Software Correctness: The Construction of Correct Software

Loops

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  (c) For any predicate ϕ, if
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- The natural numbers are defined by **induction**,
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- To **prove** properties ϕ about natural numbers we use **complete induction** (c)

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  Inductive step:

  Assume **induction hypothesis** $\;m+n \geq m$,

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  End of proof

Addition of Natural Numbers (A Refresher on Iteration)

- Instead of recursion we can use **iteration** to add two natural numbers m and n

  $k = n$
  $i = m$
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- We prefer **recursion** for **specification** – it is easy to **comprehend**
- Often **iteration** (and mutable variables) are more **efficient**
- So, we prefer to use it for **implementation**

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- Any iteration, such as,

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  can be written as a **tail-recursive** program (where the recursive call always comes last)

  $(k, i) = add(k, i)$ **where**
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- Tail-recursive functions and the corresponding iterative programs are essentially the same

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- As a consequence, we can use induction to reason about iteration as well
- But we do not need to take a detour via a tail-recursive function
- We can do it directly using **(inductive) invariants**

Example: Iteratively Counting Down to Zero

- The following program counts down to zero

```
var n: Z = randomInt()
assume(n >= 0)


while (n > 0) {


  n = n - 1

}


assert(n == 0)
```

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- Let's see what we know to be true at different locations in the program
- By reasoning about the program we want to establish that the assertion `n == 0` holds at the end of the program

Tracing Facts by Looking Forward

- At the beginning of the loop body the loop condition must be true

```
var n: Z = randomInt()
assume(n >= 0)


while (n > 0) {
  // deduce n > 0

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- This fact must be true
  because the loop condition has just been evaluated to true
  when the loop body is entered

Tracing Facts by Looking Forward

- After the loop the negation of the loop condition must be true

```
var n: Z = randomInt()
assume(n >= 0)


while (n > 0) {
  // deduce n > 0

  n = n - 1

}
// deduce n <= 0

assert(n == 0)
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Tracing Facts by Looking Forward

- After the loop the negation of the loop condition must be true

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var n: Z = randomInt()
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}
// deduce n <= 0

assert(n == 0)
```

- This fact must be true
  because the loop condition has just been evaluated to false
  when the loop is exited

Tracing Facts by Looking Forward

- From the assumption we can also determine that `n >= 0` must be true before the loop

```
var n: Z = randomInt()
assume(n >= 0)

// deduce n >= 0
while (n > 0) {
  // deduce n > 0

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}
// deduce n <= 0

assert(n == 0)
```

Tracing Facts by Looking Forward

- From the assumption we can also determine that `n >= 0` must be true before the loop

```
var n: Z = randomInt()
assume(n >= 0)

// deduce n >= 0
while (n > 0) {
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  n = n - 1

}
// deduce n <= 0

assert(n == 0)
```

- We know this because of the assumption we make

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var n: Z = randomInt()
assume(n >= 0)

// deduce n >= 0
while (n > 0) {
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}
// deduce n <= 0

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- We know this because of the assumption we make
- That's all we can deduce forward

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- From the assumption we can also determine that `n >= 0` must be true before the loop

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```

- We know this because of the assumption we make
- That’s all we can deduce forward
- Let’s consider what we can find out “looking back” from the assertion

Tracing Facts by Looking Backward

- We need to show that `n == 0`

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var n: Z = randomInt()
assume(n >= 0)

// deduce n >= 0
while (n > 0) {
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- We already know that `n <= 0` after the loop

Tracing Facts by Looking Backward

- We need to show that `n == 0`

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// deduce n >= 0
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}
// deduce n <= 0

assert(n == 0)
```

- We already know that `n <= 0` after the loop
- If we knew also that `n >= 0` we could deduce `n == 0`
- Let's add a conjecture `n >= 0`

Tracing Facts by Looking Backward

- We only conjecture that `n >= 0` should be true, but we don't know this yet

```
var n: Z = randomInt()
assume(n >= 0)

// deduce n >= 0
while (n > 0) {
  // deduce n > 0

  n = n - 1

}
// deduce n <= 0
// deduce n >= 0 ?
assert(n == 0)
```

