UNIT 3

20/01/2020 Wednesday

FINITE DIFFERENCE OPERATOR

(i) FORWARD DIFFERENCE OPERATOR Δ

- Consider the function

$$y = f(x)$$

- The values which the independent variable x takes are called arguments and corresponding values of $f(x)$ are called entries.

  The difference b/w consecutive values of x is called the interval of differencing.

- If the interval of differencing is h and x takes the value a, $a+h$, $a+2h$, --, $a+nh$... then those values are called arguments and the corresponding values of $f(x)$ is $f(a)$, $f(a+h)$, $f(a+2h)$, ---, $f(a+nh)$ are called entry.

- The differences

  $f(a+h) - f(a)$, $f(a+2h) - f(a+h)$, $f(a+3h) - f(a+2h)$, etc are called the first difference (or first forward difference) of the function $y = f(x)$ and denoted by $\Delta f(a)$, $\Delta f(a+h)$, $\Delta f(a+2h)$ .. etc

i.e.

$$\Delta f(a) = f(a+h) - f(a)$$
$$\Delta f(a+h) = f(a+2h) - f(a+h)$$
$$\Delta f(a+2h) = f(a+3h) - f(a+2h)$$
$$\vdots$$
$$\Delta f(a+nh) = f(a+(n+1)h) - f(a+nh)$$

The differences of these ~~f(x)~~ first differences are called second difference denoted by $\Delta^2 f(a)$, $\Delta^2 f(a+h)$, $\Delta^2 f(a+2h)$, ... etc

- $$\begin{aligned} \Delta^2 f(a) &= \Delta[\Delta f(a)] \\ &= \Delta[f(a+h) - f(a)] \\ &= \Delta f(a+h) - \Delta f(a) \\ &= f(a+2h) - f(a+h) - (f(a+h) - f(a)) \\ &= f(a+2h) - 2f(a+h) + f(a) \end{aligned}$$

- $$\begin{aligned} \Delta^2 f(a+h) &= \Delta[\Delta f(a+h)] \\ &= \Delta[f(a+2h) - f(a+h)] \\ &= f(a+3h) - f(a+2h) - f(a+2h) + f(a+h) \\ &= f(a+3h) - 2f(a+2h) + f(a+h) \end{aligned}$$

- The differences of second differences are called third differences denoted by
$\Delta^3 f(a)$, $\Delta^3 f(a+h)$ and so on.

These differences are shown in the following table called the difference table ~~or~~ forward difference table.

Argument x	Entry $y = f(x)$	First Difference $\Delta f(x)$ or y_1	Second Difference $\Delta^2 f(x)$ or y_2	Third Difference $\Delta^3 f(x)$ or y_3	Fourth Difference $\Delta^4 f(x)$ or y_4
a	$f(a)$				
		$\Delta f(a)$			
$a+h$	$f(a+h)$		$\Delta^2 f(a)$		
		$\Delta f(a+h)$		$\Delta^3 f(a)$	
$a+2h$	$f(a+2h)$		$\Delta^2 f(a+h)$		$\Delta^4 f(a)$
		$\Delta f(a+2h)$		$\Delta^3 f(a+h)$	
$a+3h$	$f(a+3h)$		$\Delta^2 f(a+2h)$		
		$\Delta f(a+3h)$			
$a+4h$	$f(a+4h)$				

Foward difference Table

Argument x	Entry $y = f(x)$	First Difference $\Delta f(x)$	Second Difference $\Delta^2 f(x)$	Third Difference $\Delta^3 f(x)$	Fourth Difference $\Delta^4 f(x)$
a	$f(a)$				
		$\Delta f(a)$			
$a+h$	$f(a+h)$		$\Delta^2 f(a)$		
		$\Delta f(a+h)$		$\Delta^3 f(a)$	
$a+2h$	$f(a+2h)$		$\Delta^2 f(a+h)$		$\Delta^4 f(a)$
		$\Delta f(a+2h)$		$\Delta^3 f(a+h)$	
$a+3h$	$f(a+3h)$		$\Delta^2 f(a+2h)$		
		$\Delta f(a+3h)$			
$a+4h$	$f(a+4h)$				

NOTE 1. $\Delta C = 0$: differences of a constant function are zero

2. $f(a)$ the first entry is termed as the leading term and the difference $\Delta f(a)$, $\Delta^2 f(a)$, $\Delta^3 f(a)$, ... are known as the leading differences.

3. $\Delta\{c\, f(a)\} = c\, \Delta f(a)$

4. The operator Δ is called the forward or descending difference operator

5. $\Delta f(a) \neq f(a)\Delta \Rightarrow \Delta f \neq f \Delta$

Question Construct a difference table from the following values of x and y.

x = 3.0 3.1 3.2 3.3 3.4

$y = 1/x$ = 0.33333 0.32258 0.31250 0.30303 0.29412

Argument x	Entry $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
3.0	0.33333				
		-0.01075			
3.1	0.32258		0.00067		
		-0.01008		-0.00006	
3.2	0.31250		0.00061		0.0001
		-0.00947		-0.00005	
3.3	0.30303		0.00056		
		-0.00891			
3.4	0.29412				

Theorem The n^{th} differences of a rational integral function (Polynomial) of the n^{th} degree are constant when the values of the independent variable are at equal intervals.

Proof: Let $f(x)$ be a polynomial of degree n in x be

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots\cdots + a_{n-2} x^2 + a_{n-1} x + a_n$$

(where $a_0, a_1, a_2, \ldots a_n$ are const, $a_0 \neq 0$)

$$\Delta f(x) = f(x+h) - f(x)$$
$$= a_0 \left[(x+h)^n - x^n\right] + a_1 \left[(x+h)^{n-1} - x^{n-1}\right]$$
$$+ a_2 \left[(x+h)^{n-2} - x^{n-2}\right] + \cdots\cdots$$
$$+ a_{n-2} \left[(x+h)^2 - x^2\right] + a_{n-1} \left[(x+h)^1 - x^1\right]$$

$$= a_0 \left[{}^nC_1 x^{n-1} h + {}^nC_2 x^{n-2} h^2 + \cdots\cdots {}^nC_n h^n\right]$$
$$+ a_1 \left[{}^{n-1}C_1 x^{n-2} h + {}^{n-1}C_2 x^{n-3} h^2 + \cdots\cdots\right] + \cdots$$
$$+ a_{n-2} (2 + h + h^2) + a_{n-1} h$$

$$= a_0 n h x^{n-1} + a_1' x^{n-2} + a_2' x^{n-3}$$
$$+ \cdots\cdots + a_{n-2}' x + a_{n-1}'$$

where $a_1', a_2', \ldots a_{n-2}', a_{n-1}'$ are constant and independent of x.

Thus the first difference of a rational integral function (polynomial) of n^{th} degree is a rational integral function (polynomial) of degree $n-1$.

For Second difference

$\Delta^2 f(x) = \Delta[\Delta f(x)]$

$= \Delta[a_0 nh x^{n-1} + a_1' x^{n-2} + a_2' x^{n-3} + \dots$

$\dots + \dots + a_{n-2}' x + a_{n-1}']$

$= a_0 nh[(x+h)^{n-1} - x^{n-1}] + a_1'[(x+h)^{n-2} - x^{n-2}]$

$+ \dots + a_{n-2}'(x+h-x)$

$= a_0 nh[(n-1)h x^{n-2} + {}^{n-1}C_2 x^{n-3} h^2 + \dots]$

$+ a_1'[{}^{n-2}C_1 x^{n-3} h \dots]$

$+ a_{n-2}' h$

$= a_0 n(n-1)h^2 x^{n-2} + a_2'' x^{n-3} + a_3'' x^{n-4}$

$+ \dots + b''$

Hence it is a polynomial of degree $(n-2)$.

By continuing in this manner

$\Delta^n f(x) = a_0 h^n n!$

The n^{th} differererence is therefore constant and all higher differences will be zero i.e. the $(n+1)^{th}$ and higher differences of a polynomial of n^{th} degree are zero.

