① Gauss Elimination Method

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$AX = B$$

Consider the augmented matrix

$$[AB] \xrightarrow{\text{row operation}} \text{Upper triangular matrix } [UC]$$

can be solved by back subsitution method.

Example:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Augmented Matrix

$$[AB] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

$R_2 \to R_2 - \frac{a_{21}}{a_{11}}$, $R_3 \to R_3 - \frac{a_{31}}{a_{11}}$ (Changing to triangular form)

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & b_1 \\[2em]
0 & a_{23} - \dfrac{a_{21}}{a_{11}} a_{12} & a_{23} - \dfrac{a_{21}}{a_{11}} a_{13} & b_2 - b_1 \times \dfrac{a_{21}}{a_{11}} \\[2em]
0 & a_{32} - a_{12} \times \dfrac{a_{31}}{a_{11}} & a_{33} - a_{13} \dfrac{a_{31}}{a_{11}} & b_3 - b_1 \times \dfrac{a_{31}}{a_{11}}
\end{bmatrix}
$$

Let's denote this with

$$
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\[1em]
0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\[1em]
0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)}
\end{bmatrix}
$$

$$\downarrow$$

$$
\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\[1em]
0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\[1em]
0 & 0 & a_{33}^{(3)} & b_3^{(3)}
\end{bmatrix}
$$

The elements $a_{11}^{(1)}$, $a_{22}^{(2)}$, $a_{33}^{(3)}$ - which ~~are~~ have been assumed to be ~~zero~~ non zero are called pivot elements.

In the elimination process, if any one of the pivot elements $a_{11}^{(1)}, a_{22}^{(2)}, a_{33}^{(3)} \dots a_{nn}^{(n)}$ vanishes or become very small compared to other elements in that column, then rearrange the remaining rows so as to obtain a non-vanishing ~~point~~ pivot. This method is called pivoting and it is of two types:

① Partial Pivoting (biggest element (of column) at ~~pivot~~ a_{11})
② Complete Pivoting (not in syllabus)
(- In this biggest element of matrix is brought to a_{11})

Note: If the matrix A is diagonally dominant or real, symmetric and positive definite then no pivoting is necessary

Terms:
- Diag. dominant: diag element sum > sum of other element of matrix
- Positive definite: all sub matrices (Square matrix from top left) are +ve.

Q. Solve the eq's using the Gauss elimination method

$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + x_2 + 3x_3 = 13$$

Consider the augmented matrix

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

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$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$[AB] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

It can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore \quad x_1 + x_2 + x_3 = 6$$
$$-x_2 + x_3 = 1$$
$$x_3 = 2$$

By back subsitution we get

$x_2 = 1$

$x_1 = 3$

Result: $x_1 = 3$
$x_2 = 1$
$x_3 = 2$

Q. Solve the system of equations

$$\begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$$

using Gauss elimination method with partial pivoting

Ans. $AB = \begin{bmatrix} 2 & 1 & 1 & -2 & -10 \\ 4 & 0 & 2 & 1 & 8 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1 \quad , \quad R_3 \to 2R_3 - 3R_1$

$$\begin{bmatrix} 2 & 1 & 1 & -2 & -10 \\ 0 & -2 & 0 & 5 & +28 \\ 0 & & & & \end{bmatrix}$$

Partial pivoting: $R_1 \leftrightarrow R_2$ (bringing biggest element of column at a_{11})

$$\begin{array}{c} a_{11}^{(1)} \\ \end{array} \begin{bmatrix} 4 & 0 & 2 & 1 & 8 \\ 2 & 1 & 1 & -2 & -10 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - \frac{R_1}{2}$

$R_3 \to R_3 - \frac{3}{4}R_1 \quad , \quad R_4 \to R_4 - \frac{R_1}{4}$

$$\begin{bmatrix} 4 & 0 & 2 & 1 & 8 \\ 0 & 1 & 0 & -5/2 & -14 \\ 0 & 2 & 1/2 & -3/4 & 1 \\ 0 & 3 & 3/2 & -5/4 & -7 \end{bmatrix}$$

