Chapter 12

Unbalanced government budget constraint

In this chapter, we build off of the S-period-lived agent model from Chapter 4 and the model with household taxes from Chapter 11 in order to understand how one may model a government that can run deficits for a finite number of periods. In order to understand such a model of the government's budget constraint, we add linear taxes on labor, capital, and corporate income as the source of the government's revenue. The government uses tax revenues and debt to finance spending on a government consumption good and lump sum transfers to households.

12.1 Households

The basic structure of the household's problem from Chapter 4 remains the same, but the addition of taxes and transfers alters the household's budget constraint and thus the necessary conditions describing optimal savings and labor supply.

We utilize a less general and more simple household tax structure in this chapter than in Chapter 11. Let τ^l_t and τ^k_t represent the constant tax rate on labor and capital income, respectively, and let X_t represent government transfers to households of age s in period t, we can write the household's budget constraint as the following.

$$\begin{aligned} c_{s,t} + b_{s+1,t+1} = (1 + (1-\tau^k_t)r_t)b_{s,t} + (1-\tau^l_t)w_t n_{s,t} + X_t \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0 \end{aligned} \tag{12.1}$$

Households choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{s,t+s-1}\}_{s=1}^S$, and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints,

$$\max_{\{c_{s,t+s-1},n_{s,t+s-1}\}_{s=1}^S,\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{s=1}^S \beta^{s-1}u(c_{s,t+s-1},n_{s,t+s-1}) \tag{12.2}$$

$$\begin{aligned} \text{s.t.} \quad c_{s,t} + b_{s+1,t+1} = ... \\ (1 + (1-\tau^k_t)r_t)b_{s,t} + (1-\tau^l_t)w_t n_{s,t} + X_t \end{aligned} \tag{12.1}$$

$$\text{where} \quad u\big(c_{s,t},n_{s,t}\big) = \frac{c_{s,t}^{1-\sigma}-1}{1-\sigma} + \chi^n_s b\left[1 - \left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1}{\upsilon}} \tag{12.3}$$

The set of optimal lifetime choices for an agent born in period t are characterized by the following S static labor supply Euler equations (12.4), the following $S-1$ dynamic savings Euler equations (12.5), and a budget constraint that binds in all S periods (12.1),

$$\begin{aligned} (1-\tau^l_t)w_t u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = -u_2\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S\} \\ \Rightarrow \quad (1-\tau^l_t)w_t\left(c_{s,t}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}} \end{aligned} \tag{12.4}$$

$$\begin{aligned} u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = \beta(1+(1-\tau^k_{t+1})r_{t+1})u_1\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S-1\} \\ \Rightarrow \quad \left(c_{s,t}\right)^{-\sigma} = \beta(1+(1-\tau^k_{t+1})r_{t+1})\left(c_{s+1,t+1}\right)^{-\sigma} \end{aligned} \tag{12.5}$$

$$c_{s,t}+b_{s+1,t+1} = (1+(1-\tau^k_t)r_t)b_{s,t}+\big(1-\tau^l_t\big)w_t n_{s,t}+X_t \quad \text{for} \quad s\in\{1,2,...S\}, \quad b_{1,t},b_{S+1,t+S}=0 \tag{12.1}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{s,t}$.

These $2S-1$ household decisions are perfectly identified if the household knows what prices and government transfers will be over its lifetime $\{w_u, r_u, X_u\}_{u=t}^{t+S-1}$. Let the distribution of capital and household beliefs about the evolution of the distribution of capital be characterized by (12.6) and (12.7).

$$\mathbf{\Gamma}_t \equiv \left\{b_{s,t}\right\}_{s=2}^{S} \quad \forall t \tag{12.6}$$

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u\geq 1 \tag{12.7}$$

12.2 Firms

Firms are characterized similarly to Section 2.2, with the firm's aggregate capital decision K_t governed by first order condition (12.9) and its aggregate labor decision L_t governed by first order condition (12.10). However, the addition of the corporate income tax alters the firm's profit maximization problem slightly.

The firm seeks to maximize after-tax profits and thus solves,

$$\max_{K_t,L_t}\,(1-\tau^c_t)\left(Y_t - w_t L_t\right) - (r_t+\delta)K_t + \tau^c_t\delta K_t \tag{12.8}$$

Note that the corporate income tax is levied on accounting profits. Thus wage expenses and depreciation expenses are deductible, but payments to capital are not.

In the presence of the corporate income tax, the two first order conditions that characterize firm optimization are the following.

$$r_t = (1-\tau^c_t)\left(\alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta\right) \tag{12.9}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{12.10}$$

12.3 Government

The government takes in tax revenues and uses those revenues and borrowing to finance government purchases, G_t, and total transfers, SX_t. In this model, we'll assume that these transfers are distributed lump-sum to all agents. We let D_t denote the stock of government debt at time t and R_t denote total government tax revenue. Thus we write the government budget constraint as:

$$D_{t+1} + R_t = (1 + r_t)D_t + G_t + SX_t \tag{12.11}$$

Revenues are given by:

$$R_t = \underbrace{\tau_t^c\left(Y_t - w_t L_t\right) - \tau_t^c \delta K_t}_{\text{Corporate income tax revenue}} + \underbrace{\sum_{s=1}^{S} \tau_t^l w_t n_{s,t} + \sum_{s=2}^{S} \tau_t^k r_t b_{s,t}}_{\text{Individual income tax revenue}} \tag{12.12}$$

12.3.1 Budget Closure Rule

We've specified how government debt evolves (Equation 12.11) and how revenues are determined. We need to also specify how government purchase and transfers are determined. Before we do that, we point out that that government's budget must balance over the infinite horizon. In this simple model, if government debt grows in the steady-state, then at some point, payments of interest would exceed total economic output, which is not feasible. In a richer model, with economic growth, government debt can grow, but it cannot outpace the growth of total output for the same reason noted in the previous sentence. What we do to impose this fiscal condition is to alter the path of government spending in order to hit a target debt to GDP ratio in the steady-state.

