$x + 5(x+1)x(x-3)$

(52) (1) UNIT-04

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03/02/2021
Wednesday

NUMERICAL DIFFENTIATION

(1) Consider Newtons forward formula for interpolation

$$f(x) = f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \cdots \quad (1)$$

where $u = \frac{x-a}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$

Differentiating (1) w.r.t. x

$$\frac{d}{dx}f(x) = \frac{d}{du}f(a+hu)\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta f(a) + \frac{(2u-1)}{2!}\Delta^2 f(a) + \frac{(3u^2-6u+2)}{3!}\Delta^3 f(a) + \cdots\right] \quad (2)$$

for a particular value for $x=a$, $u=0$

$$\left[\frac{d}{dx}f(x)\right]_{x=a} = \frac{1}{h}\left[\Delta f(a) - \frac{1}{2!}\Delta^2 f(a) + \frac{2}{3!}\Delta^3 f(a) - \frac{6}{4!}\Delta^4 f(a) + \cdots\right]$$

$\boxed{y = f(x)}$

$$\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta f(a) - \frac{1}{2}\Delta^2 f(a) + \frac{1}{3}\Delta^3 f(a) - \frac{1}{4}\Delta^4 f(a) \cdots\right]$$

Again diff. (2) we get $\frac{d^2}{dx^2}f(x)$

$$\left(\frac{d^2 y_x}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 \cdots\right]$$

② ③

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Q. Using the following data, find f(5)

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

② Newton's backward interpolation formula

$f(x) = f(a+nh+hu)$

$$= f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \cdots \quad ①$$

$$\frac{d}{dx}f(x) = \frac{1}{h}\left[\nabla f(a+nh) + \frac{(2u+1)}{2!}\nabla^2 f(a+nh) + \frac{(3u^2+6u+2)}{3!}\nabla^3 f(a+nh) + \cdots\right] \quad ②$$

$$\frac{d^2}{dx^2}f(x) = \frac{1}{h^2}\left[\nabla^2 f(a+nh) + \frac{(6u+6)}{3!}\nabla^3 f(a+nh) + \cdots\right]$$

at $x = a+nh$, $u=0$

Q. Find the minimum value of y from the table

x	0	1	2	3	4	5
y	0	0.25	0	2.25	16	56.25

0,2
$y=0$

Equate first derivative $=0$
& 2^{nd} derivative $= -ve$

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forward

Q. Find the first and second derivatives at $x=3.0$ from the following table

x	3.0	3.2	3.4	3.6	3.8	4.0
y	-14.000	-10.032	-5.296	-0.256	6.672	14.000

Ans = 18

Q. Given that

x	1.0	1.1	1.2	1.3	1.4	1.5
y	7.989	8.403	8.781	9.129	9.451	9.750

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.6$

Ans: 2.7416
−1.117

backward

Q. The function $y = \sin x$ is tabulated below. Find the derivative at the point $x=1$

x	0.7	0.8	0.9	1.0	1.1	1.2	1.3
y	0.644218	0.717356	0.783327	0.841471	0.891207	0.932039	0.963558

Ans: 0.540302

NOTE: In a similar way, Bessel and Stirlings formula can be used for interpolation.

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5/02/2021
Friday

NUMERICAL QUADRATURE or NUMERICAL INTEGRATION

① A General Quadrature Formula for Equidistant Ordinates

Let $I = \int_a^b y\,dx$ where $y = f(x)$

Suppose $f(x)$ is given for certain equidistant values of x say $x_0, x_0+h, x_0+2h, \ldots$ Let the range (a, b) be divided into n equal parts, each of width h so that $b-a = nh$

Let $x_0 = a$
$x_1 = x_0 + h = a+h$
$x_2 = a + 2h$
$\vdots$
$x_n = a + nh = b$

We have assumed that the $n+1$ ordinates $y_0, y_1, \ldots y_n$ are equistant

$$\therefore I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} y_x\,dx = \int_0^n y_{x_0+hu}\, h\,du$$

where, $u = \dfrac{x-x_0}{h}$, $dx = h\,du$

(using newtons forward)

$$\text{or } I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\right] du$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \ldots \text{upto } (n+1) \text{ terms}\right] \quad -①$$

This is general quadrature formula. A number of formula can be deduced by this by putting $n = 1, 2, 3, \ldots$

② Some important Approximate Quadrature Formulae

(i) The Trapezoidal Rule

Put $n=1$ in ① and neglect 2° & higher ~~degrees~~ differences.