Partial Pivoting: $R_2 \leftrightarrow R_4$ (bringing biggest of column at a_{22})

$$\begin{array}{c} a_{11}^{(1)} \\ \end{array} \begin{bmatrix} 4 & 0 & 2 & 1 & 8 \\ 0 & 3\ a_{22}^{(2)} & 3/2 & -5/4 & -7 \\ 0 & 2 & 1/2 & -3/4 & 1 \\ 0 & 1 & 0 & -5/2 & -14 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{2}{3} R_2$$

$$R_4 \rightarrow R_4 - \frac{1}{3} R_2$$

$$\begin{bmatrix} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 0 & -1/2 & 1/12 & 17/3 \\ 0 & 0 & -1/2 & -25/12 & -35/3 \end{bmatrix}$$

~~Partial pivoting~~ $\quad R_4 \rightarrow R_4 - R_3$

$$\left[\begin{array}{cccc:c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 0 & -1/2 & 1/12 & 17/3 \\ 0 & 0 & 0 & -13/6 & -52/3 \end{array}\right]$$

$a_{11}^{(1)}$, $a_{22}^{(2)}$, $a_{33}^{(2)}$, $a_{44}^{(4)}$

Using back subsitution,

$$\cancel{x_1 = 5} \quad 4x_1 + 2x_3 + x_4 = 8$$

$$3x_2 + \frac{3}{2}x_3 + \frac{-5}{4}x_4 = -7$$

$$-\frac{x_3}{2} + \frac{x_4}{12} = \frac{17}{3}$$

$$\frac{-13}{6}x_4 = \frac{-52}{3}$$

$\therefore \; x_4 = 8$

$$x_3 = 2\left[\frac{8}{12} - \frac{17}{3}\right] = \cancel{x_4\left[\frac{32-17}{12}\right]} \Rightarrow 2\left[\frac{8-68}{12}\right] = -10$$

$$3x_2 + (-15) + (-10) = -7 \Rightarrow 3x_2 = 18 \Rightarrow x_2 = 6$$

$$4x_1 + (-20) + \cancel{8} = \cancel{8} \Rightarrow x_1 = 5$$

$\therefore$ Result : $x_1 = 5, \quad x_2 = 6, \quad x_3 = -10, \quad x_4 = 8$

Q3. Solve the eq^n

$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + (3+\epsilon)x_2 + 4x_3 = 20$$
$$2x_1 + x_2 + 3x_3 = 13$$

using the gauss elimination method when ϵ is small such that $1 \pm \epsilon \approx 1$

Ans. $[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & (3+\epsilon) & 4 & 20 \\ 2 & 1 & 3 & 13 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & \epsilon & 1 & 2 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{\epsilon} R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & \epsilon & 1 & 2 \\ 0 & 0 & \left(1+\frac{1}{\epsilon}\right) & \left(1+\frac{2}{\epsilon}\right) \end{array}\right]$$

$$\therefore x_1 + x_2 + x_3 = 6$$

$$\epsilon x_2 + x_3 = 2$$

$$\left(1+\frac{1}{\epsilon}\right)x_3 = 1 + \frac{2}{\epsilon}$$

Based on these eq^n: $x_3 = \left(\frac{1 + 2/\epsilon}{1 + 1/\epsilon}\right)$

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and $x_2 = \frac{1}{\epsilon}\; 2 - \frac{(1+2/\epsilon)}{(1+1/\epsilon)}$

$x_1 = 6 - x_2 - x_3$

We won't get answer from this.

$\boxed{|-1| > \epsilon}$

Now solving using Partial Pivoting

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & \epsilon & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & \epsilon & 1 & 2 \end{array}\right]$$

$R_3 \to R_3 + \epsilon R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1+\epsilon & 2+\epsilon \end{array}\right]$$

From this $x_3 = \frac{2+\epsilon}{1+\epsilon}$

$x_2 = -1 + \frac{2+\epsilon}{1+\epsilon}$

$x_1 = 6 - x_2 - x_3$

Now even if $\epsilon = 0$, we'll get the answer

We won't get answer from this.

$> \epsilon$

wer

② Gauss - Jordan Method

The coefficient matrix is reduced to a diagonal matrix.

Eg.

