Eigen Value & Eigen Vector of a matrices

Defn:-
Given a matrix A the roots of the characteristic eqn is $|A-\lambda I|=0$. is called characteristic roots (or) Eigen values (or) Eigen roots

Then $AX=\lambda X$ for each corresponding λ is called eigen vectors. (ie) $(A-\lambda I)X=0$.

Problems:-

1. Find the eigen value & eigen vector for the matrices $A=\begin{pmatrix}1&2\\2&1\end{pmatrix}$.

Soln:-
Let $A=\begin{pmatrix}1&2\\2&1\end{pmatrix}$ The characteristic eqn of the matrix is $|A-\lambda I|=0$.

$$\left|\begin{pmatrix}1&2\\2&1\end{pmatrix}-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\right| = \begin{vmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{vmatrix}=0$$

$[(1-\lambda)(1-\lambda)$
$=1-\lambda-\lambda+\lambda^2$
$=1-2\lambda+\lambda^2$

$\Rightarrow [(1-\lambda)(1-\lambda)]-4=0$

$\lambda^2-2\lambda+1-4=0$

$\lambda^2-2\lambda-3=0$ is called the characteristic equation.

$\lambda^2-2\lambda-3=0$.

$(\lambda-3)(\lambda+1)=0$.

$\Rightarrow \lambda = 3$ & -1 are called the eigen values (or) eigen roots.

To find the Eigen Vectors:-

$$[A - \lambda I]\, x = 0$$

$$\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow (1-\lambda) x_1 + 2x_2 = 0 \quad \ldots \ldots (1)$

$2x_1 + (1-\lambda) x_2 = 0 \quad \ldots (2).$

Case (i)

When $\lambda = 3$ in (1) eqn:-

$(1-3) x_1 + 2x_2 = 0$

$-2x_1 + 2x_2 = 0$

$\Rightarrow -2x_1 = -2x_2$

$\Rightarrow -x_1 = -x_2.$

So if $x_1 = 1$ then $x_2 = +x_1$

$\boxed{x_2 = +1}.$

Now, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Case (ii)

Now $\lambda = -1$ in (1) eqn.

$(1-\lambda)x_1 + 2x_2 = 0$

$(1+1)x_1 + 2x_2 = 0.$

$2x_1 = -2x_2.$

$x_1 = -x_2.$

If $x_1 = 1$; then $x_2 = -1$.

Now $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

Now $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, are called the eigen vectors.

— x —

2. Find the eigen value & eigen vectors of

$A = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}.$

Soln:-

The characteristic eqn is $|A-\lambda I| = 0$

$$\left|\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(3-\lambda)$
$= 3-\lambda-3\lambda+\lambda^2$
$= \lambda^2-4\lambda+3$

$$\Rightarrow \left[(1-\lambda)(3-\lambda)+0\right] = 0.$$

$\lambda^2-4\lambda+3=0$ is called the characteristic eqn.

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$-3, -1$ $\quad 3$
-4

$\lambda = 3$ or 1 are called eigen values.

To find Eigen vectors:-

$$(A - \lambda I)x = 0.$$

$$\begin{pmatrix} 1-\lambda & -1 \\ 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

$$(1-\lambda)x_1 - x_2 = 0 \quad \ldots\ldots (1)$$

$$(3-\lambda)x_2 = 0 \quad \ldots\ldots (2).$$

Case(i) $\lambda = 3$ in (1) eqn:-

$$(1-3)x_1 - x_2 = 0.$$

$$-2x_1 - x_2 = 0.$$

$$\Rightarrow -2x_1 = x_2.$$

Put $x_1 = 1$:-

$$x_2 = -2x_1$$
$$= -2(1) = -2.$$

Now, $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Case(ii) $\lambda = 1$ in (1) eqn:-

$$(1-1)x_1 - x_2 = 0$$

$$0x_1 - x_2 = 0$$

$x_1 = 0 \quad ; x_2 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now $P = \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}$ are called as eigen vectors.