We get
$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$
$$= h\left[y_0 + \frac{y_1 - y_0}{2}\right] = h\left(\frac{y_0+y_1}{2}\right)$$

Illly
$$\int_{x_0+h}^{x_0+2h} y\,dx = h\left(\frac{y_1+y_2}{2}\right)$$
$$\vdots$$
$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = h\left(\frac{y_{n-1}+y_n}{2}\right)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[\frac{1}{2}(y_0+y_n) + (y_1+y_2+\cdots+y_{n-1})\right]$$

= distance between two consecutive ordinates * [mean of first and last ordinate + sum of all the intermediate ordinates]

This rule is known as the Trapezoidal Rule

NOTE: Here we have a curve through 2 points (x_0, y_0) and (x_0+h, y_0+k) only, so we suppose that y is a linear function in x, that

$y = a + bx$

(II) Simpson's One-third Rule

Put $n=2$ in (1) and neglect third & higher differences, we get

$$\int_{x_0}^{x_0+2h} y\,dx = h\left[2y_0 + 2(y_1-y_0) + \frac{\left(\frac{8}{3}-2\right)}{2}(y_2-2y_1+y_0)\right]$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Illy

$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\vdots$$

$$\int_{x_0+(n-2)n}^{x_0+nh} y\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

where n is even

Adding all these integrals, we have when n is even:

$$\boxed{\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\dots+y_{n-1}) + 2(y_2+y_4+\dots+y_{n-2})\right]}$$

This rule is known as Simpson's one-third rule.

NOTE: Since we have neglected all differences above the 2^{nd}, y must be a polynomial of 2^{nd} degree only, $y = ax^2+bx+c$

(III) Simpson's Three-eight Rule

Put $n=3$ in (1) and neglect all differences above the third, then

$$\int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0 + \frac{9}{2}(y_1-y_0) + \frac{5}{4}(y_2-2y_1+y_0) + \frac{3}{8}(y_3-3y_2+3y_1-y_0)\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

$$\text{Illy} \int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

Adding such expressions as these from x_0 to x_n where n is a multiple of 3, we have

$$\boxed{\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\dots+y_{n-1}) + 2(y_3+y_6+\dots+y_{n-3})\right]}$$

and is known as Simpson's three-eight rule.

On applying twice this rule we obtain

$$\int_0^6 u_x\,dx = \frac{3}{8}\left[(u_0+u_6) + 3(u_1+u_2+u_4+u_5) + 2u_3\right]$$

Here we have neglected all differences above the third, so that we have assumed that y is a polynomial of the third degree, that is of the form

$y = ax^3 + bx^2 + cx + d$

Example 1: Use trapezoidal rule to evaluate $\int_0^1 x^3\,dx$ considering five sub-intervals

Sol^n Dividing the interval (0,1) in 5 equal parts, each of width $h = \frac{1-0}{5} = 0.2$

the values of $f(x) = x^3$ are given below:

x	0	0.2	0.4	0.6	0.8	1.0
$f(x)$	0	0.008	0.064	0.216	0.512	1.000
	y_0	y_1	y_2	y_3	y_4	y_5