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = d_1$$
$$b_1 x_1 + b_2 x_2 + b_3 x_3 = d_2$$
$$c_1 x_1 + c_2 x_2 + c_3 x_3 = d_3$$

$$[AB] = \begin{bmatrix} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ c_1 & c_2 & c_3 & d_3 \end{bmatrix}$$

by row operation

$$= \begin{bmatrix} a_1 & a_2 & a_3 & d_1 \\ 0 & b_2' & b_3' & d_2' \\ 0 & c_2' & c_3' & d_3' \end{bmatrix}$$

by row operation

Now we'll try to make both a_2 & $c_2' = 0$

$$\begin{bmatrix} a_1 & 0 & a_3' & d_1' \\ 0 & b_2' & b_3' & d_2' \\ 0 & 0 & c_3'' & d_3'' \end{bmatrix}$$

by row operation

$$\left[\begin{array}{ccc|c} a_1 & 0 & 0 & d_1'' \\ 0 & b_2' & 0 & d_2'' \\ 0 & 0 & c_3'' & d_3'' \end{array}\right]$$

$$x_1 = \frac{d_1''}{a_1}$$

$$x_2 = \frac{d_2''}{b_2}$$

$$x_3 = \frac{d_3''}{c_3}$$

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③ Triangularisation Method

or decomposition method or factorisation method

In this method, coefficient matrix is decomposed as factorised into the product of a lower triangular matrix L and an upper triangular matrix U.

$$A = LU$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & 0 & & & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & \cdots & l_{nn} \end{bmatrix}$$

and

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & \cdots & u_{2n} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & \cdots & u_{nn} \end{bmatrix}$$

To produce a unique solution it is convenient to chose either $u_{ii} = 1$ or $l_{ii} = 1$ $(i = 1, \cdots u)$

When we chose

(i) $l_{ii} = 1$, the method is called Doolittle's method

(ii) $u_{ii} = 1$, the method is called Crout's method

in syllabus

This method fails if any of the diagonal elements l_{ii} or u_{ii} is zero. The LU decomposition is guaranteed when the matrix A is positive definite. It is only a sub... condition.

$$AX = B \quad , \quad (A = LU)$$
$$LUX = B \quad , \quad \text{Let } UX = Y$$
$$LY = B \longrightarrow y = ?$$

Q1. Solve the system of equations by Crout's method

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

$$A = LU$$

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$l_{11} = 1$, $u_{12} = 1$, $u_{13} = 1$

$l_{21} = 4$, $4 + l_{22} = 3$, $4 + l_{22}u_{23} = -1$, $\Rightarrow l_{22} = -1$, $u_{23} = 5$

$l_{31} = 3$, $3 + l_{32} = 5$, $\cancel{3} + 10 + l_{33} = \cancel{3}$

$(l_{32} = 2)$ $\quad (l_{33} = -10)$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AX = B$$

$$LUX = B, \quad \text{Let } UX = Y$$

$$\therefore LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix}_{3\times3} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{3\times1} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ 4y_1 - y_2 \\ 3y_1 + 2y_2 - 10y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$\Rightarrow y_1 = 1$$

$$y_2 = 4 - 6 = -2$$

$$3 + (-4) - 10y_3 = 4 \Rightarrow -5/10 = y_3 = -1/2$$

$$y_1 = 1, \quad y_2 = -2, \quad y_3 = -1/2$$

Now, $UX = Y$

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + x_2 + x_3 \\ x_2 + 5x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}$$

$$x_3 = -1/2$$

$$x_2 = -2 + 5\times\left(\tfrac{+1}{2}\right) = \tfrac{1}{2}$$

$$x_1 = 1$$

In this method there were 12 unknowns in U, L, Y matrix.