The converse of the above theorem is also true of the n^{th} differences of a tabulated function are constant when the values of the independent

variables

the f

Question. Evalua

(i) Δ^3

(ii) Δ^n

(iii) Δ

where

and

Ans (i) $f(x)$

If

Δ^3

(ii) Δ

variables are taken at equal intervals the function is a polynomial of degree n.

Question Evaluate

(i) $\Delta^3 (1-x)(1-2x)(1-3x)$

(ii) $\Delta^n e^{ax+b}$

(iii) $\Delta\, u_x v_x$

where the interval of differencing being unity and u_x, v_x are function of x.

Ans (i) $f(x) = (1-x)(1-2x)(1-3x)$

$= -6x^3 + 11x^2 - 6x + 1$

If $h=1$, then Δ will act like $D \left(= d/dx\right)$

$\Delta^3 f(x) = -6 \cdot 3! = -6 \times 6 = 36$

(ii) $\Delta e^{ax+b} = e^{a(x+1)+b} - e^{ax+h}$

$= e^{ax+b}(e^a - 1)$

$\Delta^2 e^{ax+b} = \Delta(\Delta e^{ax+b})$

$= \Delta e^{ax+b}(e^a - 1)$

$= (e^a - 1)\, \Delta e^{ax+b}$

$= (e^a - 1) \cdot e^{ax+b}\, (e^a - 1)$

$= (e^a - 1)^2\, e^{ax+b}$

$\vdots$

$\Delta^n e^{ax+b} = (e^a - 1)^n\, e^{ax+b}$

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(iii) $\Delta(U_x V_x) = U_{x+1} V_{x+1} - U_x V_x$

$= U_{x+1} V_{x+1} - U_{x+1} V_x + U_{x+1} V_x - U_x V_x$

$= U_{x+1} (V_{x+1} - V_x) + (U_{x+1} - U_x) V_x$

$= U_{x+1} (\Delta V_x) + (\Delta U_x) V_x$

22/01/2021

FACTORIAL NOTATION

- The product of factors of which the first factor is x and the successive factors decreases by a constant difference is called a factorial polynomial or function and is denoted by $x^{(r)}$, r being a +ve integer and is read as "x raised to the power r factorial."
- In general interval of differencing is h.
- In particular, set $x^{(0)} = 1$

define

$$x^{(r)} = x(x-h)(x-2h)(x-3h) \cdots \{x-(r-1)h\}$$

$$\Delta x^{(r)} = (x+h)^{r} - x^{r}$$
$$= (x+h)x(x-h)(x-2h) \cdots [x-(r-2)h] - x(x-h)(x-2h) \cdots [x-(r-1)h]$$

$$= \{x(x-h)(x-2h) \cdots (x-(r-2)h)\}\{\cancel{x}+\cancel{h} - \cancel{x} + rh - \cancel{h}\}$$

$$= rh\, x^{(r-1)}$$

$$\therefore \boxed{\Delta x^{(r)} = h[r\, x^{r-1}]} \quad -(1)$$

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Similarly

$$\Delta^2 x^{(r)} = h^2 \left[r(r-1) x^{(r-2)} \right]$$

- $$\Delta^n x^{(r)} = \begin{cases} h^n \left[r(r-1)(r-2)\ldots\{r-(n-1)\} \right] x^{r-n} & \text{where } r \geq n \\ 0, & \text{if } r < n \end{cases}$$

- $$\boxed{\Delta^n x^{(n)} = h^n \, n!}$$

Note : (1) For factorial notation, the operator Δ is analogous to operator D if $h = 1$. $\left(D = \frac{d}{dx} \right)$

Ques. Express $f(x) = x^3 - 2x^2 + x - 1$ into factorial notation and show that $\Delta^4 f(x) = 0$

Sol. $f(x) = x^3 - 2x^2 + x - 1$

let, $f(x) = x^3 - 2x^2 + x - 1 = x(x-1)(x-2) + Bx(x-1) + Cx + D$

Finding A, B, C, D.

Put $x = 0$, $\Rightarrow \boxed{D = -1}$, Put $x = 1$, $-1 = C + D$

$\Rightarrow \boxed{C = 0}$

Put $x = 2$, $8 - 8 + 2 - 1 = 2B + D \Rightarrow \boxed{B = 1}$

~~Put $x = 3$, $27 - 18 + 3 - 1 = A + B + D \Rightarrow A$~~

$$f(x) = x^3 - 2x^2 + x - 1 = x(x-1)(x-2) + x(x-1) - 1$$

$$f(x) = x^{(3)} + x^{(2)} - 1$$

$$\Delta f(x) = 3x^{(2)} + 2x^{(1)}$$

$$\Delta^2 f(x) = 6x^{(1)} + 2$$

$$\Delta^3 f(x) = 6$$

$$\Delta^4 f(x) = 0$$

Ques Find the function whose first difference is $2x^3+3x^2-5x+4$ (assuming, h=1)

Sol. Let $f(x)$ be the required f^n then

$$\Delta f(x) = 2x^3+3x^2-5x+4$$

Let $\Delta f(x) = 2x^3 + 3x^2 - 5x + 4$

$$= 2x(x-1)(x-2) + Bx(x-1) + Cx + D$$

At $x=0$, $\Rightarrow \boxed{D=4}$, $x=1 \Rightarrow \cancel{2}+\cancel{3}-5+\cancel{4} = C+\cancel{4}$

$\Rightarrow \boxed{C=0}$

At $x=2$, $16+12-10+\cancel{4} = 0+2B+0+\cancel{D}$

$\Rightarrow \boxed{B=9}$

$$\therefore \Delta f(x) = 2x(x-1)(x-2) + 9x(x-1) + 4$$

$$= 2x^{(3)} + 9x^{(2)} + 4$$

$$\therefore f(x) = \frac{2x^{(4)}}{4} + \frac{9x^{(3)}}{3} + \frac{4x^{(1)}}{1} + c$$

$$= \frac{1}{2}x^{(4)} + 3x^{(3)} + 4x^{(1)} + c$$

$$f(x) = \frac{1}{2}x(x-1)(x-2)(x-3) + 3x(x-1)(x-2) + 4x + c$$

similar question can made in terms of population

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Ques. Find the missing term in the following table

x	0	1	2	3	4
$f(x)$	1	3	9	-	81

Ans. Since 4 entries are given, the given function can be represented by a third degree polynomial

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1				
		2			
1	3		4		
		6		$y-19$	
2	9		$y-15$		$124-4y$
		$y-9$		$105-3y$	
3	y		$90-2y$		
		$81-y$			
4	81				

$\Delta^4 f(x) = 0 \Rightarrow 124-4y = 0 \Rightarrow \boxed{y = 31}$

$\Delta f(a+nh) = f\{a+(n+1)h\} - f\{a+nh\}$

Let $y_n = f(a+nh)$

$y_{n+1} = f\{a+(n+1)h\}$

$\Delta y_n = y_{n+1} - y_n$

- $y_{n+1} = y_n + \Delta y_n$

$y_{n+1} = (1+\Delta) y_n$

- $y_{n+2} = y_{n+1} + \Delta y_{n+1} = (1+\Delta) y_{n+1}$

$y_{n+2} = (1+\Delta)^2 y_n$

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$$y_{n+3} = (1+\Delta)^3 y_n$$

$$\vdots$$

$$\boxed{y_{n+m} = (1+\Delta)^m y_n}$$

The operator E and Δ

Let $y = f(x)$ be the function of x and let $a, a+h, a+2h, \ldots$ be consecutive values of x, then

$$\Delta f(x) = f(x+h) - f(x)$$

or $$\Delta f(a) = f(a+h) - f(a)$$

Here 1 (unity) is an operator which is such that the function on which it operates leaves the function unaltered.

Thus 1 is called Identity operator.