Tracing Facts by Looking Backward

- We only conjecture that `n >= 0` should be true, but we don't know this yet

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- Hence, it's true and all conjectures are proved

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- Together with the loop condition `n > 0` it forms the **inductive hypothesis** for the loop
- The fact `n >= 0` is central for the proof
- It is called an **(inductive) invariant** for the loop

Tracing Facts Summary

- We mention it explicitly

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var n: Z = randomInt()
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- And it describes which variables are renamed in the loop

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- Finally, we have all information needed for reasoning about the loop

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- Recall how we have used a combination of forward and backward reasoning to **reduce the gap** between **what we know** and **what we have to prove**
- Now we can describe a rule for proving while-loops correct

Invariant Rule for Proving Loops Correct

- The following program counts down to zero

```
// ... deduce I
while (C) {
  // invariant I
  //   modifies "variables modified in body"
  // deduce C
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  body
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// deduce !C
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- It describes what is allowed to change
- *Remark*. Candidates for invariants can often be found by reasoning backwards

Example: Iteratively Counting Down to Zero in Logika

```
var n: Z = randomInt()
assume(n >= 0)

Deduce(|- (n >= 0))         // invariant deduced at the beginning of the while-loop
while (n > 0) {
  Invariant(
    Modifies(n),
    n >= 0
  )
                            // The invariant is assumed to be true here
  Deduce(|- (n > 0))        // new fact from condition of while-loop
  Deduce(|- (n - 1 >= 0))   // proof by algebra
  n = n - 1
  Deduce(|- (n >= 0))       // invariant deduced at the end of the while-loop
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Deduce(|- (n <= 0))         // new fact from negated condition (and algebra)
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- The invariant is specified by

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- Let's have a brief look at termination
- We return to it later in the course

Termination of a Loop

- Let's encapsulate the iterative counting-down program in a function

```
@pure def while0(k: Z): Z = {
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  var m: Z = k
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- We can prove about this function that, should the loop terminate, then it returns `0`
- What happens if the function is called with the argument -3, that is, `while0(-3)` ?

Measuring Termination of a Loop

- If we could show that the always terminates, we would be certain the function returns 0

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- We need a method for verifying progress, like observing a *progress bar*
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Measuring Termination of a Loop

- If we could show that the always terminates, we would be certain the function returns 0

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A Measure for the Iterative Count-Down Function

- The loop decreases m in each iteration, so m appears a good measure

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    m = m - 1



  }
  return 0
}
```

- We have to prove that m decreases, but not beyond a given minimum, say, 0

A Measure for the Iterative Count-Down Function

- Let's introduce auxiliary variable for the observation

```
@pure def while0(k: Z): Z = {
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- We can deduce that the measure decreases

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@pure def while0(k: Z): Z = {
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    m = m - 1
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    Deduce(|- (measure_m_post < measure_m_pre))
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    Deduce(|- (measure_m_post < measure_m_pre))
  }
  return 0
}
```

- The fact `measure_m_post < measure_m_pre` is true at the end of the loop body

A Measure for the Iterative Count-Down Function

- We cannot deduce that the measure does not cross the minimum 0

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@pure def while0(k: Z): Z = {
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    Ensures(Res == 0)
  )
  var m: Z = k
  while (m != 0) {
    // decreases m
    val measure_m_pre = m
    Deduce(|- (measure_m_pre >= 0))
    m = m - 1
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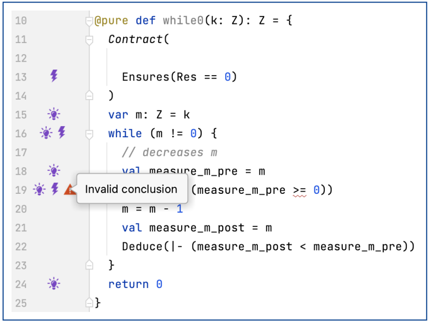

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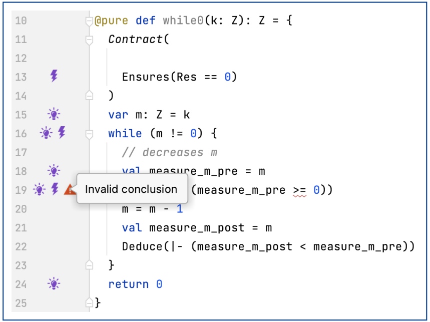

