}$$

Pf. $\alpha(t) = u^T \nabla^2 f(x + t(y-x))\, u$

$$\alpha'(t) = D^3 f(x+t(y-x))[y-x, u, u]$$

$$\leq 2 \|y-x\|_x \|u\|_x^2 \quad (*)$$

$u = y-x$ $$= 2\|y-x\|_x^3$$

$$\boxed{|\alpha'(t)| \leq 2\,\alpha(t)^{3/2}} \iff \frac{1}{2}\left|\frac{\alpha'(t)}{\alpha(t)^{3/2}}\right| \leq 1$$

$$\frac{d}{dt}\frac{1}{\sqrt{\alpha(t)}} \geq -1$$

$$\frac{1}{\sqrt{\alpha(t)}} - \frac{1}{\sqrt{\alpha(0)}} \geq -t \Rightarrow \frac{1}{\sqrt{\alpha(t)}} \geq \frac{1}{\|y-x\|_x} - t$$

$$\|y-x\|^2_{x+t(y-x)} = \alpha(t) \leq \frac{1}{\left(\frac{1}{\|y-x\|_x} - t\right)^2} = \frac{\|y-x\|_x^2}{(1-t\|y-x\|_x)^2}$$

(*)

$$\alpha'(t) \leq \underbrace{2\frac{\|y-x\|_x}{(1-t\|y-x\|_x)^2}}\alpha(t)$$

$$\left|\frac{d}{dt}\ln\alpha(t)\right| \leq -2\frac{d}{dt}\ln(1-t\|y-x\|_x)$$

$$\leq -2\ln(1-t\|y-x\|_x)$$

$$|\ln(\alpha(t)) - \ln\alpha(0)| \leq -2\ln(1 - t\|y-x\|_x)$$

$$\alpha(1) \leq \alpha(0) \cdot \frac{1}{(1-\|y-x\|)^2}$$

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<u>Lemma</u>. $y = x - \nabla^2 f(x)^{-1}\nabla f(x)$.

$r = \|\nabla f(x)\|_{\nabla^2 f(x)^{-1}} < 1$

$$\Longrightarrow \quad \|\nabla f(y)\|_{\nabla^2 f(y)^{-1}} < \frac{r^2}{1-r}$$

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Pf. Use $\nabla f(y) = \nabla f(x) + \int_0^1 \nabla^2 f(x+t(y-x))(y-x)\,dt$ and previous lemma.

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<u>Lemma</u> Suppose $\|\nabla f(x)\|_{\nabla^2 f(x)^{-1}} \leq \frac{1}{6}$

$$\Longrightarrow \quad \|x-x^*\|_{x^*}, \ \|x-x^*\|_x \leq 2\|\nabla f(x)\|_{\nabla^2 f(x)^{-1}}$$

$$f(x) \leq f(x^*) + 2\|\nabla f(x)\|^2_{\nabla^2 f(x)^{-1}}.$$

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Pf. $r = \|x - x^*\|_v \quad \left(r \leq \frac{1}{4}\right)$

$\underline{\text{Pf.}}$ $r = \|x - x^*\|_x$ $\left(r \leq \frac{1}{4}\right)$

$$\nabla f(x) = \nabla f(x) - \nabla f(x^*)$$
$$= \int_0^1 \underbrace{\nabla^2 f(x^* + t(x - x^*))}_{\succeq_r \nabla^2 f(x)(1 - (1-t)r)^2}(x - x^*)\,dt$$

$$\|\nabla f(x)\|_{\nabla^2 f(x)^{-1}} \geq \int_0^1 (1 - (1-t)r)^2 \, \underline{\|x - x^*\|_x} \, dt$$

$$\left(\frac{1}{6}\right) \geq \left(1 - r + \frac{r^2}{3}\right) \cdot r \quad \left(\Rightarrow r \leq \frac{1}{4}\right) \checkmark$$

$$\geq \frac{3r}{4}$$

$$\boxed{\|x - x^*\|_x \leq \frac{4}{3} \|\nabla f(x)\|_{\nabla^2 f(x)^{-1}}}$$

$$\|x - x^*\|_{x^*} \leq \frac{\|x - x^*\|_x}{1 - r} \leq \frac{4}{3} \cdot \frac{1}{\left(1 - \frac{1}{4}\right)} \|\nabla f(x)\|_{\nabla^2 f(x)^{-1}}$$

