1 SINGULAR VALUE DECOMPOSITION

The process of decomposing a matrix into three smaller matrices is known as factorization . Since the matrix 'A' can be factorised into the product of three matrices, the singular value decomposition of it can be represented as:

A $= U \sum . V^T$, *where* ,

$A =$ Original matrix we want to decompose

U $=$ *Left singular matrix*

$\sum$ $=$ *Diagonal matrix containing singular eigen values*

$V=$ Right singular matrix (columns are right singular vectors)

A $= \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$

$A^T = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$

A^T . A $= \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$

A^T . A -λ . I $= \begin{bmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{bmatrix}$

$det((A)^T$. A - λ . $I) = 0 - (1)$

=> $(25 - \lambda) .(25 - \lambda) - 15 . 15 = 0$

=> $625 - 25\lambda - 25\lambda + (\lambda)^2 = 0$

=> $(\lambda)^2 - 50\lambda + 400 = 0$

=> $(\lambda)^2 - 40\lambda - 10\lambda + 400 = 0$

=> $\lambda(\lambda - 40) - 10(\lambda - 40) = 0$

=> $(\lambda - 10) . (\lambda - 40) = 0$

$\lambda = 10$ *or* $\lambda = 40$

therefore, two eigen values , $\lambda = 10, 40$

substituting the value of λ *in* $- (1)$

when $\lambda = 10$

A^T . A - 10 . I $= \begin{bmatrix} 25-10 & -15 \\ -15 & 25-10 \end{bmatrix}$

$A^T . A - 10 . I = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$

$AB = 0$, *ie* , $A^T . A - 10\, I = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} = 0$

We have to convert it to a nit vector

Taking the sqare root of the sum of squares of the values

$\sqrt{(1)^2 + (1)^2} = \sqrt{(2)}$

converting to unit vector ,

Eigen vecor of eigen value , $10 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Following the similar steps to get the eigen vector for eigen value $\lambda = 40$

$A^T . A - 40\, I = \begin{bmatrix} 25-40 & -15 \\ -15 & 25-40 \end{bmatrix}$

$$A^T \,.\, A \;-\; 40\;I \;=\; \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$
$$Eigen\ vector\ of\ eigen\ value\ ,40 \;=\; \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$
$$Eigen\ vector\ ,\ V \;=\; \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$
$$\sum \;=\begin{bmatrix} \sqrt{4}0 & 0 \\ 0 & \sqrt{1}0 \end{bmatrix}$$
$$A \;=\; U\; \sum\; V^T$$
$$U \;=\; A\,.\,V[\sum]^T$$
$$A\,.\,V \;=\begin{bmatrix} -4/\sqrt{2} & 4/\sqrt{2} \\ -8/\sqrt{2} & -2/\sqrt{2} \end{bmatrix}$$
$$A\,.\,V \begin{bmatrix} -4/\sqrt{2}\,.\,1/2\sqrt{1}0 & 4/\sqrt{2}\,.\,1/2\sqrt{1}0 \\ -8/\sqrt{2}\,.\,1/2\sqrt{1}0 & -2/\sqrt{2}\,.\,1/2\sqrt{1}0 \end{bmatrix}$$
$$A\,.\,V \;=\; \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$
$$A\,.\,V\;[\sum]^T \;=\; \begin{bmatrix} -2\sqrt{(2)} & 2\sqrt{(2)} \\ -4\sqrt{(2)} & -\sqrt{(2)} \end{bmatrix}$$
$We\ have\ to\ convert\ this\ to\ unit\ vectors\ to\ get\ the\ final\ U\ matrix$
$$U \;=\; \begin{bmatrix} -2\sqrt{2}/\sqrt{4}0 & 2\sqrt{2}/\sqrt{1}0 \\ -4\sqrt{2}/\sqrt{4}0 & -\sqrt{2}/\sqrt{1}0 \end{bmatrix}$$
$$A \;=\; U\,.\,\sum\,.\,V^T \;=\; \begin{bmatrix} -1\sqrt{5} & 2\sqrt{5} \\ -2\sqrt{5} & -1\sqrt{5} \end{bmatrix} . \begin{bmatrix} \sqrt{4}0 & 0 \\ 0 & \sqrt{1}0 \end{bmatrix} . \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

