CSCI 5654 Fall 15
Assignment 2 - Solutions

Problem 1

1.

(a) $x = 0$ satisfies the constraints, so (1) is feasible. And it deduces a lower bound of the objective function, which is 0.

(b) In the initial dictionary, all the basic variables are 0, so (1) is degenerate.

(c) (Case 1) If the feasible set $S = \{0\}$, then it is reduced to a single point. (Case 2) If the feasible set S contains a positive number $\widetilde{x} > 0$, which satisfies $A\widetilde{x} \leq 0$. Then for any $\lambda > 0$, $A(\lambda\widetilde{x}) \leq 0$, so $\lambda\widetilde{x}$ is also in the feasible set. Hence, the feasible set is unbounded.

(d) Objective function $\xi = c \cdot x = \sum_{j=1}^{n} c_j x_j$, (Case 1) if $c \leq 0$, then the optimal solution is $x = 0$, the optimal value is 0. (Case 2) If there exsits $c_k > 0$, then we try to increase x_k (choose it as entering variable). Since $b = 0$, there is no constraint on x_k, so the optimal value is unbounded.

(e) The dual problem:

$$\begin{array}{ll} \text{minimize} & 0 \cdot y \\ \text{s.t.} & Ay \geq c \\ & y \geq 0 \end{array}$$

Since we are optimizing a constant, let's see the feasibility of this dual problem. (Case 1) If the feasible set of the dual is not empty, then the optimal value of the dual is 0, so the optimal value of problem (1) is 0. (Case 2) If the feasible set of the dual is empty (i.e. the dual is infeasible), then problem (1) may be unbounded or infeasible. While we have already found $x = 0$ is a feasible solution of (1), it can just be unbounded.

2.

(a) Initial dictionary:

$$\begin{aligned} \xi &= 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ x_5 &= -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\ x_6 &= -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\ x_7 &= 1 - x_1 \end{aligned}$$

Basic variables: $x_5 = 0, x_6 = 0, x_7 = 1$, all of them are ≥ 0, so this dictionary is feasible. While $x_5 = 0$ and $x_6 = 0$, so it is degenerate.

(b) (You will come back to the initial dictionary after 6 iterations)

(c) Last dictionary:

$$\begin{aligned} \xi &= 1 - 30x_2 - 42x_4 - 18x_6 - x_7 \\ x_1 &= 1 - x_7 \\ x_3 &= 1 - 3x_2 + 2x_4 + 2x_6 - x_7 \\ x_5 &= 2 - 2x_3 - 4x_4 + 5x_6 - 2x_7 \end{aligned}$$

Optimal solution: $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$, optimal value $\xi = 1$.

Problem 2

1. The first $(x_1 - x_2 \leq 3)$ and the last $(-x_1 + x_2 \leq -4)$ constraints lead to a contradiction, so the primal problem is infeasible. Hence, the dual problem may be unbounded or infeasible.

2. The dual problem:

$$\begin{array}{ll} \text{maximize} & -3y_1 - 6y_2 - 2y_3 + y_4 \\ \text{s.t.} & -y_1 + y_2 + y_3 + y_4 \le -2 \\ & y_1 - y_4 \le -3 \\ & -y_2 - y_3 \le 5 \\ & y_1, y_2, y_3, y_4 \ge 0 \end{array}$$

Initial dictionary is infeasible, so introduce auxiliary problem:

$$\begin{aligned} \zeta &= -y_0 \\ y_5 &= -2 + y_1 - y_2 - y_3 - y_4 + y_0 \\ y_6 &= -3 - y_1 + y_4 + y_0 \\ y_7 &= 5 + y_2 + y_3 + y_0 \end{aligned}$$

Last dictionary:

$$\begin{aligned} \zeta &= -5/2 - 1/2y_2 - 1/2y_3 - 1/2y_5 - 1/2y_6 \\ y_0 &= 5/2 + 1/2y_2 + 1/2y_3 + 1/2y_5 + 1/2y_6 \\ y_4 &= 1/2 + y_1 - 1/2y_2 - 1/2y_3 - 1/2y_5 + 1/2y_6 \\ y_7 &= 15/2 + 3/2y_2 + 3/2y_3 - y4 + 1/2y_5 + 1/2y_6 \end{aligned}$$

Optimal value of auxiliary problem is $-5/2 < 0$, so the dual problem is infeasible.

