PROBLEM 1



gvn, max. $c \cdot x$
s.t $Ax \leq b$
$x \geq 0$

A is a $n \times n$ matrix.
$c, x$ & $d$ are vectors of $R^n$

1) given A is invertible,

To prove:- The above L.P is feasible once we drop the constraint $x \geq 0$.

Proof:- For an L.P to be feasible there must be a point st. $Ax \leq b$ is satisfied.

Sub, x in $Ax = b$, with $A^{-1}b$,

we get;

$$Ax = A(A^{-1}b) = (AA^{-1})b = Ib = b.$$

Since $A^{-1}$ & there is a point that satisfies $Ax = b$, the L.P is feasible given the constraint $x \geq 0$ is dropped.

2) $A = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$ & $b = \begin{pmatrix} -2 \\ 9 \end{pmatrix}$ ②

a) The above L.P isn't feasible as $A^{-1}$ doesn't exist

b) <u>Proof:-</u> L.P of the above problem is infeasible using Simplex.

$$2x_1 + 2x_2 \leq -2$$

$$-2x_1 - 2x_2 \leq 9$$

adding slack variables,

$$x_3 = -2 - 2x_1 - 2x_2$$

$$x_4 = 9 + 2x_1 + 2x_2$$

Since this dictionary isn't feasible, we must perform Initialization.

$$x_0 = 2 + 2x_1 + 2x_2 + x_3$$

$$x_4 = 11 + 4x_1 + 4x_2 + x_3$$

$$z = -2 - 2x_1 - 2x_2 - x_3$$

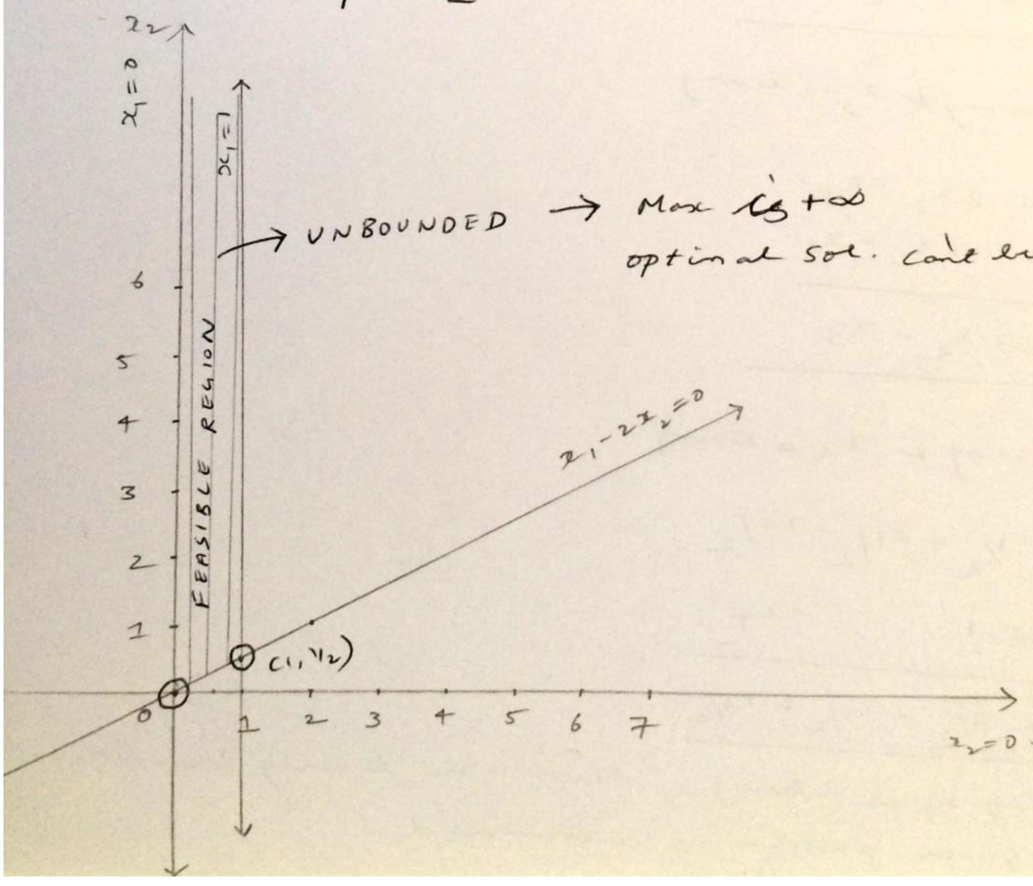
Since, no entering variable can be chosen the above auxiliary problem can't be continued. Hence, the given L.P is Infeasible.

c) rank of $A = 1$

3) gn, $A = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ & $c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

a) $$x_1 - 2x_2 \leq 0$$
$$x_1 \leq 1 \qquad x_1, x_2 \geq 0$$

max $x_1 + x_2$.




