

International Institute of Information Technology Hyderabad  
(Deemed to be University)

MA101.5 - Discrete Structures Monsoon 2024

End Semester Examination

Maximum Time : 180 Minutes

Total Marks : 100

Write detailed answers. Adequately explain your assumptions and thought process.

1. (6 points) Fill in the following blanks: x 12 = 6 points]

1. An example of a graph that is not a tree is \_\_\_\_\_  
and an example that is a tree is \_\_\_\_\_
2. The GCD  $G$  of 208 and 57 is \_\_\_\_\_  
and integers  $z$  and  $y$  such that  $208z + 57y = G$  are \_\_\_\_\_
3. The inverse of 7 in  $\mathbb{Z}_9$  is \_\_\_\_\_ and  $19 \pmod{7}$  is \_\_\_\_\_
4. If  $a_n = 2a_{n-1} + 3a_{n-2} + 2$  and  $a_0 = 1$  and  $a_1 = 2$ , then  $a_z =$  \_\_\_\_\_  
 $a_{100} =$  \_\_\_\_\_  
and  $a_n =$  \_\_\_\_\_
5. An example of a finite field of order 11 is \_\_\_\_\_  
and an example of an integral domain of characteristic 11 is \_\_\_\_\_

2. [10 points] Give an example of a group  $G$  of 48 elements, that has a subgroup  $S$  of 16 elements and answer the following questions: (2+2+1+2+3 = 10 points]

1. Give a right coset of  $S$ .
2. Is your right coset of  $S$  also a left coset?
3. All the right cosets of  $S$  partition  $G$  into how many equivalence classes?
4. What is the equivalence relation for the above equivalence class? Prove that it is indeed an equivalence relation.

[CO1, CO2, CO3, CO4, CO6]

3. [10 points] Let  $G$ , be the set of all simple undirected graphs on  $n$  nodes labelled  $1, \dots, n$ .  
Answer the following: 2+4+3+1 = 10 points]

1. What is the cardinality of  $G_n$ ?
2. Given the following binary operators on two graphs, find which of them makes  $G_n$  a group for all  $n$ ?
  - (a) Union ( $E(G \cup G') = E(G) \cup E(G')$ )
  - (b) Intersection ( $E(G \cap G') = E(G) \cap E(G')$ )
  - (c) XOR (Exclusive OR of adjacency matrices of  $G$  and  $G'$ )
  - (d) NAND (Element-wise NAND of adjacency matrices of  $G$  and  $G'$ )
3. Can you find a binary operator  $*$  (other than those listed above) for which  $G_n$  becomes a group?

4. If  $T$  is a set of all trees on  $n$  nodes, is  $(T, *)$  a subgroup of  $(G_n, *)$ ?

[CO1, CO2, CO3, CO4, CO5, CO6]

4. (10 points) Consider the recurrence relation (1.53.5 + 1.5 + 3.5 10 points)

$$a_n = A a_{n-1} - 1 + a_{n-2} \quad (1)$$

such that  $a_0 = 0, a_1 = 1$  and  $a_i$  is even integer and  $a_0 = 0, a_1 = 1$ .

1. Find  $a_2, a_3, a_4$ .

2. Prove that  $n$  divides  $a_n$ .

3. When  $n$  is prime, what is  $a_n \mod n$  for any value of  $A$ ?

In Equation 1, if  $A = 1$ , then prove that in Euclid GCD algorithm to compute the GCD( $a, c$ ) where  $rs$ , the worst case value of  $z$  (that the algorithm takes the longest time) is  $a_{n-1}$ .

[CO1, CO2, CO3, CO4]

5. (10 points) Answer the following related to (binary) trees: (2+3+3+2=10 points)

1. Is Huffman Encoding unique? Justify your answer.

2. How many paths are there in a tree between a node  $u$  and a node  $v$ ? Prove your answer.

3. Prove or disprove Every tree is 2-colourable (i.e., bipartite graph). What about its converse?

4. Convert the following arithmetic expressions to prefix and postfix notation using the corresponding tree traversal as the evaluation tree.

$$(z + yz) + (xy + zz) + ryz \quad (2)$$

[CO1, CO2, CO3, CO4]

6. (10 points) Consider a staircase with  $N$  steps. A monkey climbs  $x$  steps at a time where  $P$  is the  $k$ th prime ( $k = 1, \dots, m$ ), and finds that exactly one step remains at the end. Can you estimate  $N$  in terms of  $m$ ? (2+2+2+4=10 points)

1. Prove that if there are only  $m$  primes, then  $N > 1$  is also a prime not amongst these  $m$  primes and thus subsequently proving that there are infinitely many primes.

2. Solve for  $N$  when the monkey finds that  $-1$  steps are left (it finds itself ahead by one step for  $k = 1, \dots, n$ ).

3. Use the Chinese Remainder Theorem to exhibit an isomorphism between the rings  $\mathbb{Z}$  and  $\mathbb{Z}_a \times \mathbb{Z}_b$  where  $a, b$  are pairwise co-prime and product  $[a, b] = n$ .

[CO1, CO2, CO3, CO4]

7. (8 points) Let  $G$  be the set of non-zero complex numbers and let  $N$  be a set of complex numbers of absolute value 1 (that is,  $a + bi$  where  $a^2 + b^2 = 1$ ). Answer the following questions: [1+3+4=8 points]

1. Show that  $G$  is a group under multiplication.
2. Show that  $N$  is a normal subgroup of  $G$ .
3. Show that the quotient group  $G/N$  is isomorphic to the group of all positive real numbers under multiplication.

(C01, C02, C03, C04, C06]

8. 24 points] Prove or disprove the following (4 x 6 = 24 points]

1. Every finite integral domain is a field.
2.  $\mathbb{Z}/(p)$  is isomorphic to  $\mathbb{Z}_p$ , where  $\mathbb{Z}$  is the ring of integers,  $p$  a prime number, and  $(p)$  the ideal of  $\mathbb{Z}$  consisting of all multiples of  $p$ .
3. You have  $n$  distinct objects and you pick  $q$  of them one at a time with replacement. Then the probability that you pick an item more than once is at least  $\frac{q}{n}$ .
4. If  $R$  is a ring with unit element 1 and  $\phi$  is the homomorphism of  $R$  onto  $R'$ , then  $\phi(1)$  is the unit element of  $R'$ .
5. For all  $a$  coprime to  $n$ ,  $a^{\phi(n)} \equiv 1 \pmod{n}$  where  $\phi$  is the Euler totient.
6. A graph is a tree if and only if it has exactly  $n-1$  edges.

[C01, C02, C03, C04, C06]

9. (12 points) Write in detail about any four of the following: [4 x 3 = 12 points]

1. State and prove Lagrange's theorem
2. Three popular tree traversals
3. How to solve any second order homogeneous recurrence relation
4. Birthday paradox
5. Bound on the number of leaves of any  $m$ -ary tree of height  $h$ .
6. Analogies between Group Theory and Ring Theory

[C01, C02, C03, C04, C05, C06]