QUANTUM MECHANICS

Monsoon 2025 – CND Core – Credit 4

Instructor: Subhadip Mitra **Date:** DECEMBER 1, 2025

Time: 03 H 00 M

End Examination

Total Marks: 100

Instructions:

- **Class notes or books are not permitted. You may bring one A4 sized handwritten material (not photocopied or printed). Calculators are allowed.**
- **Clearly write any assumption you make.** Keep your answers **to the point**. Unless the logic is clear, you will **not** get any credit for a problem.
- Illegible answers will not be graded. **No 'benefit of doubt' because of bad notation/illegible handwriting etc.**

Q 1. (a) Consider an infinite square well:

$$V_0(x) = \begin{cases} 0 & \text{if } 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}$$

Suppose three delta-function bumps $V_1(x) = \alpha\delta(x-a/4)$, $V_2(x) = \alpha\delta(x-a/2)$, and $V_3(x) = \alpha\delta(x-3a/4)$ (where α is a real constant) appear simultaneously inside. Find the first-order corrections to the allowed energies and the first two non-zero terms in the expansion of the ground state wave function. Will there be any state whose energy is not perturbed?

(b) Suppose it is instead perturbed by $V(x) = \lambda\, x(a-x)$ where λ is small. Without any explicit integral, evaluate $\langle\phi_4|V|\phi_5\rangle$. Explain your answer.

(c) If the lowest-order relativistic correction to the Hamiltonian is given as

$$H' = -\frac{p^4}{8m^3c^2},$$

find the lowest-order relativistic correction to the energy levels of the one-dimensional harmonic oscillator.
Hint: First, show that $E^1 = \langle H'\rangle = -\frac{1}{2mc^2}\left[E^2 - 2E\langle V\rangle + \langle V^2\rangle\right]$. [8+4+8=20]

Q 2. Let $|x\rangle$ denote the state (wave-function) at x. We can define an infinitesimal translation operator $\hat{T}(dx)$ such that

$$\hat{T}(dx')\,|x\rangle = |x+dx'\rangle.$$

(a) What properties should such an operator satisfy? In particular, argue for
  (i) $\hat{T}^\dagger(dx')$,
  (ii) $\hat{T}^{-1}(dx')$,
  (iii) $\hat{T}(dx')\cdot\hat{T}(dx'')$ and
  (iv) $\lim_{dx'\to 0}\hat{T}(dx')$.

(b) Under what condition $\hat{T}(dx') \approx 1 - i\,\frac{\hat{K}}{\hbar}\,dx'$ satisfies all the above properties if we ignore terms of second order or higher in dx'? Is $\hat{T}$ hermitian?

(c) Show that

$$[\hat{x}, \hat{T}(dx')]\,|x'\rangle = dx'\,|x'+dx'\rangle \approx dx'\,|x'\rangle$$

and obtain $[\hat{x}, \hat{K}]$.

(d) If $\langle x|\psi\rangle = \psi(x)$ (where $|\psi\rangle$ is an arbitrary state), what is $\langle x|T(dx)|\psi\rangle$? Expand it to establish the relation between $\hat{K}$ and the momentum operator.

(e) Now derive the finite translation operator using $\lim_{N\to\infty}(1+x/N)^N = e^x$ and show that it is a unitary operator.
Hint: $e^{\hat{A}}\cdot e^{\hat{B}} = e^{(\hat{A}+\hat{B})}$ is $[\hat{A},\hat{B}] = 0$.

(f) Consider the two Hamiltonians: (a) the one-dimensional harmonic oscillator and (b) the one-dimensional free particle. Which of them (or both) will commute with the finite translation operator? (Argue briefly.)

$[4+(2+1)+(3+1)+(2+2)+(1+2)+2=20]$

Q 3. **Consider a finite square well,**

$$V(x) = \begin{cases} -V_0 & \text{for } -a < x < a \quad (V_0 > 0) \\ 0 & \text{otherwise,} \end{cases}$$

with a particle of energy $E > 0$ (scattering state).

