Solution of Ordinary Diff. Eqⁿ

TAYLOR'S SERIES METHOD

Let us consider the first order differential Eqⁿ:

$$\frac{dy}{dx} = f(x, y) \qquad -①$$

under the condition $y = 0$ for $x = x_0$

METHOD

On differentiang ① again & again, we get

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \text{etc}$$

On putting $x = x_0$ and $y = 0$ in the above equations we get the value of

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots$$

Subsituting the values of $y', y'', y''', y'''', \ldots$ in Taylor series

$$y = y_0 + (x-x_0)[y'(x_0)] + \frac{(x-x_0)^2}{2!}[y''(x_0)] + \frac{(x-x_0)^3}{3!}[y'''(x_0)] + \ldots$$

Thus we can obtain a power series for $y(x)$ in powers of $(x - x_0)$.

Example Using Taylor's series method, obtain the solution of $\frac{dy}{dx} = 3x + y^2$ and

$y = 1$, when $x = 0$.

Find the value of y for $x = 0.1$, correct to four places of decimal.

Sol.

$$\frac{dy}{dx} = 3x + y^2 \qquad -①$$

$$y(0) = 1 \qquad -②$$

Differentiating ① w.r.t 'x' we get

$$\cdot\ \frac{d^2y}{dx^2} = 3 + 2y\frac{dy}{dx} \qquad -③$$

$$\cdot\ \frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 \qquad -④$$

$$\cdot\ \frac{d^4y}{dx^4} = 2y\frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 4\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) \qquad -⑤$$

and so on.

From ① $\quad \frac{dy}{dx} = 0 + (1)^2 = 1$

From ③ $\quad \frac{d^2y}{dx^2} = 3 + 2(1)(1) = 5$

From ④ $\quad \frac{d^3y}{dx^3} = 2(1)(5) + 2(1)^2 = 12$

From ⑤ $\quad \frac{d^4y}{dx^4} = 2(1)(12) + 2(1)(5) + 4(1)(5) = 54$

We know by taylor series expansion

$$y = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \frac{(x-x_0)^4}{4!}(y'''')_0 + \qquad -⑥$$

On subsituting value in ⑥ we get

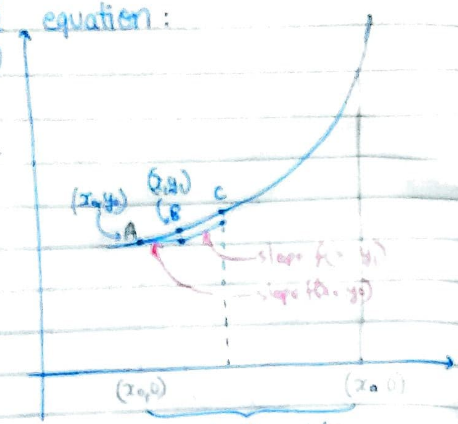
$$y = 1 + x + \frac{x^2}{2!}(5) + \frac{x^3}{3!}(12) + \frac{x^4}{4!}(54) + \ldots$$

- In a small interval, a curve is nearly a st. line. This property is used in Euler's method.

EULER'S METHOD

Consider the differential equation :

$$\frac{dy}{dx} = f(x,y) \quad -①$$



divide this into n equal parts

$$\boxed{\frac{x_n - x_0}{n} = h}$$

$x_0, x_1, \dots, x_n$ = equidistant values of x.

given → $y(x_0) = y_0$

to find → $y(x_n) = ?$

Eqⁿ of line passing through A is (and tangent to curve)

$$y - y_0 = f(x_0, y_0)(x - x_0)$$

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$\vdots$$

$$\boxed{y_{n+1} = y_n + h\, f(x_n, y_n)}$$

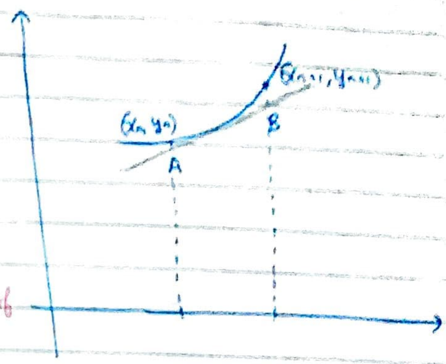
This is called Euler's algorithm.