—— x ——

Another method to find the characteristic eqn:-

1. Find the characteristic equation

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}.$$

Soln:-

$$\lambda^2 - s\lambda + |A| = 0.$$

$$|A| = \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} = 3 + 0 = 3.$$

s = Sum of the diagonal $= 1 + 3 = 4$.

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$$

diagonal

$\lambda^2 - 4\lambda + 3 = 0$ is called the characteristic eqn.

—— x ——

2. Find the eigen value & eigen vectors of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

Soln :-

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic eqn is

$$\lambda^2 - S\lambda + |A| = 0 .$$

$$|A| = 3 - 8 = -5 .$$

$S =$ sum of the diagonal $= 1 + 3 = 4 .$

$\lambda^2 - 4\lambda - 5 = 0$ is called the characteristic equation.

$$(\lambda - 5)(\lambda + 1) = 0 .$$

$\lambda = 5$ & -1 are called eigen values. $\left[\begin{matrix} -5 \\ +1 \end{matrix}\right] \begin{matrix} -5 \\ -4 \end{matrix}$

To find Eigen Vectors:-

$$(A - \lambda I)x = 0$$

$$\left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$(1-\lambda)x_1 + 4x_2 = 0 \quad \ldots \text{①}$$

$$2x_1 + (3-\lambda)x_2 = 0 \quad \ldots \text{②}.$$

Case(i) $\lambda = 5$ in ① eqn :-

$$(1-5)x_1 + 4x_2 = 0 .$$

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$-4x_1 + 4x_2 = 0$

$-4x_1 = -4x_2$

$x_1 = x_2$

Take $x_1 = 1$;

$x_2 = x_1 \Rightarrow 1$

$x_2 = 1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Case (ii) $\lambda = -1$ in ① eqn :-

$(1-(-1))x_1 + 4x_2 = 0$

$2x_1 + 4x_2 = 0$

$2x_1 = -4x_2$

$x_1 = -2x_2$

Take $x_1 = 1$:-

$-2x_2 = x_1 = 1$

$-2x_2 = 1$

$x_2 = -\frac{1}{2}$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

Now $P = \begin{pmatrix} 1 & 1 \\ 1 & -1/2 \end{pmatrix}$ are called eigen vectors.

x

Extra problems:-

1. Find the eigen value and eigen vectors

(i) $A = \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$

—x—

1. Find the eigen value & eigen vectors of the matrix $\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$

Soln:-

Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$

To find the characteristic eqn is $|A - \lambda I| = 0$.

(ie) $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$.

S_1 = sum of the diagonal elements

$= 1 + 2 - 1 = 2$.

S_2 = sum of the minors of main diagonal element.

$= \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$

$= -1 - 9 + 5 = -5$.

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$$|A| = \begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = -1-1-4 = -6.$$

Therefore the characteristic eqn is

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0.$$

To find the roots :-

If $\lambda = 1$, then $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0 \Rightarrow 1 - 2 - 5 + 6 = 0$

By synthetic division,

$$\begin{array}{c|cccc} & 1 & -2 & -5 & 6 \\ 1 & 0 & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

other roots are $\lambda^2 - \lambda - 6 = 0$

$$\begin{array}{c} -3 \\ 2 \\ \hline -1 \end{array} \Big] -6$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$\lambda = 3, -2, \& \ 1$ are the eigen values.

To find Eigen vectors :-

$$(A - \lambda I)x = 0$$

$$\begin{pmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$(1-\lambda)x_1 - x_2 + 4x_3 = 0$ ...... (1)

$3x_1 + (2-\lambda)x_2 - x_3 = 0$ ...... (2)

$2x_1 + x_2 - (1+\lambda)x_3 = 0$ ...... (3)

Case (i) $\lambda = 1$ in (1) & (2) :-

$(1-1)x_1 - x_2 + 4x_3 = 0$

$-x_2 + 4x_3 = 0$ ...... (1)

$3x_1 + (2-1)x_2 - x_3 = 0$ ...... (2)

$3x_1 + x_2 - x_3 = 0$

$$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 4 \\ 3 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-3} = \frac{x_2}{+12} = \frac{x_3}{3}$$

$\div$ divide throughout by 3 :-

$$\frac{x_1}{-1} = \frac{x_2}{+4} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ +4 \\ 1 \end{pmatrix}$$