Another way with Crout's

Computation scheme by Crout's method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Augmented matrix

$$[AB] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

The matrix of 12 unknowns so called derived matrix or auxiliary matrix is

$$\begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}$$

Q. Solve the system of eq^n by Crout's method

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$

$$[AB] = \begin{bmatrix} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

For derived matrix

#1. $l_{11} = a_{11}$

$l_{21} = a_{21}$

$l_{31} = a_{31}$

$\therefore l_{11} = 2,\ l_{21} = 8,\ l_{31} = 4$

#2. For first column:

(u_{12}, u_{13}, y_1)

divide $[a_{12}, a_{13}, b_1]$ by a_{11}

$\therefore u_{12} = \frac{1}{2},\ u_{13} = 2,\ y_1 = 6$

Now we have

$$\begin{bmatrix} 2 & 1/2 & 2 & 6 \\ 8 & \boxed{l_{22}} & & \\ 4 & \boxed{l_{32}} & & \end{bmatrix}$$

$l_{22} = a_{22} - l_{21} u_{12}$ (here)

$= -3 - 8 \times 1/2$

$= -7$

$l_{32} = a_{32} - l_{31} u_{12}$

$= 11 - 4 \times 1/2$

$= 9$

a_{32}: val in auxilary matrix at that position

l_{31}: value of I^{st} in row

u_{12}: value of I^{st} in column

Now we have

$$\begin{bmatrix} 2 & 1/2 & 2 & 6 \\ 8 & -7 & \boxed{u_{23}} & \boxed{y_2} \\ 4 & 9 & & \end{bmatrix}$$

$u_{23} = \dfrac{a_{23} - l_{21} \cdot u_{13}}{l_{22}}$

$y_2 = \dfrac{a_{23} - l_{21} u_{14}}{l_{22}}$

a_{23}: val in aux matrix at that position

l_{21}: I^{st} val in row

u_{14}: I^{st} val in column

l_{22}: Second val ~~1st~~ in row

Now we have

$$\begin{bmatrix} 2 & 1/2 & 2 & 6 \\ 8 & -7 & 2 & 4 \\ 4 & 9 & \boxed{l_{33}} & \boxed{y_3} \end{bmatrix}$$

$l_{33} = a_{33} - \underbrace{l_{31}}_{\text{first in row}} \underbrace{u_{13}}_{\text{first in column}} - \underbrace{l_{32}}_{2^{n} \text{ in row}} \underbrace{u_{23}}_{2^{nd} \text{ in column}}$

$= -1 - 8 - 18$

$= \underline{-27}$

$y_3 = \dfrac{a_{34} - l_{31} u_{14} - l_{32} u_{24}}{l_{33}}$

a_{34}: val in aux matrix

l_{31}: First in row

u_{14}: I^{st} in column

l_{32}: 2^{nd} in row

u_{24}: 2^{nd} in column

l_{33}: 3^{rd} val in row

$= \dfrac{33 - 24 - 36}{-27} = \underline{\underline{1}}$

19/01/2020
~~Tuesday~~

(4) JACOBI METHOD OF ITERATION or GAUSS JACOBI METHOD

Consider the system of eq^{ns}:

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad (1)$$

Let $|a_1| > |b_1| + |c_1|$

$|b_2| > |a_2| + |c_2|$

$|c_3| > |a_3| + |b_3|$

i.e. in each equation the coefficients of the diagonal terms are large. Hence system (1) is ready for iteration. Solving for x, y, z we get

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad (2)$$

Let x_0, y_0, z_0 be the initial approximations of the unknowns x, y, z. Subsituting those on RHS of (2) the first approximations are given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

- Second approximations will be given by subsituting the values x_1, y_1, z_1 in the RHS of (2)

- This process is continued this convergency secured

Note: In the absence of any better estimation, the initial approximation is taken as
$x_0 = 0,\ y_0 = 0,\ z_0 = 0$

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Example 1 Solve by Jacobi iteration method the system $8x - 3y + 2z = 20$; $6x + 3y + 12x = 35$;
$4x + 11y - z = 33$
so that the diagonal elements are dominant in the coefficient matrix

Real ans
$x = 3.016$
$y = 1.985$
$z = 0.9118$

Ans. We'll write equation in the form :

$$\left.\begin{aligned} x &= \frac{1}{8}(20 + 3y - 2z) \\ y &= \frac{1}{11}(33 - 4x + z) \\ z &= \frac{1}{12}(35 - 6x - 3y) \end{aligned}\right\} \quad (3)$$