Define an operator E, called displacement operator, by the eq^n

$$\boxed{E\, f(a) = f(a+h)}$$

The operator E is also called as shift operator, since it results in the next value of the function.

$$\Delta f(a) = f(a+h) - f(a)$$

$$\Rightarrow \quad = E\,f(a) - f(a)$$

$$\Rightarrow E\,f(a) = f(a) + \Delta f(a)$$

$$= (1+\Delta)\,f(a)$$

$$\therefore \quad \boxed{E \equiv 1+\Delta}$$

$$E^2 f(a) = E\{E\,f(a)\}$$

$$= E\,f(a+h) = f(a+2h)$$

$$\vdots$$

$$E^n f(a) = f(a+nh)$$

- The inverse operator E^{-1} is defined as

$$\boxed{E^{-1} f(x) = f(x-h)}$$

- $\boxed{E^0 = 1}$

Properties of two operators Δ & E.

1. (i) distributive (ii) Commutative (iii) index law

(i) $\Delta\left[f(x) + g(x) + \dots\right] = \Delta f(x) + \Delta g(x) + \dots$

$$E\left[f(x) + g(x)\right] = E\,f(x) + E\,g(x)$$

(ii) Commutative w.r.t constant only

$$\Delta c f(x) = c\,\Delta f(x)$$

$$\& \quad E\left[c\,f(x)\right] = c\,E\,f(x)$$

(iii) $\Delta^m \{\Delta^n f(x)\} = \Delta^{m+n} f(x)$

$$\& \quad E^m \{E^n f(x)\} = E^{m+n} f(x)$$

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(2) If m is a +ve integer, then we can define symbol Δ^{-m} such that

$$\Delta^m [\Delta^{-m} f(x)] = f(x)$$

IIlly $E^m [E^{-m} f(x)] = f(x)$

(3) $\boxed{\Delta E = E \Delta}$

$$\Delta E\{f(x)\} = E\Delta\{f(x)\}$$

BACKWARD DIFFERENCE OPERATOR ∇

The operator ∇ is called the backward or ascending operator or ascending difference operator and is defined by

$$\nabla f(a+h) = f(a+h) - f(a)$$

so that $\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$$\Rightarrow \boxed{\nabla \equiv 1 - E^{-1}}$$

$$\begin{aligned}
\nabla^2 f(x) &= \nabla\{\nabla f(x)\} = \nabla\{f(x) - f(x-h)\} \\
&= \nabla f(x) - \nabla f(x-h) \\
&= f(x) - f(x-h) - f(x-h) + f(x-2h) \\
&= f(x) - 2f(x-h) + f(x-2h)
\end{aligned}$$

and so on.

~~The operator D & ∇~~

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The Operator D and ∇

Relations b/w the operators Δ, E, ∇, D

Taylor series

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

$$E f(x) = f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \cdots$$

$$= \left(1 + hD + \frac{h^2}{2!} D^2 \cdots\right) f(x)$$

$$= e^{hD} f(x)$$, h interval of differencing

$\therefore$ $E \equiv e^{hD}$

$\therefore$ $\boxed{E \equiv 1 + \Delta \equiv e^{hD}}$

from this

$$hD = \log E = \log(1+\Delta)$$

$$hD = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - + \cdots$$

$\therefore$ $\boxed{\nabla = \Delta E^{-1}}$

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Other operators

(I) Central difference operator δ :

It is defined as

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

(II) Averaging operator μ :

It is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

Relation between operators

(i) $\delta = E^{1/2} - E^{-1/2}$

(II) $\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

(III) $\delta = \Delta E^{-1/2}$

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$= \Delta f\left(x-\frac{h}{2}\right)$$

$$= \Delta E^{-1/2} f(x)$$

$$\Delta f\left(x-\frac{h}{2}\right)$$
$$= f\left(x-\frac{h}{2}+h\right) - f\left(x-\frac{h}{2}\right)$$
$$= f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\Rightarrow \boxed{\delta = \Delta E^{-1/2}}$$

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(iv) $\delta = \nabla E^{1/2}$

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$= \nabla f\left(x+\frac{h}{2}\right)$$

$$= \nabla E^{1/2} f(x)$$

$$\Rightarrow \boxed{\delta = \nabla E^{1/2}}$$

25/01/2020
Monday

Q. Obtain the estimate of the missing figure in the following Table

x	2.0	2.1	2.2	2.3	2.4	2.5	2.6
$y=f(x)$	0.135	-	0.111	0.100	-	0.082	0.074

Sol^n (M1) older one as difference table.

(M2) Since five values are given we assume that a polynomial of fourth degree can be fitted so that fifth difference is zero.

$$\Delta^5 y_n = 0 \quad \text{for all } n$$

$$(E-1)^5 y_n = 0$$

$$(E^5 - {}^5C_1 E^4 + {}^5C_2 E^3 - {}^5C_3 E^2 + {}^5C_4 E^1 - 1) y_n = 0$$

$$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_n = 0$$

For $n = 2$,

$$E^5 y_{2.0} - 5E^4 y_{2.0} + 10E^3 y_{2.0} - 10E^2 y_{2.0} + 5E y_{2.0} - y_{2.0} = 0$$

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$\Rightarrow y_{2.5} - 5y_{2.4} + 10y_{2.3} - 10y_{2.2} + 5y_{2.1} - y_{2.0} = 0$ — (i)

At $n = 2.1$

$E^5 y_{2.1} - 5E^4 y_{2.1} + 10E^3 y_{2.1} - 10E^2 y_{2.1} + 5E y_{2.1} - y_{2.1} = 0$

$\Rightarrow y_{2.6} - 5y_{2.5} + 10y_{2.4} - 10y_{2.3} + 5y_{2.2} - y_{2.1} = 0$

$$\boxed{y_{2.1} = 0.123 \quad , \quad y_{2.4} = 0.090}$$

Interpolation

Interpolation means insertion or filling up intermediate terms of a series. It is the technique of estimating the value of a function under a certain assumptions for any intermediate value of the independent variable when the values of the function corresponding to a number of the values of the variable are given. The process of computing the value of a function outside the range of a given values of the variable is called extrapolation.

① Newton's - Gregory Formula for Forward Interpolation

Let $y = f(x)$ denote a function which takes the value $f(a), f(a+h), \dots f(a+nh)$ for $n+1$ equidistant values $a, a+h, a+2h, \dots a+nh$. of independent variable x and let $P_n(x)$ be a polynomial of degree n in x.

Then, $P_n(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h)$
$+ A_3(x-a)(x-a-h)(x-a-2h) + \dots$
$+ A_n(x-a)(x-a-h)(x-a-2h) \dots (x-a-(n-1)h)$ — (1)

We chose the coefficients $A_0, A_1, A_2, \ldots A_n$ so that :

$$P_n(a) = f(a) \quad , \quad P_n(a+h) = f(a+h) \ldots$$
$$P_n(a+nh) = f(a+nh)$$

Put $x = a, a+h, \ldots\ldots, a+nh$, we get

$$\boxed{A_0 = f(a)}$$

$\Rightarrow f(a+h) = A_0 + A_1 h$

$$\Rightarrow \boxed{A_1 = \frac{f(a+h) - f(a)}{h} = \frac{1}{h} \Delta f(a)}$$

$\Rightarrow f(a+2h) = A_0 + A_1 2h + A_2 2h^2$

$\Rightarrow f(a+2h) - 2h \cdot \frac{1}{h} \Delta f(a) - f(a) = A_2 2h^2$

$$\Rightarrow \boxed{A_2 = \frac{f(a+2h) - 2f(a+h) + f(a)}{2h^2}}$$

$$\text{or} \quad \boxed{A_2 = \frac{1}{2h^2} \Delta^2 f(a)} \quad (3)$$

Similarly, $A_3 = \dfrac{1}{3!\,h^3} \Delta^3 f(a)$

$\vdots$

$$A_n = \frac{1}{n!\,h^n} \Delta^n f(a) \quad (2)$$

Subsituting values in (1) :

$$\boxed{P_n(x) = f(a) + \frac{(x-a)}{h} \Delta f(a) + \frac{(x-a)(x-a-h)}{2!\,h^2} \Delta^2 f(a) + \ldots\ldots + \frac{(x-a)(x-a-h)\ldots(x-a-(n-1)h)}{n!\,h^n} \Delta^n f(a)} \quad (2)$$