Case (ii) $\lambda = 3$ in (1) & (2) :-

$-2x_1 - x_2 + 4x_3 = 0$

$3x_1 - x_2 - x_3 = 0$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -1 & 4 \\ -1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 4 \\ 3 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix}}$$

$$\frac{x_1}{5} = \frac{x_2}{\cancel{2+12}\ +10} = \frac{x_3}{+5}$$

$\div$ by 5:-

$$\frac{x_1}{1} = \frac{x_2}{+2} = \frac{x_3}{1}.$$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ +2 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = -2$ in ① & ② eqn:-

$$3x_1 - x_2 + 4x_3 = 0$$
$$3x_1 + 4x_2 - x_3 = 0$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -1 & 4 \\ 4 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & 4 \\ 3 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & -1 \\ 3 & 4 \end{vmatrix}}$$

$$\frac{x_1}{-15} = \frac{x_2}{+15} = \frac{x_3}{15}$$

$\div 15$:-

$$\frac{x_1}{-1} = \frac{x_2}{+1} = \frac{x_3}{1}$$ Then $X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ +1 \\ 1 \end{pmatrix}$

Now $P = \begin{pmatrix} -1 & 1 & -1 \\ +4 & +2 & +1 \\ 1 & 1 & 1 \end{pmatrix}$ are the eigen Vectors.

2. Find all the eigen values & eigen vectors of the matrix.

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{pmatrix}$$

Soln:-

The characteristic eqn is $|A-\lambda I| = 0$

(ie) $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$.

S_1 = sum of the diagonal $= -2+1+0 = -1$.

S_2 = sum of the minor of the diagonal $= \begin{vmatrix} 1 & 6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$

$= -12 - 3 - 6 = -21$.

$$|A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{vmatrix} = 45.$$

Then, $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$.

Find root:-

Check, $\lambda = 1 \Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 1+1-21-45 \neq 0$

$\lambda = -1 \Rightarrow -1+1+21-45 = -24 \neq 0$

$\lambda = 2 \Rightarrow 8+4-42-45 = -75 \neq 0$

$\lambda = -2 \Rightarrow -8+4+42-45 = -1 \neq 0$.

$\lambda = 3 \Rightarrow 27+9-63-45 = -72 \neq 0$

$\lambda = -3 \Rightarrow -27+9+63-45 = 0$.

hence $\lambda = -3$ then by synthetic division

$$\begin{array}{r|rrrr} & 1 & 1 & -21 & -45 \\ -3 & 0 & -3 & 6 & +45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

Now, $\lambda^2 - 2\lambda - 15 = 0$

$(\lambda - 5)(\lambda + 3) = 0$

$\lambda = 5, -3, -3$ are the eigen values. $\quad \begin{array}{c} -5 \\ 3 \\ -2 \end{array} \Big] -15$

To find Eigen vectors :-

$(A - \lambda I)(x) = 0$.

$$\left[\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & 6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$(-2-\lambda)x_1 + 2x_2 - 3x_3 = 0 \quad \ldots (1)$

$2x_1 + (1-\lambda)x_2 + 6x_3 = 0 \quad \ldots (2)$

$-x_1 - (2)x_2 - \lambda x_3 = 0 \quad \ldots (3)$

Case(i) if $\lambda = -3$ then (1) & (2) eqn becomes

$x_1 + 2x_2 - 3x_3 = 0$

$2x_1 + 4x_2 + 6x_3 = 0$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & 6 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ 4 & 6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -3 \\ 2 & 6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}}$$

$$\frac{x_1}{24} = \frac{-x_2}{12} = \frac{x_3}{0}$$

$\div 12$:-

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0}.$$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

Case (ii) here $\lambda = -3$ (twice exist so we take (2) & (3) eqn)

$$2x_1 + 4x_2 + 6x_3 = 0$$
$$-x_1 \cancel{+} 2x_2 + 3x_3 = 0$$

$$\begin{pmatrix} 2 & 4 & 6 \\ -1 & -2 & 3 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 6 \\ -1 & 3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{24} = \frac{-x_2}{12} = \frac{x_3}{0}$$

