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Bisection Method

A numerical method in mathematics to find a root of a given function.

Well known mathematical prop

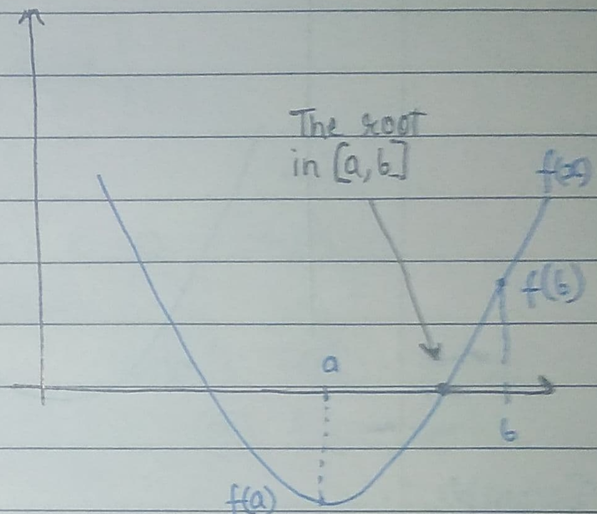
If $f(x)$ is continuous on interval $[a, b]$ and sign of $f(a) \neq$ sign of $f(b)$, then :

then there is a value $c \in [a, b]$ such that

$f(c) = 0$

i.e. there is a root c in interval $[a,b]$.

Example



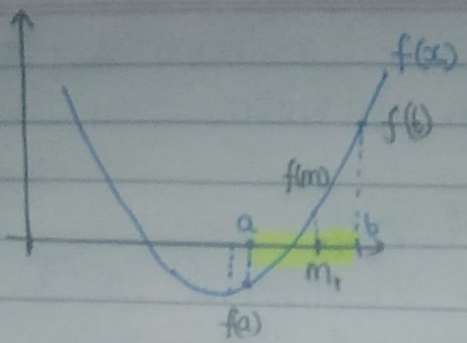
*. The bisection method is an successive approximation method that narrows down an interval that contains a root of the function $f(x)$

- Given: an initial interval $[a,b]$ that contains a root.

- It will cut the interval into 2 halves & check which half interval contains a root of the function

- It will keep cutting the interval in halves until the resulting interval is extremely small.

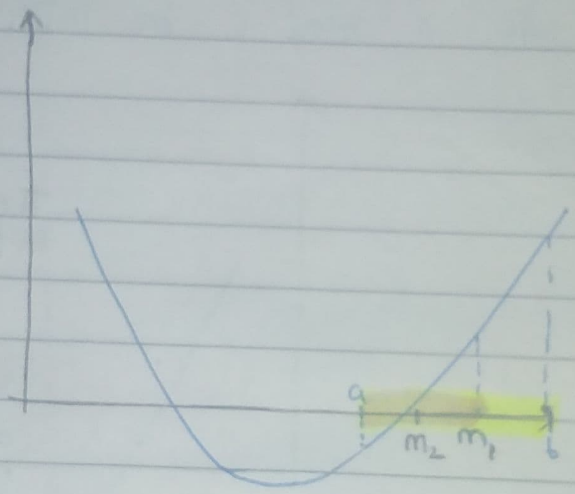
Example 1:



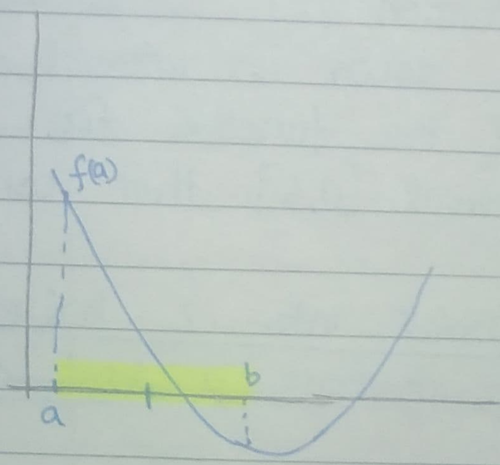
Since $f(a)\,f(m_1) < 0$,

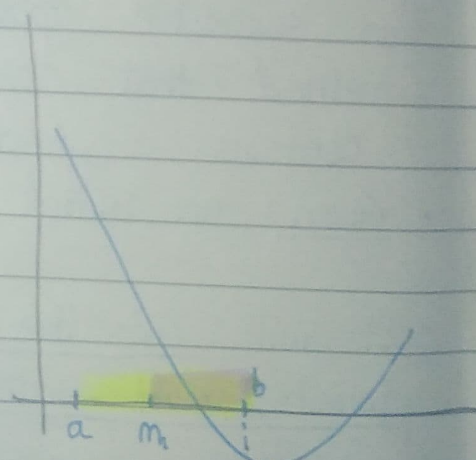
Now $m_2 = \frac{a + m_1}{2}$

$I_1 = m_1 - a$
$= \frac{b-a}{2}$



Example 2.