2 LINEAR DISCRIMINANT ANALYSIS

It is a dimensional reduction technique used as a pre-processing step in pattern classification and machine learning application . It's goal is to project a feature space (N- dimensional data) on to a smaller subspace K (K$\leq (n-1)$)

$Example\ :$

$Given\ ,\ X_1 \;=\; [(x_1,y_1)] \;=\; [(4,1),(2,4),(2,3),(3,6),(4,4)]$

$X_2 \;=\; [(x_2,y_2)] \;=\; [(9,10),(6,8),(9,5),(8,7),(10,8)]$

$Compute\ within\ class\ scatter\ matrix\ (S_w)$

$S_w \;=\; S_1 \;+\; S_2$

$S_1\ :\ Co-variance\ of\ C_1$

$S_2\ :\ Co-variance\ of\ C_2$

$S_1 \;=\; \sum (X_1 \;-\; \mu_1)\,.\,(X_1 \;-\; \mu_1)^T$

$\mu_1 \;=\; [(4+2+2+3+4)/5 \quad (1+4+3+6+4)/5]$

$\mu_1 \;=\; [3.00 \;\; 3.60]$

$\mu_2 \;=\; [(9+6+9+8+10)/5 \quad (10+8+5+7+8)/5]$

$\mu_2 \;=\; [8.4 \;\; 7.60]$

$X_1 \; - \; \mu_1 \; = \; \begin{bmatrix} -1 & -1 & -1 & 0 & 1 \\ -2.6 & 0.4 & 0.6 & 2.4 & 0.4 \end{bmatrix}$

Seperately , calculate $(X_1 - \mu_1) \; . \; (X_1 - \mu_1)^T$

First matrix $= \begin{bmatrix} 1 \\ -2.6 \end{bmatrix} . \begin{bmatrix} 1 & -2.6 \end{bmatrix} = \begin{bmatrix} 1 & -2.6 \\ -2.6 & 6.76 \end{bmatrix}$

Second matrix $= \begin{bmatrix} -1 \\ 0.4 \end{bmatrix} . \begin{bmatrix} -1 & 0.4 \end{bmatrix} = \begin{bmatrix} 1 & -0.4 \\ -0.4 & 0.16 \end{bmatrix}$

Third matrix $= \begin{bmatrix} -1 \\ 0.6 \end{bmatrix} . \begin{bmatrix} -1 & 0.6 \end{bmatrix} = \begin{bmatrix} 1 & -0.6 \\ -0.6 & 0.36 \end{bmatrix}$

Fourth matrix $= \begin{bmatrix} 0 \\ 2.4 \end{bmatrix} . \begin{bmatrix} 0 & 2.4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5.76 \end{bmatrix}$

Fifth matrix $= \begin{bmatrix} 1 \\ 0.4 \end{bmatrix} . \begin{bmatrix} 1 & 0.4 \end{bmatrix} = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 0.16 \end{bmatrix}$

Adding all these 5 *matrices together we get ,*

$S_1 \; = \; \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 13.2/5 \end{bmatrix}$

$=> \; S_1 \; = \; \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 2.64 \end{bmatrix}$

similarly ,

$S_2 \; = \; \sum (X_2 \; - \; \mu_2) \; . \; (X_2 \; - \; \mu_2)^T$

$X_2 \; - \; \mu_2 \; = \; \begin{bmatrix} 9 - 8.4 & 6 - 8.4 & 9 - 8.4 & 8 - 8.4 & 10 - 8.4 \\ 10 - 7.6 & 8 - 7.6 & 5 - 7.6 & 7 - 7.6 & 8 - 7.6 \end{bmatrix}$

$X_2 \; - \; \mu_2 \; = \; \begin{bmatrix} 0.6 & -2.4 & 0.6 & -0.4 & 1.6 \\ 2.4 & 0.4 & -2.6 & -0.6 & 0.4 \end{bmatrix}$

