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Homework 0

① we need to prove all the three properties of metric space.

(i) $\forall$ ~~some~~ $x, y \in \mathbb{R}$

$|x-y| \geq 0$ (modulo operator is to make negative numbers positive)

when $x = y$;

$d(x,y) = 0 = |x-y|$

when $d(x,y) = 0$

$|x-y| = 0 \Rightarrow x = y$

(ii) $d(x,y) = |x-y| = |(-1)(y-x)|$

$= |y-x|$

$= d(y,x)$

$\therefore d(x,y) = d(y,x)$

(iii) $d(x,y) = |x-y|$

$= |(x-z) + (z-y)|$

$\leq |x-z| + |z-y|$

$= d(x,z) + d(z,y)$

$\therefore d(x,y) \leq d(x,z) + d(z,y)$

All the three properties are satisfied. Therefore $d(x,y) = |x-y|$ is a metric space over $\mathbb{R}$.

② (i) The distance b/w two words x and y will lie b/w 0 and n (both included). This means that

$d(x,y) \geq 0$

$d(x,y)=0$ when x word is equal to word y.

when x is equal to word y then there will be no place in the word where different letters are used which gives $d(x,y)=0$.

(ii) let at m places in the word x and y different letters are present which means

$$d(x,y)=m$$

similarly if we look from y ~~and x~~ with respect to x, the same places will have different letters which also gives $d(y,x)=m$

$$\therefore \quad d(x,y) = d(y,x)$$

(iii) think of $d(x_i,y_i)$ for position i in the word as

$$d(x_i,y_i) = \begin{cases} 1 ; & (x_i \neq y_i) \\ 0 ; & x_i = y_i \end{cases}$$

let there be some $z = \{z_1 \ldots z_n\}$

$$d(x_i,z_i) = \begin{cases} 1 ; & x_i \neq z_i \\ 0 ; & x_i = z_i \end{cases}$$

$$d(z_i,y_i) = \begin{cases} 1 ; & z_i \neq y_i \\ 0 ; & z_i = y_i \end{cases}$$

If $d(x_i,y_i)=1$ then there can be three conditions if we have z_i

either $x_i = z_i$ and $y_i \neq z_i$ OR
$x_i \neq z_i$ and $y_i = z_i$ OR $\Big\} \Rightarrow d(x_i,y_i)=1$
$x_i \neq z_i$ and $y_i \neq z_i$ $\Rightarrow d(x_i,y_i) \leq 2$

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from above conditions we can say that

~~$d(x_i, z_i$~~

$d(x_i, y_i) \leq d(x_i, z_i) + d(z_i, y_i)$

[with all the three conditions]

applying summation on both side we have

$$\sum_{i=1}^{n} d(x_i, y_i) \leq \sum_{i=1}^{n} \left(d(x_i, z_i) + d(z_i, y_i) \right)$$

$$d(x, y) \leq \sum_{i=1}^{n} d(x_i, z_i) + \sum_{i=1}^{n} d(z_i, y_i)$$

$$\therefore \; d(x, y) \leq d(x, z) + d(z, y)$$

This shows that (X, d) is a metric space

(3). given $d(x, y) = \|x - y\|$
and $(X, \|.\|)$ $\in$ is a normed linear space

(i) from the properties of norm

$\|x - y\| \geq 0 \Rightarrow d(x, y) \geq 0$

and when

$d(x,y) = \|x - y\| = 0$ when $x - y = 0 \Rightarrow x = y$

also when $x = y$; $x - y = 0 \Rightarrow \|x - y\| = 0$

$d(x, y) = 0$

(ii) from the property ~~of~~ 2, from norm

$\|\lambda z\| = |\lambda| \, \|z\|$, we have

$d(x, y) = \|x - y\| = \|(-1)(y - x)\|$

here $\lambda = -1$ and $z = y - x$

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$$d(x,y) = |(-1)| \cdot \|y-x\|$$
$$d(x,y) = \|y-x\|$$
$$\therefore \quad d(x,y) = d(y,x) \qquad \{d(y,x) = \|y..$$

(iii) we know from property 3 of norm that

$$\|m+n\| \le \|m\| + \|n\| \quad \text{for } \cancel{\text{some}} \text{ all } m \,\&\, n$$

given
$$d(x,y) = \|x-y\| = \|(x-z)+(z-y)\|$$
let
$$m = (x-z) \text{ and } n = (z-y)$$
then using the norm property

$$d(x,y) \le \|x-z\| + \|z-y\|$$

$$= d(x,z) + d(z,y)$$

$$\therefore d(x,y) \le d(x,z) + d(z,y)$$

This shows that $d(x,y) = \|x-y\|$ is a metric on X.