We assume that transfers are a constant fraction of GDP in all periods:

$$SX_t = \alpha_X Y_t \tag{12.13}$$

Given this path for transfers and revenues, G_t adjusts to stabilize the debt to GDP ratio. Let α_D be the target debt to GDP ratio in the steady state. In the initial periods, we assume that the ratio of government spending to GDP remains constraint, $G_t = \alpha_G Y_t$. At some future period, t_{G1}, government spending begins to adjust to move towards this target debt to GDP ratio. At period t_{G2}, government spending is adjusted to hit the target debt to GDP ratio, if it has not already been reached. Letting ρ_G be the parameter that describes how quickly the convergences to the steady state debt to GDP ratio takes place and, we right the law of motion for government spending as:

$$G_t = \begin{cases} \alpha_G Y_t & \text{if} \quad t < t_{G1} \\ [\rho_G \alpha_D Y_t + (1-\rho_G) D_t] - (1+r_t) D_t - SX_t + R_t & \text{if} \quad t_{G1} \leq t < t_{G2} \\ \alpha_D Y_t - (1+r_t) D_t - SX_t + R_t & \text{if} \quad t \geq t_{G2} \end{cases} \tag{12.14}$$

This law of motion for the fiscal variables, together with the amount of debt in the initial period of the time path, will allow us to solve for the values government spending and debt at each period. We set the initial debt level to match the debt to GDP ratio in the economy in the year we want the model to being. Let α_{D0} represent the debt to GDP ratio in the initial period. Thus $D_1 = \alpha_{D0} Y_1$.

A few notes on this closure rule are in order. First, we chose to adjust government spending because of the simplicity of doing so. Since government spending does not enter into the household's utility function, it's level does not affect the solution of the household problem. This simplifies the model solution significantly. That said, one could chose to adjust taxes or transfers to close the budget (or a combination of all of the government fiscal policy levers). Second, since government spending is doing all of the lifting to hit the target debt to GDP ratio, it is possible that government spending is forced to be less than zero to make this happen. This would be the case if tax revenues bring in less than is needed to financed transfers and interest payments on the national debt. None of the equations we've specified above preclude that result, but it does raise conceptual difficulties. Namely, what does it mean for government spending to be negative? Is the government selling off pubic assets? We caution those using this budget closure rule to consider carefully how the budget is closed in the long run given their parameterization. We'll also note that such difficulties present themselves across all budget closure rules when analyzing tax or spending proposals that induce structural budget deficits.

12.4 Market Clearing

Four markets must clear in this model: the labor market, the capital market, the bond market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t = \sum_{s=1}^{S} n_{s,t} \quad \forall t \tag{12.15}$$

$$K_t + D_t = \sum_{i=2}^{S} b_{s,t} \quad \forall t \tag{12.16}$$

$$\begin{split} &Y_t = C_t + I_t + G_t \quad \forall t \\ &\qquad \text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \end{split} \tag{12.17}$$

The goods market clearing equation (12.17) is redundant by Walras' Law.

In this model households hold two different assets—capital and government debt. Both are risk free and thus will yield the same rate of return in equilibrium.[1] Given this, we do not differentiate between the household's holdings of capital and government debt. Rather, $b_{s,t}$ represents total assets held by a household of age s at time t and is potentially a mix of capital and government debt.

12.5 Equilibrium

An equilibrium is found when:

i. Households optimize according to equations (12.4) and (12.5).

ii. Firms optimize according to (12.9) and (12.10).

[1] The assumption of equivalent return on capital and debt could be relaxed, and a wedge could be placed between the return on government debt and private capital.

iii. Government debt evolves according to (12.11) and (12.14).

iv. Markets clear according to (12.15) and (12.16).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

We can characterize the equilibrium solution by solving for the amount of government debt, D_t^d, as a function of the household's decisions and then substituting the market clearing conditions (12.15) and (12.16) into the firm's optimal conditions (12.9) and (12.10) to solve for the equilibrium wage and interest rate as functions of the distribution of capital.

To find debt, we use the government budget constraint as characterized in (12.11) along with the exogenous laws of motion describing government spending and transfers.

$$
\begin{aligned}
D_{t+1} &= (1+r_t)D_t + G_t + X_t - R_t \\
&= (1+r_t)D_t + \alpha_G Y_t + \alpha_X Y_t - \tau_t^c\left(Y_t - w_t L_t\right) - \tau_t^c \delta K_t^d - \sum_{s=1}^{S} \tau_t^l w_t n_{s,t} - \sum_{s=1}^{S} \tau_t^k r_t b_{s,t} \\
&= (1+r_t)D_t + (\alpha_G + \alpha_X - \tau_t^c)\left[A_t\left(\sum_{s=1}^{S} b_{s,t} - D_t\right)^{\alpha}\left(\sum_{s=1}^{S} n_{s,t}\right)^{1-\alpha}\right] + \ldots \\
&\qquad \tau_t^c w_t \sum_{s=1}^{S} n_{s,t} - \tau_t^c \delta\left(\sum_{s=1}^{S} b_{s,t} - D_t\right) - \sum_{s=1}^{S} \tau_t^l w_t n_{s,t} - \sum_{s=1}^{S} \tau_t^k r_t b_{s,t}
\end{aligned}
\tag{12.18}
$$

Note that the initial debt to GDP ratio is an exogenous initial condition and the steady-state target debt to GDP ratio is also exogenous. Thus the path for debt is entirely pinned down by the distribution of savings and labor supply, $D_t(\mathbf{\Gamma}_t)$.

