Stochastic Modelling of Financial Derivatives

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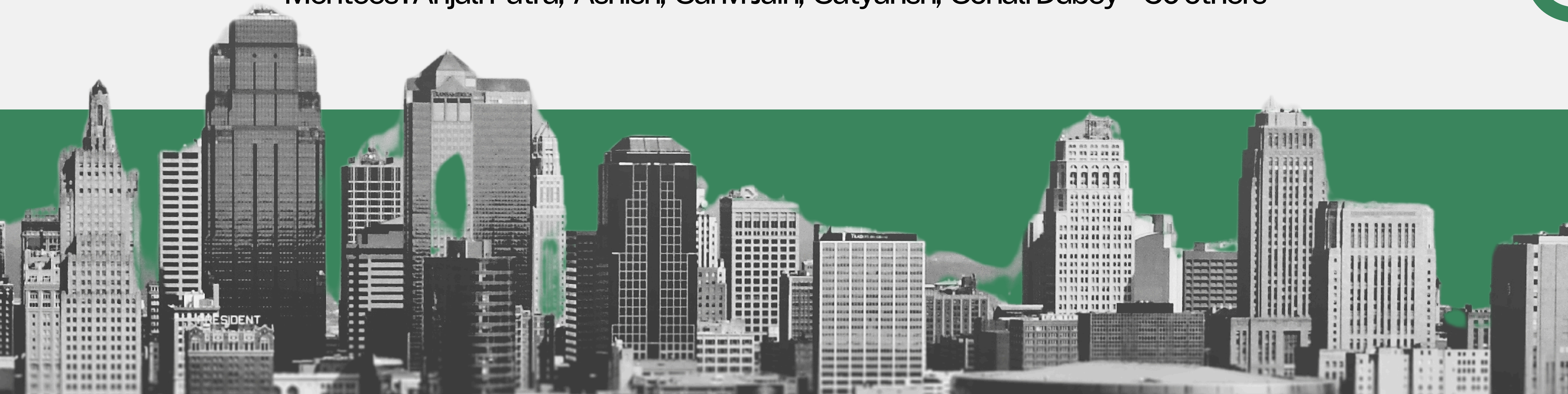


INTRODUCTION

This project provides a comprehensive introduction to financial derivatives and their mathematical modelling. It began with the fundamentals of probability theory, then gradually progressed to more advanced concepts in financial mathematics. Stochastic processes such as Brownian Motion and Geometric Brownian Motion are used to model market behavior.

Further, core models like the Black-Scholes-Merton formula are derived and explored through Monte Carlo Simulations. The project will conclude with the calibration of the Heston stochastic volatility model using real market data, demonstrating the practical application of theoretical models in real-world financial analysis.

WEEK 1:

Topics Covered:

- Probability Theory Refresher:
  - Random variables (discrete & continuous)
  - Expectation, variance, conditional expectation
- Bayes' Rule & Law of Total Probability
- Standard Distributions:
  - Binomial, Poisson, Normal
- Python Fundamentals:
  - Implementing probability problems
  - Simulations and combinatorics with math, random

Concepts learnt applied in assignment:

1. **Derangement Problem (2 Marks)**
   N letters are to be put in N separate envelopes. Assuming an envelope can hold only a single letter. What is the probability that at least one letter is in the correct envelope? Find an approximation of this probability for $N = 50$.

5. **Expectation of Statistic (8 Marks)**
   From N identical lotteries with prizes, $1, 2, ..., N$, $n \leq N$ tickets are drawn with replacement. You are allowed to keep only the maximal prize ticket. Let M = prize money obtained. Find $\mathbb{E}(M)$.