→ UNBOUNDED → Max is $+\infty$
optimal sol. can't be determined

b) $x_1 - 2x_2 = 0$

$x_1 = 1$ (4)

$z = x_1 + x_2$

Adding slack variables, new dictionary is

$x_3 = -x_1 + 2x_2$

$x_4 = 1 - x_1$

$z = x_1 + x_2$

'$x_1$' is entering & '$x_3$' is leaving.

$x_1 = 2x_2 - x_3$

$x_4 = 1 - 2x_2 + x_3$

$z = 3x_2 - x_3$

'$x_2$' is entering & '$x_4$' is leaving.

$x_2 = 1/2 + x_3/2 - x_4/2$

$x_1 = 1 \quad - x_4$

$z = 3/2 + x_3/2 - x_4/2$

Now choosing '$x_3$' as entering, there is no leaving variables,

$\Rightarrow$ the given problem is unbounded.

PROBLEM 2

1) a) Let $x_1$ be no. of batches of pancakes.

$x_2$ be no. of batches of Waffles.

objective, max $Z = 6x_1 + 5x_2$

Constraints
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + 2x_2 \leq 18$$
$$2x_1 + 2x_2 \leq 20$$
$$x_1, x_2 \geq 0.$$

b) Let $x_1$ be no. of hours tutoring Geometry Student

$x_2$ be no. of hours baby sitting.