By trapezoidal rule, we have

$$\int_0^1 x^3\,dx = \frac{h}{2}\left[(y_0+y_5) + 2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}\left[(0.0+1.00) + 2(0.008+0.064+0.216+0.512)\right]$$

$$= 0.1 \times 2.6$$

$$= 0.26$$

Example 2: Evaluate $\int_0^1 \frac{dx}{1+x^2}\,dx$ using

(i) Simpson's 1/3 rule taking $h = 1/4$

(ii) Simpson's 3/8 rule taking $h = 1/6$

~~(iii)~~

Sol. (i) The values of $f(x) = \frac{1}{1+x^2}$ at $x = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$

are given below:

x	0	1/4	1/2	3/4	1
$f(x)$	1	0.9412	0.8000	0.6400	0.5000
	y_0	y_1	y_2	y_3	y_4

By Simpson's 1/3 rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_4) + 4(y_1+y_3) + 2y_2\right]$$

$$= \frac{1}{12}\left[(1+0.5000) + 4(0.9412+0.6400) + 2(0.8000)\right]$$

$$= 0.7854$$

$$\left(\text{Also } \int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x\Big]_0^1 = \tan^{-1}1 = \frac{\pi}{4} = 0.7854\right)$$

(ii) The value of $f(x) = \frac{1}{1+x^2}$ at $x = 0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1$

are given below:

x	0	1/6	1/3	1/2	2/3	5/6	1
$f(x)$	1	0.9730	0.9000	0.8000	0.6923	0.5902	0.5000

By Simpson's 3/8 rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right]$$

$$= \frac{1}{16}\left[(1+0.5) + 3(0.9730+0.9000+0.6923+0.5902) + 2(0.8000)\right]$$

$$= 0.7854$$

$$\left(\text{Also } \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} = 0.7854\right)$$

Example 3. The speed, v metres per second of a car, t seconds after it starts is shown in following table

t	0	12	24	36	48	60	72	84	96	108	120
v	0	3.60	10.08	18.90	21.60	18.54	10.26	5.40	4.50	5.40	9.00

Using Simpson's rule, find the ~~difference~~ distance travelled by car in 2 minutes.

Using 1/3 rd,
My ans: 1236.96

ROMBERG INTEGRATION

If h be the interval of differencing then trapezoidal rule gives

$$\int_{x_0}^{x_0+h} y\,dx = \frac{h}{2}(y_0 + y_1)$$

Since this method gives only rough value of the integral a modification is needed which is done with the help of function error.

Suppose the error given by the trapezoidal method is Ah^2 when A is const.

Evaluate I by trapezoidal method

$$I = \int_{x_0}^{x_n} y\,dx$$

Let h_1, h_2 be two different sub interval of h so the exact value

$$I = I_1 + \text{error} = I_1 + Ah_1^2 \quad -(1)$$

$$I = I_2 + \text{error} = I_2 + Ah_2^2 \quad -(2)$$

$$I_1 + Ah_1^2 = I_2 + Ah_2^2$$

$$A = \frac{I_2 - I_1}{h_1^2 - h_2^2}$$

$$I = I_1 + \left(\frac{I_2 - I_1}{h_1^2 - h_2^2}\right) h_1^2$$

$$= \frac{I_1 h_1^2 - I_1 h_2^2 + I_2 h_1^2 - I_1 h_1^2}{h_1^2 - h_2^2}$$

$$= \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}$$

$$= \frac{I_2 h_1^2 - I_1 h_2^2}{h_1^2 - h_2^2}$$

which will be a better approximate to I than I_1 or I_2

For particular values of $h_1 = h$ and $h_2 = \frac{h}{2}$

$$I = \frac{I_1 \frac{h^2}{4} - I_2 h^2}{\frac{h^2}{4} - h^2} = \frac{I_1 - 4I_2}{-3}$$

$$= \frac{4I_2 - I_1}{3}$$

$$= I_2 + \frac{1}{3}(I_2 - I_1)$$

This formula gives us much improved value of I. The above process can be improved further by giving values of h_1 and h_2 till we arrive at two values of I which are very near to each other. This process of improvement is known as Romberg method.

Example. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using Romberg's method correct to 4 decimal places. Hence find an approximate value of π.

