EE5603: Concentration Inequalities, Fall 2022 (12)
Indian Institute of Technology Hyderabad
HW 0, Assigned: Sunday 21.08.2022. 90 points.
Due: Friday 26.08.2022 at 11:59 pm.

1. Find the moment generation function of $X \sim \mathcal{N}(0, \sigma^2)$. (5)
2. Use the Markov inequality to prove the following (with appropriate assumptions):
   (a) The Chebyshev inequality. (5)
   (b) The Chernoff bound. (5)
3. (a) Give an example where the Markov inequality is tight. (5)
   (b) Give an example where the Chernoff bound is tighter than the Chebyshev inequality. (5)
4. (a) Use the basic inequalities to prove the weak law of large numbers (LLN). (5)
   (b) Illustrate the efficacy of the LLN with a Python program using at least four different types of distributions. (5)
5. Definition: A real valued random variable X is said to be σ^2-sub-Gaussian if there exists a positive number σ such that $\mathbb{E}[e^{\lambda X)}] \leq e^{\frac{\sigma^2\lambda^2}{2}}$ for every $\lambda \in \mathbb{R}$. Show that the following random variables are σ^2-sub-Gaussian and find σ for each of them.
   (a) A Rademacher random variable. (5)
   (b) A Gaussian random variable with zero mean and variance σ^2. (5)
   (c) A random variable that is zero mean and bounded in the interval $[a, b]$. (5)
6. Prove that for a σ^2-sub-Gaussian random variable with mean μ, $P[|X - \mu| \geq t] \leq \exp(\frac{-t^2}{2\sigma^2})$ for all $t > 0$. (5)
7. If X_i are independent, mean-zero, σ_i^2-sub-Gaussian random variables, show that $\sum\limits_{i=1}^{n} X_i$ is $\sum\limits_{i=1}^{n} \sigma_i^2$-sub-Gaussian. (5)
8. If X_i are bounded random variable (in $[a_i, b_i]$), show that $P(\frac{1}{n}\sum\limits_{i=1}^{n}(X_i - \mathbb{E}[X_i]) \geq t) \leq \exp\Big(-\frac{2n^2t^2}{\sum\limits_{i=1}^{n}(b_i-a_i)^2}\Big)$ and $P(\frac{1}{n}\sum\limits_{i=1}^{n}(X_i - \mathbb{E}[X_i]) \leq -t) \leq \exp\Big(-\frac{2n^2t^2}{\sum\limits_{i=1}^{n}(b_i-a_i)^2}\Big)$ for all $t > 0$. (5)
9. Definition: A random variable X with mean μ is sub-exponential if there are non-negative parameters (ν, b) such that $\mathbb{E}[e^{\lambda(X-\mu)}] \leq e^{\frac{\nu^2\lambda^2}{2}}$ for all $|\lambda| < \frac{1}{b}$. Show that:
   (a) The exponential random variable with parameter λ is sub-exponential. (5)
   (b) The χ^2-random variable is sub-exponential. (5)
10. Suppose that X is a (ν, b)-sub-exponential random variable with mean μ, derive the tail bound $P[X \geq \mu + t]$ for all $t > 0$. (5)
11. If X_i are independent, (ν, b)-sub-exponential random variables, then $\sum\limits_{i=1}^{n} X_i$ is $(\sum\limits_{i=1}^{n} \nu_i, b_*)$-sub-exponential where $b_* = \max_i b_i$. (5)
12. Bernstein bound: If X is a random variable with mean μ and variance σ^2 and satisfies the condition $|\mathbb{E}[(X-\mu)^k]| \leq \frac{1}{2}k!\sigma^2 b^{k-2}$, it is said to satisfy the Bernstein condition with parameter b. For such a random variable show that $\mathbb{E}[e^{\lambda(X-\mu)}] \leq e^{\frac{\lambda^2\sigma^2/2}{1-b|\lambda|}}]$ for all $|\lambda| < \frac{1}{b}$. (5)