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current range

$[a,b] = I_0 \Rightarrow b-a = \text{length}$

$\frac{a+b}{2} = m_1$ then $f(a)\cdot f(m_1) < 0$

or $f(m_1)\cdot f(b) < 0$

$[m_1,b] = I_1$ (Suppose $f(m_1)\cdot f(b) < 0$)

length $b - m_1 = b - \frac{a+b}{2} = \frac{b-a}{2}$

$\therefore \boxed{\text{length of } I_1 = \frac{b-a}{2}}$

Now if $f(m_1)\cdot f(b) < 0$ then

$\frac{m_1+b}{2} = m_2$ and $f(m_1)\cdot f(m_2) < 0$

or $f(m_2)\cdot f(b) < 0$

Suppose

$f(m_2)\cdot f(b) < 0$

$I_2 = [m_2, b]$

$\therefore$ length of $I_2 = b - m_2 = b - \frac{m_1+b}{2}$

$= \frac{b-m_1}{2} = \frac{b + \frac{a+b}{2}}{2}$

$\boxed{\text{length of } I_2 = \frac{b-a}{2^2}}$

and so on

$\boxed{\text{length of } I_n = \frac{b-a}{2^n}}$

Question: Perform ~~finite~~ five iteration of the bisection method to obtain the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0$$

Ans. ans = 0.2016

$f(0) = 1$

$f(1) = 1 - 5 + 1 = -3$

$\therefore f(0) \cdot f(1) < 0$

Hence there is a root b/w $[0, 1]$

Start:

$a = 0, f(0) = 1$

$b = 1, f(b) = -3$ | $I_0 = 1$

#1 Iteration 1:

$m_1 = \left(\frac{a+b}{2}\right) = \frac{1}{2} = 0.5$

$f(m_1) = 0.125 - 2.5 + 1 = 0.875 = +ve$

Since, $f(m_1) \cdot f(b) < 0$

#2 New range $= [0.5, 1]$ | $I_1 = 0.5$

$m_2 = \frac{0.5 + 1}{2} = 0.75$

$f(m_2) = -2.328 = -ve$

$f(m_1) \cdot f(m_2) < 0$

$\therefore$ New range $[m_1, m_2] = [0.5, 0.75]$

$m_3 = 0.625$

$f(m_3) = -1.88 = -ve$

$\therefore$ New range $= [m_1, m_3] = [0.5, 0.625]$

$m_4 = 0.5625$

$f(m_4) = -1.634 = -ve$

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~~New Range [0.5, 0.5625~~

Start:

$a = 0$, $f(0) = 1$ +ve

$b = 1$, $f(1) = -3$ -ve

#0 Range $[0, 1]$

$m_1 = 0.5$

$f(m_1) = 0.125 - 2.5 + 1 = -1.375$ (-ve)

$\therefore f(a) \cdot f(m_1) < 0$

#1. Range $= [a, m_1] = [0, 0.5]$

$m_2 = 0.25$

$f(m_2) = -0.2343$ (-ve)

$\therefore f(a) \cdot f(m_2) < 0$

#2 Range $= [0, 0.25]$

$m_3 = 0.125$

$f(m_3) = 0.3769$ (+ve)

$\therefore f(m_3) \cdot f(m_2) < 0$

#3. Range $= [0.125, 0.25]$

$m_4 = 0.1875$

$f(m_4) = 0.069$ (+ve)

#4 Range $= [m_4, m_2] = [0.1875, 0.25]$

$m_5 = 0.21875$

$f(m_5) = -0.08328 = $ (-ve)

#5. Range $= [m_4, m_5] = [0.1875, 0.21875]$

$m_6 = 0.203125$

$f(m_6) = -0.007244$ -ve (≈ 0)

After 6 iterations, the root can be 0.203125

$\left(rad \times \frac{180}{\pi}\right)$ $\frac{\pi}{180}$

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Q2. Perform 5 iteration of the bisection method to obtain root of the eqn:

$$f(x) = \cos(x) - xe^{x} = 0$$

rad

(Ans = 0.515625)

$f(0) = 1 - 0 = 1$ (+ve)

$f(1) = 0.540 - 2.718 = -2.178$ (-ve)

Start

$a = 0$ $f(0) = 1$

$b = 1$ $f(1) = -2.178$

#0. Range $[0, 1]$

$m_1 = 0.5$

$f(m_1) = 0.0532$ (= +ve)

#1. Range = $[m_1, b] = [0.5, 1]$

$m_2 = 0.75$

$f(m_2) = -0.856$ (-ve)

#2. Range = $[m_1, m_2] = [0.5, 0.75]$

$m_3 = 0.625$

$f(m_3) = -0.3567$ (-ve)

#3. Range = $[m_1, m_3] = [0.5, 0.625]$

$m_4 = \cancel{-0.14129}\ 0.5625$

$f(m_4) = -0.14129$ (-ve)