We can then use the firm's first order conditions together with the market clearing conditions to show that the equilibrium interest rate and wage rates are functions of the distributions of savings and labor supply.

$$w_t(\mathbf{\Gamma}_t): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^S b_{s,t} - D_t(\Gamma_t)}{\sum_{s=1}^S n_{s,t}}\right)^{\alpha} \quad \forall t \tag{12.19}$$

$$r_t(\mathbf{\Gamma}_t): \quad r_t = (1-\tau_t^c)\left(\alpha A\left(\frac{\sum_{s=1}^S n_{s,t}}{\sum_{s=2}^S b_{s,t} - D_t(\Gamma_t)}\right)^{1-\alpha} - \delta\right) \quad \forall t \tag{12.20}$$

Now (12.19), (12.20), and the budget constraint (12.1) can be substituted into household Euler equations (12.4) and (12.5) to get the following $(2S-1)$-equation system. Extended across all time periods, this system completely characterizes the equilibrium.

$$\begin{split}(1-\tau_t^l)w_t(\mathbf{\Gamma}_t)\left((1-\tau_t^l)w_t(\mathbf{\Gamma}_t)n_{s,t} + \Bigl[1+(1-\tau_t^k)r_t(\mathbf{\Gamma}_t)\Bigr]b_{s,t} + X_t - b_{s+1,t+1}\right)^{-\sigma} = \\ \chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \\ \text{for} \quad s\in\{1,2,...S\} \quad \text{and} \quad \forall t\end{split} \tag{12.21}$$

$$\begin{split}\left((1-\tau_t^l)w_t(\mathbf{\Gamma}_t)n_{s,t} + \Bigl[1+(1-\tau_t^k)r_t(\mathbf{\Gamma}_t)\Bigr]b_{s,t} + X_t - b_{s+1,t+1}\right)^{-\sigma} = \\ \beta\Bigl[1+(1-\tau_t^k)r_{t+1}(\mathbf{\Gamma}_{t+1})\Bigr]\times ... \\ \left((1-\tau_t^l)w_{t+1}(\mathbf{\Gamma}_{t+1})n_{s+1,t+1} + \Bigl[1+(1-\tau_t^k)r_{t+1}(\mathbf{\Gamma}_{t+1})\Bigr]b_{s+1,t+1} + X_{t+1} - b_{s+2,t+2}\right)^{-\sigma} \\ \text{for} \quad s\in\{1,2,...S-1\} \quad \text{and} \quad \forall t\end{split} \tag{12.22}$$

The system of S nonlinear static equations (12.21) and $S-1$ nonlinear dynamic equations (12.22) characterizing the lifetime labor supply and savings decisions for each household $\{n_{s,t+s-1}\}_{s=1}^S$ and $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ is not identified. Each individual knows the current distribution of capital $\mathbf{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\mathbf{\Gamma}_{t+1} = \{\{b_{s+1,t+1}\}_{s=1}^{S-1}\}$ and a number of subsequent periods for all agents alive in those subsequent periods. We also need to solve for a policy function for the individual $b_{s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 12.1 (Steady-state equilibrium). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$, labor supply $\{\bar{n}_s\}_{s=1}^S$, and savings $\{\bar{b}_s\}_{s=2}^S$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (12.4) and (12.5),

ii. firms optimize according to (12.9) and (12.10),

iii. government debt satisfies (12.11)

iv. markets clear according to (12.15) and (12.16).

The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{s,t} = \phi_s\left(\mathbf{\Gamma}_t\right)$, and let the equilibrium policy functions for savings be represented by $b_{s+1,t+1} = \psi_s\left(\mathbf{\Gamma}_t\right)$. The arguments of the functions (the state) may change overtime causing the labor and savings levels to change over time, but the function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 12.2 (Non-steady-state functional equilibrium). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as stationary allocation functions of the state $\left\{n_{s,t} = \phi_s\big(\mathbf{\Gamma}_t\big)\right\}_{s=1}^S$, $\left\{b_{s+1,t+1} = \psi_s\big(\mathbf{\Gamma}_t\big)\right\}_{s=1}^{S-1}$ and and stationary price functions $w(\mathbf{\Gamma}_t)$ and $r(\mathbf{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (12.7), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{\Gamma}_{t+u} = \mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (12.4) and (12.5),

iii. firms optimize according to (12.9) and (12.10),

iv. government debt evolves according to (12.11) and (12.14),

v. markets clear according to (12.15) and (12.16).

12.6 Solution Method

In this section we characterize computational approaches to solving for the steady-state equilibrium from Definition 12.1 and the transition path equilibrium from Definition 12.2.

12.6.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 12.1. The parameters needed for the steady-state solution of this model are $\left\{S,\beta,\sigma,\tilde{l},b,\upsilon,\{\chi^n_s\}_{s=1}^S,A,\alpha,\delta,\bar{\tau}^l,\bar{\tau}^k,\bar{\tau}^c,\alpha_X,\alpha_D\right\}$, where S is the number of periods in an individual's life, $\left\{\beta,\sigma,\tilde{l},b,\upsilon,\{\chi^n_s\}_{s=1}^S\right\}$ are household utility function parameters, $\{A,\alpha,\delta\}$ are firm production function parameters, and $\{\bar{\tau}^l,\bar{\tau}^k,\bar{\tau}^c,\alpha_X,\alpha_D\}$ are parameters that describe steady-state fiscal policies. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$c_{s,t}=\bar{c}_s,\quad n_{s,t}=\bar{n}_s,\quad b_{s,t}=\bar{b}_s\quad\forall s,t \tag{12.23}$$

From the market clearing conditions (12.15) and (12.16) and the firms' first order equations (12.9) and (12.10), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t = \bar{r}, \quad w_t = \bar{w}, \quad B_t = \bar{B}, \quad D_t = \bar{D}, \quad K_t = \bar{K}, \quad G_t = \bar{G}, \quad X_t = \bar{X}, \quad L_t = \bar{L} \quad \forall t \tag{12.24}$$

The steady-state is characterized by the steady-state versions of the set of $2S-1$ Euler equations over the lifetime of an individual (after substituting in the budget constraint) and the $2S-1$ unknowns $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$\begin{split}(1-\bar{\tau}^l)\bar{w}\Bigl([1+(1-\bar{\tau}^k)\bar{r}]\bar{b}_s + (1-\bar{\tau}^l)\bar{w}\bar{n}_s + \bar{X} - \bar{b}_{s+1}\Bigr)^{-\sigma} = ... \\ \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s=\{1,2,...S\}\end{split} \tag{12.25}$$

$$\begin{split}\Bigl([1+(1-\bar{\tau}^k)\bar{r}]\bar{b}_s + (1-\bar{\tau}^l)\bar{w}\bar{n}_s + \bar{X} - \bar{b}_{s+1}\Bigr)^{-\sigma} = ... \\ \beta(1+(1-\bar{\tau}^k)\bar{r})\Bigl([1+(1-\bar{\tau}^k)\bar{r}]\bar{b}_{s+1} + (1-\bar{\tau}^l)\bar{w}\bar{n}_{s+1} + \bar{X} - \bar{b}_{s+2}\Bigr)^{-\sigma} \\ \text{for} \quad s=\{1,2,...S-1\}\end{split} \tag{12.26}$$