\# Let. $x_0 = y_0 = z_0 = 0$
Subsituting these in (1)
$x_1 = 2.5$, $y_1 = 3$, $z_1 = 2.917$

\# 2^{nd} iteration:
$x_2 = 2.896$, $y_2 = 2.356$, $z_2 = \cancel{0.6874}$ 0.916

\# 3^{rd} : $x_3 = \cancel{3.21}$ 3.154 , $y_3 = \cancel{2.0094}$ 2.03 , $z_3 = 0.88$

\# 4^{th} : $x_4 = 3.041$, $y_4 = 1.933$, $z_4 = 0.832$

\# 5^{th} : $x_5 = 3.017$, $y_5 = 1.9698$, $z_5 = 0.913$

\# 6^{th} : $x_6 = 3.010$, $y_6 = 1.985$, $z_6 = 0.9157$

\# 7^{th} : $x_7 = 3.015$, $y_7 = 1.9887$, $z_7 = 0.9154$

Stopping here since consistent to 2 decimal places in 6^{th} & 7^{th} iteration

$\therefore$ $\boxed{\text{Result : } x = 3.015,\ y = 1.9887,\ z = 0.9154}$

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d the
$12x = 35$;

ne dominant

Real ans
$x = 3.016$
$y = 1.985$
$z = 0.9118$

:

(3)

2.917

0.916
~~0.6874~~

0.88

0.832

0.913

0.9157

0.9154

imal

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Though in book example, they went until 12th iteration, where 4 decimal digits were consistent

(5) GAUSS-SEIDEL ITERATIVE METHOD

- This is modification of Gauss-Jacobi method.
- eq (1) and (2) will be considered here too.
- Only change is in ~~appr~~ initial approximation & subsequent subsitutions.

\# We start with initial approximation x_0, y_0, z_0.
\# Subsitute y_0 and z_0 in first eq^n of (2) to get x_1
\# Now subsitute x_1 & z_0 in 2nd eq^n of (2) to get y_1
\# Then subsitute x_1 & y_1 in 3rd eq^n of (2) to get z_1

This process continues till values of x, y, z are obtained to desired degree of accuracy.

- This method is more rapid in convergence than Gauss-Jacobi method and rate of convergence is twice of that of Gauss Jacobi.

Example 2: Solve example 1 by Gauss Seidel method.

- Let $x_0, y_0, z_0 = 0$
- $x_1 = 2.5$, $y_1 = 2.0909$, $z_1 = 1.143939$
- $x_2 = 2.99810$, $y_2 = 2.01377$, $z_2 = 0.9141728$
- $x_3 = 3.026620$, $y_3 = 1.982517$, $z_3 = 0.907727$
- x_4
- x_5
- $x_6 = 3.0167678$, $y_6 = 1.9858913$, $z_6 = 0.9118099$
- $x_7 = 3.0167568$, $y_7 = 1.9858894$, $z_7 = 0.9118159$

Example 3: Solve using Gauss-Seidel iteration method : $10x - 2y - z - w = 3$; $-2x + 10y - z - w = 15$; $-x - y + 10z - 2w = 27$; $-x - y - 2z + 10w = -9$

⑥ RELAXATION METHOD

Consider the eqⁿ

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Conditions: for convergence

$|a_1| \geq |b_1| + |c_1|$

$|b_2| \geq |a_2| + |c_2|$

$|c_3| \geq |a_3| + |b_3|$

with strict inequality for atleast one row.

We define the residuals R_1, R_2, R_3 by the relations

Errors or residuals

$$R_1 = d_1 - a_1x - b_1y - c_1z$$
$$R_2 = d_2 - a_2x - b_2y - c_2z$$
$$R_3 = d_3 - a_3x - b_3y - c_3z$$

To make $R_1 = 0$, we'll make change only in x and for $R_2 = 0$, change in y & for $R_3 = 0$, change in z.

For start, we assume $x = y = z = 0$ and calculate initial residuals. These residuals are reduced ~~by~~ step by step by giving increments to the variables.

At each step, the numerically largest residual is reduced to almost zero. To reduce a particular residue, say r_2, by α, then y should be increased by α / b_2.