3. Weak duality theorem: $c \cdot x \le b \cdot y$, where x is feasible for primal problem, y is feasible for dual. Suppose the dual problem is feasible, then we can find an upper bound of the primal's objective function, which contradicts with the fact that the primal is unbounded. Hence, the dual is infeasible.

4. Check the LP in the hint. Both the primal and dual are infeasible, so the inverse is not true.

Problem 3

1.

(a) The dual problem:

$$\begin{array}{ll} \text{maximize} & -6y_1 + y_2 - 6y_3 - 3y_4 - 6y_5 + 3y_6 \\ \text{s.t.} & 2y_1 + 3y_2 - 9y_3 - y_4 - 7y_5 + 5y_6 \le 1 \\ & -7y_1 - y_2 + 4y_3 + y_4 + 3y_5 - 2y_6 \le 2 \\ & y_1, y_2, y_3, y_4, y_5, y_6 \ge 0 \end{array}$$

(b) It is easier to solve the primal, which has only two variables, so we can easily use graphic approach to solve it. The optimal solution $x_1 = 3/5, x_2 = 0$, the optimal value $-3/5$

2.

(a) The primal's initial dictionay:

$$\begin{aligned} \xi &= -x_1 - 2x_2 \\ x_3 &= 6 + 2x_1 - 7x_2 \\ x_4 &= -1 + 3x_1 - x_2 \\ x_5 &= 6 - 9x_1 + 4x_2 \\ x_6 &= 3 - x_1 + x_2 \\ x_7 &= 6 - 7x_1 + 3x_2 \\ x_8 &= -3 + 5x_1 - 2x_2 \end{aligned}$$

The dual's initial dictionay:

$$\begin{aligned} \zeta &= -6y_1 + y_2 - 6y_3 - 3y_4 - 6y_5 + 3y_6 \\ y_7 &= 1 - 2y_1 - 3y_2 + 9y_3 + y_4 + 7y_5 - 5y_6 \\ y_8 &= 2 + 7y_1 + y_2 - 4y_3 - y_4 - 3y_5 + 2y_6 \end{aligned}$$

(b) The primal's initial dictionay is infeasible, the dual's is feasible.
Associate matrices:

$$P = \begin{pmatrix} 0 & -1 & -2 \\ 6 & 2 & -7 \\ -1 & 3 & -1 \\ 6 & -9 & 4 \\ 3 & -1 & 1 \\ 6 & -7 & 3 \\ -3 & 5 & -2 \end{pmatrix}, D = \begin{pmatrix} 0 & -6 & 1 & -6 & -3 & -6 & 3 \\ 1 & -2 & -3 & 9 & 1 & 7 & -5 \\ 2 & 7 & 1 & -4 & -1 & -3 & 2 \end{pmatrix}$$

(c) $P = -D^T$

3.

(a) Last dictionary:

$$\begin{aligned} \xi &= -3/5 - 12/5x_2 - 1/5x_8 \\ x_1 &= 3/5 + 2/5x_2 + 1/5x_8 \\ x_3 &= 36/5 - 31/5x_2 + 2/5x_8 \\ x_4 &= 4/5 + 1/5x_2 + 3/5x_8 \\ x_5 &= 3/5 + 2/5x_2 - 9/5x_8 \\ x_6 &= 12/5 + 3/5x_2 - 1/5x_8 \\ x_7 &= 9/5 + 1/5x_2 - 7/5x_8 \end{aligned}$$

The optimal solution of the primal: $x_1 = 3/5, x_2 = 0$, the optimal value of the primal: $\xi = -3/5$.

(b) The optimal solution of the dual: $y_i = -c^*_{n+i}$, so $y_1 = 0, y_2 = 0, y_3 = 0, y_4 = 0, y_5 = 0, y_6 = -c^*_8 = 1/5$, the optimal value of the dual: $\zeta = 3/5$.

(c) Hint: Denote previous results as x^* and y^*
(1) Check x^* is the feasible solution of the primal.
(2) Check y^* is the feasible solution of the dual.
(3) Check $c \cdot x^* = b \cdot y^*$, where $c = (-1, -2), b = (6, -1, 6, 3, 6, -3, 0)$