$\div 12$:- $\quad \frac{x_1}{2} = \frac{-x_2}{1} = \frac{x_3}{0}$ $\quad =0 \,|$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$ (so value twice exist. answer twice exist).

case (iii) $\lambda = 5$ in ① & ② eqn:-

$$-7x_1 + 2x_2 - 3x_3 = 0$$

$$2x_1 - 4x_2 + 6x_3 = 0$$

$$\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & 6 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & 6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -7 & -3 \\ 2 & 6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{0} = \frac{x_2}{+36} = \frac{x_3}{24}$$

$\div\ 12$:- $\quad \dfrac{x_1}{0} = \dfrac{x_2}{+3} = \dfrac{x_3}{2}$

$$X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ +3 \\ 2 \end{pmatrix}.$$

Then, $P = \begin{pmatrix} 2 & 2 & 0 \\ -1 & -1 & +3 \\ 0 & 0 & 2 \end{pmatrix}$ are the eigen vectors.

---x---

3. Find the eigen value & vectors of

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}.$$

Soln:-

The characteristic eqn is of the form $|A - \lambda I|$.

(ie) $\lambda^3 - S_1\lambda + S_2\lambda^2 - |A| = 0$.

S_1 = sum of the diagonal $= 1 + 5 + 1 = 7$.

S_2 = sum of the minor of the diagonal [(ie) $a_{11} + a_{22} + a_{33}$

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$$

$$= 4 + (-8) + 4 = 0.$$

$|A| = 36.$

Then, $\lambda^3 - 7\lambda^2 + 36 = 0$ is the characteristic eqn.

To find root :-

$$\begin{array}{c|cccc} 3 & 1 & -7 & 0 & 36 \\ & 0 & 3 & -12 & -36 \\ \hline & 1 & -4 & -12 & \boxed{0} \end{array}$$

Check for
$\lambda = 1, -1$
$\lambda = 2, -2$
Then, $\lambda = 3$
in $\lambda^3 - 7\lambda^2 + 36 = 0$

Now, $\lambda^2 - 4\lambda - 12 = 0$

$(\lambda - 6)(\lambda + 2) = 0$

-6, 2 | -12, -4

$\lambda = 6, -2, 3$ are eigen values.

To find eigen roots :-

$(A - \lambda I)(x) = 0$

Then
$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$(1-\lambda)x_1 + x_2 + 3x_3 = 0 \quad \ldots (1)$

$x_1 + (5-\lambda)x_2 + x_3 = 0 \quad \ldots (2)$

$3x_1 + x_2 + (1-\lambda)x_3 = 0 \quad \ldots (3)$

Case (i) If $\lambda = 6$ in ① & ② eqn:-

$$-5x_1 + x_2 + 3x_3 = 0$$

$$x_1 - 6x_2 + x_3 = 0$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-8} = \frac{x_3}{4}$$

$\div$ by 4 :- $\dfrac{x_1}{1} = \dfrac{x_2}{2} = \dfrac{x_3}{1}$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = -2$ in ① & ② eqn:-

$$3x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 7x_2 + x_3 = 0$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-20} = -\frac{x_2}{0} = \frac{x_3}{+20}$$

$\div$ by 20 :- then,

$$\Rightarrow \frac{x_1}{-1} = \frac{-x_2}{0} = \frac{x_3}{1}$$

then

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Case (iii) :- When $\lambda = 3$ in ① & ② :-

$$-2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{-5} = -\frac{x_2}{-5} = \frac{x_3}{-5}$$

$\div$ by 5 :-

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Now $P = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ are the eigen vectors

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4. Find the eigen value & eigen vectors of $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$.

Soln :-

The characteristic eqn is $|A-\lambda I| = 0$ (10)

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

S_1 = sum of the diagonal $= 6+3+3 = 12$.

S_2 = sum of the minor of the diagonal $= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & +3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

$= 8 + (+14) + 14 = 36$

$|A| = 32$.

Now, $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ is the characteristic eqn.