Q1)
Answer:
let D_N be the derangement of N.
Prob. that no envelope has its letter
$= \frac{D_N}{N!}$
∴ Prob. that at least one letter is in correct env. $= 1 - \frac{D_N}{N!}$
as $N \to \infty$ or $N \to$ large number
$\left(1 - \frac{D_N}{N!}\right) \to 1 - \frac{1}{e} \approx 0.632$
∴ for $N = 50$
approx. $P = 0.632$

Q5)
Ans. M = prize money obtained i.e. the maximal prize ticket kept.
P(all tickets are less than $x-1$)
$= \left(\frac{x-1}{N}\right)^n$
P(all tickets are less than x)
$= \left(\frac{x}{N}\right)^n$
∴ $P(M=x) = \left(\frac{x}{N}\right)^n - \left(\frac{x-1}{N}\right)^n$
∴ $E(M) = \sum_{x=1}^{N} x \cdot P(M=x)$
$E(M) = \sum_{x=1}^{N} x \cdot \left[\left(\frac{x}{N}\right)^n - \left(\frac{x-1}{N}\right)^n\right]$

Concepts learnt applied in assignment:

1. **Royal Revenge (20 Marks)**
   Edmond Dantes is on a maze consisting of the integer lattice points, $G = \{(x, y) : 0 \leq x, y \leq n\}$. He starts off at the origin $(0, 0)$ and he wants to reach his fiancée Mercedes up at (n, n). At every second he can take one step to the right moving from the point (u, v) to $(u + 1, v)$ or a step above to $(u, v + 1)$. However, there is a river running between the lines $y = x$ and $y = x+1$, so if he goes above the diagonal he will fall into the river. Let P_n denote the number of paths from $(0, 0) \rightarrow (n, n)$ that do *not* cross the diagonal i.e. the path never goes above the line $y = x$.

   Your mission should you choose to accept it, is to write a program that computes the value of $P_n \bmod (10^9 + 7)$ for $n = 1, 2, \ldots, 100$.

   Hint: It is easy to make a recurrence for the answer. However, if you try to write a recursive function to do this, then you run the risk of exploding your personal computational gizmo. This approach will compute the same answer repeatedly. *You might want to store all the answers that you compute in a matrix and look up the answer again whenever needed.*

   Bonus: What is the asymptotic behaviour for P_n? Can you come up with a closed-form expression for P_n?

```python

# Edmond takes total 2*n steps, n right and n up
# let x = number of right steps till now and y = number of up steps till now
# For our condition of not crossing the diagonal, y <= x
# This is the same as the condition of finding the number of valid n parenthesis
# So, the answer for n, is Catalan number n

mod = 10**9 + 7
findtill = 100
dp = [0] * (findtill + 1)
dp[0] = 1

for n in range(1, findtill + 1):
    numerator = 2 * (2 * n - 1)
    denominator = n + 1

    # Recurrence relation used is C_n = ((2 * (2*n -1))/(n+1)) * C_n-1
    # for caculating modular inverse, fermat's little theorem is used

    inv_denominator = pow(denominator, mod - 2, mod)
    dp[n] = (dp[n-1] * numerator) % mod
    dp[n] = (dp[n] * inv_denominator) % mod

for n in range(1, 101):
    print(f"Number of paths P{n} = {dp[n]}")
```

Other key concepts learnt through the assignment:

- Monty Hall Problem (Showman): Applied Bayes' Rule to evaluate expected value if switching
- Moments & Expectations: Constructed random variables with divergent or bounded moments
- True/False Questions: Reinforced concepts like the chain rule, conditional independence, etc.
- Expectation via CDF: Proved $E[X] = \int_0^{\infty} (1 - F(x))\, dx$

Royal Revenge Maze Problem: Python + Probability

- Learnt an efficient way to calculate the Catalan Numbers
- Learned to break down a combinatorics problem using recursion and use python programming to calculate it using dynamic programming and memoization
- Understood asymptotic behavior via approximation

WEEK 2:

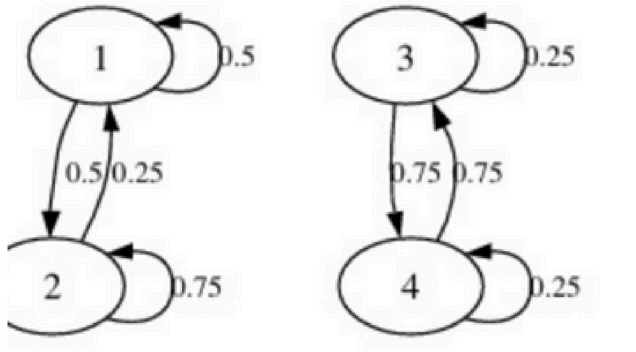
Topics Covered:

- Introduction to Stochastic Processes mainly Markov Chains
- Martingales and fair games
- Random scaled walk, limiting case
- Types of derivatives: forwards, futures, European/American options, Asian options, passport options

Concepts learnt applied in assignment:

. **Two-State Loop (10 Marks)**
Consider the Markov chain shown below, with state space $\{1, 2, 3, 4\}$, where the labels next to arrows indicate the probabilities of those transitions.