Seperately , calculate $(X_2 - \mu_2) \; . \; (X_2 - \mu_2)^T$

First matrix $= \begin{bmatrix} 0.6 \\ 2.4 \end{bmatrix} . \begin{bmatrix} 0.6 & 2.4 \end{bmatrix} = \begin{bmatrix} 0.36 & 1.44 \\ 1.44 & 5.76 \end{bmatrix}$

Second matrix $= \begin{bmatrix} -2.4 \\ -0.4 \end{bmatrix} . \begin{bmatrix} -2.4 & -0.4 \end{bmatrix} = \begin{bmatrix} 5.76 & -0.96 \\ 0.96 & 0.16 \end{bmatrix}$

Third matrix $= \begin{bmatrix} 0.6 \\ -2.6 \end{bmatrix} . \begin{bmatrix} 0.6 & -2.6 \end{bmatrix} = \begin{bmatrix} 0.36 & -1.56 \\ -1.56 & 6.76 \end{bmatrix}$

Fourth matrix $= \begin{bmatrix} -0.4 \\ -0.6 \end{bmatrix} . \begin{bmatrix} -0.4 & -0.6 \end{bmatrix} = \begin{bmatrix} 0.16 & 0.24 \\ 0.24 & 0.36 \end{bmatrix}$

Fifth matrix $= \begin{bmatrix} 1.6 \\ 0.4 \end{bmatrix} . \begin{bmatrix} 1.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 2.56 & 0.64 \\ 0.64 & 0.16 \end{bmatrix}$

Adding all these 5 *matrices together we get ,*

$S_2 \; = \; \begin{bmatrix} 9.2/5 & -0.2/5 \\ -0.2/5 & 13.2/5 \end{bmatrix}$

$=> \; S_2 \; = \; \begin{bmatrix} 1.84 & -0.04 \\ -0.04 & 2.64 \end{bmatrix}$

Within class scatter matrix , $S_w \; = \; S_1 \; + \; S_2 \; = \; \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 2.64 \end{bmatrix} \; + \begin{bmatrix} 1.84 & -0.04 \\ -0.04 & 2.64 \end{bmatrix}$

$$S_W = \begin{bmatrix} 2.64 & -0.44 \\ -0.44 & 5.28 \end{bmatrix}$$

$$Between\ class\ scatter\ matrix\ S_B = (\mu_1 - \mu_2)\ .\ (\mu_1 - \mu_2)^T$$

$$(\mu_1 - \mu_2) = \begin{bmatrix} -3 - 8.4 \\ 3.2 - 7.6 \end{bmatrix}$$

$$Therefore,\ (\mu_1 - \mu_2) = \begin{bmatrix} -5.4 \\ -4.4 \end{bmatrix}$$

$$(\mu_1 - \mu_2)\ .\ (\mu_1 - \mu_2)^T = \begin{bmatrix} -5.4 \\ -4.4 \end{bmatrix}\ .\ \begin{bmatrix} -5.4 & -4.4 \end{bmatrix}$$

$$(\mu_1 - \mu_2)\ .\ (\mu_1 - \mu_2)^T = \begin{bmatrix} 29.16 & 23.76 \\ 23.76 & 19.36 \end{bmatrix}$$

$$(S_w)^{-1} = 1/[(2.64)\ .\ (5.28) - (0.44)\ .\ (0.44)] \begin{bmatrix} 5.28 & 0.44 \\ 0.44 & 2.64 \end{bmatrix}$$

$$=> (S_w)^{-1} = 1/(13.93 - 0.1936) \begin{bmatrix} 5.28 & 0.44 \\ 0.44 & 2.64 \end{bmatrix}$$

$$=> (S_w)^{-1} = 1/(13.73) \begin{bmatrix} 5.28 & 0.44 \\ 0.44 & 2.64 \end{bmatrix}$$

$$=> (S_w)^{-1} = \begin{bmatrix} 0.38 & 0.03 \\ 0.03 & 0.19 \end{bmatrix}$$

$$S_B = (S_w)^{-1}\ .\ (\mu_1 - \mu_2) = \begin{bmatrix} 0.38 & 0.03 \\ 0.03 & 0.19 \end{bmatrix} . \begin{bmatrix} -5.4 \\ -4.4 \end{bmatrix} = \begin{bmatrix} -2.1 \\ -0.9 \end{bmatrix}$$