where both $\bar{w}$ and $\bar{r}$ are functions of the distribution of labor supply and savings as shown in (12.19) and (12.20). We solve this system of equations using the solution method proposed in Section 4.6.1. Here, we need to update the method for the new specification of fiscal policy. In particular, we need to know government transfers to households when solving the household problem and we need to know government debt for the asset market clearing condition. We use a bisection method in the outer loop, were we solve for the steady-state equilibrium interest rate, $\bar{r}$, and total transfers, $\bar{X}$. Using the firm's first order conditions this guess at $\bar{r}^i$ implies a wage, $\bar{w}$. In the inner loop, we solve the household's problem given $\bar{r}^i$, $\bar{w}$, and $\bar{X}^i$ using a root finder to solve jointly for the 2S-1 unknowns in the household problem.

i. Make a guess for the steady-state interest rate $\bar{r}^i$ and transfers, $\bar{X}^i$.

   (a) The value $\bar{r}^i$ will imply a wage rate $\bar{w}$ from (12.9) and (12.10). The values $\{\bar{r}^i, \bar{w}, \bar{X}^i\}$ are necessary to solve the households' problems.

ii. Given $\bar{r}^i$, $\bar{w}$, and $\bar{X}^i$, solve for the steady-state household's lifetime decisions using a root finder on the household first order conditions to solve for the $2S-1$ unknowns in the household problem, $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_s\}_{s=2}^S$.

iii. Given solution for optimal household decisions $\{\bar{n}_s\}_{s=1}^S$, and $\{\bar{b}_s\}_{s=2}^S$ based on the guesses for the interest rate $\bar{r}^i$ and total transfers $\bar{X}^i$, solve for new implied values of $\bar{r}^{i'}$ and $\bar{X}^{i'}$.

(a) We can solve for $\bar{L}$ as a function of optimal household labor supply $\{\bar{n}_s\}_{s=1}^S$ using the labor market clearing condition (12.15).

(b) We can solve for a $\bar{Y}$ consistent with the initial guess of transfers through assumption (12.13).

$$\bar{Y} = \frac{S\bar{X}^i}{\alpha_X} \tag{12.27}$$

(c) Steady-state debt $\bar{D}$ is a fixed percentage α_D of GDP through assumption (12.14).

$$\bar{D} = \alpha_D \bar{Y} \tag{12.28}$$

(d) The aggregate capital stock can be determined using household savings and aggregate debt in the capital market clearing condition (12.16).

$$\bar{K} = \sum_{s=2}^{S} \bar{b}_s - \bar{D} \tag{12.29}$$

(e) With aggregate capital $\bar{K}$ and labor $\bar{L}$, we can solve for the implied value of the interest rate $\bar{r}^{i'}$ using the firm's first order condition for capital demand (12.9).

$$\bar{r}^{i'} = (1-\tau^c)\left[\alpha A\left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} - \delta\right] \tag{12.30}$$

(f) We can solve for an updated value of transfers by calculating an updated value of GDP $\bar{Y}^{i'}$ given the implied values for $\bar{K}$ and $\bar{L}$ in the firms' production function and assumption (12.13).

$$\bar{X}^{i'} = \alpha_X\left(\frac{A\left(\bar{K}\right)^\alpha\left(\bar{L}\right)^{1-\alpha}}{S}\right) \tag{12.31}$$

iv. Update the guess for the steady-state interest rate $\bar{r}^{i+1}$ and transfers $\bar{X}^{i+1}$ until the implied values $\{\bar{r}^{i'}$ and $\bar{X}^{i'}$ equal the initial guesses $\{\bar{r}^i, \bar{X}^i\}$

(a) The bisection method characterizes the updated guess for the outer-loop steady-state variables $\{\bar{r}^{i+1}$ and $\bar{X}^{i+1}$ as a convex combination of the initial guess $\{\bar{r}^i, \bar{X}^i\}$ and the value implied by household and firm optimization $\{\bar{r}^{i'}, \bar{X}^{i'}\}$, where the weight put on the new value $\{\bar{r}^{i'}, \bar{X}^{i'}\}$ is given by $\xi \in (0,1]$. The value for ξ must sometimes be small—between 0.05 and 0.2—for certain parameterizations of the model to solve.

$$\left[\bar{r}^{i+1}, \bar{X}^{i+1}\right] = \xi\left[\bar{r}^{i'}, \bar{X}^{i'}\right] + (1-\xi)\left[\bar{r}^i, \bar{X}^i\right] \quad \text{for} \quad \xi \in (0,1] \tag{12.32}$$

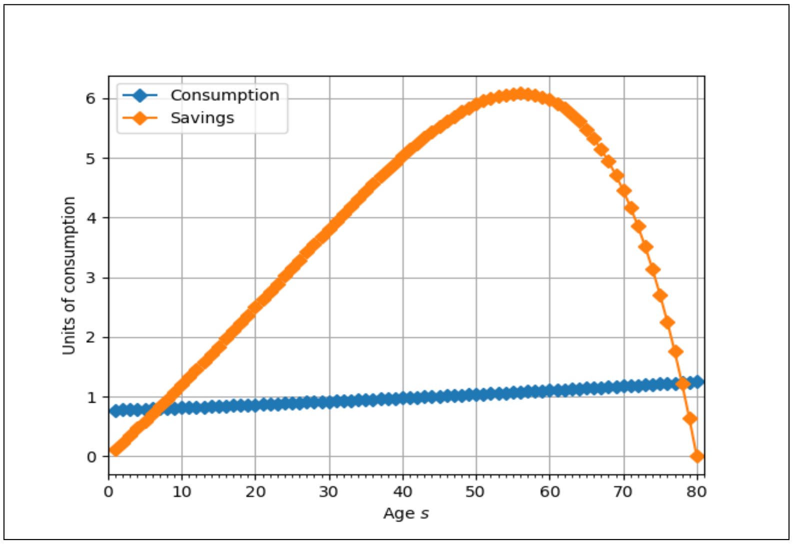
(b) Let $\|\cdot\|$ be a norm on the space of feasible interest rate values r and transfer values X. We often use a sum of squared errors or a maximum absolute error. Check the distance between the initial guess and the implied values as in (11.47). If the distance is less than some tolerance `SS_toler` > 0, then the problem has converged. Otherwise, update the value of the interest rate according to (12.32) and repeat steps (ii) through (v).

$$\texttt{SS\_dist} \equiv \left\|\left[\bar{r}^{i'}, \bar{X}^{i'}\right] - \left[\bar{r}^i, \bar{X}^i\right]\right\| \tag{12.33}$$

Define the updating of aggregate variable values $(\bar{r}^i, \bar{X}^i)$ in step (iv) indexed by i as the "outer loop" of the fixed point solution. Although computationally intensive, the bisection method described above is the most robust solution method we have found.