~~(4)~~
(6)

$$k(x,x') = (\langle x \cdot x' \rangle + c)^m \quad -(1)$$

given $x, x' \in \mathbb{R}^d$, therefore we can write
x as $(x_1, \dots, x_d)$
and x' as $(x'_1, \dots x'_d)$

$$\langle x \cdot x' \rangle = x_1 x'_1 + x_2 x'_2 \dots, x_d x'_d$$

putting above result in equation (1)

$$k(x, x') = (x_1 x_1' + \cdots x_d x_d' + c)^m$$

expanding above polynomial using multinomial theorem, given below.

$$* \;\; \phi(x_1 \cdots x_d)^m = \binom{m}{n_1 \ldots n_d} \prod_{t=1}^{d} x_t^{n_t}$$

$$k(x, x') = \sum_{\substack{n_1+n_2 \cdots n_{d+1} = m \\ n_1, n_2 \cdots n_{d+1} \geq 0}} \binom{m}{n_1 \ldots n_{d+1}} \prod_{t=1}^{d} (x_t x_t')^{n_t} \, c^{n_{d+1}}$$

$$= \sum_{\substack{n_1+n_2 \cdots n_{d+1} = m \\ n_1 \cdots n_{d+1} \geq 0}} \left[\sqrt{\binom{m}{n_1 \ldots n_{d+1}}} \prod_{t=1}^{d} (x_t)^{n_t} \sqrt{c^{n_{d+1}}}\right] \left[\sqrt{\binom{m}{n_1 \ldots n_{d+1}}} \prod_{t=1}^{d} (x_t')^{n_t} \sqrt{c^{n_{d+1}}}\right] \quad \text{— (2)}$$

~~let $\phi(x)$ =~~ In above summation according to multinomial theorem, we have $\binom{m+d}{d}$ terms. let $l_{m,d} = \binom{m+d}{d}$

~~let $\phi(x)$ = (a~~

let $\phi(x) = (a_1 \ldots a_l \ldots a_{l_{m,d}})$

where ~~a~~

$$a_l = \sqrt{\binom{m}{n_1 \cdots n_{d+1}}} \prod_{t=1}^{d} (x_t)^{n_t} \sqrt{c^{n_{d+1}}}$$

putting this in equation (2) we get

$k(x, x') = \langle \phi(x), \phi(x') \rangle$

which is what the definition of kernel means.

Therefore $k(x, x') = (\langle x . x' \rangle + c)^m$ is a valid kernel.

(5) let there be n kernels each with corresponding feature map represented as ϕ_i where $i \in \{1, 2 \dots n\}$

and kernels are represented as k_i $\forall i \in \{1, \dots n\}$

$k_i(x, y) = \langle \phi_i(x), \phi_i(y) \rangle$

let ϕ denote the concatenation of $\phi_1, \phi_2 \dots \phi_n$ which is a feature map for kernel k.

$\phi(x) = (\phi_1(x), \phi_2(x) \dots \phi_n(x))$

we know that

$k(x, y) = \langle \phi(x), \phi(y) \rangle$

$= \langle (\phi_1(x) \dots \phi_n(x)), (\phi_1(y) \dots \phi_n(y)) \rangle$

$= \langle \phi_1(x), \phi_1(y) \rangle + \dots$
$\langle \phi_n(x), \phi_n(y) \rangle$

$k(x, y) = k_1(x, y) + \dots k_n(x, y)$

This shows that sum of kernels is also a valid kernel.