If we take

$$\frac{x-a}{h} = u \quad \Rightarrow \quad x = a + hu \qquad (3)$$

then (2) can be written as

$$P_n(x) = P_n(a+hu) = f(a) + u\,\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a)$$

$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \cdots$$

$$+ \frac{u(u-1)(u-2)\cdots(u-(n-1))}{n!}\Delta^n f(a) \qquad (4)$$

or in factorial notation

$$P_n(x) = f(a) + u^{(1)}\Delta f(a) + \frac{1}{2!}u^{(2)}\Delta^2 f(a) + \frac{1}{3!}u^{(3)}\Delta^3 f(a) \qquad (5)$$

$$+ \cdots + \frac{1}{n!}u^{(n)}\Delta^n f(a)$$

Another Method

$$x = a + hu, \quad u = \frac{x-a}{h} \qquad -1 \leq u \leq 1$$

$$f(x) = f(a+hu) = E^{u} f(a) = (1+\Delta)^u f(a) \qquad -(6)$$

$$= f(a) + u\,\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$+ \cdots + \frac{u(u-1)(u-2)\cdots(u-(n-1))}{n!}\Delta^n f(a) \qquad (7)$$

Newtons - Gregory formula for Backward interpolation

Let,

$$P_n(x) = A_0 + A_1(x-a-nh) + A_2(x-a-nh)\cdot(x-a-nh+h)$$

$$+ \cdots + A_n(x-a-nh)(x-a-nh+h)\cdots(x-a-h) \qquad -(8)$$

Choose the coefficients $A_0, A_1, A_2, \ldots A_n$ in such a way that

$P_n(a+nh) = f(a+nh), \ldots\ldots, P_n(a) = f(a)$

Putting successively in (8), $x = a+nh$, $x = a+nh-h$, ……

we get

$f(a+hn) = A_0$

$$f(a+nh-h) = f(a+nh) + A_1(-h)$$
$$A_1 = \frac{f(a+nh) - f(a+nh-h)}{h}$$
$$= \frac{1}{h}\nabla f(a+nh)$$

Similarly we get

$$A_2 = \frac{1}{2!\,h^2}\nabla^2 f(a+nh)$$

$\vdots$

$$A_n = \frac{1}{n!\,h^n}\nabla^n f(a+nh) \quad (6)$$

$$\boxed{\begin{aligned} P_n(x) &= f(a+nh) + \frac{(x-a-nh)}{h}\nabla f(a+nh) + \\ &\quad + \frac{(x-a-nh)(x-a-nh+h)}{2!\,h^2}\nabla^2 f(a+nh) + \dots \\ &\quad \dots + \frac{(x-a-nh)\dots(x-a-h)}{n!\,h^n}\nabla^n f(a+nh) \end{aligned}} \quad (7)$$

If we take

$$u = \frac{x - (a+nh)}{h}$$

or $x = a + nh + hu$ (8)

then

$$\boxed{\begin{aligned} P_n(x) &= P_n(a+nh+hu) \\ &= f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) \\ &\quad + \dots + \frac{u(u+1)(u+2)\dots(u+n-1)}{n!}\nabla^n f(a+nh) \end{aligned}}$$

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Other methods

$$f(x) = f(a + nh + hu)$$

$$= E^{u} f(a+nh) \qquad \left(\text{using } \nabla = 1 - E^{-1}\right)$$

$$= (1-\nabla)^{-u} f(a+nh)$$

$$= f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \dots + \frac{u(u+1)(u+2)\dots(u+n-1)}{n!}\nabla^4 f(a+nh) \qquad (9)$$

NOTE (1) Newton's forward difference formula is used for interpolating the values of the function near the beginning of a set of tabulated value.

(2) Newton's backward difference formula is used for interpolating the values of the function near the end of a set of tabulated values.

Q1. Calculate approximate value of $\sin x$ for $x = 0.54$ and $x = 1.36$ using following table

x	0.5	0.7	0.9	1.1	1.3	1.5
$\sin x$	0.47943	0.64422	0.78333	0.89121	0.96356	0.99749

Ans. Difference Table :

x	$\sin x$	I^{st} diff	II^{nd} diff	III^{rd} diff	IV^{th} diff	V^{th} diff
0.5	0.47943					
		0.16479				
0.7	0.64422		-0.02568			
		0.13911		-0.00555		
0.9	0.78333	~~0.13911~~	-0.03123		0.00125	
		0.10788		-0.00430		0.00016
1.1	0.89121		-0.03553		0.00141	
		0.07235		-0.00289	~~0.00141~~	
1.3	0.96356		-0.03842			
		0.03393				
1.5	0.99749					

(i) <u>here</u>, we'll use forward interpolation

$a = 0.5$, $h = 0.2$, $x = 0.54$

$$u = \frac{x-a}{h} = \frac{0.54 - 0.5}{0.2} = 0.2$$

Newton's forward formula (Eq. 4)

$$f(\overset{x}{0.54}) = f(a + hu)$$

$$= f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$

$$+ \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 f(a)$$

$$= 0.47943 + (0.2 \times 0.16479) + \frac{0.2\,(0.2-1)}{2!}(-0.02568)$$

$$+ \frac{0.2\,(0.2-1)(0.2-2)}{3!} \times -0.00555$$

$$+ \frac{0.2\,(0.2-1)(0.2-2)(0.2-3)}{4!} \times 0.00125$$

$$\underline{f(0.54) = 0.977849}$$

(ii) For $f(x)$ at $x = 1.36$, we'll use backward interpolation
(since we know $\nabla^n f(1.5)$ upto $n = 5$)

$\therefore$ ~~$x = 1.36$, $a = 1.5$, $n = 5$, $h = 0.2$~~

~~$u = \frac{x-(a+nh)}{h} = \frac{1.36-(1.5+1)}{0.2} = -5.7$~~

~~Using Newton's backward interpolation formula (Eq 10)~~

~~$f(1.36) = f(a+nh+hu) = f(1.5+1-1.14)$~~

~~$= f(a+nh) + u\nabla f(a+nh)$~~

$x = 1.36$, $a = 0.5$, $n = 5$, $h = 0.2$

$a+nh = 1.5$ $\quad\therefore\quad u = \frac{1.36-1.5}{0.2} = -0.7$

Using Newton's backward interpolation formula (Eq 10)

$$f(1.36) = f(a+nh+hu) = f(0.5+1-0.14)$$

$$\Rightarrow \quad = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$

$$+ \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a+nh)$$

$$+ \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 f(a+nh)$$

$$\Rightarrow \quad = f(1.5) + (-0.7)\nabla f(1.5) + \frac{(-0.7)(0.3)}{2}\nabla^2 f(1.5)$$

$$+ \frac{(-0.7)(0.3)(1.3)}{6}\nabla^3 f(1.5) + \frac{(-0.7)(0.3)(1.3)(2.3)}{24}\nabla^4 f(1.5) + \frac{(-0.7)(0.3)(1.3)(2.3)}{120}$$

$$= 0.99749 + (-0.023751) + (0.0040341) + (0.00013149\,5) + (-0.0000328$$

$$f(1.36) = 0.9778649$$

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Q2. Find the cubic polynomial $f(x)$ which takes the values $f(0) = -4$, $f(1) = -1$, $f(2) = 2$, $f(3) = 11$, $f(4) = 32$, ~~$f(5) = 1$~~ $f(5) = 71$.
Find $f(6)$ and $f(2.5)$.