- Having instrumented the program with the measure, we can use Logika for proof support
- In fact, we do not know whether `m >= 0` when the loop is first entered

A Measure for the Iterative Count-Down Function

- A pre-condition can constrain `k` such that `m` is at least `0`

```
@pure def while0(k: Z): Z = {
  Contract(
    Requires(k >= 0),
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  while (m != 0) {
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Logika Verified
Programming logic proof is accepted

A Recursive Count-Down Function

- Similarly, to iteratively counting down, we can count down recursively

```
@pure def count0(k: Z): Z = {
  Contract(

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  if (k == 0) {
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  } else {


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}
```

- We need a measure on a function parameters
  that is bounded below when the function is entered
  and decreased at each recursive call

A Measure for the Recursive Count-Down Function

- Parameter `k` seems like a good candidate for a measure (It's the only candidate, of course)

```
@pure def count0(k: Z): Z = {
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```

- Let's introduce two auxiliary variables `measure_k_entry` and `measure_k_call` to observe progress

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- The function instrumented for observation of the measure:

```
@pure def count0(k: Z): Z = {
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  // decreases k
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```

- Indeed, we have `measure_k_call < measure_k_entry`

A Measure for the Recursive Count-Down Function

- We cannot deduce that the measure is bounded below by 0

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  )
  // decreases k
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  Deduce(|- (k >= 0))
  if (k == 0) {
    return k
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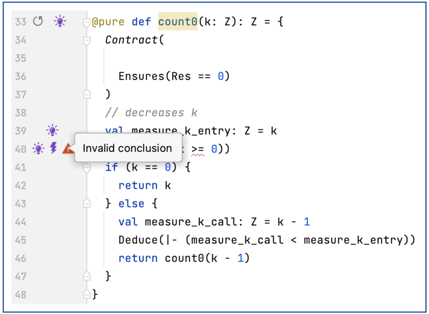

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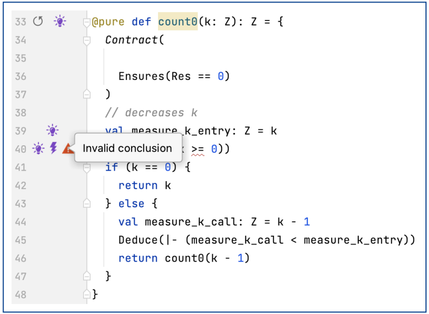

- As in the case of the iterative version, this can be achieve by means of a pre-condition

A Measure for the Recursive Count-Down Function

- With the precondition added, the termination proof succeeds

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- What we consider a specification is a **matter of perspective**

Mathematical Specification of the Factorial

- The factorial of a natural number `n` is usually specified based on their inductive definition

```
@strictpure def fac_rec_spec(n: Z): Z = n match {
  case 0 => 1
  case m => m * fac_rec_spec(m - 1)
}
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- *Remark*. We use the type of integer numbers for simplicity.

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- The Scala expression

```
e match {
  case p1 => r1
  ...
  case pn => r1
}
```

matches expression `e` with the first possible `pi` and returns the corresponding `ri`

Induction Rules the Factorial

- We formulate inductive rules for proving properties about the factorial

```
// Base case
@pure def fac_rec_spec_0() {
  Contract(
    Ensures(fac_rec_spec(0) == 1)
  )
}

// Inductive case
@pure def fac_rec_spec_step(n: Z) {
  Contract(
    Requires(n > 0),
    Ensures(fac_rec_spec(n) == n * fac_rec_spec(n - 1))
  )
}
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}
```

- Using the recursive definition of `fac_rec_spec` and the inductive numbers these are straightforward to derive

Induction Rules the Factorial

- We formulate inductive rules for proving properties about the factorial

```
// Base case
@pure def fac_rec_spec_0() {
  Contract(
    Ensures(fac_rec_spec(0) == 1)
  )
}

// Inductive case
@pure def fac_rec_spec_step(n: Z) {
  Contract(
    Requires(n > 0),
    Ensures(fac_rec_spec(n) == n * fac_rec_spec(n - 1))
  )
}
```