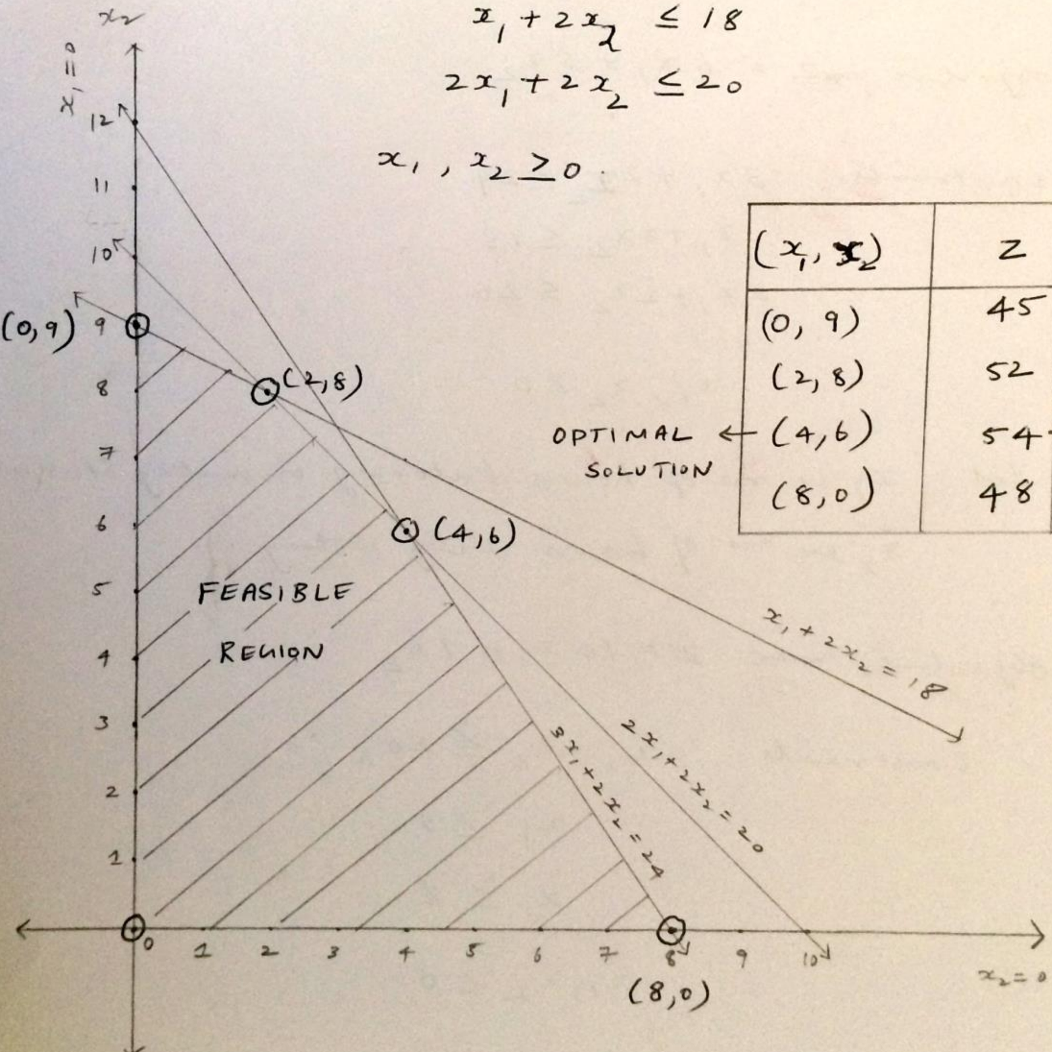
objective, max $Z = 10x_1 + 7x_2$

Constraints
$$x_1 + x_2 \leq 20$$
$$x_1 \geq 3$$
$$x_1 \leq 8$$
$$x_1, x_2 \geq 0.$$

## 2) a) (i) Graphical Method

$$\max.\ Z = 6x_1 + 5x_2$$

such that
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + 2x_2 \leq 18$$
$$2x_1 + 2x_2 \leq 20$$
$$x_1, x_2 \geq 0.$$




| $(x_1, x_2)$ | Z |
|---|---|
| (0, 9) | 45 |
| (2, 8) | 52 |
| OPTIMAL SOLUTION ← (4,6) | 54 → OPTIMAL VALUE |
| (8,0) | 48 |

(ii) Simplex Method



$$\max \quad z = 6x_1 + 5x_2$$

Such that,
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + 2x_2 \leq 18$$
$$2x_1 + 2x_2 \leq 20.$$
$$x_1, x_2 \geq 0.$$

The L.P is in standard form.

→ Adding slack variables, new dictionary is

$$x_3 = 24 - 3x_1 - 2x_2$$
$$x_4 = 18 - x_1 - 2x_2$$
$$x_5 = 20 - 2x_1 - 2x_2$$

---

$$z = 6x_1 + 5x_2$$

---

'$x_1$' is entering variable & '$x_3$' is leaving variable, new dictionary is,

$$x_1 = 8 - 2/3\,x_2 - x_3/3$$
$$x_4 = 10 - 4/3\,x_2 + x_3/3$$
$$x_5 = 4 - 2/3\,x_2 + 2x_3/3$$

---

$$z = 48 + x_2 - 2x_3$$

---

'$x_2$' is entering Variable & '$x_5$' is leaving Variable. 8

new dictionary is:

$$x_2 = 6 + x_3 - 3/_2\, x_5$$

$$x_4 = 2 - 4/_3\, x_3 + 2x_5$$

$$x_1 = 4 - x_3 + x_5$$

$$z = 54 - x_3 - 3/_2\, x_5$$

Since no entering Variable can be chosen, this is our final Solution.

Setting nonbasic Variables $x_3$ & $x_5 = 0$, we get.

$$x_1 = 4 \;;\; x_2 = 6 \;\; \& \;\; z = 54.$$

∴ To feed as many people as possible

24 batches of pancakes & 30 batches of Waffles must be made.

