FALL 2025: EPSILON FACTORS READING GROUP

ORGANIZERS: MLADEN DIMITROV, SANYAM GUPTA

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1. Introduction

First and foremost, we offer a general introduction to the notion of *ϵ-factors*, hoping that this will also motivate the theme of the reading group. L-functions and their associated ϵ-factors form an essential part of the *Langlands program*, a vast network of conjectures and theorems that has profoundly influenced modern number theory and representation theory.

The Langlands program, first articulated by Robert P. Langlands in his celebrated 1967 letter to André Weil, can be viewed as a far-reaching generalization of *class field theory*. Classical class field theory gives a complete description of the abelian extensions of a number field F in terms of the multiplicative group $F^\times$ and its idèle class group. The Langlands program extends this vision to the *non-abelian setting*, predicting deep connections between

- representations of global Galois groups (or, more precisely, of the global Weil group), and
- automorphic representations of reductive algebraic groups over global fields.

In the local case, the correspondence matches smooth irreducible representations of $\mathrm{GL}_n(F)$ with n-dimensional representations of the local Weil group $\mathcal{W}_F$.

1.1. Class Field Theory and the Weil Group.

Let us briefly recall the framework of local class field theory, which serves as the foundation of the Langlands program.

Local Setup. Let F be a non-Archimedean local field with residue field k of cardinality $q = p^n$ (for some $n \geq 1$) and prime p. Thus F is either a finite extension of $\mathbb{Q}_p$ or the field of formal Laurent series $\mathbb{F}_{p^r}((t))$. Let $\Omega_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$ denote the absolute Galois group, where F^{sep} is a separable closure of F. Let F^{unr} be the maximal unramified extension of F inside F^{sep}. Galois theory gives the fundamental short exact sequence:

$$1 \longrightarrow \mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{unr}}) \longrightarrow \Omega_F \longrightarrow \mathrm{Gal}(F^{\mathrm{unr}}/F) \cong \mathrm{Gal}(\overline{k}/k) \longrightarrow 1.$$

The subgroup

$$I_F := \mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{unr}})$$

is called the *inertia subgroup* of Ω_F. The quotient $\mathrm{Gal}(\overline{k}/k)$ is isomorphic to the profinite completion $\widehat{\mathbb{Z}}$, and is topologically generated by the *Frobenius automorphism* $x \mapsto x^q$.

The Weil Group. The *Weil group* $\mathcal{W}_F$ is defined as the preimage of the copy of $\mathbb{Z}$ generated by the Frobenius element inside $\mathrm{Gal}(F^{\mathrm{unr}}/F)$ under the natural surjection

$$\Omega_F \twoheadrightarrow \mathrm{Gal}(F^{\mathrm{unr}}/F).$$

We endow $\mathcal{W}_F$ with the topology induced by the discrete topology on $\mathbb{Z}$ and the profinite topology on I_F.

Local Reciprocity. Local class field theory provides a unique injective homomorphism with dense image:

$$\mathrm{rec}_F : F^\times \longrightarrow \Omega_F^{\mathrm{ab}} \tag{1}$$

such that for every uniformizer $\pi \in F^\times$, the restriction of $\mathrm{rec}_F(\pi)$ to F^{unr} is the *geometric Frobenius* (i.e. it induces the inverse of Frobenius on the residue field). Moreover, the image of rec_F coincides with the abelianization of the Weil group:

$$\mathrm{rec}_F : F^\times \xrightarrow{\sim} \mathcal{W}_F^{\mathrm{ab}}.$$

This immediately yields the fundamental bijection

$$\{\text{continuous characters } \xi : \mathcal{W}_F \to \mathbb{C}^\times\} \longleftrightarrow \{\text{continuous characters } \chi : F^\times \to \mathbb{C}^\times\}. \tag{2}$$

Global Reciprocity. For a global field K (number field or global function field) there is a global reciprocity map

$$\mathrm{rec}_K : \mathbb{A}_K^\times / K^\times \longrightarrow \Omega_K, \tag{3}$$

where $\mathbb{A}_K^\times$ denotes the idèle group of K and $\Omega_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ its absolute Galois group. This map is compatible with the local maps (1). One can also define a global Weil group W_K, allowing for a global analogue of the bijection (2).