Sol^n By taking $h = 0.5, 0.25, 0.125$ respectively let us evaluate the given integral using trapezoidal rule

(i) when $h = 0.5$

x	0	0.5	1
y	1	0.8	0.5

$$I = \frac{0.5}{2}\left[(1 + 0.5) + 2(0.8)\right] = 0.775$$

(ii) When $h = 0.25$

x	0	0.25	0.5	0.75	1
y	1	0.9412	0.8	0.64	0.5

$$I = \frac{0.25}{2}\left[(1+0.5) + 2(0.9412 + 0.8 + 0.64)\right]$$

$$= 0.7828$$

(iii) When $h = 0.125$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

$$I = \frac{0.125}{2}\left[(1+0.5) + 2(0.9846 + 0.9412 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5664)\right]$$

$$= 0.78475$$

Now, we have three values of the given definite integral

Let $I_1 = 0.775$

$I_2 = 0.7828$

$I_3 = 0.78475$

Applying the formula $I = I_2 + \frac{1}{3}(I_2 - I_1)$

to the pair I_1, I_2 & I_2, I_3 we get

$$I_1^* = 0.7828 + \frac{1}{3}(0.7828 - 0.775) = 0.7854$$

$$I_2^* = 0.78475 + \frac{1}{3}(0.78475 - 0.7828) = 0.7854$$

Since these two are the same, we conclude that the value of the integral $= 0.7854$

i.e. $\int_0^1 \frac{1}{1+x^2}\,dx = 0.7854$

We know $\int_0^1 \frac{1}{1+x^2} = \frac{\pi}{4}$ $\therefore$ $\frac{\pi}{4} = 0.7854$ $\Rightarrow$ $\pi = 3.1416$

08/02/2021
Monday

Cote's formula

Consider Lagrange's formula, when $x_0, x_1, \ldots, x_n$ are equidistant point and $x_n = x_0 + nh$

$$P(x) = \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)} f(x_1)$$

$$+ \cdots\cdots + \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

$$P(x) = \sum_{k=0}^{n} l_k(x) f(x_k) \quad -(1)$$

where

$$l_k(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$f(x_k)$ is the value of the function at x_k

but $x_k = x_0 + kh$.

Also let $x = x_0 + ph$ —(2)

$dx = h\,dp$ —(3)

$x - x_0 = ph$

$x - x_1 = x_0 + ph - x_0 - h = (p-1)h$

$x - x_2 = (p-2)h$

$\vdots$

$x_k - x_0 = x_0 + kh - x_0 = kh$

$x_k - x_1 = x_0 + kh - x_0 - h = (k-1)h$ ... and so on

$$\therefore l_k(x) = \frac{p(p-1)(p-2)\cdots\cdots(p-k+1)(p-k-1)\cdots\cdots(p-n)}{k(k-1)\cdots\cdots(1)(-1)\cdots\cdots(k-n)}$$

Integrating (1) both side with x

Then $$\int_{x_0}^{x_n} P(x)\,dx = \int_{x_0}^{x_n} \sum_{k=0}^{n} l_k(x)\, f(x_k)\,dx$$

$$= h\int_0^n \sum_{k=0}^{n} l_k\, f_k\, dp$$

$$= nh\left\{\frac{1}{n}\sum_{k=0}^{n} f_k \int_0^n l_k\, dp\right\}$$

$$= nh\sum_{k=0}^{n} {}^nC_k\, f_k$$

where ${}^nC_k = \frac{1}{n}\int_0^n l_k\, dp \qquad (0 \le k \le n)$

are called coles ~~no~~ number.