- Using the recursive definition of `fac_rec_spec` and the inductive numbers these are straightforward to derive
- Logika “knows” these rules

Recursive Implementation of the Factorial

- Following the mathematical definition very closely, we implement the factorial recursively

```
@pure def fac_rec(n: Z): Z = {
  Contract(
    Requires(n >= 0),
    Ensures(Res == fac_rec_spec(n))
  )
  if (n == 0) {
    return 1
  } else {
    return n * fac_rec(n - 1)
  }
}
```

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- Because the implementation is so close to the mathematical definition, Logika can prove it without further information
- Specifications should be as “obvious” as possible
- Let’s implement `fac_rec` iteratively

Iterative Implementation of the Factorial

- The iterative implementation `fac_rec` (see post-condition)

```
@pure def fac_it(n: Z): Z = {
  Contract(
    Requires(n >= 0),
    Ensures(Res == fac_rec(n))
  )
  var x: Z = 1
  var m: Z = 0;
  while (m < n) {
    Invariant(
      Modifies(x, m),
      (x == fac_rec(m)),                    // invariant maintained until m == n,
      (m <= n),                             // yielding post-condition x == fac_rec(n).
      (m >= 0)                              // m >= 0 needed for pre-condition of fac_rec
    )
    Deduce(|- (m < n))                      // new fact from condition (forward)
    Deduce(|- (x * (m + 1) == fac_rec(m + 1)))  // proved fact (backward conjecture)
    m = m + 1
    Deduce(|- (x * m == fac_rec(m)))        // proved fact (backward conjecture)
    x = x * m
    Deduce(|- (x == fac_rec(m)))            // proved fact (backward conjecture)
  }
  Deduce(|- (m >= n))                       // new fact from negation of condition (forward).
  Deduce(|- (m <= n))                       // replace n by loop index variable m to obtain
  return x                                  // invariant x == fac_rec(m) from post-condition
}
```

Recursive Implementation of the Factorial

- We have proved

```
@pure def facimp(n: Z) {
  Contract(
    Requires(n >= 0),
    Ensures(fac_it(n) == fac_rec(n))
  )
}
```

Recursive Implementation of the Factorial

- We have proved

```
@pure def facimp(n: Z) {
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    Requires(n >= 0),
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that is,
given `n >= 0`,
the recursive specification and the iterative implementation are inter-replaceable

Recursive Implementation of the Factorial

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that is,
given `n >= 0`,
the recursive specification and the iterative implementation are inter-replaceable

- We have implemented the factorial function correctly

Exercise 1

(a) Prove that `fac_rec` terminates
(b) Prove that `fac_it` terminates

Mathematical Specification of the Fibonacci Number

- The fibonacci number for `n` is specified as follows

```
@strictpure def fib_rec_spec(n: Z): Z = n match {
  case 0 => 0
  case 1 => 1
  case m => fib_rec_spec(m - 1) + fib_rec_spec(m - 2)
}
```

Exercise 2

(a) State the inductive rules for `fib_rec_spec`
(There are three of them!)

(b) Implement the recursive function `fib_rec` computing the fibonacci number

(c) Prove that `fib_rec` terminates using Logika

Exercise 3

```
@pure def fib_it(n: Z): Z = {
  Contract(
    Requires(n >= 0),
    Ensures(Res == fib_rec(n))
  )
  if (n == 0) {
    return 0
  } else if (n == 1) {
    return 1
  } else {
    var x: Z = 0
    var y: Z = 1
    var m: Z = 1;
    while (m < n) {
      Invariant(
        Modifies(x, y, m),
        (x == fib_rec(m - 1)),
        (y == fib_rec(m)),
        (m <= n),
        (m >= 0)
      )
      ...
    }
    return y
  }
}
```

(a) Complete the implementation of the iterative version `fib_it`
- Do not introduce any additional variable
- Hint: Look at the previous example of in-place number swapping

(b) Prove your implementation correct using Logika

(c) Prove that your implementation terminates using Logika

Summary

- We have reviewed induction, recursion and iteration
- We have inductively reasoned about while-loops
- We have inductively reasoned about recursive functions
- We have considered termination verification for loops and recursive functions
- We have learned about a method to develop programs from specifications