Assignment #2 STA457H1F/2202H1F

Due Friday October 11, 2024

Instructions: Solutions to problems 1–3 are to be submitted on Quercus (PDF files only). You are strongly encouraged to do problems 4 through 6 but these are **not** to be submitted for grading.

1. In Assignment #1, you looked at daily Canadian/U.S. dollar exchange rates ($US/$CAN) from Jan. 2, 1997 to Dec. 29, 2000; these data are given in the file `dollar.txt` on Quercus. Analyze the data on the log-scale:

```
> dollar <- scan("dollar.txt")
> dollar <- ts(log(dollar))
```

Exploratory analysis of this time series indicates that the time series is non-stationary with stationary first differences.

(a) Carry out the Augmented Dickey-Fuller (ADF) test of the null hypothesis that the data have a unit root for various lags. (To use the R function `adf.test`, you must load the R package `tseries`; `adf.test` can then be used as follows:

```
> library(tseries)
> adf.test(x,k=5)
```

where `x` is a vector or time series.) Do the results of the ADF test indicate that the time series is non-stationary?

(b) Carry out Ljung-Box and Bartlett tests on the first differences and absolute first differences of the data. Based on the results of these tests is a white noise model valid for either of these time series?

An R function for the Bartlett test is available on Quercus in the file `bartlett.txt`; it can be loaded into R as follows:

```
> source("bartlett.txt")
```

A cumulative periodogram plot can be obtained by `bartlett(x,plot=T)` where `x` is the name of the time series.

(c) Fit AR(1) and ARMA(1,1) models to the absolute first differences. Which of these two models is better?

2. Hourly data on tide heights at Sooke Basin (B.C.) are given in the file `tides.txt` on Quercus. Read the data into R as follows:

```
> tides <- scan("tides.txt")
> tides <- ts(tides,frequency=24)
```

The argument `frequency=24` indicates that we are making 24 (i.e. hourly) measurements per day using "day" as the unit of time.

As mentioned in lecture, this time series is driven by a number of periodic forces. Thus we would expect the underlying spectral density function of the process to have peaks at the appropriate frequency.

A simple, if somewhat crude, method of estimating the spectral density function is to fit an AR model to the time series and then estimate the spectral density function from the estimated AR model (where, for example, we select the model order using AIC).

(a) Fit an AR model to the time series using AIC to choose the model order; you can use either `ar.yw` or `ar.burg`. The maximum order can be defined by using the argument `order.max`. Then use the R function `spec.ar` to plot the estimated spectral density function.

(b) Do the residuals from your AR model look like white noise? Carry out both graphical and formal white noise tests. Note that if an AR(p) model is fit to the data then the first p residuals will be undefined. If `r` is the output of `ar.yw` or `ar.burg` then the residuals can be defined as follows:

```
> resid <- r$resid[(p+1):length(tides)]
```

If the residuals are not well-modeled by a white noise process, how might this impact the validity of the spectral density function estimate in part (a)? (Hint: Compare the variance of the residuals from the AR model to the variance of the time series itself.)

3. Suppose that $\{X_t\}$ is a stationary stochastic process with spectral density function $f_X(\omega)$ and define $\{Y_t\}$ by

$$Y_t = \alpha \sum_{u=0}^{\infty} (1-\alpha)^u X_{t-u}$$

where $0 < \alpha < 1$.

(a) Show that $\{Y_t\}$ can be defined recursively by

$$Y_t = \alpha X_t + (1-\alpha) Y_{t-1}.$$

(Note that this is exponential smoothing.)

(b) Define $f_Y(\omega) = |\Psi(\omega)|^2 f_X(\omega)$ be the spectral density function of $\{Y_t\}$. Evaluate $|\Psi(\omega)|^2$ and determine for which frequencies $|\Psi(\omega)|^2$ is largest and smallest. What is the effect of varying α?

4. (a) Suppose that $\{X_t\}$ be an MA(q) process

$$X_t = \mu + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \cdots + \beta_q \varepsilon_{t-q}$$

and define $\bar{X}_n$ to be the sample mean of $X_1, \cdots, X_n$. Find the limit of $n\text{Var}(\bar{X}_n)$. For what values of $\beta_1, \cdots, \beta_q$ is this limit 0? What can you say about the MA(q) process in this case?

(b) Suppose that $\{X_t\}$ is a stationary and invertible process, written in the AR(∞) form

$$X_t = \mu + \sum_{s=1}^{\infty} b_s (X_{t-s} - \mu) + \varepsilon_t.$$

Give an expression for the limit of $n\text{Var}(\bar{X}_n)$. Can this limit ever equal 0?

(In (a) and (b), assume that the variance of the white noise process $\{\varepsilon_t\}$ is σ^2.)

5. Define $\{X_t\}$ and $\{N_t\}$ to be independent stationary processes with spectral density functions $f_X(\omega)$ and $f_N(\omega)$ respectively and define

$$Y_t = X_t + N_t.$$

We want to reconstruct $\{X_t\}$ by filtering $\{Y_t\}$; more precisely, define

$$\widehat{X}_t = \sum_{u=-\infty}^{\infty} c_u Y_{t-u}.$$

and suppose that $\widehat{X}_t$ is chosen to minimize

$$E[(X_t - \widehat{X}_t)^2] + \lambda E[(\nabla^d \widehat{X}_t)^2]$$

for some integer $d \geq 1$ and "tuning" parameter $\lambda > 0$. (Note that this is a sort of hybrid between the Wiener and Hodrick-Prescott filters.)

(a) Use the fact that

$$E[(\nabla^d \widehat{X}_t)^2] = \int_0^1 4^d \sin^{2d}(\pi\omega)|\Gamma(\omega)|^2 (f_X(\omega) + f_N(\omega)) d\omega.)$$

to show that the transfer function for the optimal filter is

$$\Gamma(\omega) = \frac{f_X(\omega)}{(f_X(\omega) + f_N(\omega))(1 + 4^d \lambda \sin^{2d}(\pi\omega))}.$$

(b) Suppose that $d = 2$ and $\lambda = 1$. If $\{X_t\}$ is an AR(1) process

$$X_t = 0.9 X_{t-1} + \varepsilon_t$$

with $E(\varepsilon_t) = 0$, $\text{Var}(\varepsilon_t) = 1$ and $\{N_t\}$ is a white noise process with $E(N_t) = 0$ and $\text{Var}(N_t) = 1$, find an approximation to the optimal filter of part (a). (Hint: consider a finite length filter.)

6. Suppose that $\{X_t\}$ is the MA(2) process

$$X_t = \varepsilon_t + 2\beta\varepsilon_{t-1} + \beta^2\varepsilon_{t-2}$$

where $\{\varepsilon_t\}$ is a zero mean white noise process.

(a) For what values of β is $\{X_t\}$ invertible? For what value of β does $\{X_t\}$ have an MA unit root?

(b) For the values of β found in part (a), give the invertible representation of $\{X_t\}$.