Sol^n:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-4			
		3		
1	-1		0	
		3		6
2	2		6	
		9		6
3	11		12	
		21		6
4	32		18	
		39		
5	71			

Newton's forward : $u = \frac{x-a}{h} = \frac{x-0}{1} = x$

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)\Delta^2 f(a)}{2!} + \frac{u(u-1)(u-2)\Delta^3 f(a)}{3!}$$

$$= -4 + (x \times 3) + \frac{x(x-1)}{2!} \times 0 + \frac{x(x-1)(x-2)}{3!} \times 6$$

$$= -4 + 3x + x(x^2 - 3x + 2)$$

$$= -4 + 3x + x^3 - 3x^2 + 2x$$

$$\underline{f(x) = x^3 - 3x^2 + 5x - 4}$$

Newton's backward

$$u = \frac{x-a}{h} = \frac{6-5}{1} = 1$$

$$f(6) = f(5) + u\nabla f(5) + \frac{u(u+1)}{2!}\nabla^2 f(5) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(5)$$

$$= 71 + (1\times 39) + 18 + 6$$

$$= 71 + 39 + 24$$

$$f(6) = 134$$

Newton's forward

$$u = \frac{x-a}{h} = \frac{2.5-2}{1} = 0.5$$

$$f(2.5) = f(2) + u\Delta f(2) + \frac{u(u-1)}{2!}\Delta^2 f(2) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(2)$$

$$= 2 + (0.5\times 9) + \frac{0.5\times -0.5}{2!}\times 12 + \frac{0.5(0.5-1)(0.5-2)}{3!}\times 6$$

$$= 2 + 4.5 - 1.5 + 0.375$$

$$f(2.5) = 5.375$$

Q3. Find the cubic polynomial in x for the following data

x	0	1	2	3	4	5
y	-3	3	11	27	57	107

ans: $x^3 - 2x^2 + 7x - 3$

Solⁿ.

Difference Table:

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x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-3			
		6		
1	3		2	
		8		6
2	11		8	
		16		5
3	27		14	
		30		6
4	57		20	
		50		
5	107			

Newton's forward

$$u = \frac{x-a}{h} = \frac{x-0}{1} = x$$

$$f(x) = f(a) + u\,\Delta f(a) + \frac{u(u-1)\Delta^2 f(a)}{2!} + \frac{u(u-1)(u-2)\Delta^3 f(a)}{3!}$$

$$= -3 + x \times 6 + \frac{x(x-1)}{2} \times 2 + \frac{x(x-1)(x-2)}{6} \times 6$$

$$= -3 + 6x + x^2 - x + x^3 - 3x^2 + 2x$$

$$f(x) = x^3 - 2x^2 + 7x - 3$$

$y_0 = f(a)$, $y_1 = f(a+h)$, $y_2 = f(a+2h)$

$y_{-1} = f(a-h)$, $y_{-2} = f(a-2h)$

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Central Difference formula

Let $\cdots y_{-2}, y_{-1}, y_0, y_1, y_2, \cdots$ be the value of a function $y = f(x)$ for $x =$
$\cdots a-2h, a-h, a, a+h, a+2h, \cdots$

Then

$$\delta = E^{1/2} - E^{-1/2}$$

$$\Rightarrow \delta E^{1/2} = E - 1 = \Delta$$

$\Rightarrow$ $\Delta = \delta E^{1/2}$, $\Delta^2 = \delta^2 E$, $\Delta^3 = \delta^3 E^{3/2}$

$$\Delta y_{-2} = \delta E^{1/2} y_{-2} = \delta y_{-3/2}$$

$$\Delta y_{-1} = \delta E^{1/2} y_{-1} = \delta y_{-1/2}$$

$$\Delta y_0 = \delta E^{1/2} y_0 = \delta y_{1/2}$$

$$\Delta y_1 = \delta E^{1/2} y_1 = \delta y_{3/2}$$

$$\Delta y_2 = \delta E^{1/2} y_2 = \delta y_{5/2}$$

Similarly,

$$\Delta^2 y_{-2} = \delta^2 E y_{-2} = \delta^2 y_{-1}$$

$$\Delta^2 y_{-1} = \delta^2 E y_{-1} = \delta^2 y_0$$

$$\Delta^2 y_0 = \delta^2 E y_0 = \delta^2 y_1$$

$$\Delta^2 y_1 = \delta^2 E y_1 = \delta^2 y_2$$

and so on further

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$$\Delta^3 y_{-2} = \delta^3 E^{3/2} y_{-2} = \delta^3 y_{-1/2}$$
$$\Delta^3 y_{-1} = \delta^3 E^{3/2} y_{-1} = \delta^3 y_{1/2}$$
$$\Delta^3 y_0 = \delta^3 E^{3/2} y_0 = \delta^3 y_{3/2}$$

and so on

Hence central difference table is expressed as

	x	$f(x)$	$\Delta f(x)$ or $\delta f(x)$	$\Delta^2 f(x)$ or $\delta^2 f(x)$	$\Delta^3 f(x)$ or $\delta^3 f(x)$	$\Delta^4 f(x)$ or $\delta^4 f(x)$
	x_{-2}	y_{-2}				
			$\Delta y_{-2} = \delta y_{-3/2}$			
$x = a-h \rightarrow$	x_{-1}	y_{-1}		$\delta^2 y_{-1} = \Delta^2 y_{-2}$		
			$\delta y_{-1/2} = \Delta y_{-1}$		$\delta^3 y_{-1/2} = \Delta^3 y_{-2}$	
$x = a \rightarrow$	x_0	y_0		$\delta^2 y_0 = \Delta^2 y_{-1}$		$\delta^4 y_0 = \Delta^4 y_{-2}$
			$\delta y_{1/2} = \Delta y_0$		$\delta^3 y_{1/2} = \Delta^3 y_{-1}$	
$x = a+h$	x_1	y_1		$\delta^2 y_1 = \Delta^2 y_0$		
			$\delta y_{3/2} = \Delta y_1$			
	x_2	y_2				

Gauss Forward Interpolation formula

The Newton Gregory forward interpolation formula is:

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots \quad -(1)$$

where $u = \dfrac{x-a}{h}$

$$\therefore \Delta^2 y_0 = \Delta^2 \underbrace{E y_{-1}} = \Delta^2 (1+\Delta) y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$({}^uC_2 + {}^uC_3)\Delta^3 y_{-1} = \frac{u(u-1)}{2} + \frac{u(u+1)(u-2)}{6}$$

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Similarly

$$\Delta^3 y_{-1} = \Delta^3 E y_{-2} = \Delta^3 (1+\Delta) y_{-2},$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

and $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc.

Subsituting the values of $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0 \ldots$ in (1)

$$y_u = y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 (\Delta^2 y_{-1} + \Delta^3 y_{-1}) + {}^uC_3 (\Delta^3 y_{-1} + \Delta^4 y_{-1}) + {}^uC_4 (\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

$$= y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-1} + {}^{u+1}C_4 \Delta^4 y_{-1} + \ldots$$

$$\boxed{y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-1} + \ldots}$$

$$y_u = y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 (\Delta^2 y_{-1} + \Delta^3 y_{-1}) + {}^uC_3 (\Delta^3 y_{-1} + \Delta^4 y_{-1}) + {}^uC_4 (\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

$$= y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-1} + {}^uC_3 (\Delta^4 y_{-2} + \Delta^5 y_{-2}) + {}^uC_4 (\Delta^4 y_{-2} + \Delta^5 y_{-2} + \Delta^5 y_{-1}) \ldots$$

$$= y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-1} + {}^{u+1}C_4 \Delta^4 y_{1} + \ldots$$

$$\boxed{y_u = y_0 + u\Delta y_0 + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots}$$

$$\boxed{\begin{aligned} y_u = y_0 &+ u\,\delta y_{1/2} + \frac{u(u-1)}{2!}\delta^2 y_0 + \\ &+ \frac{(u+1)u(u-1)}{3!}\delta^3 y_{+1/2} \\ &+ \frac{(u+1)u(u-1)(u-2)}{4!}\delta^4 y_0 + \cdots \end{aligned}}$$

This formula involves odd differences below the central line $(x = x_0)$ and even differences on the line as

Δ odd

Δ even

centered line

$$\begin{array}{ccccccccc} y_0 & & \Delta^2 y_{-1} & & \Delta^4 y_{-2} & & \Delta^6 y_{-3} & & \Delta^8 y_{-4} \\ & \Delta y_0 & & \Delta^3 y_{-1} & & \Delta^5 y_{-2} & & \Delta^7 y_{-3} & \end{array}$$

i.e. Δ^4, Δ^5 should have y_{-2} and so on.