#4. Range $[m_1, m_4] = [0.5, 0.5625]$

$m_5 = 0.53125$

$f(m_5) = -0.041512$ (-ve)

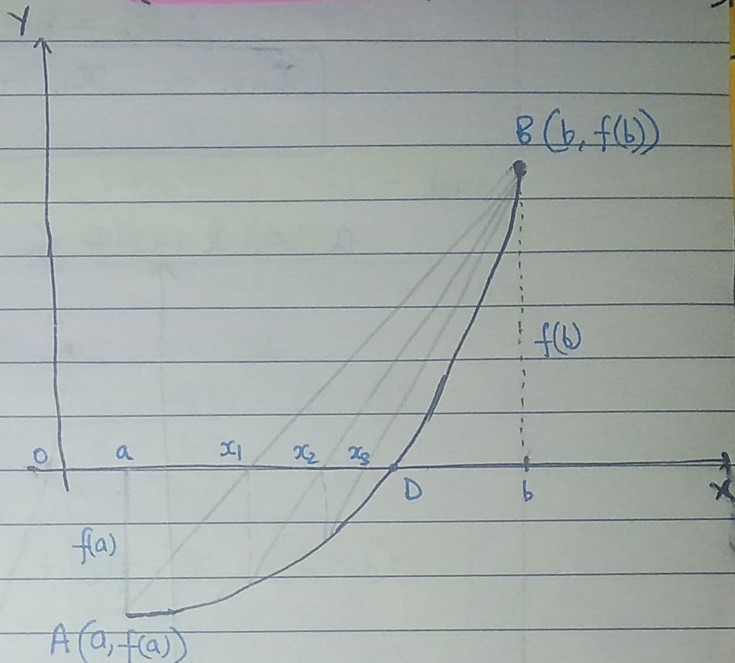
#5. Range = $[m_1, m_5] = [0.5, 0.53125]$

$\boxed{m_6 = 0.515625}$

Regula - Falsi Method (Method of Chords) (or Method of False Position)

To find the root l of the eq^n $f(x) = 0$ in the interval $[a, b]$ assume that $f(a) < 0$ and $f(b) > 0$ so that

$$f(a) \cdot f(b) < 0$$



Eq^n of chord AB

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

If passes through C and let the coordinate of C be $(x_1, 0)$ then

$$- f(a) = \frac{f(b) - f(a)}{b - a}(x_1 - a)$$

$$\Rightarrow \boxed{x_1 = a - \frac{f(a)}{f(b) - f(a)}(b - a)}$$

If $f(a) < 0$, then the end point b is fixed and the successive approximations $\boxed{x_0 = a}$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f(b) - f(x_n)}(b - x_n)}$$

(for $n = 0, 1, 2, \ldots$) form a bounded increasing monotonic sequences and

$$x_0 < x_1 < x_2 < x_3 \cdots < x_n < l < b$$

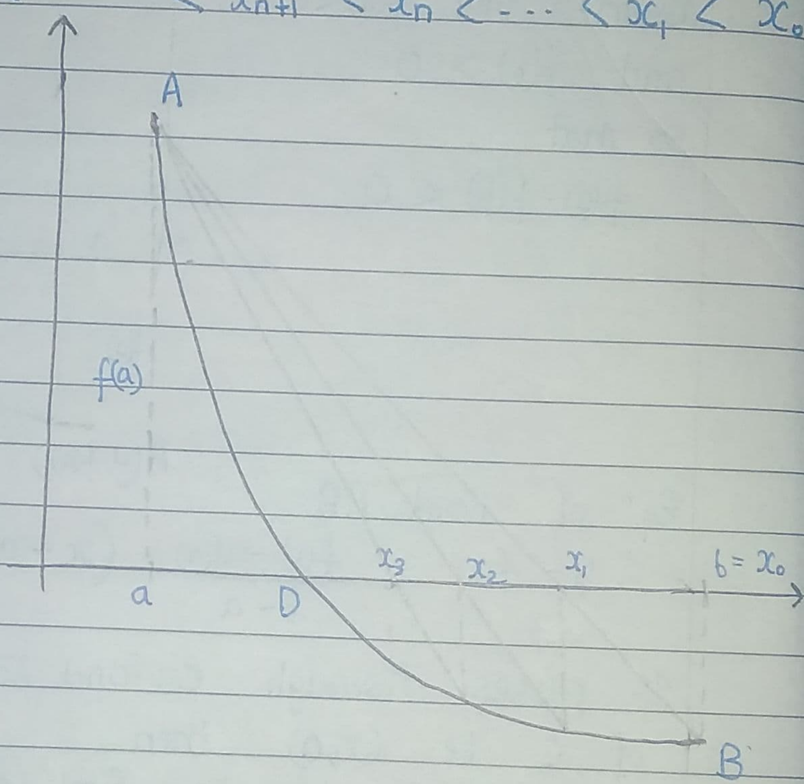
and if $f(a) > 0$, end point 'a' is fixed and successive approximation are

$$x_0 = b$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(a)} (x_n - a)}$$

and

$$a < l \cdots < x_{n+1} < x_n < \cdots < x_1 < x_0$$



Note:

1. Fix the end point for which sign of f and f'' are same
2. Successive approximations x_n lie on the side of the root l where sign of f is opposite to the ~~calculation~~ sign of f''

Calculation Regula Falsi Method when b is fixed

- $x_1 = a - \frac{f(a)}{f(b) - f(a)} (b - a)$
- $a \Rightarrow x_0$
- $x_1 = x_0 - \frac{f(x_0)}{f(b) - f(x_0)} (b - x_0)$

$$x_1 \Rightarrow x_2$$

$$x_2 = x_1 - \frac{f(x_1)}{f(b) - f(x_1)}(b - x_1)$$

$$\vdots$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f(b) - f(x_n)}(b - x_n)}$$

Calculation: when a is fixed

$$y - f(a) = \frac{f(a) - f(b)}{a - b}(x - b)$$

If passes through $(x_1, 0)$

$$-f(a) = \frac{f(a) - f(b)}{a - b}(x_1 - b)$$

$$\Rightarrow x_1 = b - \frac{f(b)}{f(a) - f(b)}(a - b)$$

$$\Rightarrow \boxed{x_1 = b - \frac{f(b)}{f(b) - f(a)}(b - a)}$$

$(b = x_0)$ $b \longrightarrow x_0$

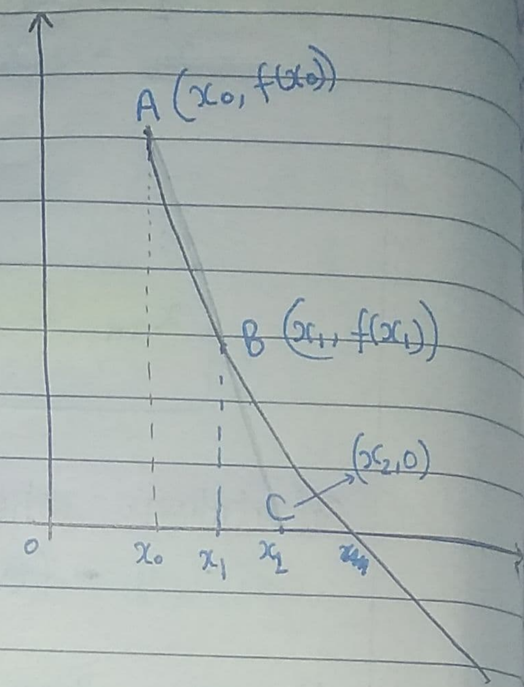
$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f(x_0) - f(a)}(x_0 - a)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f(x_1) - f(a)}(x_1 - a)$$

$$\vdots$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(a)}(x_n - a)}$$

Secant Method



Eq^n of AB

$$y - f(x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}(x - x_1)$$

If it passes through $(x_2, 0)$ then

$$x_2 = x_1 - \frac{x_0 - x_1}{f(x_0) - f(x_1)} f(x_1)$$

$$\Rightarrow \boxed{x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)} \quad *$$

$\vdots$

$$\boxed{x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)} \quad **$$

Q1. Use the secant and Regula-Falsi methods to determine the roots of the eq^n

(i) $\cos x - xe^x = 0$

(ii) $x^3 - 5x + 1 = 0$

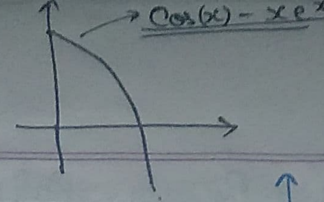
Ans (i) $\cos x - xe^x = 0$

~~(A) Regula Falsi Method~~

at $x = 0$, $f(x) = 1$ (+ve)

$x = 1$, $f(x) = -2.178$ (-ve)

$\therefore$ This will be the case where A will be fixed (Regula Falsi)



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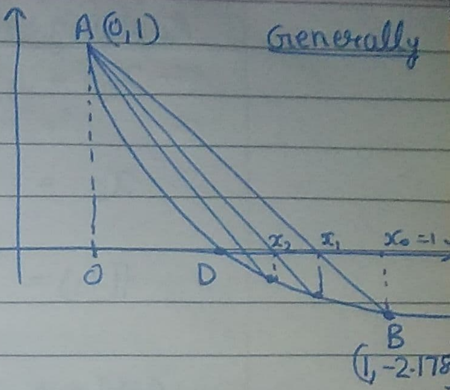