$(x - x^*) \nabla^2 f(x^*)(x - x^*)$

$$< 2 \|\nabla f(x)\|_{\nabla^2 f(x)^{-1}}.$$

$$f(x) = f(x^*) + \underbrace{\langle \nabla f(x^*), x - x^* \rangle}_{=0} + \int_0^1 (1-t)\,(x-x^*)^T \underline{\nabla^2 f(x^* + t(x-x^*))}(x-x^*)\,dt$$

$$\leq f(x^*) + \int_0^1 \frac{1-t}{(1-(1-t)r)^2} \|x-x^*\|_x^2 \,dt$$

$$\leq f(x^*) + \frac{1}{r^2}\left(\frac{r}{1-r} + \log(1-r)\right) \|x-x^*\|_x^2$$

$$\leq f(x^*) + \left(\frac{1}{2} + r\right) \|x-x^*\|_x^2$$

$$\leq f(x^*) + \left(\frac{1}{2}+r\right)\left(\frac{4}{3}\right)^2 \cdot \|\nabla f(x)\|^2_{\nabla^2 f(x)^{-1}}.$$

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$$\min_K \; c^T x \qquad \phi(x) \to \infty \qquad x \to \partial K$$

$$\ln x \to \infty \qquad x \to 0 \qquad \boxed{x \geq 0}$$

$$\min \; \phi_t(x) = t \cdot c^T x + \phi(x)$$

$\phi_t(x) = t \cdot c^T x + \phi(x)$

$\boxed{\text{increase } t}$ $\qquad x_t = \arg\min_x \phi_t(x)$

<u>Plan</u>. $x$ close to $x_1$

Repeat: 1) move closer to $x_t$

2) $t \leftarrow (1+h)t$

<u>Def</u>. $\phi$ is a $\nu$-selfconcordant barrier for $K$ if $-\phi$ is convex, self-concordant

- $\phi \to \infty$ as $x \to \partial K$
- $\|\nabla \phi(x)\|^2_{\nabla^2 \phi(x)^{-1}} \le \nu$.

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<u>Thm</u> $\exists \phi$ which achieves $\nu = n$ $\forall K$.

$-\sum_i \ln A_i x - b_i \qquad \nu = m$

$Ax = b$
$x \geq 0$

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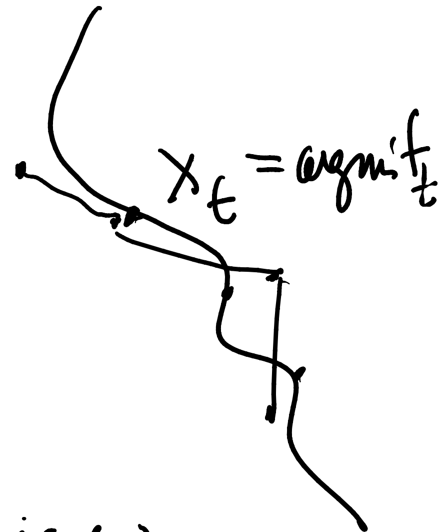
$\boxed{\text{IPM}}$ $\quad f_t(x) = t\,c^T x + \phi(x)$




$- \; t_0$

while $\quad t \le \dfrac{\nu + \sqrt{\nu}}{\varepsilon}$

(1) $\quad x \leftarrow x - \nabla^2 f_t(x)^{-1} \nabla f_t(x)$

(2) $\quad t \leftarrow t(1+h) \qquad h = \dfrac{1}{9\sqrt{\nu}}$

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<u>Lema</u> $\quad \langle \nabla\phi(x), y - x \rangle \le \nu \qquad \forall\, y, x \in K.$

<u>Th</u>. $\quad \boxed{\langle c, x_t \rangle \le \langle c, x^* \rangle + \dfrac{\nu}{t}}$

$$x^* = \underset{K}{\text{argm}}\; c^T x$$

$$tc + \nabla\phi(x_t) = 0$$

$$\langle c, x_t \rangle - \langle c, x^* \rangle =$$

$$\frac{1}{t} \langle \nabla\phi(x_t), x^* - x_t \rangle \le \frac{\nu}{t}.$$

$$\frac{1}{t}\langle \nabla\phi(x_t), x^{\cdot} - x_t \rangle - \frac{\quad}{t}$$

for $x$ s.t. $\|tc + \nabla\phi(x)\|_{\nabla^2\phi(x)^{-1}} \le \frac{1}{6}$

$$\langle c, x\rangle \le \langle c, x^*\rangle + \frac{\nu + \sqrt{\nu}}{t}.$$

$\varepsilon$-OPT $\qquad t = O\left(\frac{\nu}{\varepsilon}\right).$

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**Thm** $O\left(\sqrt{\nu}\,\log\left(\frac{\nu}{\varepsilon}\|c\|_{\nabla^2 f(x_0)^{-1}}\right)\right)$ iterations.