optimal Solution $(4, 6)$

optimal Value $= 54$

b) (i) Graphical Method



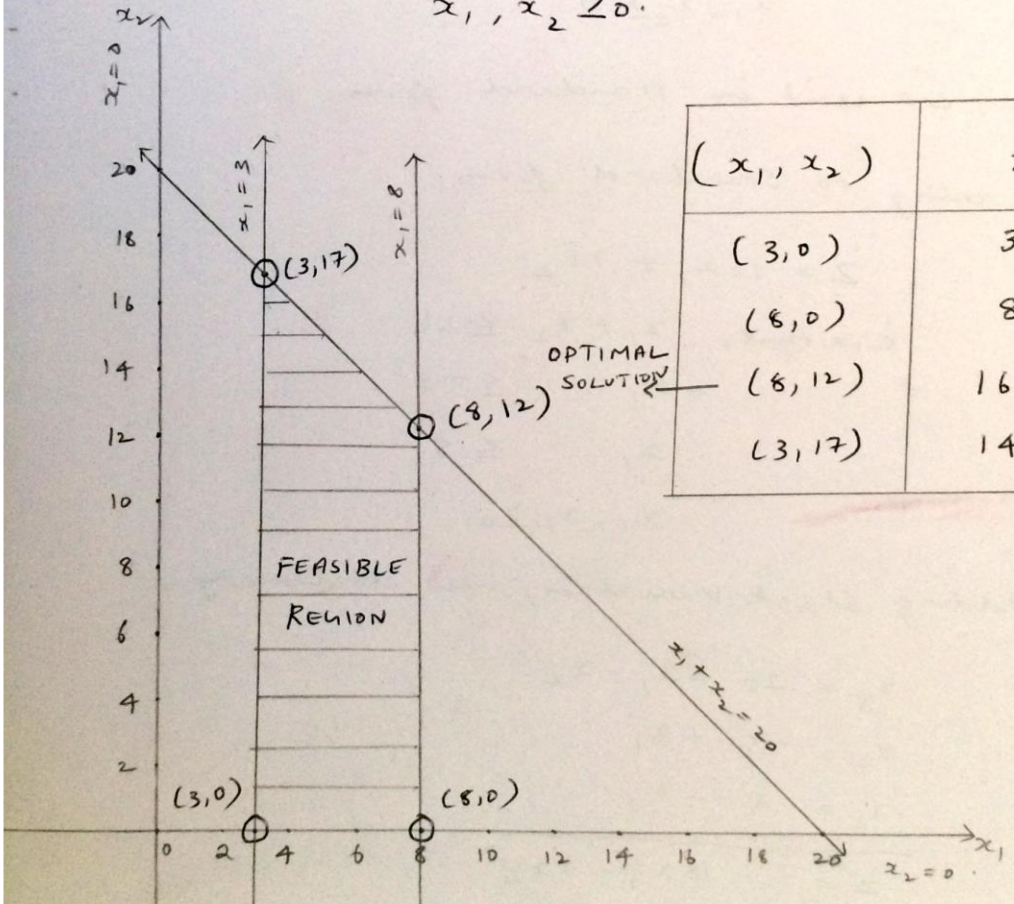
max. $Z = 10x_1 + 7x_2$

Such that $x_1 + x_2 \leq 20$

$x_1 \geq 3$

$x_2 \leq 8$

$x_1, x_2 \geq 0.$




| $(x_1, x_2)$ | Z |
|---|---|
| (3,0) | 30 |
| (8,0) | 80 |
| (8,12) | 164 |
| (3,17) | 149 |

OPTIMAL SOLUTION ← (8,12)

164 → OPTIMAL VALUE

(ii) Simplex Method

$$\max \quad z = 10x_1 + 7x_2$$

Such that, $x_1 + x_2 \leq 20$

$x_1 \geq 3$

$x_2 \leq 8$

$x_1, x_2 \geq 0.$

The L.P isn't in standard form.

→ Converting to standard form,

$$z = 10x_1 + 7x_2$$

Such that, $x_1 + x_2 \leq 20$

$-x_1 \leq -3$

$x_1 \leq 8$

$x_1, x_2 \geq 0.$

→ Adding slack variables, new dictionary is:

$$\begin{aligned} x_3 &= 20 - x_1 - x_2 \\ x_4 &= -3 + x_1 \\ x_5 &= 8 - x_1 \\ \hline z &= 10x_1 + 7x_2 \end{aligned}$$

Since the initial dictionary is infeasible we must first convert to a feasible one by performing initialization & solving the auxiliary problem. 

## Initialization

$$\max \; -x_0$$

such that,

$$x_1 + x_2 + x_3 - x_0 = 20$$

$$-x_1 + x_4 - x_0 = -3$$

$$x_1 + x_5 - x_0 = 8$$

$x_1, x_2, x_3, x_4, x$
$x_0 \geq 0$

$$z = -x_0$$

new dictionary is,

$$x_0 = 3 - x_1 + x_4$$

$$x_3 = 23 - 2x_1 - x_2 + x_4$$

$$x_5 = 11 - 2x_1 + x_4$$

$$z = -3 + x_1 - x_4$$

'$x_1$' is the entering variable & '$x_0$' is the leaving variable; the new dictionary is:

$x_1 = 3 + x_4 - x_0$

$x_3 = 17 - x_2 - x_4 + 2x_0$

$x_5 = 5 - x_4 + x_0$

---

$z = -x_0$

---

Since the z has reached optimum value, the auxiliary problem is solved. Now, since we have got a feasible dictionary we can start solving, new dictionary is after removing $x_0$,

$$
\begin{aligned}
x_1 &= 3 + x_4 \\
x_3 &= 17 - x_2 - x_4 \\
x_5 &= 5 - x_4 \\
\hline
z &= 30 + 7x_2 + 10x_4
\end{aligned}
$$

'$x_4$' is the entering variable & '$x_5$' is the leaving variable, new dictionary is.

$$
\begin{aligned}
x_4 &= 5 - x_5 \\
x_1 &= 8 - x_5 \\
x_3 &= 12 - x_2 + x_5 \\
\hline
z &= 80 + 7x_2 - 10x_5
\end{aligned}
$$

'$x_2$' is the entering variable & '$x_3$' is the leaving variable; new dictionary is

$$\begin{aligned} x_2 &= 12 - x_3 + x_5 \\ x_4 &= 5 \quad\quad - x_5 \\ x_1 &= 8 \quad\quad - x_5 \\ \hline z &= 164 - 7x_3 - 3x_5 \end{aligned}$$

Since no entering can be chosen, this is our final solution.

