AI21MTECH12003

classmate

Date

Page

Homework - 2

2 (a)

$$Z = f(X_1 \ldots X_n)$$
$$E_i(Z) = \mathbb{E}[Z \mid X_1 \ldots X_i]$$
$$\Delta_i = E_i(Z) - E_{i-1}(Z)$$

$$\sum_{i=1}^{n} \Delta_i = \mathbb{E}_n(Z) - \cancel{E_{n-1}(Z)} + \cancel{E_{n-1}(Z)} - \cancel{E_{n-2}(Z)}$$
$$\ldots \cancel{E_1[Z]} - \cancel{E[Z]}\ \mathbb{E}[Z]$$

$$\therefore \sum_{i=1}^{n} \Delta_i = Z - \mathbb{E}[Z]$$

$$\mathrm{Var}(Z) = \mathbb{E}[(Z - \mathbb{E}[Z])^2]$$ (definition variance)

$$= \mathbb{E}\left[\left(\sum_{i=1}^{n} \Delta_i\right)^2\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} \Delta_i^2 + \sum_{j>i}^{n} \Delta_i \Delta_j\right]$$ (breaking the internal term)

$$= \mathbb{E}\left[\sum_{i=1}^{n} \Delta_i^2\right] + \mathbb{E}\left[\sum_{j>i}^{n} \Delta_i \Delta_j\right]$$ (linearity of expectation)

$$\mathbb{E}\left[\sum_{j>i} \Delta_i \Delta_j\right] = \mathbb{E}\left[\sum_{j>i}\left(\left(\mathbb{E}[Z \mid X_1 \ldots X_i] - \mathbb{E}[Z \mid X_1 \ldots X_{i-1}]\right)\left(\mathbb{E}[Z \mid X_1 \ldots X_j] - \mathbb{E}[Z \mid X_1 \ldots X_{j-1}]\right)\right)\right]$$

$$= \mathbb{E}\left[\sum_{j>i} \mathbb{E}\left[\Delta_i \Delta_j \mid X_i \ldots X_{j-1}\right]\right]$$

$$= \mathbb{E}\left[\sum_{j>i} \Delta_i \left(\mathbb{E}\left[\mathbb{E}(Z \mid X_1 \ldots X_j) \mid X_1 \ldots X_{j-1}\right] - \mathbb{E}[Z \mid X_1 \ldots X_{j-1}]\right)\right]$$

$$= \mathbb{E}\left[\sum_{j>i} \Delta_i \left(\mathbb{E}[Z \mid X_1 \cdots X_{j-1}] - \mathbb{E}[Z \mid X_1 \cdots X_{j-1}]\right)\right]$$

$$= 0$$

$$Var(Z) = \mathbb{E}\left[\sum_{i=1}^{n} \Delta_i^2\right]$$

② (b)

$$\Delta_i = \mathbb{E}_i[Z] - \mathbb{E}_{i-1}(Z)$$

$$= \mathbb{E}_i[Z] - \mathbb{E}[\mathbb{E}[Z \mid X_1 \cdots X_{i-1}, X_{i+1} \cdots X_n] \mid X_1 \cdots X_i]$$

$(\mathbb{E}[\mathbb{E}[Z \mid AB] \mid AC] = \mathbb{E}[Z$

$$= \mathbb{E}_i[Z - \mathbb{E}[Z \mid X_1 \cdots X_{i-1}, X_{i+1} \cdots X_n]]$$

(using given variation

$$= \mathbb{E}_i(Z - \mathbb{E}^i(Z))$$

$$\Delta_i^2 = \left[\mathbb{E}_i[Z - \mathbb{E}^i[Z]]\right]^2$$

using jensens inequality

$$\Delta_i^2 \leq \mathbb{E}_i\left[(Z - \mathbb{E}^i[Z])^2\right]$$

① from efron-stein inequality

$$Var(Z) \leq \sum_{i=1}^{n} \mathbb{E}_i\left[(Z - \mathbb{E}^i[Z])^2\right]$$

$$= \sum_{i=1}^{n} \mathbb{E}_i[Var_i(Z)]$$

$$\leq \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}\left[\left(\frac{d}{dx_i} f(x_1 \cdots x_i \cdots x_n)\right)^2 \mid \right.\right.$$

(from Gaussian poincare inequality

$$\mathrm{Var}_i(z) \le \mathbb{E}[(f'(x))^2]$$

given from the bounded difference property $\nabla_i = c_i$ for $i \in [1, n]$

$$\mathrm{Var}(z) \le \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}\left[c_i^2 f(x_1 \ldots x_i \ldots x_n)^2 / x^{-1}\right]\right]$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[c_i^2 \mathrm{Var}_i(x)\right]$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[c_i^2 p_i (1-p_i)\right]$$

($\mathrm{Var}_i(x) = p_i(1-p_i)$ Bernoulli distribution fixing everything else other than x_i)

$$\mathrm{Var}(z) \le \sum_{i=1}^{n} c_i^2 p_i (1-p_i)$$