Figure 12.1 shows the steady-state distribution of individual consumption and savings in an 80-period-lived agent model with parameter values listed above the line in Table 12.3 in Section 12.7. Figure 12.2 shows the steady-state distribution of individual labor supply

Figure 12.1: Steady-state distribution of consumption $\bar{c}_s$ and savings $\bar{b}_s$



by age. The left side of Table 12.1 gives the resulting steady-state values for the prices and aggregate variables.

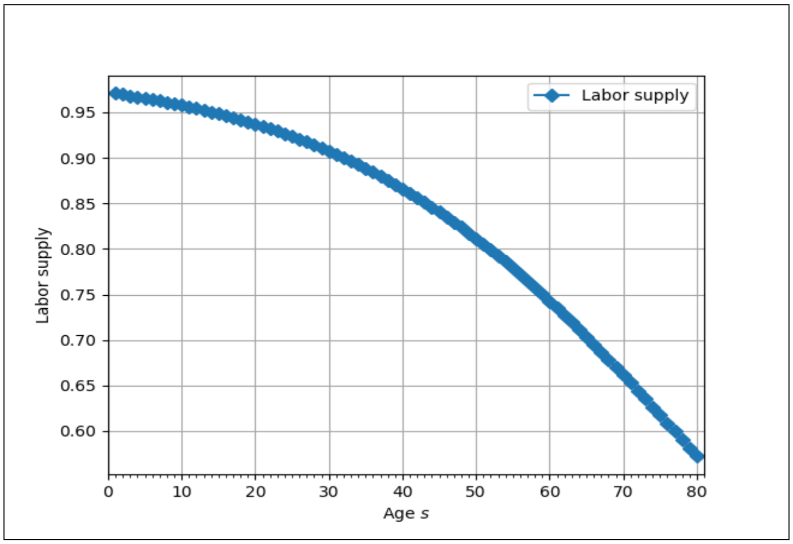
As a final note, it is important to make sure that all of the characterizing equations are satisfied in order to verify that the steady-state has been found. In this model, we must check the $2S-1$ Euler errors from the labor supply and savings decisions, the final period savings decision (should be zero), the two firm first order conditions, and the three market clearing conditions (including the goods market clearing condition, which is redundant by Walras law). The right side of Table 12.1 shows the maximum errors in all these characterizing conditions. Because all Euler errors are smaller than 4.5e-16, the final period individual savings is less than 6.1e-13, and the resource constraint error is less than 1.8e-06, we can be confident that we have successfully solved for the steady-state.

12.6.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the S-period-lived agent model with endogenous labor is similar to the method described in Section 4.6.2.

To solve for the transition path (non-steady-state) equilibrium from Definition 12.2, we

Figure 12.2: Steady-state distribution of labor supply $\bar{n}_s$



must know the parameters from the steady-state problem $\left\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta, \bar{\tau}^l, \bar{\tau}^k, \bar{\tau}^c, \alpha_X, \alpha_D\right\}$, the steady-state solution values $\{\bar{r}, \bar{Y}\}$, initial distribution of savings $\boldsymbol{\Gamma}_1$, and TPI parameters $\{T1, T2, \xi\}$. Tables 12.3 and 12.1 show a particular calibration of the model and a steady-state solution. In addition, we need parameters that describe fiscal policy over the time path:$\{\alpha_G, t_{g1}, t_{G2}, \rho_G, \alpha_{D0}, \boldsymbol{\tau}^l, \boldsymbol{\tau}^k, \boldsymbol{\tau}^c\}$. The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is the following.

i. Choose a period $T1$ in which the initial guess for the time paths of aggregate capital and aggregate labor will arrive at the steady state and stay there. Choose a period $T2$ upon which and thereafter the economy is assumed to be in the steady state. You must have the guessed time path hit the steady state before individual optimal decisions will hit their steady state.

ii. Guess initial time paths for the real interest rate $\boldsymbol{r}^i = \{r^i_1, r^i_2, ...r^i_{T1}\}$ and aggregate transfers $\boldsymbol{Y}^i = \{Y^i_1, Y^i_2, ...Y^i_{T1}\}$. Both of these time paths will have to be extended with their respective steady-state values so that they are $T2 + S - 1$ elements long. This is the time-path length that will allow you to solve the lifetime of every individual alive in period $T2$.

iii. Use the firms' first order conditions to solve form the time path of wage rates, $\boldsymbol{w}^i$,

Table 12.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Equilibrium error	Value
$\bar{r}$	0.082	Max. absolute savings Euler error	7.44e-11
$\bar{w}$	1.037	Max. absolute labor supply Euler error	1.47e-11
$\bar{K}$	252.648	Absolute final period savings $\bar{b}_{S+1}$	-1.16e-13
$\bar{L}$	66.423	Resource constraint error	4.20e-08
$\bar{Y}$	106.019		
$\bar{C}$	79.293		
$\bar{D}$	42.408		
$\bar{G}$	14.094		
$\bar{X}$	10.602		
$\bar{R}$	28.187		

given the time path for interest rates, $\boldsymbol{r}^i$.

iv. Output, together with the fiscal rule for government transfers from (**??**) yield the path for government trantsfers, $\boldsymbol{X}^i$, which in turn gives the path for transfers per household: $\boldsymbol{x}^i = \frac{\boldsymbol{X}^i}{S}$.

v. Given time paths $\boldsymbol{r}^i$, $\boldsymbol{w}^i$ and $\boldsymbol{x}^i_s$, solve for the lifetime labor supply $n_{s,t}$, and savings $b_{s+1,t+1}$ decisions of all households alive in periods $t = 1$ to $t = T2$.