$\therefore$

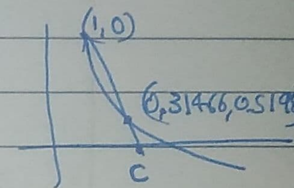
$$x_1 = b - \frac{f(b)}{f(b) - f(a)}(b-a)$$

$$= 1 - \frac{(-2.178)}{-2.178 - 1}(1)$$

$$= 0.31466$$

$$f(x_1) = 0.5198$$

Now applying Secant Method

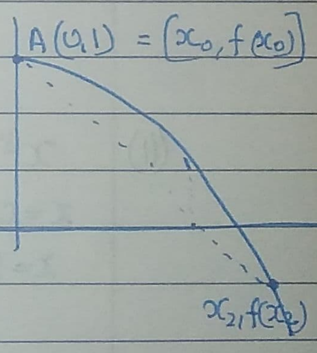


$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 0.31466 - \frac{(0.31466 - 0)}{(0.5198 - 1)} \times 0.5198$$

$$= 0.6552$$

$$f(x_2) = -0.46894$$



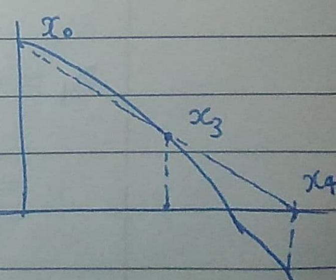
Now applying Regula Falsi

$$x_3 = x_2 - \frac{f(x_2)}{f(x_2) - f(a)}(x_2 - a)$$

$$= 0.6552 - \frac{(-0.46894)}{(-0.46894 - 1)}(0.6552 - 0)$$

$$= 0.4460$$

$$f(x_3) = 0.2054$$



Now applying Secant method

$$x_4 = 0.4460 - \frac{(0.4460 - 0)}{(0.2054 - 1)} \times (0.2054)$$

x_4 = ~~1.03246~~ 0.56128 and $f(x_4)$ = ~~2.386~~ -0.13733

Applying Regula Falsi

$$x_5 = \cancel{1.03246}\ 0.56128 - \frac{(\cancel{-2.386})\ -0.13733}{(\cancel{-2.386}\ -0.13733 - 1)} (\cancel{1.03246}\ 0.56128 - 0)$$