Note :- In Euler's method, the curve of actual solution is approximated by a sequence of short lines.

- The process is very slow.

- If h is not properly chosen, the curve ABC.... of short lines representing numerical solution deviates significantly from the curve of actual solution. Hence, improvement of this method is called Modified Euler.

MODIFIED EULER



$$\frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0$$

$$y(x_n) = ?$$

average of 2 slopes

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

This is from Euler's Iˢᵗ formula

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + h\, f(x_n, y_n))\right]$$

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Example. Using Euler's method, find an approximate value of y corresponding to $x=2$, given that $\frac{dy}{dx} = x + 2y$ and $y=1$ when $x=1$

Sol. $f(x,y) = x + 2y$

(taking $h=0.1$)

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$
$$= y_n + 0.1\,(x+2y)$$

Method: In colum 3, we record the value of $x+2y$ and in colum 4, we enter the sum of the value of y and the product of 0.1 with the value of $x+2y$. This value entered in 4th column is transfered to 2nd column for the next calculation.

x	y	$x+2y = \frac{dy}{dx}$	$dy + 0.1\left(\frac{dy}{dx}\right) = \text{new } y$
1.0	1.00	3.00	1.0 + 0.1(3) = 1.30
1.1	1.30	3.70	1.3 + 0.1(3.7) = 1.67
1.2	1.67	4.54	1.67 + 0.1 (4.54) = 2.12
1.3	2.12	5.54	2.12 + 0.1 (5.54) = 2.67
1.4	2.67	6.74	2.67 + 0.1 (6.74) = 3.34
1.5	3.34	8.18	3.34 + 0.1 (8.18) = 4.16
1.6	4.16	9.92	4.16 + 0.1 (9.92) = 5.15
1.7	5.15	12.00	5.15 + 0.1 (12.0) = 6.35
1.8	6.35	14.50	6.35 + 0.1 (14.50) = 7.80
1.9	7.80	17.50	7.80 + 0.1 (17.50) = 9.55
2.0	9.55		

Thus approximate value of y is 9.55

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Questions

1. Using Euler's method, find an approximate value of y corresponding to $x=1$, given that $\frac{dy}{dx} = x + y$ and $y=1$ when $x=0$.
   Ans. ~~[illegible]~~ 1.35?
2. Using Euler's method, find an approximate value of y corresponding to $x=1.4$, given $\frac{dy}{dx} = x^{1/2}$ and $y=1$ when $x=1$
   Ans. 1.49857
3. Using Euler's method, find an approximate value of y corresponding to $x=1.6$, given $dy/dx = y^2 - y/x$ and $y=1$ when $x=1$.

Example Apply Euler's modified method to solve $dy/dx = x+3y$ subject to $y(0)=1$ and hence find an approximate value of y when $x=1$.

Sol. $f(x,y) = x + 3y$

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$
$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

This gives, $y_{n+1} = y_n + 0.1\,(x_n + 3y_n)$

$$y_{n+1} = y_n + 0.05\left[(x_n + 3y_n) + (x_{n+1} + 3y_{n+1})\right]$$

The following table shows the computation work

#	x_n	y_n	$x_n + 3y_n$	Euler's formula y_{n+1}	x_{n+1}	$x_{n+1} + 3y_{n+1}$	Euler's modified y_{n+1}
0	0.0	1	3	1.3	0.1	4	1.35
1	0.1	1.35	4.15	1.765	0.2	5.495	1.832
2	0.2	1.832	5.695	2.402	0.3	7.506	2.492
3	0.3	2.492	7.776	3.270	0.4	10.21	3.391
4	0.4	3.391	10.573	4.448	0.5	13.844	4.612
5	0.5	4.612	14.336	6.046	0.6	18.738	6.266
6	0.6	6.266	19.398	8.206	0.7	25.318	8.502
7	0.7	8.502	26.206	11.123	0.8	34.169	11.521
8	0.8	11.521	35.363	15.057	0.9	46.071	15.593

9	0.9	15.593	47.679	20.361	1.0	62.083	21.081
10	1.0	21.081					

Hence the required value of y at $x=1$ is 21.081
The exact sol^n gives $y = 21.873$ for $x=1$.
The error is 0.792, i.e., 3.6%

Ans Procedure. We calculated y_{n+1} by Euler's formula i.e.
$y_{n+1} = y_n + 0.1\,(x_n + 3y_n)$ and entered in 5th column.
In 7th column we record the sum i.e. $x_{n+1} + 3y_{n+1}$.
Then we computed the value of y_{n+1} by Euler's modified formula, i.e.,

$$y_{n+1} = y_n + \frac{0.1}{2}\left[(x_n + 3y_n) + (x_{n+1} + 3y_{n+1})\right]$$

and entered in 8th column.

