FIRST TALK

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1. Locally Profinite Groups

Definition 1. *A topological group G is called locally profinite if it is Hausdorff, locally compact, totally disconnected and has a basis of neighborhoods of the identity consisting of open compact subgroups.*

Equivalently, A locally profinite group is a topological group G such that every open neighbourhood of the identity in G contains a compact open subgroup of G. In fact, we can write

$$G = \operatorname{proj}\lim_{K} G/K$$

where the limit runs over all open normal subgroups K of G.

Definition 2. *A local field F is a field with a non-trivial absolute value $|\cdot|$, that is locally compact under the topology induced by $|\cdot|$ and whose topology is not discrete.*

If $(F,|\cdot|)$ is a valued field, then we say that it is archimedean if the absolute value $|\cdot|$ is archimedean, i.e., if for every $x, y \in F$ with $|x| < |y|$, there exists an integer n such that $|nx| > |y|$. Otherwise, we say that $(F,|\cdot|)$ is non-archimedean.

We say that F is a non-archimedean local field if the topology on F is induced by a non-archimedean absolute value. Examples of non-archimedean local fields are finite extensions of $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$. Examples of archimedean local fields are $\mathbb{R}$ and $\mathbb{C}$.

Let F be a non-Archimedean local field. Thus F is the field of fractions of a discrete valuation ring $\mathcal{O}$. Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$ and $k = \mathcal{O}/\mathfrak{p}$ the residue class field. We will always assume that k is finite, and we will generally denote the cardinality $|k|$ by q.

Let π be a prime element of F, that is, an element satisfying

$$\pi\mathcal{O} = \mathfrak{p},$$

where $\mathfrak{p}$ is the maximal ideal of the ring of integers $\mathcal{O}$ of F. Every element $x \in F^\times$ admits a unique factorization

$$x = u\pi^n,$$

with $u \in \mathcal{O}^\times = U_F$ a unit and $n \in \mathbb{Z}$. We write $n = \nu_F(x)$ for the normalized valuation of x.

The field F carries the absolute value

$$\|x\| = q^{-n} = q^{-\nu_F(x)},$$

where $q = |\mathcal{O}/\mathfrak{p}|$. We set $\|0\| = 0$, so that $\|x\| > 0$ for $x \neq 0$. This absolute value defines a metric under which F is complete, making F a topological field.

For each $n \in \mathbb{Z}$, the fractional ideal

$$\mathfrak{p}^n = \pi^n\mathcal{O} = \{x \in F : \|x\| \leq q^{-n}\}$$

Date: June 25, 2025.

is an open additive subgroup of F, and the collection $\{\mathfrak{p}^n\}_{n\in\mathbb{Z}}$ forms a fundamental system of neighborhoods of 0.

Because F is complete and the residue field $k = \mathcal{O}/\mathfrak{p}$ is finite, the canonical map

$$\mathcal{O} \longrightarrow \varprojlim_n \mathcal{O}/\mathfrak{p}^n$$

is a topological isomorphism. Each quotient $\mathcal{O}/\mathfrak{p}^n$ is finite, hence the inverse limit is compact. Moreover, each fractional ideal $\mathfrak{p}^n$ is topologically isomorphic to $\mathcal{O}$ and is therefore compact. It follows that the additive group $(F, +)$ is *locally profinite*, and F is the union of its compact open subgroups.

The same reasoning shows that the multiplicative group $F^\times$ is locally profinite. Standard arguments then imply that for every $n \geq 1$, the groups

$$F^n, \quad M_n(F), \quad \mathrm{GL}_n(F), \quad \mathrm{SL}_n(F), \quad \mathrm{SO}_n(F), \quad \mathrm{GO}_n(F), \quad \mathrm{Mp}_n(F)$$

are all locally profinite as well.

2. Characters of Locally Profinite Groups

Definition 3. *Let G be a locally profinite group. A character of G is a continuous homomorphism $\chi : G \to \mathbb{C}^\times$. We say that a character is unitary if its image lies in the unit circle $S^1 \subset \mathbb{C}^\times$.*

For a local field F we write $\hat{F}$ for the group of unitary characters of the additive group $(F, +)$. [1]

References

1. Colin J. Bushnell and Guy Henniart, *The local langlands conjecture for gl(2)*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006.