HW2

Problem 2

Magnetron tube has an exponential distribution probability distribution.

(a) Likelihood function of rate parameter β is

$$L(\beta, x_1 \dots x_n) = \beta^n \exp\left(-\beta \sum_{j=1}^{n} x_j\right)$$

(b) MLE of β; $\hat{\beta} = \dfrac{n}{\sum_{j=1}^{n} x_j} = \dfrac{100}{691.74} = 0.14456$

(c) So the uniform prior: $P(\beta) = 1$.

Posterior distribution of β.

$$P(\beta | x) \propto \underbrace{\beta^n \exp\left(-\beta \sum_{j=1}^{n} x_j\right)}_{\text{Likelihood}} \underbrace{1}_{\text{Uniform Prior}}$$

$$P(\beta | x) \propto \beta^n \exp\left(-\beta \sum_{j=1}^{n} x_j\right), \ \beta > 0$$

which is the kernel of Gamma distribution $\text{Gamma}\left(n+1, b = \sum_{j=1}^{n} x_j\right)$

Posterior Mean $= \dfrac{n+1}{\sum_{j=1}^{n} x_j} = \dfrac{101}{691.74} = 0.146$

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(d) So, when the prior is Gamma distribution

$$p(\beta) \propto \beta^{a-1} \exp(-\beta b),$$

Posterior distribution of β is

$$p(\beta|x) \propto \underbrace{\beta^n \exp\left(-\beta \sum_{j=1}^{n} x_j\right)}_{\text{Likelihood}} \underbrace{\beta^{a-1} \exp(-b\beta)}_{\text{Gamma prior}}$$

$$\propto \beta^{n+a-1} \exp\left(-\beta\left(b + \sum_{j=1}^{n} x_j\right)\right), \beta > 0$$

which is kernel of Gamma $\left(n+a, b+\sum_{j=1}^{n} x_j\right)$

So, posterior distribution of β under Gama prior is

$$\beta|x \sim \text{Gama}\left(n+a, b+\sum_{j=1}^{n} x_j\right)$$

$$\text{Posterior Mean} = \frac{n+a}{b+\sum x_j} = \frac{100+a}{b+691.74}$$