Questions

1. Using Euler's modified formula, find an approximate value of y when $x = 0.3$, given that $dy/dx = x+y$ and $y=1$ when $x=0$.
ans. 1.3997

2. Using Euler's modified formula, find an approximate value of y when $x = 0.06$, given that $dy/dx = x^2+y$ and $y(0) = 1$, taking the interval 0.02.
Ans. 1.0619

3. Using Euler's modified formula, solve $dy/dx = 1-2xy$ given $y=0$ at $x=0$ from $x=0$ to 0.6 taking the interval $h=0.2$.
Ans. 0.4748

RUNGE'S FORMULA

Euler's modified formula is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

or
$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h,\ y_n + hf_n)\right] \quad -①$$

Let
$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + h,\ y_n + hf(x_n, y_n))$$
$$= hf(x_n + h,\ y_n + k_1)$$

Putting the values of k_1 and k_2 in ① we get

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

This is known as Runge's formula of order 2.

Example Apply Runge's formula of order 2 approximate value of y when $x = 1.1$, given $dy/dx = 3x + y^2$ and $y = 1.2$ when $x = 1$

Sol^n. Here we have $x_0 = 1$, $y_0 = 1.2$, $h = 0.1$

$f(x, y) = 3x + y^2$

$f(x_0, y_0) = 3(1) + (1.2)^2 = 4.44$

$k_1 = hf(x_0, y_0) = 0.1 \times 4.44 = 0.444$

$k_2 = hf(x_0 + h,\ y_0 + k_1) = 0.1\,f(1.1,\ 1.2 + 0.444) = 0.1\,f(1.1,\ 1.644)$
$= 0.1\left[3 \times 1.11 + (1.644)^2\right]$
$= 0.600$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$\therefore y_1 = 1.2 + \frac{1}{2}(0.444 + 0.600)$$

$= 1.722$

Questions

1. Apply Runge's formula of second order to find approximate value of y when $x = 1.1$ given that $dy/dx = x - y$ and $y = 1$ when $x = 1$. Ans. 1.005?

2. Apply Runge's formula of second order to find approximate value of y when $x = 0.02$ given that $\frac{dy}{dx} = x^2 + y$ and $y(0) = 1$ ans. 1.0202?

RUNGE'S FORMULA (THIRD ORDER)

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_4 + k_3)$$

where, $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$k_3 = hf(x_0 + h, y_0 + k_2)$

$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

This is the Runge's Formula (third order) with an error of the order h^4.

Example. Using Runge's Formula (third order), solve the differential eqn $dy/dx = x - y$ subject to $y = 1$ when $x = 1$.

Sol. $f(x, y) = x - y$

here $h = 0.1$

$x_0 = 1$

$y_0 = 1$

$k_1 = hf(x_0, y_0) = 0.1(x - y) = 0.1(1 - 1) = 0$

$k_2 = hf(x_0 + h, y_0 + k_1)$
$= 0.1 f(1.1, 1 + 0) = 0.1(1.1 - 1)$
$= 0.01$

$k_3 = hf(x_0 + h, y_0 + k_2)$
$= 0.1 f(1.1, 1.01) = 0.1(1.1 - 1.01)$
$= 0.009$

$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$
$= 0.1 f(1.05, 1 + 0/2)$
$= 0.1(1.05 - 1)$
$= 0.005$

$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_4 + k_3)$

$y(0.1) = 1 + \frac{1}{6}(0 + 0.02 + 0.009)$

$= 1. + 0.004833$

$= 1.004833$

RUNGE'S - KUTTA FORMULA (FOURTH ORDER)

A fourth order Runge's - Kutta formula for solving the differential equation is

$$y = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$k_4 = hf(x_0 + h, y_0 + k_3)$

This is known as Runge-Kutta Formula. The error in this formula is of the order h^5.

This method have greater accuracy.

No derivatives are required to be tabulated.