It can be seen that

$${}^nC_k = {}^nC_{n-k} \quad \text{and} \quad \sum_{k=0}^{n} {}^nC_k = 1$$

CASE 1:

Let $n=1$, then

$$\int_{x_0}^{x_1} P(x)\, dx = h\sum_{k=0}^{1} {}^1C_k\, l_k\, f_k$$

$$= h\left({}^1C_0 f_0 + {}^1C_1 f_1\right) \quad -①$$

- ${}^1C_0 = \frac{1}{1}\int_0^1 l_0\, dp$

$$l_0 = \frac{x - x_1}{x_0 - x_1}$$

$$= \frac{\cancel{x_0} + ph - \cancel{x_0} - h}{\cancel{x_0} - \cancel{x_0} - h} = \frac{p-1}{-1}$$

$$= -\int_0^1 (p-1)\, dp = -\left[\frac{p^2}{2} - p\right]_0^1$$

$$= -\left[\frac{1}{2} - 1\right] = \frac{1}{2}$$

$${}^1C_1 = \frac{1}{1}\int_0^1 l_1\, dp$$

$$l_1 = \frac{x - x_0}{x_1 - x_0}$$

$$= \frac{\cancel{x_0} + ph - \cancel{x_0}}{\cancel{x_0} + h - \cancel{x_0}} = p$$

$${}^1C_1 = \int_0^1 p\, dp = \left[\frac{p^2}{2}\right]_0^1 = \frac{1}{2}$$

From eq ①,

$$\int_{x_0}^{x_1} P(x)\, dx = h\left(\frac{f_0}{2} + \frac{f_1}{2}\right)$$

which is nothing but trapezoidal rule.

CASE 2:

put $n = 2$

$$\int_{x_0}^{x_2} P(x)\, dx = 2h\sum_{k=0}^{2} {}^2C_k\, l_k\, f_k$$

$$= 2h\left[{}^2C_0 f_0 + {}^2C_1 f_1 + {}^2C_2 f_2\right]$$

$${}^2C_0 = \frac{1}{2}\int_0^2 l_0\, dp$$

$$l_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(p-1)(p-2)}{(-1)(-2)}$$

$${}^2C_0 = \frac{1}{2}\int_0^2 \frac{(p-1)(p-2)}{2}\, dp = \frac{1}{4}\int_0^2 (p^2 - 3p + 2)\, dp$$

$$= \frac{1}{4}\left[\frac{p^3}{3} - \frac{3p^2}{2} + 2p\right]_0^2$$

$$= \frac{1}{4}\left[\frac{8}{3} - 6 + 4\right] = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$$

$$^2C_1 = \frac{1}{2}\int_0^2 l_1 \, dp$$

$$l_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{p(p-2)}{1\times(-1)}$$

$$^2C_1 = \frac{1}{2}\int_0^2 \frac{p(p-2)}{-1}\,dp = \frac{4}{6}$$

$$^2C_2 = \frac{1}{2}\int_0^2 \frac{p(p-1)}{2.1} = \frac{1}{6}$$

Hence

$$\int_{x_0}^{x_2} f(x)\,dx = 2h\left[\frac{1}{6}f_0 + \frac{4}{6}f_1 + \frac{1}{6}f_2\right]$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right]$$

which is Simpson's 1/3 formula

CASE III :

Put $n = 1/3$

$$^3C_0 = \frac{1}{3}\int_0^3 \frac{(p-1)(p-2)(p-3)}{(-1)\,(-2)\,(-3)}\,dp = +\frac{1}{8}$$

$$^3C_1 = \frac{1}{3}\int_0^3 \frac{p(p-2)(p-3)}{1\,(-1)\,(-2)}\,dp = \frac{3}{8}$$

$$^3C_2 = \frac{1}{3}\int_0^3 \frac{p(p-2)(p-3)}{1\,(-1)\,(-2)}\,dp = \frac{3}{8}$$

$$^3C_3 = \frac{1}{3}\int_0^3 \frac{p(p-1)(p-3)}{2\times 1\times(-1)}\,dp = \frac{1}{8}$$

Hence

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + f_3\right)$$

which is Simpson's three-eight rule.

After this error in thee three rule.

Properties of Cote's numbers

(i) $^nC_k = {}^nC_{n-k}$ (ii) $\sum_{k=0}^{n} {}^nC_k = 1$