(a) **Show that the probability of the particle reflecting back is nonzero in general.**

(b) What happens if $E \gg V_0$ or $E \to 0$? Show that there are some energies for perfect transmission (*transmission resonance*, this is why you get a very large transmission when you scatter low-energy electrons through noble-gas atoms).

(c) We say that *the absolute value of potential does not matter, only the difference matters. Hence, if we add a constant to the overall potential, nothing changes.* Is this true in Quantum Mechanics? If so, how do we see that? If not, why not?

(d) Suppose, now, the particle has $E < 0$ (i.e., it is in a bound state). What will happen if we add a purely imaginary constant (say, $-i\Gamma$) to the potential instead of a real constant? Will it affect the probability of finding the particle? If so, how?

(e) Consider a periodic potential, i.e., $V(x+\lambda) = V(x)$. Show that the wave function at $(x_0+\lambda)$ is proportional to $\psi(x_0)$ up to a constant (i.e., x-independent) phase.

[3+3+4+5+5=20]

Q 4. Consider a box of volume V containing free electron gas (assume the total number of atoms to be N with each one contributing q electrons). The normalized wave functions are given as

$$\psi_{n_x,n_y,n_z} = \sqrt{\frac{8}{V}} \sin\left(\frac{n_x \pi}{l_x} x\right) \sin\left(\frac{n_y \pi}{l_y} y\right) \sin\left(\frac{n_z \pi}{l_z} z\right)$$

where $V = l_x l_y l_z$. The allowed energies are

$$E_{n_x,n_y,n_z} = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)$$

where the wave vector $\vec{k} = (k_x, k_y, k_z)$ with $k_i = n_i \pi / l_i$.

(a) Show that the Fermi energy is $E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$ where ρ is the free electron density. How is it related to the chemical potential?

(b) The total energy $E_{tot} \propto V^{-2/3}$. Find the proportionality constant and the degeneracy pressure.

Suppose you have three particles and three distinct one-particle states ($\psi_a(x)$, $\psi_b(x)$, $\psi_c(x)$) are available.

(c) Construct all possible three-particle states if they are *(i)* distinguishable particles, *(ii)* identical fermions and *(iii)* identical bosons?

Suppose a particle is in a superposition state:

$$|\Psi\rangle = \frac{1}{\sqrt{3}}|2,-1\rangle + \frac{1}{\sqrt{3}}|2,0\rangle + \frac{1}{\sqrt{3}}|2,2\rangle,$$

where the states are labelled with the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ as $|\ell, m\rangle$.

(d) Compute $\langle L^2 \rangle$ and $\langle L_z \rangle$.

[4+6+5+5=20]

Q 5. A spinning electron constitutes a magnetic dipole. Its dipole moment is proportional to the spin,

$$\vec{\mu} = \gamma \vec{S}$$

where γ is the gyromagnetic ratio. If you put it in a magnetic field $\vec{B}$, it feels a torque. The energy associated with the torque is $-\vec{\mu}\cdot\vec{B}$.

(a) If the magnetic field is constant $\vec{B} = B_0\hat{z}$, then show that $<\vec{S}>$ gets tilted and it processes about the field with a constant frequency.

(b) If $\vec{B} = B_0 \cos(\omega t)\hat{z}$ (where ω is a constant) and the electron starts out in the spin-down state in the x direction, i.e.,

$$\chi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

then obtain $\chi(t)$ by solving the time-dependent Schrödinger equation

$$i\hbar\frac{\partial \chi}{\partial t} = \mathbf{H}\chi,$$

where $\mathbf{H}$ is the Hamiltonian matrix

(c) Calculate $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$ if the field in the last question were $\vec{B} = B_0 \cos(\omega t)\hat{x}$ instead?

[7+8+5=20]