It requires only functional values at some selected points on the sub interval.

Example Apply Runge-Kutta method to find an approximate value of y when $x = 0.2$ given that $dy/dx = x+y$, $y=1$ when $x=0$

Sol^n Let $h = 0.1$

Here $x_0 = 0$, $y_0 = 1$, $f(x,y) = x+y$

Now $k_1 = h\,f(x_0,y_0) = 0.1\,(0+1) = 0.1$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\,f(0 + 0.05,\ 1 + 0.005)$$
$$= 0.1\,[0.05 + 1.05]$$
$$= 0.11$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$
$$= 0.1\,f(0 + 0.05,\ 1 + 0.055)$$
$$= 0.1\,(0.05 + 1.055)$$
$$= 0.1105$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3)$$
$$= 0.1\,f(0 + 0.1,\ 1 + 0.1105)$$
$$= 0.1\,f(0.1,\ 1.1105) = 0.1\,(0.1 + 1.1105)$$
$$= 0.12105$$

According to Runge Kutta (Fourth order) formula

$$y = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$y_{0.1} = 1 + \frac{1}{6}(0.1 + 0.22 + 0.221 + 0.12105)$$
$$= 1 + \frac{1}{6}(0.66205)$$
$$= 1.11034$$

FOR THE SECOND STEP :

$x_0 = 0.1$

$y_0 = 1.11034$

$h = 0.1$

$$k_1 = h\,f(x_0,y_0) = 0.1\,(0.1 + 1.11034)$$
$$= 0.121034$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.1\,f(0.1 + 0.05,\ 1.11034 + 0.060517)$$
$$= 0.1\,(0.15 + 1.170857)$$
$$= 0.1320857$$

$k_3 =$

$= 0.132638$

$k_4 =$

$= 0.144298$

$$y_{0.2} = 1.2428$$

12/02/2020

Milne's Predictor - Corrector Method

Milne's Predictor-corrector method is the method in which first ~~predict~~ predict a value of y_{n+1} by using a certain formula and then correct this value by using a more accurate formula. Thus an initial crude estimate is refined.

Consider the diffential eq^n $\frac{dy}{dx} = f(x,y)$,

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x,y), \\ y(x_0) &= y_0 \end{aligned}\right\} \quad (1)$$

We first find numerical values of

$y_0 = y(x_0)$

$y_1 = y(x_0+h)$

$y_2 = y(x_0+2h)$

$y_3 = y(x_0+3h)$

Using (1):

$$y_0' = \left(\frac{dy}{dx}\right)_{(x_0,y_0)} = f(x_0, y_0)$$

$$y_1' = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = f(x_1, y_1)$$

$$y_2' = f(x_2, y_2)$$

For equi-spaced arguments $x_0, x_0+h, x_0+2h, \dots,$ by Newtons forward interpolation

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \dots$$

$$y' = y_0 + u\Delta y_0' + \frac{u(u-1)}{2!}\Delta^2 y_0' + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0' + \dots$$

$$h = \frac{x-x_0}{h}$$

$$\int_{x_0}^{x_0+4h} y'\,dx = \int_{x_0}^{x_0+4h}\left[y_0' + u\Delta y_0' + \frac{u(u-1)}{2!}\Delta^2 y_0' + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0' + \dots\right]dx$$

$$\left(u = \frac{x-x_0}{h} \Rightarrow dx = h\,du\right)$$

$$\Rightarrow y_4 - y_0 = h\int_0^4\left[y_0' + u\Delta y_0' + \frac{u(u-1)}{2!}\Delta^2 y_0' + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0'\right]du$$

$$\Rightarrow y_4 = y_0 + h\left(4y_0' + 8\Delta y_0' + \frac{20}{3}\Delta^2 y_0' + \frac{8}{3}\Delta^3 y_0'\right)$$

$$= y_0 + h\left[4y_0' + 8(E-1)y_0' + \frac{20}{3}(E-1)^2 y_0' + \frac{8}{3}(E-1)^3 y_0'\right]$$

$$= y_0 + h\left[4y_0 + 8(y_1' - y_0') + \frac{20}{3}(E^2-2E+1)y_0' + \frac{8}{3}(E^3 - 3E^2 + 3E - 1)y_0'\right]$$

$$= y_0 + h\left[4y_0 + 8(y_1' - y_0') + \frac{20}{3}(y_2' - 2y_1' + y_0) + \frac{8}{3}(y_3' - 3y_2' + 3y_1' - y_0')\right]$$

$$y_4 = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_2')$$

$$\boxed{y_{n+1,p} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n)}$$

If we had integrated from x_0 to $x_0 + 2h$

$$\int_{x_0}^{x_0+2h} y\,dx$$

$$\boxed{y_2 = y_0 + \frac{h}{3}\left(y_0' + 4y_1' + y_2'\right)}$$

$$\boxed{y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(f_{n-1} + 4f_n + f_{n+1}\right)}$$