When all the residuals have been reduced to almost 0, then the increment in x, y, and z are added seperately to give the desired solution.

Note: Do verify answer after getting x, y and z.

(8) operation Table:

x	y	z	r_1	r_2	r_3
1	0	0	$-a_1$	$-a_2$	$-a_3$
0	1	0	$-b_1$	$-b_2$	$-b_3$
0	0	1	$-c_1$	$-c_2$	$-c_3$

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Q. Solve by relaxation method the eq^{ns}:

$$10x - 2y - 2z = 6$$
$$-x + 10y - 2z = 7$$
$$-x - y + 10z = 8$$

Defining residuals

$$R_1 = 6 - 10x + 2y + 2z \quad \text{—(1)}$$
$$R_2 = 7 + x - 10y + 2z \quad \text{—(2)}$$
$$R_3 = 8 + x + y - 10z \quad \text{—(3)}$$

Operation table

	R_1	R_2	R_3
x	-10	1	1
y	2	-10	1
z	2	2	-10

(Try to make largest residue = 0)

Relaxation table:

	R_1	R_2	R_3
$x = y = z = 0$ or $S_x = S_y = S_z$	6	7	8 largest
$S_z = 0.8$ ↑ ($x=0, y=0, z=0.8$)	7.6	8.6 largest	0
$S_y = 0.86$ ↑ ($x=0, y=0.86, z=0.8$)	9.32 largest	0	0.86
$S_x = 0.93$ ↑ ($x=0.93, y=0.86, z=0.8$)	0.02	0.93	1.79
$S_z = 0.18$ ↑ ($x=0.93, y=0.86, z=0.98$)	0.38	1.29	-0.01
$S_y = 0.13$ ↑	0.64	-0.01	0.12
$S_x = 0.06$ ↑	0.04	0.05	0.18

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All R_1, R_2, R_3 close to 0. $\therefore$ Stopping!

$x = \Sigma S_x = 0.93 + 0.06 = 0.99$
$y = \Sigma S_y = 0.99$
$z = \Sigma S_z = 0.98$

Q2 Solve by relaxation method, the eqⁿ

$9x - y + 2z = 9$
$x + 10y - 2z = 15$
$2x - 2y - 13z = -17$

Ans
$x = 0.918$
$y = 1.649$
$z = 1.14523$

Ans. Defining residuals

$R_1 = 9 - 9x + y - 2z$
$R_2 = 15 - x - 10y + 2z$
$R_3 = -17 - 2x + 2y + 13z$

Operation table

	r_1	r_2	r_3
x	-9	-1	-2
y	1	-10	2
z	-2	2	13

Relaxation table

		R_1	R_2	R_3
#1	$x = y = z = 0$, $S_x = S_y = S_z = 0$	9	15	-17
#2	$S_z = \frac{+17}{+13} = 1.30$ $\therefore x = 0, y = 0, z = 1.30$	6.4	17.6	-0.1
#3	$S_y = \frac{17.6}{+10} = 1.76$ $x = 0, y = 1.76, z = 1.30$	8.16	0	3.42

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		R_1	R_2	R_3
#4.	$S_x = \frac{8.16}{9} = 0.91$ $x = 0.91,\ y = 1.76,\ z = 1.30$	-0.03	-0.91	1.6
#5.	$S_z = \frac{1.6}{-13} = -0.12$ $x = 0.91,\ y = 1.76,\ z = 1.18$	0.21	-1.15	0.04
#6.	$S_y = \frac{-1.15}{10} = -0.12$ $x = 0.91,\ y = 1.66,\ z = 1.18$	0.11	-0.15	-0.16
#7.	$S_z = \frac{-0.16}{-13} = 0.01$ $x = 0.91,\ y = 1.66,\ z = 1.19$	0.09	-0.13	-0.03
#8.	$S_y = \frac{-0.13}{10} = -0.01$ $x = 0.91,\ y = 1.65,\ z = 1.19$	0.08	-0.03	-0.05

Residuals are close to 0, $\therefore$ Stopping!

$$x = \Sigma S_x = 0.91$$
$$y = \Sigma S_y = 1.65$$
$$z = \Sigma S_z = 1.19$$