   (a) Given the time paths for the interest rate $\boldsymbol{r}^i$, wage $\boldsymbol{w}^i$, transfers $\boldsymbol{x}^i$ and the period-1 distribution of savings (wealth) $\boldsymbol{\Gamma}_1$, solve for the lifetime decisions $n_{s,t}$, and $b_{s,t}$ of each household alive during periods 1 and $T2$. This is done using the method outlined in steps (ii) of the steady-state computational algorithm outlined in Section 12.6.1.

vi. Use time path of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households optimal decisions given $\boldsymbol{r}^i$ and $\boldsymbol{Y}^i$ to compute total tax revenue using (12.12). Call this path of tax revenues $\boldsymbol{R}^i$

vii. By putting the time path for aggregate output, $\boldsymbol{Y}^i$, transfers, $\boldsymbol{X}^i$, and revenues, $\boldsymbol{R}^i$, into the budget closure rule from (12.14), we can find the time path for government debt, $\boldsymbol{D}^i_t$, and government spending, $\boldsymbol{G}^i$.

viii. Use time path of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households optimal decisions given $\boldsymbol{r}^i$ and $\boldsymbol{Y}^i$ to compute paths for aggregate savings and aggregate labor $\boldsymbol{B}^i$ and $\boldsymbol{L}^i$.

ix. Using the asset market clearing condition, (**??**), find the path for aggregate capital, $\boldsymbol{K}^i = \boldsymbol{B}^i - \boldsymbol{D}^i$.

x. Using the labor market clearing condition (12.15) find the path for aggregate labor supply, $\boldsymbol{L}^i$.

xi. Using the paths for aggregate capital and labor, together with the firm's first order condition for its choice of capital and the firm production function, find the newly implied paths of real interest rates and outputs, $\boldsymbol{r}^{i'}$ and $\boldsymbol{Y}^{i'}$, respectively.

xii. Compare the distance of the time paths of the new implied paths for interest rates and output $\left(\boldsymbol{r}^{i'},\boldsymbol{Y}^{i'}\right)$ versus the initial paths for interest rates and output $\left(\boldsymbol{r}^{i},\boldsymbol{Y}^{i}\right)$.

$$\texttt{dist} = \left\|\left(\boldsymbol{r}^{i'},\boldsymbol{Y}^{i'}\right) - \left(\boldsymbol{r}^{i},\boldsymbol{Y}^{i}\right)\right\| \geq 0 \tag{12.34}$$

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock and aggregate labor $(\boldsymbol{r}^{i},\boldsymbol{Y}^{i})$. Common norms to use are the L^2 and the L^∞ norms.

(a) If the distance is less than or equal to some tolerance level $\texttt{dist} \leq \texttt{TPI\_toler} > 0$, then the fixed point, and therefore the equilibrium transition path, has been found.

(b) If the distance is greater than some tolerance level, then update the guess for a new set of initial time paths to be a convex combination current initial time paths and the implied time paths.

$$\left(\boldsymbol{r}^{i+1},\boldsymbol{Y}^{i+1}\right) = \xi\left(\boldsymbol{r}^{i'},\boldsymbol{Y}^{i'}\right) + (1-\xi)\left(\boldsymbol{r}^{i},\boldsymbol{Y}^{i}\right) \quad \text{for} \quad \xi \in (0,1] \tag{12.35}$$

The 6 panels of Figure 12.3 show the equilibrium time paths of the interest rate r_t, wage w_t, and aggregate variables K_t, L_t, Y_t, and C_t. The three panels of Figure 12.4 show the transition paths of the distributions of consumption $c_{s,t}$, labor supply $n_{s,t}$ and savings $b_{s,t}$.

Figure 12.3: Equilibrium transition paths of prices and aggregate variables

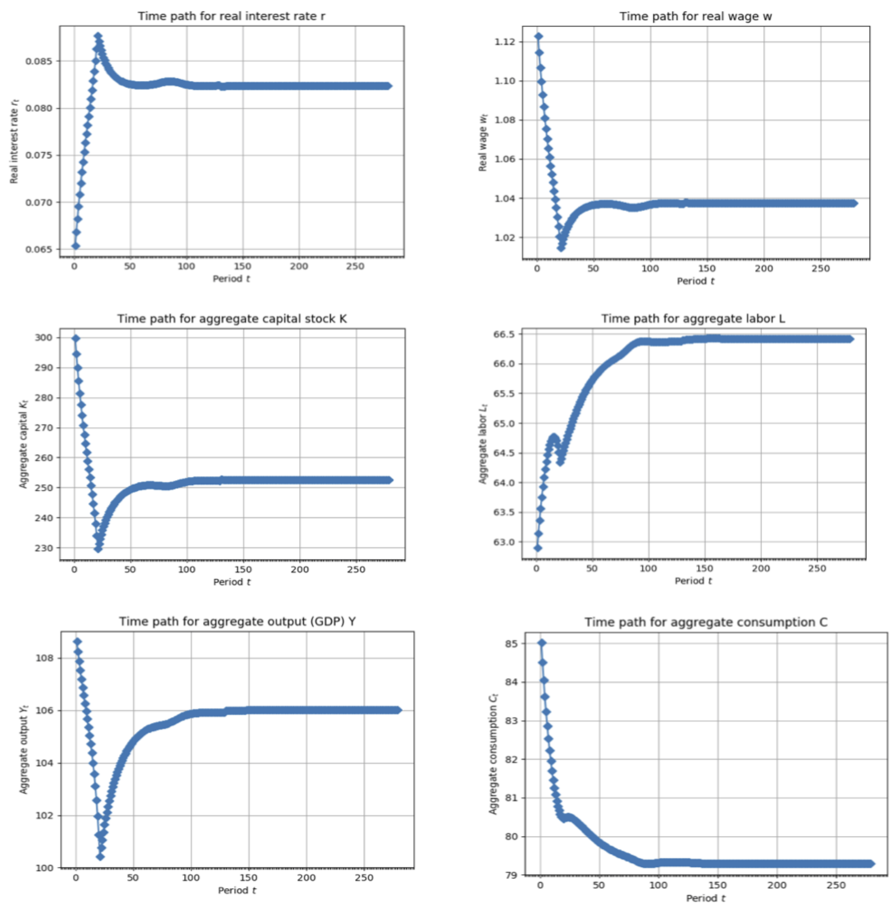


Figure 12.4: Equilibrium transition paths of distributions of consumption, labor supply, and savings