(a) Write down the transition matrix Q for this chain.
(b) Which states (if any) are recurrent? Which states (if any) are transient?
(c) Find two different stationary distributions for the chain.



Two - State Loop

State space - $\{1, 2, 3, 4\}$

a) Transition Matrix

$$Q = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0.5 & 0.5 & 0 & 0 \\ 2 & 0.25 & 0.75 & 0 & 0 \\ 3 & 0 & 0 & 0.25 & 0.75 \\ 4 & 0 & 0 & 0.75 & 0.25 \end{array}$$

The chain divides into two closed communicating classes:

Class $\{1,2\}$ and $\{3,4\}$.

Each of these is finite and irreducible, so all 4 states are recurrent

b) No transient states

c) A stationary distribution of a Markov chain is a probability distribution that remains unchanged in the Markov chain as time progresses.

It is represented as a row vector π, whose entries are probabilities summing to 1, and given Q, it satisfies $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$

$\pi = \pi P$ $\quad \sum_i \pi_i = 1 \quad \pi_i \geq 0$

$\pi = \pi Q$

∵ The given chain is irreducible, there is not a unique π, but a one-parameter family.

Within each class, we can find invariant measures.

Class $\{1, 2\}$:

$\pi_1 = 0.5\pi_1 + 0.25\pi_2$
$\pi_2 = 0.5\pi_1 + 0.75\pi_2$

Solving these two eqns,

$0.5\pi_1 = 0.25\pi_2 \Rightarrow \pi_2 = 2\pi_1$

One unnormalized solution can be $\pi_1 = 1$ & $\pi_2 = 2$.

Class $\{3, 4\}$:

$\pi_3 = 0.25\pi_3 + 0.75\pi_4$
$\pi_4 = 0.75\pi_3 + 0.25\pi_4$

Solving,

$0.75\pi_3 = 0.75\pi_4 \Rightarrow \pi_3 = \pi_4$

$\pi_3 = \pi_4 = 1$ is one unnormalized solⁿ.

Any convex combination of these two "subchain" distributions is global stationary distⁿ.

$\omega \in [0,1]$ and $\pi_1 = \frac{\omega}{3}$, $\pi_2 = \frac{2\omega}{3}$, $\pi_3 = \frac{1-\omega}{2}$, $\pi_4 = \frac{1-\omega}{2}$

$\sum \pi_i = 1$ and $\pi Q = \pi$.

Concepts learnt applied in assignment:

Put-Call Parity: Risk-Free Rate (10 marks)
The prices of European call and put options on a non-dividend-paying stock with 12 months to maturity, a strike price of \$120, and an expiration date in 12 months are \$20 and \$5, respectively. The current stock price is \$130. What is the implied risk-free rate?

Put-Call Parity : Risk-Free Rate

→ Using the European put-call parity for a non-dividend stock:

$$C - P = S - Ke^{-rT}$$

T: 12 months , strike Price (K) = 120
C = \$20   P = \$5
Current Stock Price (S) = \$130

Plugging in the values,

$\Rightarrow 20 - 5 = 130 - 120 \times e^{-r.1}$

$\Rightarrow 15 = 130 - 120 \times e^{-r}$

$\Rightarrow e^{-r} = \frac{115}{120}$, 0.9583

Taking log on both sides,

$\Rightarrow -r = \ln(0.9583)$

$\Rightarrow -r = -0.0429$

$\Rightarrow r = 0.043$

Other key concepts learnt :

Futures Contracts & Mark-to-Market:

- Exchange-Traded vs. OTC: Futures are standardized and daily-settled; forwards settle only at maturity.
- Daily P/L: Each day i, gain/loss = $f_i - f_{i-1}$
- Cumulative P/L (Telescoping Sum):

$$\sum_{i=1}^{n}(f_i - f_{i-1}) = f_n - f_0$$

which under $f_0 = S_0e^{rT}$ and $f_n = S_T$ gives $S_T - S_0e^{rT}$

- Economic Equivalence: In deterministic-rate settings, total futures P/L = forward P/L despite different settlement timing.
- **No-Arbitrage:** Prices adjust so you can't lock in a guaranteed, risk-free profit.
- **Risk-Neutral Valuation**: Assume all assets grow at the risk-free rate, then **price = discounted average** of future payoffs.
- **Early-Exercise Insights:** (For American options) when — and when not — it's optimal to exercise before expiry.