1.2. **Towards the Langlands Correspondence.** Note that $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$, $F^\times = \mathrm{GL}_1(F)$, and $\mathbb{A}_K^\times = \mathrm{GL}_1(\mathbb{A}_K)$. Langlands' idea was to generalize the correspondence (2) to GL_n for $n \geq 1$, and then further to arbitrary reductive groups. Such correspondences are conjectured to satisfy several key properties:

- **Compatibility of L- and ϵ-factors:** To every continuous character $\chi : F^\times \to \mathbb{C}^\times$ one attaches local factors
$$L(s, \chi), \qquad \epsilon(s, \chi, \psi),$$
as established in Tate's thesis for GL_1. Via local class field theory, χ corresponds to a 1-dimensional Weil representation
$$\rho_\chi : \mathcal{W}_F \longrightarrow \mathbb{C}^\times,$$
and representation theory of $\mathcal{W}_F$ defines local factors
$$L(s, \rho_\chi), \qquad \epsilon(s, \rho_\chi, \psi).$$
Local class field theory asserts that
$$L(s, \chi) = L(s, \rho_\chi), \qquad \epsilon(s, \chi, \psi) = \epsilon(s, \rho_\chi, \psi),$$
providing perfect compatibility of local constants in the case $n = 1$. This property is required to hold in general for the local Langlands correspondence for GL_n.
- **Local-global compatibility:** The local correspondences should be compatible with the global correspondence under localization at a place.
- **Functoriality:** The correspondence should be natural with respect to morphisms of reductive groups $G \to H$.

Restricting to the local situation and the case $G = \mathrm{GL}_n$, the two sides of the conjectural correspondence are:

- isomorphism classes of n-dimensional Weil–Deligne representations of $\mathcal{W}_F$ (the Galois side);
- isomorphism classes of irreducible smooth admissible representations of $\mathrm{GL}_n(F)$ (the automorphic side).

1.3. **Historical Development of L- and ϵ-Factors.** The study of local factors has its roots in the work of Riemann and Dedekind on zeta functions of number fields. Tate's thesis (1950) gave a uniform, adelic definition of L-functions for characters of GL_1 and established their analytic continuation and functional equation, introducing the local ϵ-factors in the process.

Building on Tate's work, Langlands introduced local factors for GL_n in the late 1960s, using his theory of Eisenstein series and the global-to-local functional equation. Subsequently, Deligne formulated a general theory of local constants for representations of the Weil group, providing a representation-theoretic foundation and proving that Tate's ϵ-factors coincide with those coming from Galois representations under local class field theory.

Today, the equality of local factors on both the Galois and automorphic sides is one of the cornerstones of the local Langlands correspondence.

1.4. **Aim of This Reading Group.** Having provided the general background and motivation, the first main aim of this reading group is to develop the machinery of L-functions and ϵ-factors attached to finite-dimensional, semisimple, smooth representations of $\mathcal{W}_F$.

2. Structure of the reading group

2.1. **Talk 2 (September 25): Tate's Thesis** GL_1**.**

- Give the definition of Local fields F and additive/multiplicative characters of local fields following [1, §1.1 - 1.4, §1.6-1.8]. Give a crash course on Haar measures on locally profinite groups following [1, §3.1-3.4] including $(F, +)$ and $(F^\times, \cdot)$ as explicit examples.
- Introduce the local zeta integrals for $\mathrm{GL}_1(F)$ and prove the corresponding functional equation as in the theorem in [1, §23.1-23.2]. Cover the rest of the material in section 23 till [1, §23.4].
- Following [1, §23.5-23.7] computation of ϵ-factors leading to Theorem in §23.5 and relation to Gauss sums in [1, §23.6].

2.2. **Talk 3 (October 02): Archimedean Weil-Deligne Representations.** In this talk, we will study the Archimedean components of the Langlands correspondence, focusing on the Weil-Deligne representations associated with real and complex places. We will explore their construction, properties, and the role they play in the local Langlands correspondence.

2.3. **Talk 4 : Representations of the Weil group I.** This talk addresses the Galois side of the Langlands correspondence, namely representations of the Weil group $\mathcal{W}_F$ (F is local non-archimedean). We study their first properties and define the local Artin L-functions associated with them.

- Recall the facts about Galois theory contained in [1, §28.1-28.3].
- Define the Weil group WF and discuss its first properties ([1, §28.4-28.5]).
- Give the first results concerning the $\mathbb{C}$-valued representations of $\mathcal{W}_F$ contained in [1, §28.6-28.7] including the proofs if time permits.
- Recall quickly the statements of local class field theory [1, §29.1]
- Following [1, §29.2-29.4], define the local Artin L-function associated to finite-dimensional, smooth, semisimple representations of $\mathcal{W}_F$ and state the theorem in [1, §29.4] concerning the local constants $\epsilon(\rho, s, \psi)$.

References

1. Colin J. Bushnell and Guy Henniart, *The local langlands conjecture for gl(2)*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006.