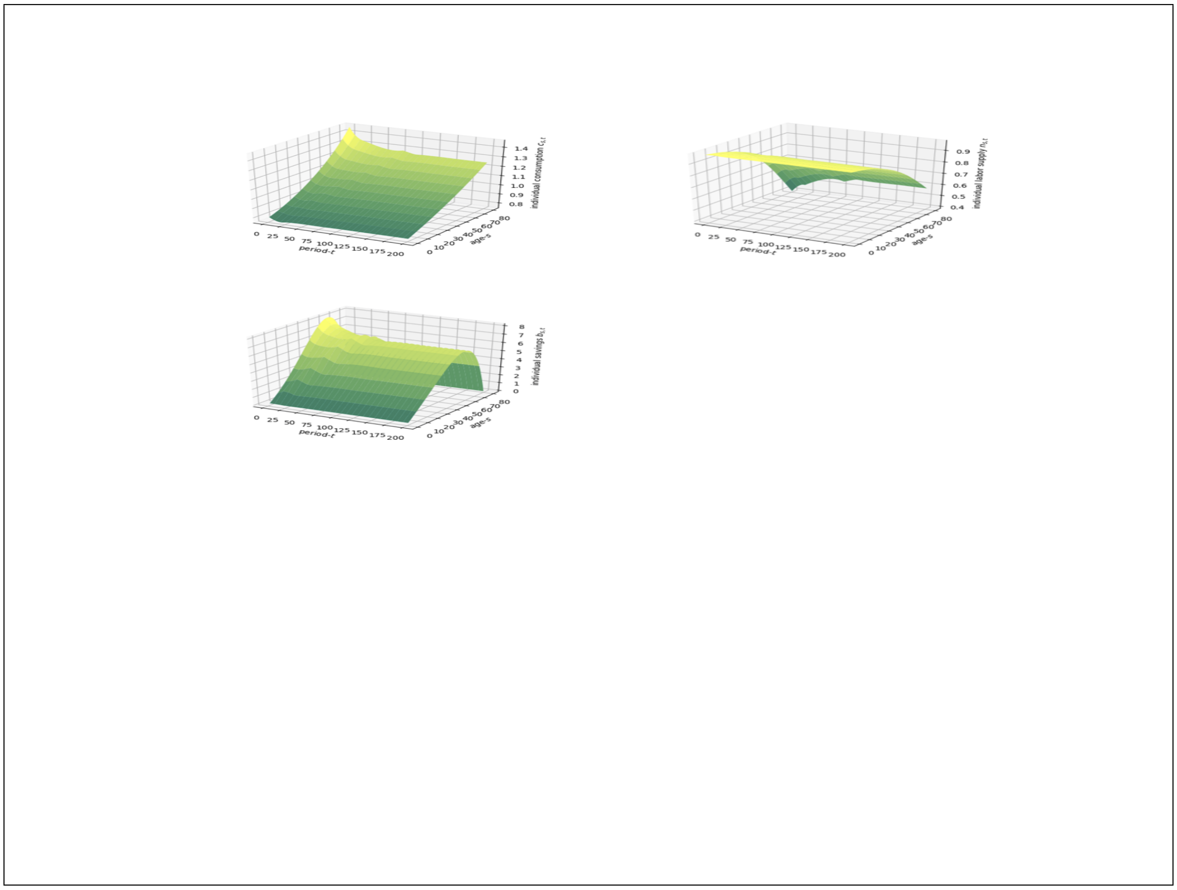


Figure 12.5: Equilibrium transition paths of government tax revenue, transfers, spending, and debt as percentages of output

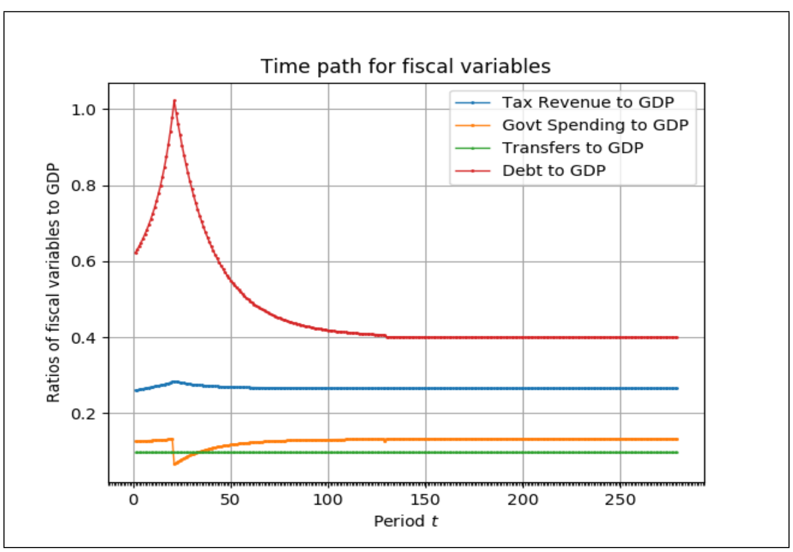


Table 12.2: Maximum absolute errors in characterizing equations across transition path

Description	Value
Maximum absolute labor supply Euler error	4.87e-13
Maximum absolute savings Euler error	8.07e-16
Maximum absolute final period savings $\bar{b}_{S+1,t}$	0.00
Maximum absolute resource constraint error	3.20e-08

The time path of fiscal aggregates are shown in Figure 12.5. In all of the time paths, a sharp kink is evident at t_{G1}, when the budget closure rule begins. Table 12.2 shows the maximum absolute Euler errors, end-of-life savings, and resource constraint errors across the transition path. All of these should be zero in equilibrium. The fact that none of them is greater than 2.0e-12 in absolute value is evidence that we have successfully solved for the non-steady-state equilibrium transition path of the model.

