EE5603: Concentration Inequalities, Fall 2022 (12)

Indian Institute of Technology Hyderabad

HW 2, 20 points. Assigned: Monday 05.09.2022.

Due: Wednesday 07.09.2022 at 11:59 pm.

1. Assume that the random variables $X_1, \ldots, X_n$ are independent and binary {-1, 1}-valued with $P\{X_i = 1\} = p_i$ and that $f : \{-1, 1\}^n \longrightarrow \mathbb{R}$ has the bounded differences property with constants $c_1, \ldots, c_n$. Show that if $Z = f(X_1, \ldots, X_n)$, $\mathrm{Var}(Z) \leq \sum_{i=1}^{n} c_i^2 p_i(1 - p_i)$. (10)

2. Efron-Stein inequality: Recall the notation and formulation from class. $X_1, \ldots, X_n$ are independent random variables that take values from the set $\mathcal{X}$, $f : \mathcal{X}^n \longrightarrow \mathbb{R}$ is a square-integrable function, $Z = f(X_1, \ldots, X_n), E_i(Z) = E[Z|X_1, \ldots, X_i], E^i(Z) = \int_{x_i \in \mathcal{X}} f(X_1, \ldots, x_i, \ldots X_n) dP(x_i), \Delta_i = E_i(Z) - E_{i-1}(Z)$. Prove the following equalities and inequalities that were claimed to be true in class:

   (a) $\mathrm{Var}(Z) = \sum_{i=1}^{n} \Delta_i^2$. (5)

   (b) $\Delta_i^2 \leq E_i((Z - E^i(Z))^2)$. (5)