$$x_5 = \cancel{0.30492}\ 0.4935$$

$$f(x_5) = \cancel{0.54024}\ 0.072$$

Applying Secant

$$\cancel{x_6 = 0.30492 - \frac{0.30492 - 0}{(0.54024 - 1)}}$$

$$x_6 = 0.4935 - \frac{0.4935 - 0}{\cancel{0.54}\ (0.072 - 1)} (0.072)$$

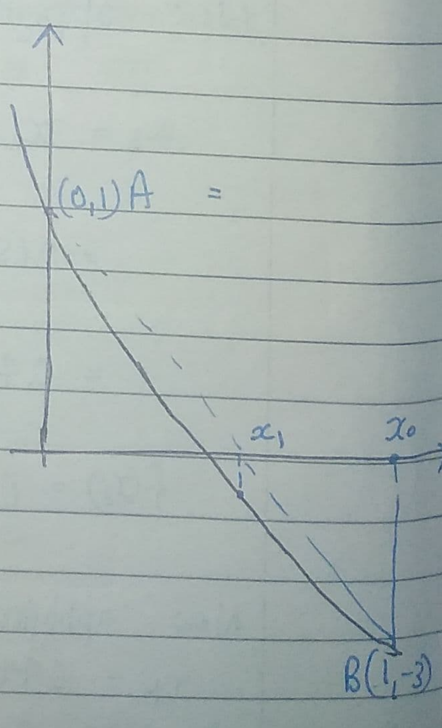
$$= 0.531954$$

$$f(x_6) = -0.0437$$

This is significantly small to stop
∴ ans (after 6 iterations) = 0.531954

(ii) $x^3 - 5x + 1$

$x = 0, \quad f(0) = 1$
$x = 1, \quad f(1) = -3$



Applying Regula Farsi

$$x_1 = 1 - \frac{-3}{-3 - 1} \times (1 - 0)$$

$$= 0.25$$

$$f(x_1) = -0.234375 \quad (-ve)$$

Again Regula Falsi

$$x_2 = 0.25 - \frac{(-0.234375)}{(-0.234375-1)}(0.25-0)$$

$$x_2 = 0.20253$$

$$f(x_2) = -0.00435$$

#3. One more Regula falsi

$$x_3 = 0.20253 - \frac{-0.00435}{(-0.00435-1)} \times (0.20253-0)$$

$$\underline{x_3 = 0.20165281}$$

$$f(x_3) = -0.0000640$$

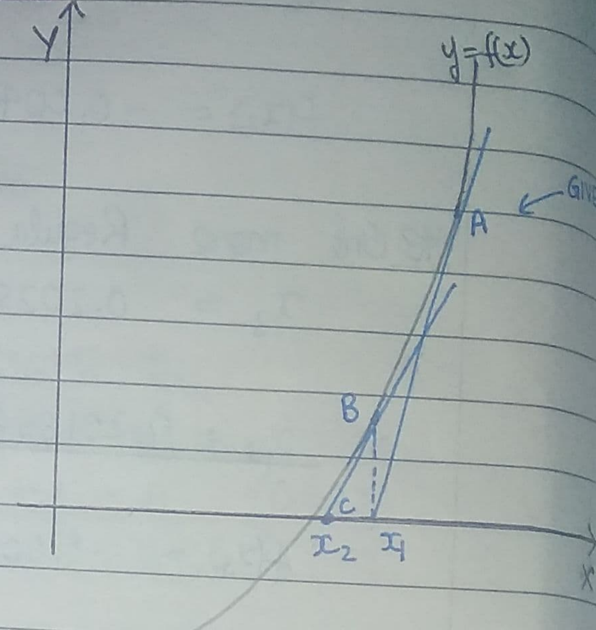
We can stop here, ans (root) is $\underline{0.20165281}$

[actual ans = 0.201639, only -0.0068 % deviation]

Formulae are imp to learn

NEWTON'S METHOD
or Newton - Raphson Method
or Method of Tangent

Geometrically, Newton's method is equivalent to replacing a small arc of the curve $y = f(x)$ by a tangent line drawn at a point on the curve.



Let $\alpha = x + h$

where α is approximate root
h is a small quantity

$f(\alpha) = 0 \Rightarrow f(x+h) = 0$

$f(x) + h f'(x) = 0$ (Expanding by Taylor Series)

$\Rightarrow h = -\dfrac{f(x)}{f'(x)}$

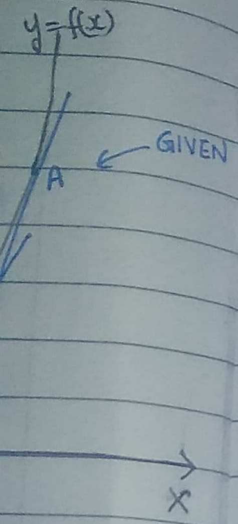
then

$$\alpha = x + h = \boxed{x - \frac{f(x)}{f'(x)} = x_1}$$

In general,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

NOTE: ① A root α of the eq^n $f(x) = 0$ can be computed to any degree of accuracy if a good initial approximation x_0 is chosen for which $f(x_0) \cdot f''(x_0) > 0$



(11) Newton's method converge slow if f' is small (fails when $f' = 0$)

Q1. Find the real root of the following eq^n correct to three decimal places, using Newton - Raphson method

$$x^3 - 2x - 5 = 0$$

ans: 2.09461

Since there is no starting point A given, we'll find one as midpoint of 2 points for which $(f(x) = +, -)$ as in bisection method.

$f(2) = 8 - 4 - 5 = -1 = -ve$

$f(3) = 27 - 6 - 5 = 16 = +ve$

$\therefore$ Let $A = 2.5$ and $f(A) = 5.625$ (+ve)

$f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2$

$x_1 = x - \dfrac{f(x)}{f'(x)} = 2.5 - \dfrac{f(2.5)}{f'(2.5)}$

$= 2.5 - \dfrac{5.625}{16.75}$

$= 2.16418$

$x_2 = 2.16418 - \dfrac{f(2.16418)}{f'(2.16418)}$

$= 2.097136$

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$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 2.094555$

Correct Ans: 2094551482

could've Stopped here.

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$\boxed{x_4 = 2.094551}$

Q2. By using Newton Raphson method, find the root of $x^4 - x - 10 = 0$ which is near to $x = 2$ correct to 3 places of decimal.

Ans. Starting with $x = 2$
$f(x) = 16 - 2 - 10 = 4$ (+ve)

$f(x) = x^4 - x - 10$
$f'(x) = 4x^3 - 1$

(Correct ans:?) for when 3 decimal digits stops changing STOP

$x_1 = x - \frac{f(x)}{f'(x)}$

$= 2 - \frac{f(2)}{f'(2)} = 1.05098$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.749203$

$x_3 = x_2 - \frac{f(x_1)}{f'(x_1)} = 2.872968$

$x_4 = 2.284234$

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$x_5 = 1.9641326$

$x_6 = 1.8645518$

$x_7 = 1.855661$

$\boxed{x_8 = 1.855584}$

verifying:

$x_8^4 - x - 10$

$= 1.29 \times 10^{-5}$

(≈ 0)

Q3. Using Newton-Raphson method, compute the real root of the following eq^n

(i) $e^x = 3x$ lying b/w 0 and 1

(ii) $x \log_{10} x - 1.2 = 0$

(i) $e^x - 3x = 0$

~~let~~ $f(0) = 1$

$f(1) = e - 3 = -ve$

let point A (starting) have $x = 0.5$

$f(0.5) = 0.1487$

$f(x) = e^x - 3x$

$f'(x) = e^x - 3$

$x_1 = x - \dfrac{f(x)}{f'(x)}$

$x_1 = 0.61005$

$x_2 = 0.618997$

$x_3 = 0.6190612$

$\boxed{x_4 = 0.6190612}$

This will be our ans!