Note: It is used to interpolate the values of y for $0 < u < 1$

Gauss's Backward Interpolation Formula

NEWTONS forward interpolation formula:

$$y_u = y_0 + {}^uC_1 \Delta y_0 + {}^uC_2 \Delta^2 y_0 + {}^uC_3 \Delta^3 y_0 + {}^uC_4 \Delta^4 y_0 + \cdots$$

$$\Delta y_0 = \Delta E y_{-1} = (\Delta + \Delta^2)\, y_{-1}$$

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$\vdots$

and so on

Similarly

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \quad \text{etc.}$$

$$y_u = y_0 + {}^uC_1(\Delta y_{-1} + \Delta^2 y_{-1}) + {}^uC_2(\Delta^2 y_{-1} + \Delta^3 y_{-1})$$
$$+ {}^uC_3(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + {}^uC_4(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \dots$$

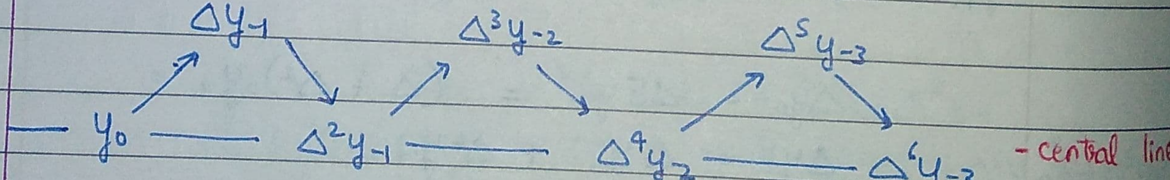
$$= y_0 + {}^uC_1 \Delta y_{-1} + {}^{u+1}C_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-1}$$
$$+ {}^{u+1}C_4 \Delta^4 y_{-1} + \dots$$

$$= y_0 + {}^uC_1 \Delta y_{-1} + {}^{u+1}C_2 \Delta y_{-1} + {}^{u+1}C_3(\Delta^3 y_{-2} + \Delta^4 y_{-2})$$
$$+ {}^{u+1}C_4(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \dots$$

$$= y_0 + {}^uC_1 \Delta y_1 + {}^{u+1}C_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-2}$$
$$+ {}^{u+2}C_4 \Delta^4 y_{-2}$$

$$\boxed{y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \dots}$$

The formula involves odd differences above the central line and even differences on the central line.



It is useful when $-1 < u < 0$

$$\boxed{y_u = y_0 + u\delta y_{-1/2} + \frac{(u+1)u}{2!}\delta^2 y_0 + \frac{(u+1)u(u-1)}{3!}\delta^3 y_{-1/2} + \frac{(u+2)(u+1)u(u-1)}{4!}\delta^4 y_0 + \dots}$$

01/02/2020

Stirling's

Taking the average of the two Gauss formula:

$$y_u = y_0 + \frac{u}{2}\left(\Delta y_0 + \Delta y_{-1}\right) + \frac{u}{2!}\left(\frac{u_{-1} + u_{+1}}{2}\right)\Delta^2 y_{-1}$$
$$+ \frac{(u+1)u(u-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$
$$+ \frac{(u+1)u(u-1)}{4!}\left(\frac{u_{-2} + u_{+2}}{2}\right)\Delta^4 y_{-2} + \cdots$$

$$\boxed{y_u = y_0 + \frac{u}{2}\left(\Delta y_0 + \Delta y_{-1}\right) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots}$$

It can easily be written with the help of following table

coef	1	u	$\frac{u^2}{2!}$	$\frac{u(u^2-1)}{3!}$	$\frac{u^2(u-1)}{4!}$
	y_0	Δy_{-1} Δy_0	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$ $\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$

as it is multiply

inka mean lete hai!

★★
★ **NOTE:** Stirling formula may be used when

$$-\frac{1}{4} < u < \frac{1}{4}$$

The mean of odd differences above and below the central line and even differences on the central line.

Bessel's Interpolation Formula

Gauss's backward formula with origin transferred to until takes the term

$$y_u = y_1 + (u-1)\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0$$

$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-1}$$

$$+ \dots$$

Taking mean of this and Gauss's forward formula, we obtain

$$\boxed{\begin{aligned} y_u = {} & \frac{1}{2}(y_0 + y_1) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) \\ & + \frac{(u-\frac{1}{2})u(u-1)}{3!}\Delta^3 y_{-1} + (u+1)u(u-1)(u-2)\left[\frac{\Delta^4 y_{-1} + \Delta^4 y}{2}\right. \\ & + \dots \end{aligned}}$$

for $u = 1/2$, term containing odd differences vanish and therefore the formula is most suitable when $u = \frac{1}{2}$

Q1. Given

θ	:	$0°$	$5°$	$10°$	$15°$	$20°$	$25°$	$30°$
$\tan\theta$	:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Using Stirling's formula, show that $\tan 16° = 0.28671$

Sol. Taking origin at $15°$ and $h = 5$

$$x = \frac{\theta - 15}{5} \qquad \text{When } \theta = 16°, \; x = \frac{1}{5} = 0.2$$

The difference table is as follows

x	$y_x = \tan x$	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
-3	0.0000				
		0.0875			
-2	0.0875		0.0013		
		0.0888		0.0015	
-1	0.1763		0.0028		0.002
		0.0916		0.0017	
0	0.2679		0.0045		0.000
		0.0916		0.0017	
1	0.3640		0.0062		0.009
		0.1023		0.0026	
2	0.4663		0.0088		
		0.1111			
3	0.5774				

Stirling's formula is

$$y_x = y_0 + x \cdot \frac{1}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{x^2}{2!}\Delta^2 y_{-1}$$

$$+ \frac{x(x^2-1)}{3!} \cdot \frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{x^2(x^2-1)}{4!}\Delta^4 y_{-2} + \ldots$$

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Putting $x = 0.2$, we get

$$y_{0.2} = 0.2679 + 0.2 \times \frac{1}{2}(0.0961 + 0.0916) + \frac{(0.2)^2}{2} \times 0.0045$$

$$+ \frac{(0.20)(0.04-1)}{6} \times \frac{1}{2}(0.0017 + 0.0017) + \ldots$$

$$= 0.2679 + 0.01877 + 0.0009 - 0.00005 = 0.28671$$

$\therefore$ $\tan 16° = 0.28671$

Q2. Use Bessel's formula to find the value of y if $x = 3.75$, given

x :	2.5	3	3.5	4.0	4.5	5.0
y :	24.145	22.043	20.225	18.644	17.262	16.047

Sol. Taking the origin at 3.5 and $h = 0.5$

$u = \frac{x - 3.5}{0.5}$ when $x = 3.75$, $u = \frac{0.25}{0.5} = \frac{1}{2}$

The difference table

u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
-2	24.145				
		-2.102			
-1	22.043		0.284		
		-1.818		-0.047	
0	20.225		0.237		0.009
		-1.581		-0.038	
1	18.644		0.199		0.006
		-1.382		-0.032	
2	17.262		0.167		
		-1.215			
3	16.047				

Bessel's formula is

$$y_u = \frac{1}{2}(y_1 + y_0) + (u - 1/2)\Delta y_0 + \frac{u(u-1)}{2!}\cdot\frac{1}{2}(\Delta^2 y_0 + \Delta^2 y_{-1})$$

$$+ \frac{(u-1/2)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\cdot\frac{1}{2}(\Delta^4 y_{-1} + \Delta^4 y_{-2})$$

$$+ \ldots$$

Putting $u = \frac{1}{2}$, we get

$$y_{1/2} = \frac{1}{2}(18.644 + 20.225) + 0 +$$

$$+ \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2} \times \frac{1}{2} \times (0.237 + 0.199) + 0$$

$$+ \frac{(3/2)(1/2)(-1/2)(-3/2)}{24} \times \frac{1}{2} \times (0.009 + 0.006)$$

$$= 19.4345 - 0.0272 + 0.0001$$

$$\simeq 19.407$$

$\therefore$ When $x = 3.75$, $y = 19.407$

Lagrange's Interpolation Formula For Unequal Intervals

- Let $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$ be the values of the function $y = f(x)$ corresponding to the $(n+1)$ arguments $x_0, x_1, x_2, \dots, x_n$, not necessarily equally spaced.

