Endsem: Probability and Statistics (100 marks)

Cheat Sheet 2:

- The probability density function (PDF) of a Gaussian random variable $X \sim N(\mu, \sigma^2)$ is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}.$$

- The probability mass function (PMF) of a Poisson random variable $X \sim \text{Poisson}(\lambda)$ is:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots.$$

- The PDF of a Gamma random variable $X \sim \text{Gamma}(\alpha, \beta)$ is:

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0,$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$ is the Gamma function.

Each question: 10 marks

1. Write the pdf/pmf and derive the moment generating function for the following:
   (a) Geometric with parameter p.
   (b) Exponential with parameter λ.

2. Give a Monte Carlo method to estimate π. You are given access to only samples from Uniform. Give justification why this works. (Hint: use the uniform to pick points in a unit square and count hits inside the unit circle.)

3. (Erlang) Let $Z = X_1 + X_2$ where X_i is an Exponential random variable with parameter λ for $i = 1, 2$. Furthermore, X_1 and X_2 are independent. Derive the pdf of Z and also derive an expression for its mean using the pdf.

4. Let Z be a discrete random variable taking values in $\{1, \ldots, n\}$ with

$$\mathbb{P}(Z = i) = p_i, \qquad i = 1, \ldots, n.$$

   Conditioned on $Z = i$, let X be distributed as Y_i. Thus we define $X = Y_Z$.
   Find $E[X]$ (5 mks). Also prove that $f_X(x) = \sum_{i=1}^n p_i\, f_{Y_i}(x)$. (5mks)

5. Let $\mathcal{D} = \{x_1, \ldots x_n\}$ denote i.i.d samples from a uniform random variable $U[a, b]$ where a and b are unknown. Find an MLE estimate for the unknown parameters a and b.

6. Let $\mathcal{D} = \{x_1, \ldots x_n\}$ denote i.i.d samples from a Poisson random variable with unknown parameter γ. Find an MLE estimate for the unknown parameter γ. (5mks) What is its Mean Squared Error (MSE) (5mks)?

7. Let $X_n \sim \text{Uniform}\left(5 - \frac{1}{n}, 5 + \frac{1}{n}\right)$.
   (a) Show that $X_n \xrightarrow{d} 5$. (5mks)

(b) Compute $P(|X_n - 5| > \varepsilon)$ explicitly and show that it converges to 0 as $n \to \infty$. (5mks)

8. Hitting probabilities: let F_{ij} denote the probability of the Markov chain ever returning to state i having started in state j. For a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$ and transition matrices given below find F_{ii} for $i = 1, 2, 3$. From the values of F_{ii}, deduce which states are transient and recurrent. $P = \begin{bmatrix} 0.1 & 0.9 & 0 \\ .2 & .6 & .2 \\ 0 & 0 & 1 \end{bmatrix}$ Hint: (Express F_{11} in terms of F_{21}. F_{21} further can be expressed as a recursion.)

9. Conjugate prior: Suppose $D = \{x_1, \ldots, x_n\}$ is a data set consisting of independent samples from an exponential random variable with unknown parameter λ^*. Now assume a prior model $\Lambda \sim Gamma(\alpha, \beta)$ on the unknown parameter λ^*. (Density of gamma is given in the cheat sheet) Obtain an expression for the posterior distribution on λ^*. (7 mks). What is the MAP estimate for λ^*? (3mks).?

10. 5 Marks each

(a) Given an examples of a 3 state Markov chain that has a stationary distribution but does not have a limiting distribution. Obtain its stationary distribution. (Hint: You saw a two state example in class)

(b) Define the Mean square error of an estimator. Explain the bias-variance tradeoff.