This formula is called Milne's corrector formula

Example Given $dy/dx = x^2(1+y)$ and $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, evaluate $y(1.4)$ by Milne's predictor-corrector method

Sol^n Here

$y' = f(x,y) = x^2(1+y)$; $\quad h = 0.1$

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$

$y_0 = 1$, $y_1 = 1.233$, $y_2 = 1.548$, $y_3 = 1.979$

~~Using Milne's~~ $y_1' = f(x_1, y_1) = (1.1)^2 * (1 + 1.233) = 2.7019 = f_1$

$y_2' = f(x_2, y_2) = (1.2)^2 * (1 + 1.548) = 3.6691 = f_2$

$y_3' = f(x_3, y_3) = (1.3)^2 * (1 + 1.979) = 5.0345 = f_3$

Using Milne's predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$\Rightarrow y_{4p} = 1 + \frac{4}{3} \times (0.1) \times (5.4038 - 3.6691 + 10.0690)$$

$$= 2.5738$$

Now $x_4 = 1.4$, $y_4 = 2.5738$

$\therefore y_4' = f(x_4, y_4) = (1.4)^2 (1 + 2.5738) = 7.0046 = f_4$

Using Milne's Corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4\right)$$

$$\Rightarrow y_{4,c} = 1.548 + \frac{0.1}{3}(3.6691 + 20.1380 + 7.0046)$$

$$= 2.5750$$

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Hence $y(1.4) = 2.5750$

Example 2. Use Milne's method to find $y(0.3)$ from $y' = x^2 + y^2$, $y(0) = 1$. Find the initial values $y(-1)$, $y(1)$ and $y(2)$ from Taylor Series method.

$y(0.3) = 1.4385$

Sol^n. Given eq^n is $y' = x^2 + y^2$

Differentiate repeatedly w.r.t x, we get

$$y'' = 2x + 2yy'$$
$$y''' = 2 + 2(yy'' + y'^2)$$

$$y(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0)$$

$$= 1 + x + x^2 + \frac{4x^3}{3}$$

$$y(-0.1) = 1 - 0.1 + (-0.1)^2 + \frac{4(-0.1)^3}{3} - \cdots$$

$$= 0.9087$$

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)^2}{3} - \cdots$$

$$= 1.1113$$

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{4(0.2)^3}{3} - \cdots$$

$$= 1.2506$$

Now $x_{-1} = -0.1$, $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$

$h = 0.1$

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$y_{-1} = 0.9087$, $y_0 = 1$, $y_1 = 1.1113$, $y_2 = 1.2506$

$y_0' = f(x_0, y_0) = 0^2 + 1^2 = 1 = f_0$

$y_1' = f(x_1, y_1) = (0.1)^2 + (1.1113)^2 = 1.2449 = f_1$

$y_2' = f(x_2, y_2) = (0.2)^2 + (1.2506)^2 = 1.6040 = f_2$

Using Milne's predictor formula

$$y_{3,p} = y_{-1} + \frac{4h}{3}(2f_0 - f_1 + 2f_2)$$

$$= 0.9087 + \frac{4 \times 0.1}{3}(2 - 1.2449 + 3.2080)$$

$$= 1.4371$$

Since $x_3 = 0.3$, $y_3 = 1.4371$

$$y_3' = f(x_3, y_3) = (0.3)^2 + (1.4371)^2 = 2.1552 = f_3$$

Using Milne's corrector formula,

$$y_{3,c} = y_1 + \frac{h}{3}(f_1 + 4f_2 + f_3)$$

$$= 1.1113 + \frac{0.1}{3}(1.2449 + 6.4160 + 2.1552)$$

$$= 1.4385$$

Hence $y(0.3) = 1.4385$