- Suppose that $P_n(x)$ is a polynomial in x of degree n. Then

$$P_n(x) = A_0(x-x_1)(x-x_2)\dots(x-x_n) + A_1(x-x_0)(x-x_2)\dots(x-x_n) + \dots + A_n(x-x_0)(x-x_1)\dots(x-x_{n-1}) \quad -(1)$$

where A's are constants. We determine the $(n+1)$ constants $A_0, A_1, \dots A_n$ so as to make

$$P_n(x_0) = f(x_0),$$
$$P_n(x_1) = f(x_1), \dots$$
$$\vdots$$
$$P_n(x_n) = f(x_n)$$

To ~~define~~ determine A_0, put $x = x_0$ and $P_n(x_0) = f(x_0)$ so that

$$f(x_0) = A_0(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)$$

$$\Rightarrow A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}$$

Similarly

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$$

$$\boxed{A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})}}$$

Subsituting the values of A's in (1), we get

$$\begin{aligned} P_n(x) = {} & \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) \\ & + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1) \\ & + \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} f(x_2) \\ & + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n) \qquad -(2) \end{aligned}$$

which is Lagrange's interpolation formula.

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Q1. Using Lagrange's interpolation formula, find the values of y when $x = 10$, from the following table:

x :	5	6	9	11
y :	12	13	14	16

Sol. Here $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$

$y_0 = 12$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$

Lagrange's interpolation formula is

$$f(x) = y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$
$$+ y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$f(10) = \frac{(12)(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} + \frac{(13)(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}$$
$$+ (14)\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} + (16)\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}$$

$$= 12\frac{(4)\times(1)\times(-1)}{(-1)(-4)(-6)} + 13\frac{(5)(1)(-1)}{(1)(-3)(-5)}$$
$$+ (14)\frac{(5)(4)(-1)}{(4)(3)(-2)} + \frac{(16)(5)(4)(1)}{(6)(5)(2)}$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \frac{44}{3}$$

$$= 14\frac{2}{3}$$

(41)

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Q2. Using Lagrange's formula, find the form of the function $f(x)$ given that:

x :	0	2	3	6
$f(x)$:	659	705	729	804

Sol. Here $x_0 = 0$, $x_1 = 2$, $x_2 = 3$, $x_3 = 6$;

$y_0 = 659$, $y_1 = 705$, $y_2 = 729$, $y_3 = 804$;

Using Lagrange's Formula,

$$f(x) = y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$
$$+ y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$\Rightarrow f(x) = (659)\frac{(x-2)(x-3)(x-6)}{(-2)(-3)(-6)} + (705)\frac{(x-0)(x-3)(x-6)}{(2)(-1)(-4)}$$
$$+ (729)\frac{(x-0)(x-2)(x-6)}{(3)(1)(-3)} + (804)\frac{(x-0)(x-2)(x-3)}{(6)(4)(3)}$$

$$\Rightarrow f(x) = \frac{-659}{36}(x^3 - 11x^2 + 36x - 36) + \frac{705}{8}(x^3 - 9x^2 + 18x)$$
$$-81(x^3 - 8x^2 + 12x) + \frac{67}{6}(x^3 - 5x^2 + 6x)$$

$$\Rightarrow f(x) = \frac{1}{72}(-x^3 + 29x^2 + 1602x + 47448)$$

Q3 Use Lagrange's interpolation formula, prove that

$$32 f(1) = -3f(-4) + 10 f(-2) + 30 f(2) - 5 f(4)$$

02/02/2021
Tuesday

Interpolation with unequal spaced points

The classical polynomial interpolating formulae discussed until now are limited to case with equally spaced intervals of independent variables. Now we'll discuss interpolation formulae with unequally spaced values of the argument.

(A) DIVIDED DIFFERENCES

Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the values of a function f corresponding to the arguments $x_0, x_1, \ldots, x_n$ where the intervals $x_1 - x_0,\ x_2 - x_1, \ldots, x_n - x_{n-1}$ are not necessarily equally spaced.

Then the first divided differences of f for the arguments $x_0, x_1, x_2, \ldots$ are defined by :

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and so on.

The second divided difference (divided difference of order 2) of f for three arguments x_0, x_1, x_2 is defined by

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

and similarly, the divided difference of order n is defined by

$$f(x_0, x_1, \ldots x_n) = \frac{f(x_1, x_2, \ldots, x_n) - f(x_0, x_1, \ldots x_{n-1})}{x_n - x_0}$$

$$= \frac{f(x_0, x_1, x_2, \ldots x_{n-1}) - f(x_1, x_2, \ldots, x_n)}{x_0 - x_n}$$

A divided difference table can be constructed for any set of tabulated values of $f(x)$.

Divided Difference Table

Argument x	Entry $f(x)$	First Div. diff. $\Delta' f(x)$	Second Div diff $\Delta'^2 f(x)$	Third Div diff. $\Delta'^3 f(x)$
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		
x_3	$f(x_3)$			

Divided Differences when two or more arguments are same or coincident

If two of the arguments coincide, the divided difference can be given a meaning. The meaning assigned is obtained by taking the limit.
Thus

$$f(x_0, x_0) = \lim_{\epsilon \to 0} \cancel{f(x_0, x_0+\epsilon)} - f(x_0+\epsilon, x_0)$$

$$= \lim_{\epsilon \to 0} \frac{f(x_0+\epsilon) - f(x_0)}{(x_0+\epsilon) - x_0} = f'(x_0) \quad -(4)$$

if $f(x)$ is differentiable at x_0.

Again, $f(x_0, x_0, x_0) = \lim\limits_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} f(x_0, x_0+\epsilon, x_0+\epsilon')$ — (5)

$$= \lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \frac{f(x_0, x_0+\epsilon) - f(x_0+\epsilon, x_0+\epsilon')}{x_0 - (x_0+\epsilon')}$$

$$= \lim_{\epsilon' \to 0} \frac{f'(x_0) - f(x_0, x_0+\epsilon')}{-\epsilon'}$$

$$= \lim_{\epsilon' \to 0} \frac{f'(x_0) - \dfrac{f(x_0+\epsilon') - f(x_0)}{\epsilon'}}{-\epsilon'}$$

$$= \lim_{\epsilon' \to 0} \frac{\epsilon' f'(x_0) - f(x_0+\epsilon') + f(x_0)}{-\epsilon'^2}$$

$$= \frac{1}{2!} f''(x_0)$$

by applying D' Hospitals rule twice.