Setting non-basic variables $x_3$ & $x_5 = 0$, we get

$x_1 = 8$; $x_2 = 12$ & $z = 164$.

$\therefore$ Kayla should teach Geometry for 8 hours & babysit for 12 hours.

The maximum weekly earnings of Kayla is \$164

optimal solution — $(8, 12)$

optimal value — $164$

## PROBLEM 3

given, $\min \; x_1 + 2x_2 - x_3$

$$\text{s.t} \quad x_1 - x_2 = 1 \quad \rightarrow (1)$$

$$x_1 - 3x_2 + x_3 \geq 2 \quad \rightarrow (2)$$

$$x_1 \geq 0,\; x_2 \geq 0,\; x_3 \leq 0.$$

1) a) To prove; if the above L.P is feasible then $x_3 \geq 1 + 2x_2$.

Proof:- In order that above LP is feasible. there must exist atleast one solution solving all the constraints

Sub (1) in (2), we get.

$x_1 = 1 + x_2$ in $x_1 - 3x_2 + x_3 \geq 2$

$$\Rightarrow 1 + x_2 - 3x_2 + x_3 \geq 2$$

$$= x_3 \geq 1 + 2x_2.$$

Hence proved.

b) To prove, the above L.P is infeasible (15)

Proof:-

From the above assumption, inorder that L.P to be feasible

$$x_3 \geq 1 + 2x_2 . \rightarrow (1)$$

but, we know that $x_2 \geq 0$ & $x_3 \leq 0$

$\Rightarrow$ $x_3$ is Zero or negative.

whereas $x_2$ is zero (or) positive.

$\Rightarrow$ assuming $x_3 = 0$ & $x_2 = 0$ & sub in (1)

we get $0 \geq 1$, which isn't possible

$\Rightarrow$ That the given L.P is infeasible

2) a) standard form of,

$$\begin{aligned} \min \quad & x_1 + 2x_2 - x_3 \\ \text{s.t} \quad & x_1 - x_2 = 1 \\ & x_1 - 3x_2 + x_3 \geq 2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \leq 0 \end{aligned}$$

Standard form

let $y = -x_3$.

So the L.P is

$$\left.\begin{aligned} \max \quad & -x_1 - 2x_2 + (-y) \\ \text{st} \quad & x_1 - x_2 \leq 1 \\ & -x_1 + x_2 \leq -1 \\ & -x_1 + 3x_2 + y \leq 2 \\ & x_1, x_2, y \geq 0. \end{aligned}\right\} \to \text{Standard Form of given L.P.}$$

b) To prove, the above L.P is infeasible using Simplex algorithm.

$$\begin{aligned} \max \quad & -x_1 - 2x_2 - y \\ \text{st} \quad & x_1 - x_2 \leq 1 \\ & -x_1 + x_2 \leq -1 \\ & -x_1 + 3x_2 + y \leq 2 \\ & x_1, x_2, y \geq 0 \end{aligned}$$

Adding slack variables,

$$x_3 = 1 - x_1 + x_2$$
$$x_4 = -1 + x_1 - x_2$$
$$x_5 = 2 + x_1 - 3x_2 - y$$
$$z = -x_1 - 2x_2 - y$$

Since, we willn't be able to proceed as the dictionary is infeasible, so we must convert it to a feasible one using initialization;

new dictionary is

$$x_0 = 1 - x_1 + x_2 + x_4$$
$$x_3 = 2 - 2x_1 + 2x_2 + x_4$$
$$x_5 = 3 \quad - 2x_2 + x_4 - y.$$
$$z = -1 + x_1 - x_2 - x_4$$

'$x_0$' is the leaving variable, & '$x_1$' is the entering vari.

$$x_1 = 1 + x_2 + x_4 - x_0.$$
$$x_3 = \quad - x_4 + 2x_0 \qquad z = -x_0.$$
$$x_5 = 3 - 2x_2 + x_4 \quad - y$$

The new feasible dictionary is (18)

$$x_1 = 1 + x_2 + x_4$$
$$x_3 = -x_4$$
$$x_5 = 3 - 2x_2 + x_4 - y.$$

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$$z = -1 - 3x_2 - x_4 - y.$$

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Since we aren't able to find an entering variable, the given L.P is infeasible.