(4) Given

inner product

$$<(x_1, x_2), (y_1, y_2)>_{H_1 \oplus H_2} = <x_1, y_1>_{H_1} + <x_2, y_2>_{H_2}$$

— (1)

first we will prove all the properties of inner products then completeness of $H_1 \oplus H_2$ space

(i) $x_1, y_1 \in H_1$ & $x_2, y_2 \in H_2$

$$<(x_1, x_2), (y_1, y_2)> = <x_1, y_1> + <x_2, y_2>$$

(from def (1))

(H₁ and H₂ follow the properties of inner products)

$$= <y_1, x_1> + <y_2, x_2>$$

$$<(x_1, x_2), (y_1, y_2)> = <(y_1, y_2), (x_1, x_2)>$$

(ii) $<(x_1, x_2), \alpha(y_1, y_2) + \beta(z_1, z_2)>$

$\alpha, \beta \in \mathbb{R}$

$$= <(x_1, x_2), (\alpha y_1 + \beta z_1, \alpha y_2 + \beta z_2)>$$

$$= <x_1, \alpha y_1 + \beta z_1> +$$

$$<x_2, \alpha y_2 + \beta z_2>$$

(From definition (1))

$$= \alpha <x_1, y_1> + \beta <x_1, z_1>$$
$$+ \alpha <x_2, y_2> + \beta <x_2, z_2>$$

(H₁ and H₂ follow inner products properties)

$= \alpha < (x, x_2), (y_1 y_2) > +$
$\beta < (x, x_2), (z_1 z_2) >$

(from def ①)

$< (x_1 x_2), \alpha (y_1, y_2) + \beta (z_1, z_2) >$

$= \alpha < (x, x_2), (y_1 y_2) > +$
$\beta < (x_1 x_2), (z_1 z_2) >$

(iii) $< (x_1 x_2), (x_1 x_2) >$
$= < x_1 x_1 > + < x_2 x_2 >$
> 0

(H_1 and H_2 individualy follow properties of inner products)

either $< x_1 x_1 > > 0$ or $< x_2, x_2 > > 0$
so $(x_1 x_2) \neq (0, 0)$

lets prove convergence of the cauchy sequence in $H_1 \oplus H_2$.
let $\{(x_n, y_n)\} \in H_1 \oplus H_2$ denote some cauchy sequence ~~in~~.
Thus x_n and y_n individually converge to x and y in H_1 and H_2.
let $(x, y) \in H_1 \oplus H_2$ ~~and~~

$d(x_n, x)$ ~~$d\,x$~~ $= \| x_n - x \|_{H_1} \to 0$
$d(y_n, y) = \| y_n - y \|_{H_2} \to 0$

let $\| x_n - x \| =$ ~~$< x_n, x >^{1/2}$~~ $< (x_n - x), (x_n - x) >^{1/2}$

$$d((x_n, y_n), (x, y))$$
$$= \|(x_n, y_n) - (x, y)\|_{H_1 \oplus H_2}$$
$$= \left[\langle (x_n, y_n) - (x, y), (x_n, y_n) - (x, y)\rangle\right]^{1/2}$$
$$\cancel{= \langle (x_n - x)(y_n - y)\rangle}$$
$$= \left[\langle (x_n - x, y_n - y), (x_n - x, y_n - y)\rangle\right]^{1/2}$$
$$= \left[\langle (x_n - x), (x_n - x)\rangle + \langle (y_n - y), (y_n - y)\rangle\right]^{1/2}$$
$$= \left[\|x_n - x\|_{H_1}^2 + \|y_n - y\|_{H_2}^2\right]^{1/2}$$
$$d((x_n, y_n), (x, y)) = \left[\|x_n - x\|_{H_1}^2 + \|y_n - y\|_{H_2}^2\right]^{1/2} = 0$$

This terms inside root converges to 0

Therefore $(x_n, y_n) \longrightarrow (x, y)$

$\cancel{d((x_n, y_n), (x, y)) \longrightarrow 0}$

This proves the completeness of the ~~Hilbert~~ space ~~space~~ $H_1 \oplus H_2$.
which means $H_1 \oplus H_2$ is a hilbert space.
with inner product as
$$\langle (x_1, x_2), (y_1, y_2)\rangle_{H_1 \oplus H_2} = \langle x_1, y_1\rangle_{H_1} + \langle x_2, y_2\rangle_{H_2}$$