12.7 Calibration

Use the following parameterization of the model for the problems below. Assume that agents are born at age 21 and die at age 100 (80 years of life). Your time dependent parameters can be written as functions of S, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is $\beta = 0.96^{80/S}$. Assume initially that $S = 80$. Let the annual depreciation rate of capital be 0.05. Then the model period depreciation rate is $\delta = 1 - (1 - 0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 3$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$.

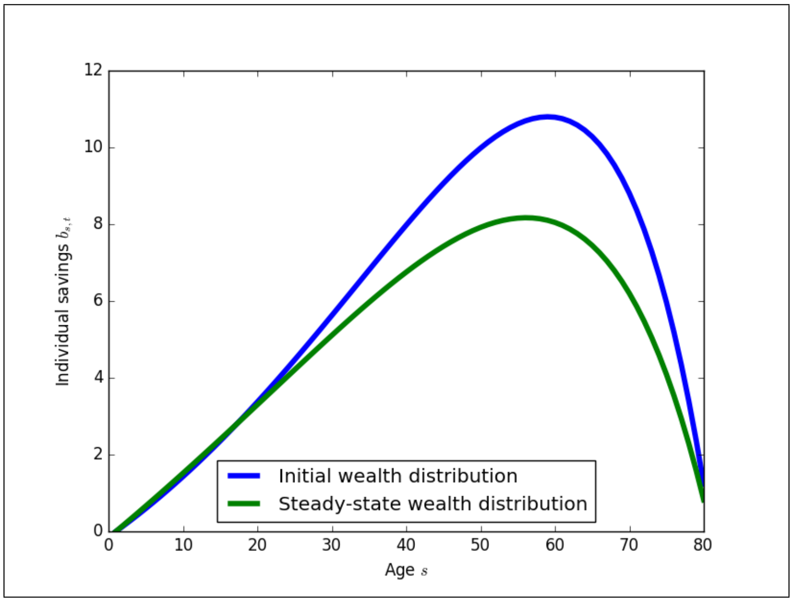
Assume that each individual's time endowment in each period is $\tilde{l} = 1$. Initially, let $\chi^n_s = 1$ for all s. However, you could calibrate these values to match the empirical distribution of annual hours by age in the United states (see Exercise 4.5).

Table 12.3: Calibrated parameter values for simple endogenous labor model with government debt financing

Parameter	Description	Value
S	Number of periods in individual life	80
β	Per-period discount factor	0.96
σ	Coefficient of relative risk aversion	2.5
$\tilde{l}$	Time endowment per period	1.0
b	Elliptical disutility of labor scale parameter	0.501[a]
υ	Elliptical disutility of labor shape parameter	1.554[a]
$\{\chi^n_s\}_{s=1}^S$	Disutility of labor relative scale factor by age	1.0
A	Total factor productivity	1.0
α	Capital share of income	0.35
δ	Per-period depreciation rate of capital	0.05
α_X	Ratio of government transfers to GDP	0.10
α_G	Ratio of government spending to GDP up to t_{G1}	0.12
α_D	Steady-state ratio of government debt to GDP	0.40
$\bar{\tau}^l$	Steady-state marginal tax rate on labor income	0.25
$\bar{\tau}^k$	Steady-state marginal tax rate on capital income	0.30
$\bar{\tau}^c$	Steady-state marginal tax rate on corporate income	0.15
$\boldsymbol{\Gamma}_1$	Initial distribution of savings (wealth)	(see Fig. 4.7)
$T1$	Time period in which initial path guess hits steady state	160
$T2$	Time period in which the model is assumed to hit the steady state	200
ξ	TPI path updating parameter	0.2
α_{D0}	Ratio of government spendign to GDP in $t = 1$	0.59
t_{G1}	Period when budget closure rule begins	20
t_{G2}	Period when budget closure rule ends	128
ρ_G	Rate at which debt to GDP adjusts to steady-state ratio	0.05
$\boldsymbol{\tau}^l$	Time path of tax rate on labor income	$0.25\ \forall\ t$
$\boldsymbol{\tau}^k$	Time path of tax rate on capital income	$0.30\ \forall\ t$
$\boldsymbol{\tau}^c$	Time path of tax rate on corporate income	$0.15\ \forall\ t$

[a] The calibration of b and υ is based on matching the marginal disutility of labor supply of a constant Frisch elasticity of labor supply functional form with a Frisch elasticity of 0.8. See Evans and Phillips (2017).

Figure 12.6: Initial vs. steady-state distributions of wealth (savings) $b_{s,t}$



12.8 Exercises

Exercise 12.1. Not all fiscal paths are achievable. For example, if tax revenue is low enough relative to outlays, then the debt burden may become unsustainable before the budget closure rule can kick in. Modify the fiscal parameters to see you can "break" the model. In particular:

i. Can you find a debt to GDP ratio (α_D) such that government spending is negative in the steady-state? How would you interpret this?

ii. Hold the rules for X, G, debt, and taxes constant, but push out the time until the closure rule kicks in (i.e., increase t_{G1}). Can you make this period far enough into the future that the debt burden becomes unsustainable and the time path equlibrium cannot be found?

iii. Once you've found broken the model by moving out t_{G1}, can you adjust α_G and α_X (holding constant at the new t_{G1}) so that the model no longer breaks?

Exercise 12.2. We've illustrated one of many ways to close the government's budget above. Try to implement another. Specifically, assume that government spending remains constant as a fraction of GDP for all time, but that government transfers, X_t, adjust to close the budget.

i. Outline the solution algorithm for the steady-state. How would this have to change?

ii. Now consider the time path. What would have to change from the algorithm outlined above? Think about the closure rule above and notice the time path for government spending. Would such a discrete jump in transfers impose and problems computationally? Is it realistic? What would happen to the time paths of other economic aggregates if X did have such a discrete jump? Given your answers to these questions, sketch out a budget closure rule that closes the government's budget by adjusting transfers.

Harder Modify the code to implement this alternative budget closure rule.