To find roots :-

$$\begin{array}{r|rrrr} 8 & 1 & -12 & 36 & -32 \\ & 0 & 8 & -32 & 32 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

Then $\lambda = 8$, $\lambda^2 - 4\lambda + 4 = 0$.

$(\lambda-2)(\lambda-2) = 0$.

-2, -2 ; $+4$; -4

$\lambda = 8$, & $\lambda = +2$ (twice) are the eigenvalues

To find eigen vectors :-

$(A-\lambda I)(x) = 0$.

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$(6-\lambda)x_1 - 2x_2 + 2x_3 = 0 \quad \ldots \quad (1)$$

$$-2x_1 + (3-\lambda)x_2 - x_3 = 0 \quad \ldots \quad (2)$$

$$2x_1 + x_2 + (3-\lambda)x_3 = 0 \quad \ldots \quad (3).$$

Case(i) $\lambda = 8$ in (1) & (2) eqn :-

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 - 5x_2 - x_3 = 0$$

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x_1}{12} = \frac{-x_2}{6} = \frac{x_3}{6}$$

$\div$ by 6 :- $\quad \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}.$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Case(ii) $\lambda = +2$ in (1) & (2) eqn :-

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \end{pmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{+0} = \frac{-x_2}{+0} = \frac{x_3}{0}$$

Special case :-

Here all 3 eqns are same. So it cannot have the value 0. since it has rank 3.

$$4x_1 - 2x_2 + 2x_3 = 0 \quad \ldots (4)$$
$$-2x_1 + x_2 - x_3 = 0 \quad \ldots (5)$$
$$2x_1 - x_2 + x_3 = 0 \quad \ldots (6).$$

Choose $x_1 = 0$ in (4) eqn :-

$$-2x_2 + 2x_3 = 0$$
$$-2x_2 = -2x_3$$
$$x_2 = x_3$$

Put $x_2 = 1$ $\Rightarrow$ then $x_2 = x_3 = 1$.

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Next, Choose $x_2 = 0$ in (4) eqn :-

$$4x_1 + 2x_3 = 0$$
$$4x_1 = -2x_3$$
$$2x_1 = -x_3$$

Now, $x_1 = 1$ then $2 = -x_3 \Rightarrow x_3 = -2$.

$$X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

Now $P = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$ are the eigen vectors.

Properties of Eigen values & its vectors:-

1 Sum of the eigen value = sum of the diagonal element.

2 Product of the eigen value = Its determinant value.

3 Every square matrix and its transpose has the same eigen values.

* If $\lambda_1, \lambda_2 \ldots \lambda_n$ are eigen values of A, then

(i) the inverse A^{-1} has the eigen value

$= \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$.

(ii) the matrix $A+kI$ has eigen values as $k+\lambda_1, k+\lambda_2 \ldots + k+\lambda_n$.

(iii) the matrix A^2 has eigen values as $\lambda_1^2, \lambda_2^2, \lambda^2 \ldots \lambda_n^2$.

(iv) the matrix kA has the eigen value as $k\lambda_1, k\lambda_2 \ldots k\lambda_n$.

Some problems using properties:-

1. Find the sum & product of the eigen values $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

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using property (1):-

sum of the eigen value = sum of the diagonal element.

= 2+2+1

Sum of the eigenvalue = 5.

product of the eigen value = Its determinant (Property (2))

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2(2)+1(1)+1(0)$$

= 3.

—— x ——

2. If 2 & 3 are eigen values of A. Find the eigen value of A^{-1} & A^3. & Where $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$

Soln:-

Here $\lambda_1 = 2$, $\lambda_2 = 3$ & $\lambda_3 = ?$

Now, using property (1)

sum of the eigen value = sum of the diagonal element.

$\lambda_1 + \lambda_2 + \lambda_3 = 3 - 3 + 7$

$2 + 3 + \lambda_3 = 7$

$5 + \lambda_3 = 7$

$\Rightarrow \lambda_3 = 7 - 5 = 2.$

Now, $\lambda_3 = 2$

Then, Find A^{-1} :-

The inverse of A has the eigen value

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ (using property (i).

So $A^{-1} = \frac{1}{2}, \frac{1}{3}, \frac{1}{2}$.