WEEK-3

Topics Covered

Random Walk

- Symmetric Random Walk
- Increments Property
- Martingales Property
- Limiting Distribution

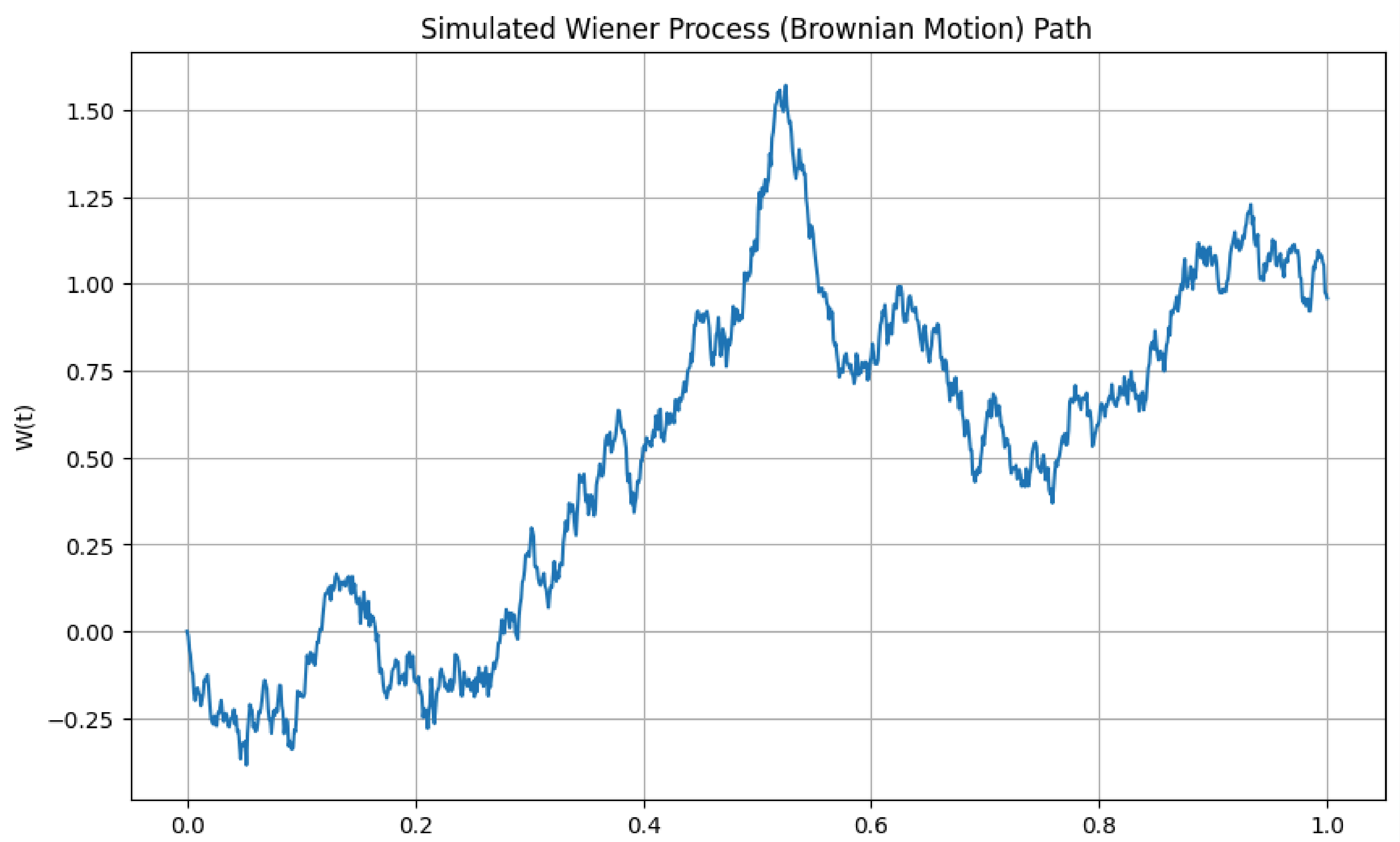
Brownian Motion

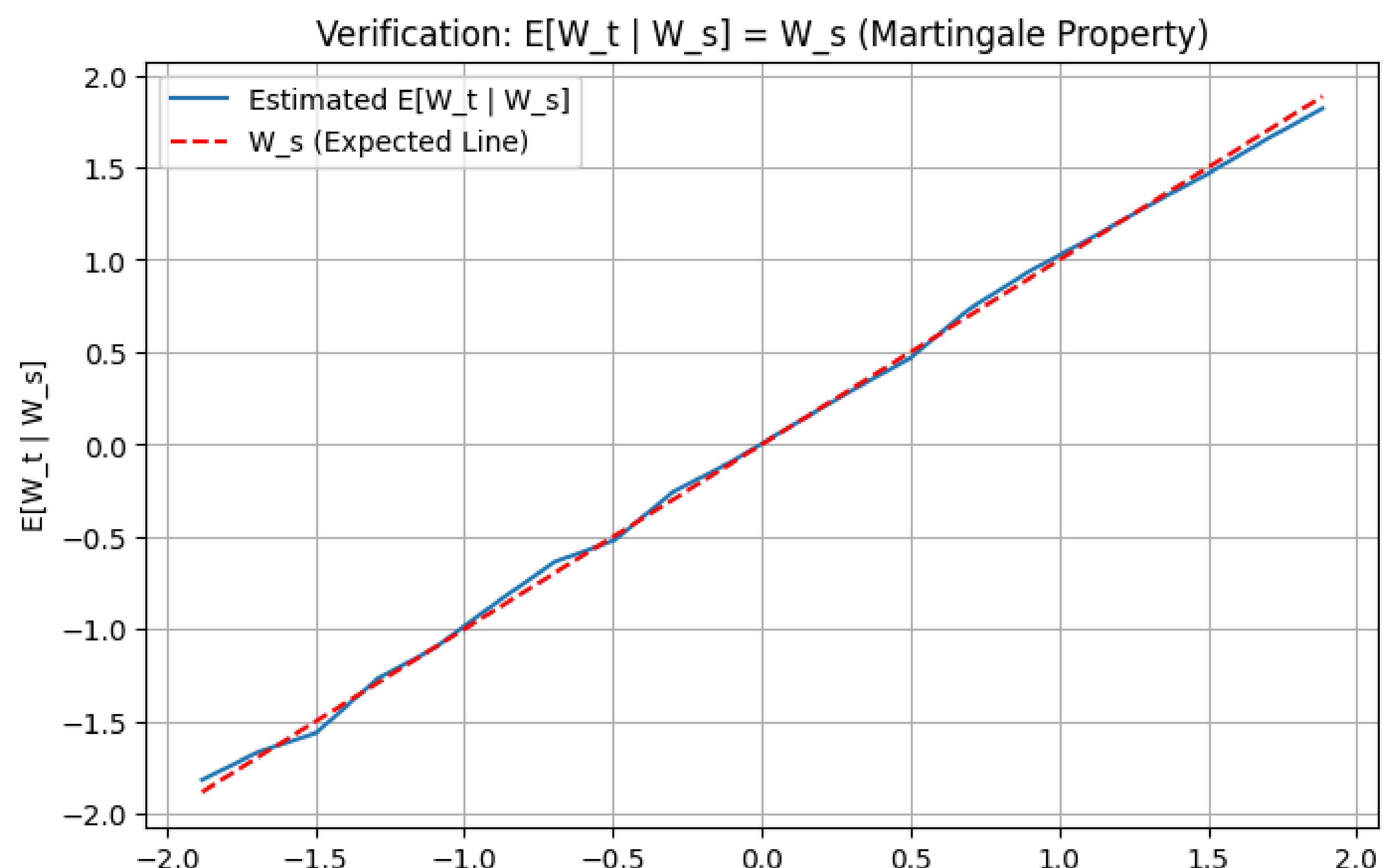
- Distribution of Brownian Motion
- Martingales Property
- Volatality for geometric Brownian Motion
- Markov Property

Programming Applications

Simulation of a single path of a one-dimensional Wiener process over the interval [0,T].

