# Assignment #4 STA 457H1S/2202H1S

Due Monday December 2, 2024

**Instructions:** Solutions to problems 1 and 2 are to be submitted on Quercus (PDF files only). You are strongly encouraged to do problems 3 and 4 but these are **not** to be submitted for grading.

1. Daily stock prices (adjusted for stock splits and dividends) for Barrick Gold (from January 12, 1995 to November 14, 2016) are given in the filw `barrick.txt` on Quercus; the data are already transformed by taking logs and you should analyze them on this scale.

(a) Fit ARIMA(0,1,1) and ARIMA(0,1,2) models to the data. Do these models seem to fit the data adequately? Which model do you prefer and why? (You may want to do an ADF test to see if the time series needs to be differenced to make it stationary but the non-stationary should be quite clear.)

(b) Using the residuals from your preferred model from part (a), fit ARCH($m$) models for $m = 1, 2, 3, 4, 5$. Which model seems to be the best?

(c) Repeat part (b), using GARCH(1, $s$) models for $s = 1, 2, 3$. Are any of these models an improvement over the best ARCH model from part (b)?

**Note:** The R package `fGarch` contains the function `garchFit`, which can be used to fit both ARCH and GARCH models.

2. Consider a stationary bivariate process $\{(X_t, Y_t)\}$. If $\gamma_{xy}(s) = \mathrm{Cov}(X_t, Y_{t+s})$ is the cross-covariance function then we can define the cross-spectral density function (cross-spectrum) as

$$f_{xy}(\omega) = \sum_{s=-\infty}^{\infty} \gamma_{xy}(s) \exp(2\pi\iota\omega s).$$

The cross-spectrum can be complex-valued.

(a) Show that the cross-covariance function can be recovered from the cross-spectrum as

$$\gamma_{xy}(s) = \int_0^1 f_{xy}(\omega) \exp(-2\pi\iota\omega s)\, d\omega.$$

(b) Suppose we want to predict $Y_t$ using $\{X_t\}$:

$$\widehat{Y}_t = \sum_{s=-\infty}^{\infty} a_s X_{t-s}$$

where the $a_s$'s are chosen to minimize the prediction MSE $E[(Y_t - \widehat{Y}_t)^2]$. Show that the $a_s$'s satisfy the equations

$$\sum_{s=-\infty}^{\infty} a_s \gamma_x(u-s) = \gamma_{xy}(u)$$

for $u = 0, \pm 1, \pm 2, \cdots$ where $\gamma_x(s)$ is the autocovariance function of $\{X_t\}$.

(c) Let $\Gamma(\omega)$ be the transfer function of the "optimal" coefficients $\{a_s\}$ as defined in part (b):

$$\Gamma(\omega) = \sum_{s=-\infty}^{\infty} a_s \exp(2\pi\iota\omega s).$$

If $\{(X_t, Y_t)\}$ has cross-spectrum $f_{xy}(\omega)$ and $\{X_t\}$ has spectral density function $f_x(\omega)$, determine $\Gamma(\omega)$ in terms of $f_{xy}(\omega)$ and $f_x(\omega)$. (Hint: Multiply both sides of the equation in part (b) by $\exp(2\pi\iota\omega u)$ and sum over $u$ from $-\infty$ to $\infty$.)

(d) Suppose that $f_x(\omega) = 4(\omega - 1/2)^2$ and $f_{xy}(\omega) = (\omega - 1/2)^4$. Use the method from Problem 3 of Assignment #1 to obtain the values of $a_s$ for $s = -20, \cdots, 20$.

3. Suppose that $\{\varepsilon_t\}$ is a white noise process whose conditional variances

$$\sigma_t^2 = E(\varepsilon_t^2 | \varepsilon_{t-1}, \varepsilon_{t-2}, \cdots)$$

follow a GARCH(1,1) model:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

where $\alpha_1, \beta_1 \geq 0$ and $\alpha_0 > 0$. Assume that both $\{\sigma_t^2\}$ and $\{\varepsilon_t^2\}$ are stationary processes.

(a) Show that $E(\sigma_t^2) = E(\varepsilon_t^2)$ if both expectations are finite.

(b) Show that $\alpha_1 + \beta_1 < 1$ implies that $E(\sigma_t^2)$ is finite and $E(\sigma_t^2) = \alpha_0/(1 - \alpha_1 - \beta_1)$.

(c) Find conditions on $\alpha_1$ and $\beta_1$ so that $E(\sigma_t^4)$ is finite. (Hint: Note that $\sigma_t^4 = (\alpha_0 + \alpha_1 \varepsilon_t^2 + \beta_1 \sigma_{t-1}^2)^2$ and $E(\sigma_t^4) \geq [E(\sigma_t^2)]^2$.)

4. Consider a VAR($p$) process

$$\boldsymbol{X}_t = \Phi_1 \boldsymbol{X}_{t-1} + \cdots + \Phi_p \boldsymbol{X}_{t-p} + \boldsymbol{\varepsilon}_t$$

where $\Phi_1, \cdots, \Phi_p$ are $k \times k$ matrices. The $k$-variate process $\{\boldsymbol{X}_t\}$ is stationary if

$$\det\left(I - z\Phi_1 - z^2\Phi_2 - \cdots - z^p\Phi_p\right) \neq 0$$

for all $z$ with $|z| \leq 1$.

(a) Suppose that $\{\boldsymbol{a}^T \boldsymbol{X}_t\}$ is a stationary (univariate) AR($p$) process for some $\boldsymbol{a} \neq \boldsymbol{0}$. What can be said about $\boldsymbol{a}$?

(b) Suppose that $k = p = 2$ and define

$$\boldsymbol{X}_t = \Phi_1 \boldsymbol{X}_{t-1} + \Phi_2 X_{t-2} + \boldsymbol{\varepsilon}_t$$

where

$$\begin{aligned} \Phi_1 &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \\ \Phi_2 &= \begin{pmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{pmatrix} \end{aligned}$$

(i) Show that each component of $\boldsymbol{X}_t$ is integrated, that is, non-stationary with stationary first differences.

(ii) Find a vector $\boldsymbol{a}$ such that $\{\boldsymbol{a}^T \boldsymbol{X}_t\}$ is stationary.