verifying

$e^{x_4} - 3x_4$

$= 9.9 \times 10^{-8}$

$\approx 10^{-7} \to 0$

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(ii) $x \log_{10} x - 1.2 = 0$

at $x=1$, $f(1) = -1.2$ (= -ve)
$x=10$, $f(10) = 8.8$ (= +ve)

Let starting Point A have $x = 5$

$f(x) = x \log x - 1.2$

$f'(x) = \cancel{x} \times \frac{1}{\cancel{x}} + \log x$

$= 1 + \log x$

$x_1 = x - \frac{f(x)}{f'(x)}$

$x_1 = 3.6492698$

$x_2 = 3.1041168$
$x_3 = 2.88491$
$x_4 = 2.7976327$
$x_5 = 2.76310$
$x_6 = 2.7494876$
$x_7 = 2.744125592$
$x_8 = 2.7420152$
$x_9 = 2.741184776$
$x_{10} = 2.740858$
$\boxed{x_{11} = 2.740729482}$

verifying
$f(x_{11}) = 0.000072$
$\to 0$

Stopping here (because first three decimal places stopped changing).

$\therefore$ Answer is $x = 2.740729482$

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Rate of Convergence of Secant Method

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})\, f(x_n)}{f(x_n) - f(x_{n-1})}$$

If $x_n = \alpha + e_n$

where α = actual root & $f(\alpha) = 0$

and e_n is the error.

then

$$\alpha + e_{n+1} = (\alpha + e_n) - \frac{[(\alpha + e_n) - (\alpha + e_{n-1})]\, f(\alpha + e_n)}{f(\alpha + e_n) - f(\alpha + e_{n-1})}$$

$$e_{n+1} = e_n - \frac{(e_n - e_{n-1})\left[f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) \ldots\right]}{\left[\left(f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \ldots\right) - \left(f(\alpha) + \frac{e_{n-1}}{1!} f'(\alpha) + \frac{e_{n-1}^2}{2!} f''(\alpha)\right)\right]}$$

$$= e_n - \frac{\cancel{(e_n - e_{n-1})}\left[e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \ldots\right]}{\cancel{(e_n - e_{n-1})}\left[f'(\alpha) + \frac{(e_n + e_{n-1})}{2!} f''(\alpha) \ldots\right]}$$

$$= e_n - \frac{e_n \cancel{f'(\alpha)}\left[1 + \frac{e_n}{2!}\frac{f''(\alpha)}{f'(\alpha)} + \ldots\right]}{\cancel{f'(\alpha)}\left[1 + \frac{(e_n + e_{n-1})}{2!}\frac{f''(\alpha)}{f'(\alpha)} \ldots\right]}$$

$$= e_n - e_n\left[1 + \frac{e_n}{2!}\frac{f''(\alpha)}{f'(\alpha)} + \ldots\right]\left[1 + \frac{(e_n + e_{n-1})}{2!}\frac{f''(\alpha)}{f'(\alpha)} + \ldots\right]^{-1}$$

$$= e_n - e_n\left[1 + \frac{e_n}{2!}\frac{f''(\alpha)}{f'(\alpha)} + \ldots\right]\left[1 - \frac{(e_n + e_{n-1})}{2!}\frac{f''(\alpha)}{f'(\alpha)} + \ldots\right]$$

$$= e_n - e_n\left[1 - \frac{(\cancel{e_n} + e_{n-1})}{2!}\frac{f''(\alpha)}{f'(\alpha)} + \frac{\cancel{e_n}}{2!}\frac{f''(\alpha)}{f'(\alpha)} + \ldots\right]$$

$$= \cancel{e_n} - e_n\left[\cancel{1} - \frac{e_{n-1}}{2!}\frac{f''(\alpha)}{f'(\alpha)}\right]$$

0.000072 → 0

$$e_{n+1} = \frac{e_n e_{n-1}}{2!} \frac{f''(\alpha)}{f'(\alpha)}$$

$$= e_n e_{n-1} e_1$$

[Since, $e_1 = \frac{1}{2!} \frac{f''(\alpha)}{f'(\alpha)}$]

By definition

- $e_n = c\, e_{n-1}^{p}$

  $e_{n-1} = \left(\frac{e_n}{c}\right)^{1/p}$

- $e_{n+1} = c\, e_n^{p}$

$$c e_n^{p} = c_1 e_n \left(\frac{e_n}{c}\right)^{1/p}$$

$$e_n^{p} = c_1 \cdot e_n^{1+1/p} \cdot c^{-(1+1/p)}$$

$$p = 1 + \frac{1}{p}$$

$$\Rightarrow p^2 - p - 1 = 0$$

$$\Rightarrow p = \frac{1 \pm \sqrt{1+4}}{2} = 1.618$$

Rate of Convergence of Regula Falsi Method

$e_{n+1} = c\, e_0 e_n$, $a = \alpha + e_0$

$e_{n+1} = c_1 e_n$ (here $p = 1$)

Hence Regula Falsi has linear rate of convergence.