Find A^3 :-

The matrix A^3 has eigen value as

$\lambda_1^3, \lambda_2^3, \lambda_3^3$ (using property (iii)

Now $A^3 = 2^3, 3^3, 2^3$.

---

3. If two of the eigen values are -1 & 2. It is a 3X3 matrix whose determinant is equal to 4. Find the third value?

Soln:-

Given $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = ?$

$|A| = 4$.

Product of the diagonal = determinant [by property 2.

$\lambda_1 \lambda_2 \lambda_3 = |A|$.

$(2)(-1)\lambda_3 = 4$.

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$$-2\lambda_3 = 4$$

$$\lambda_3 = \frac{4}{-2}$$

$$\boxed{\lambda_3 = -2}$$

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Extra problems:-

1. Find the eigen values & eigen vectors of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

2. Find the eigen value & eigen vectors of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

3. If 2 & 3 are eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Find the third eigen value and also its inverse A^{-1} & A^2?

4. If 1, 1, 5 are eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ find the eigen value of $5A$?

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Cayley hamilton theorem :-

Given a square matrix of order 'n×n' and the identity matrix of same order then the determinant $|A - \lambda I| = 0$ is a polynomial of a degree 'n' called the characteristic equation of the matrix A.

Statement :-

Every square matrix satisfies its characteristic equation $|A - \lambda I| = 0$. (statement only)

Note :-

cayley hamilton theorem is useful in finding the inverse of a matrices.

Problem :-

Using cayley hamilton find the characteristic eqn, find A^{-1} and also verify the theorem.

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$

soln :-

$$\lambda^2 - S\lambda + |A| = 0$$

$S =$ sum of the diagonal $= 1 + 3 = 4$

$|A| = \overset{3}{\cancel{4}} + 2 = 5.$

Then, $\lambda^2 - 4\lambda + 5 = 0 \quad \ldots (1)$ is the characteristic eqn.

[redacted]

$|A| = 40.$

Now, $\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$ is the characteristic eqn. .... (1)

Cayley hamilton statement :-
Every square matrix satisfy its Characteristic equation.

Now, $\lambda^2 - 4\lambda + 5 = 0$

Can be written as $A^2 - 4A + 5I = 0$ .. ②.

$$A^2 = A \times A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 8 \\ 4 & 7 \end{pmatrix} \quad \dots ③$$

$$4A = 4\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ -4 & 12 \end{pmatrix} \quad \dots ④$$

$$5I = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \quad \dots ⑤$$

Apply ③, ④, ⑤ in ② eqn :-

$A^2 - 4A + 5I$

$$= \begin{pmatrix} -1 & 8 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} 4 & 8 \\ -4 & 12 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So it is verified.

Find A^{-1} using equation :-

post multiply by A^{-1} :-

$$A^{-1} \cdot A^2 - 4A^{-1} \cdot A + 5A^{-1} I = 0 I.$$

using cayley hamilton,

$$A^3 + A^2 - 18A - 40 = 0 \quad \ldots \text{(2)}$$

<u>Verify:-</u>

$$A^3 = A^2 \times A$$

$$A^2 = \begin{pmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{pmatrix} \quad \ldots\ldots \text{(3)}$$

$$A^3 = \begin{pmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{pmatrix} \quad \ldots\ldots \text{(4)}$$

$$18A = \begin{pmatrix} 18 & 36 & 54 \\ 36 & -18 & 72 \\ 54 & 18 & -18 \end{pmatrix} \quad \ldots \text{(5)}$$

$$40I = \begin{pmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{pmatrix}$$

$$A^3 + A^2 - 18A - 40I = \begin{pmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{pmatrix}$$

$$+ \begin{pmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{pmatrix} - \begin{pmatrix} 18 & 36 & 54 \\ 36 & -18 & 72 \\ 54 & 18 & -18 \end{pmatrix} - \begin{pmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ hence verified.

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Extra problem:-

1. Using cayley hamilton theorem. find the characteristic eqn. Find A^{-1} & verify the theorem.

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}$$

2. Using cayley hamilton theorem. Find the characteristic eqn. Find A^{-1} & verify the theorem.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

—— x ——