Processing in this way, we find that

$$f(x_0, x_0, \ldots, x_0) = \frac{1}{r!} f^{r}(x_0) \qquad \text{— (5')}$$

where the $(r+1)$ arguments are each equal to x_0.
Thus if all the $(r+1)$ arguments are equal to x_0, then r^{th} divided difference is equal to the r^{th} derivative of $f(x)$ at x_0 divided by $r!$

Finally, we also get

$$f(x, x, x_0, x_1, \ldots x_n) = \frac{d}{dx} f(x, x_0, x_1, \ldots x_n)$$

NOTE: The limiting value of a divided difference, which arises when two or more of the arguments coincide is called a confluent divided difference arising from the confluence of the arguments in question.

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Further, we observe that

- $$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f(x_1, x_0)$$

- $$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{1}{(x_2 - x_0)}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

and in general

$$f(x_0, x_1, \dots x_n) = \frac{f(x_0)}{(x_0 - x_1)\dots(x_0 - x_n)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)}$$

$$\dots + \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})}$$

Hence divided differences are symmetrical in their arguments.

- For any function f, the value of the divided difference remains unaltered when any of the arguments involved are interchanged.
- Thus, value of the divided difference depends only on the value of the argument involved and not on the order in which they are taken.
- Thus,

$$f(x_0, x_1) = f(x_1, x_0)$$

$$f(x_0, x_1, x_2) = f(x_2, x_1, x_0) = f(x_1, x_0, x_2)$$

Theorem The n^{th} divided differences of a polynomial of the n^{th} degree are constant.

PROOF: Consider the function $f(x) = x^n$.

The first divided difference

$$f(x_r, x_{r+1}) = \frac{f(x_{r+1}) - f(x_r)}{x_{r+1} - x_r} = \frac{x_{r+1}^n - x_r^n}{x_{r+1} - x_r}$$

$$= x_{r+1}^{n-1} + x_r x_{r+1}^{n-2} + \dots$$

$$\dots + x_r^{n-2} x_{r+1} + \dots + x_r^{n-1}$$

is a homogeneous polynomial of degree $n-1$ in x_r, x_{r+1}.

Similarly it can be shown that second divided differences are homogenous polynomials of degree $n-2$.

Proceeding by mathematical induction, it can be shown that divided difference of n^{th} order is a polynomial of degree $n-n=0$ and so is constant

For a polynomial of the n^{th} degree with leading term $a_0 x^n$, the divided difference of terms except the leading term are zeros. So the n^{th} divided differences of this polynomial are constant and of value a_0

Remark Let the arguments be equally spaced so that $x_1 - x_0 = x_2 - x_1 = \dots = x_n - x_{n-1} = h$. Then

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{h} = \frac{\Delta f_0}{h}$$

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{1}{2h}\left[\frac{\Delta f_1}{h} - \frac{\Delta f_0}{h}\right]$$

$$= \frac{1}{2h^2}\Delta^2 f_0 = \frac{1}{2!}\frac{1}{h^2}\Delta^2 f_0$$

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and in general

$$f(x_0, x_1 \ldots x_n) = \frac{1}{n!}\frac{1}{h^n}\Delta^n f_0$$

If the tabulated function is a polynomial of n^{th} degree then $\Delta^n f_0$ would be constant and hence the n^{th} divided difference would also be constant.

NEWTON'S FUNDAMENTAL (DIVIDED DIFFERENCE) FORMULA

Let $f(x_0), f(x_1), \ldots f(x_n)$ be the values of a function f corresponding to the arguments $x_0, x_1, \ldots x_n$, where the intervals $x_1 - x_0$, $x_2 - x_1, \ldots, x_n - x_{n-1}$ are not necessarily equally spaced. By the definition of divided differences, we have

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

~~and so~~

$$f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad - (1)$$

Further,

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\Rightarrow \quad f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1) \quad - (2)$$

Illly $\Rightarrow$ $f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2) \quad - (3)$

and in general

$$f(x, x_0, \ldots, x_{n-1}) = f(x_0, x_1, \ldots x_n) + (x - x_n) f(x, x_0, x_1, \ldots, x_n) \quad - (4)$$

(2) $\times (x - x_0)$

$+$ (3) $\times (x - x_0)(x - x_1)$

$+$ (4) $\times (x - x_0)(x - x_1) \ldots (x - x_{n-1})$

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1}) f(x_0, x_1, \dots x_n) + R$$

where $R = (x-x_0)(x-x_1)\dots(x-x_n) f(x, x_0, \dots x_n)$

This formula is called Newton's divided difference formula. The last term R is the remainder term after $(n+1)$ terms.

Remark If we consider the case of equal spacing, then we have

$$f(x_0, x_1, \dots x_n) = \frac{1}{h^n n!} \Delta^n f_0$$

and so on.

$$f(x) = f(x_0) + \frac{(x-x_0)\Delta f_0}{h} + \frac{(x-x_0)(x-x_1)\Delta^2 f_0}{h^2 2!} + \dots$$

$$= f_0 + \frac{x_0 + ph - x_0}{h} \Delta f_0 + \frac{(x_0+ph-x_0)(x_0+ph-x_1)}{h^2 2!} \Delta^2 f_0 + \dots$$

$$= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \dots$$

which is nothing but Newton's forward difference formula

Example. Find a polynomial satisfied by $(-4, 1245)$, $(-1, 33)$, $(0, 5)$, $(2, 9)$ and $(5, 1335)$

Sol^n. The divided difference table based on the given nodes is:

x	y				
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

In fact,

$$f(x_0,x_1) = \frac{f(x_0)-f(x_1)}{x_0-x_1} = \frac{1245-33}{-3} = -404$$

$$f(x_1,x_2) = \frac{f(x_1)-f(x_2)}{x_1-x_2} = \frac{28}{-1} = -28$$

$$f(x_2,x_3) = \frac{f(x_2)-f(x_3)}{x_2-x_3} = \frac{5-9}{2} = 2$$

$$f(x_3,x_4) = \frac{f(x_3)-f(x_4)}{x_3-x_4} = \frac{9-1335}{2-3} = 442$$

$$f(x_0,x_1,x_2) = \frac{f(x_0,x_1)-f(x_1,x_2)}{x_0-x_2} = \frac{-404+28}{-4} = 94$$

$$f(x_1,x_2,x_3) = \frac{f(x_1,x_2)-f(x_2,x_3)}{x_1-x_3} = \frac{-28-2}{-3} = 10$$

$$f(x_2,x_3,x_4) = \frac{f(x_2,x_3)-f(x_3,x_4)}{x_2-x_4} = \frac{2-442}{-5} = 88$$

$$f(x_0,x_1,x_2,x_3) = \frac{f(x_0,x_1,x_2)-f(x_1,x_2,x_3)}{x_0-x_3} = \frac{94-10}{-6} = 14$$

$$f(x_1,x_2,x_3,x_4) = \frac{f(x_1,x_2,x_3)-f(x_2,x_3,x_4)}{x_1-x_4} = \frac{10-88}{-6} = 13$$

$$f(x_0,x_1,x_2,x_3,x_4) = \frac{f(x_0,x_1,x_2,x_3)-f(x_1,x_2,x_3,x_4)}{x_0-x_4} = \frac{-14-13}{-9} = 3$$

Putting these values in Newton's fundamental formula, we have

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3)$$
$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4)$$
$$= 1245 - 404(x+4) + 94(x+4)(x+1)$$
$$-14(x+4)(x+1)x + 3(x+4)(x+1)x(x-2)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

Example: Using the table given below, find $f(x)$ as a polynomial in x

x	-1	0	3	6	7
$f(x)$	3	-6	39	822	1611

Sol^n: The divided difference table for the given data is shown below

	x	$f(x)$				
x_0	-1	3				
			-9			
x_1	0	-6		6		
			15		5	
x_2	3	39		41		1
			261		13	
x_3	6	822		132		
			789			
x_4	7	1611				

Putting these values in the Newton's divided difference formula, we have

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3)$$
$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4)$$

$= 8 - 9(x+1) + 6(x+1)x + 5(x+1)x(x$

$+ 1(x+1)x(x-1)(x-6)$

$= x^4 - 8x^3 + 5x^2 + - 6$