Show that for any $t \geq 0$, $E[W_t|F_s] = W_s$ for $0 \leq s \leq t$. Conclude that Brownian motion is a martingale.





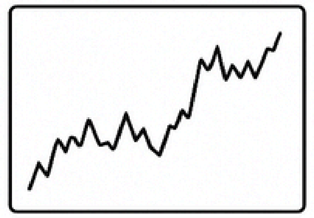
Other Problems

Let $0 \leq s < t$. Show that $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$, and that increments over non-overlapping intervals are independent.

Show that for standard Brownian motion, $E[W_sW_t] = \min(s, t)$ for $s, t \geq 0$.

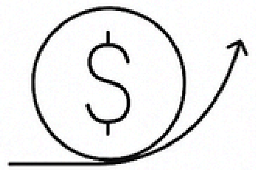
Motivation

Classical calculus breaks down for Brownian motion paths (nowhere differentiable)



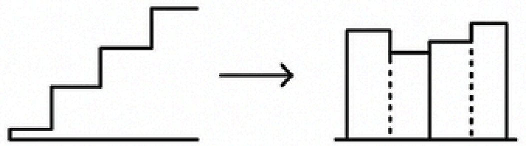
Idea

Define an integral for random processes to model asset prices in continuous time



Construction

Start with simple stepwise processes (piecewise constant holdings)	Integrate with respect to increments of Brownian motion	Sum of stochastic 'gains' gives the Itō integral



$\rightarrow I_t = \int_0^0 H_s\, dW_s$

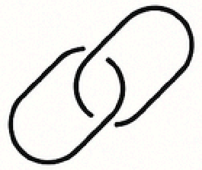
MARTINGALE PROPERTY AND ITÔ ISOMETRY

Martingale Property $\bar{E}\,[I_t \mid \bar{F}_s] = I_s$ for $s < t$

No free profit via predictable strategies

Itô Isometry $\bar{E}\left(\left[\int_0 [H_s dW_s\right)^{T}\right] = \bar{\bar{E}}$

$= \int_0^T H_s^2\, ds\,]$



Links variance of integral to squared integrand

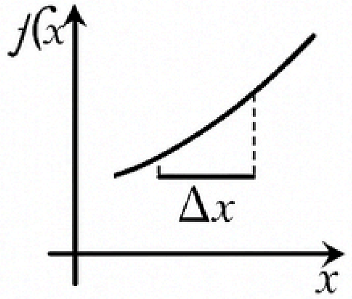
WEEK 4:
Ito's Integral, Black-Scholes Model, Monte Carlo Simulations

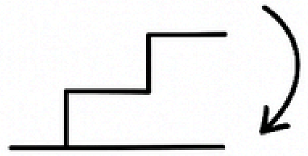
EXTENDING TAYLOR'S THEOREM TO STOCHASTIC CALCULUS

$$df(X_t) = f'(X_t)\,dX_t \frac{1}{2} f''(X_t)\; d\langle X \rangle_t$$

Example:

$$df(W_t^{\,2}) = 2W_t\,dW_t \,{}^{1}_{+} f''(x)\,dt$$





$f'(x)\Delta x + \frac{1}{\Delta x\, x^2} f(\prime)$ quadratic variation

BLACK-SCHOLES-MERTON (BSM) MODEL AND MONTE CARLO SIMULATIONS

BSM Formula

$C = S_0 N(d_1 - Ke^{-rT}N(d_2)$

Key Features

- Analytical solution
- Assumes lognormal distribution of asset prices
- Uses closed-form expressions

Use Cases

- Option pricing
- Risk management
- Valuation of complex derivatives

Monte Carlo Simulations

- Simulate multiple asset price paths
- Calculate payoffs for each path
- Average payoffs
- Discount to present value

Comparison

	BSM	Monte Carlo
Approach:	Analytical	Numerical
Flexibility:	Limited	High

Thank you !