$e_1 = \frac{1}{2!}\frac{f''(\alpha)}{f'(\alpha)}$

Method

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Rate of convergence of Newton - Rapson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\cancel{\alpha} + e_{n+1} = \cancel{\alpha} + e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_n)}$$

$$e_{n+1} = e_n - \frac{\left[\overset{=0}{f(\alpha)} + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \cdots\right]}{\left[f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) + \cdots\right]}$$

$$= e_n - e_n \frac{\left[1 + \frac{e_n}{2!}\frac{f''(\alpha)}{f(\alpha)}\right]}{\left[1 + e_n \frac{f''(\alpha)}{f'(\alpha)} + \frac{e_n^2}{2!}\frac{f'''(\alpha)}{f'(\alpha)} + \cdots\right]}$$

$$= e_n - e_n \left[1 + \frac{e_n}{2!} f''(\alpha)\right]\left[1 - e_n \frac{f''(\alpha)}{f'(\alpha)}\right]$$

$$e_{n+1} = \cancel{e_n} - e_n\left[\cancel{1} - e_n \frac{f''(\alpha)}{f'(\alpha)} + \frac{e_n}{2}\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]$$

$$e_{n+1} \approx \frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

$$e_{n+1} = C e_n^2$$

where $C = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$

$$p = 2$$

Hence Newton's method has 2^{nd} order convergence.

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Fixed Point method / Iteration method

Let $f(x) = 0$ — ①
can be written as
$\phi(x) = x$ — ②

where $|\phi'(x)| < 1$ [CONDITION]

- Let first approximate root be
$x_1 = a$

- Second approximate root be x_2
Put x_1 in LHS of ②

$\phi(x_1) = x_2$

and then third approximate root
$\phi(x_2) = x_3$

Generally, $\boxed{x_{n+1} = \phi(x_n)}$

Q. Use this method to solve $x = e^{-x}$ starting with $x = 1$. Perform 4 iterations taking reading upto 4 decimals.

Ans. $x = e^{-x}$

$\phi(x) = x = e^{-x}$

$|\phi'(x)| = |-e^{-x}| = |e^{-x}|$

at $x = 1$, $|\phi'(x)| = e^{-1} < 1$ $\therefore$ CONDITION SATISFIED!!

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Now $x_{n+1} = \phi(x_n)$

Given $x_1 = 1$

$\therefore x_2 = \phi(x_1) = \phi(1) = e^{-1} = 0.367879$

$x_3 = e^{-x_2} = 0.6922$

$x_4 = e^{-x_3} = 0.50047$

$x_5 = e^{-x_4} = 0.6062$

Q. Find the real root of the equation $x^3 + x^2 - 1 = 0$, by iteration method.

Ans. $x^3 + x^2 - 1 = 0$

$f(0) = -1$
$f(1) = 1$ } So the initial point be (from bisection method) $= \frac{0+1}{2} = 0.5$

$\boxed{x_1 = 0.5}$

$x^3 + x^2 - 1 = 0$

$x^3 = 1 - x^2$

$x = (1 - x^2)^{1/3}$

$\therefore$ Let $\phi(x) = x = (1 - x^2)^{1/3}$

$|\phi'(x)|_{x=0.5} = \left| \frac{1}{3} \times (1 - x^2)^{-2/3} \times -2x \right|_{x=0.5} = 0.403 < 1$

$\therefore$ CONDITION IS SATISFIED

$x_2 = (1 - x_1^2)^{1/3} = 0.90856$

$x_3 = \; = 0.55883$

$x_4 = \; = 0.88267$

$x_5 = \; = 0.60448$

$x_6 = 0.85934$

$x_7 = 0.63950$

$x_8 = 0.83921$

$x_9 = 0.66237$

$x_{10} = 0.8223530$

$x_{11} = 0.68664$

$x_{12} = 0.80851$

$x_{13} = 0.70224$

$x_{14} = 0.79731$

$x_{15} = 0.71419$

$x_{16} = 0.788334$

$x_{17} = 0.72338$

$x_{18} = 0.781186$

$x_{19} = 0.730456$

$x_{20} = 0.77552$

$x_{21} = 0.73592$

$x_{22} = 0.77106$

$= 0.74014$

$= 0.76754$

$= 0.74342$

$x_{21} = 0.76478$

$x_{22} = 0.7459$

$x_{23} = 0.76262$

$x_{24} = 0.747938$

$x_{25} = 0.760928$

⋮

$x_{58} = 0.75477$

$x_{59} = 0.7549$

[Stopping since first 3 digit stopped changing]

[$f(x_{59}) = 0.00028$]

Answer is converging very slowly