# ENERGY STABLE SEMI-IMPLICIT SCHEMES FOR ALLEN-CAHN AND FRACTIONAL CAHN-HILLIARD EQUATIONS

XINYU CHENG

ABSTRACT. It is well known that the Cahn-Hilliard equation facilitates the study of phase separation of a two-phase or a multiple-phase mixture and other material science phenomena, while the Allen-Cahn equation models the competition of crystal grains with different orientations in an annealing process. An important property of the solutions to those two equations is that the energy functional decreases in time. To study these solutions, researchers have developed different numerical schemes to give accurate approximations, since analytic solutions are only available in a very few simple cases. However, not all schemes satisfy the energy-decay property, which is an important standard to determine whether the scheme is stable. In recent work, a semi-implicit scheme for the Cahn-Hilliard equation was developed satisfying the energy-decay property. In this paper, we extend this semi-implicit scheme to the Allen-Cahn equation and fractional Cahn-Hilliard equation also with a rigorous proof of similar energy-decay property. Results in 2 and 3 spatial dimensions are shown. Moreover, this semi-implicit scheme is practical and can be applied to more general diffusion equations while preserving the energy-decay stability.

## 1. Introduction

In this work we will consider two phase field models: Allen-Cahn (AC) and Cahn-Hilliard (CH) equations. The (AC) model was developed in [3] by Allen and Cahn to study the competition of crystal grain orientations in an annealing process separation of different metals in a binary alloy; while the (CH) was introduced in [8] by Cahn and Hilliard to describe the process of phase separation of different metals in a binary alloy. These equations are presented as:

$$\begin{cases} \partial_t u = \nu\Delta u - f(u), & (x,t) \in \Omega \times (0,\infty) \\ u(x,0) = u_0 \end{cases}, \tag{AC}$$

and

$$\begin{cases} \partial_t u = \Delta(-\nu\Delta u + f(u)), & (x,t) \in \Omega \times (0,\infty) \\ u(x,0) = u_0 \end{cases}, \tag{CH}$$

where $u(x,t)$ is a real valued function and values of $u$ in $(-1,1)$ represent a mixture of the two phases, with $-1$ representing the pure state of one phase and $+1$ representing the pure state of the other phase. Vector position $x$ is in the spatial domain $\Omega$, which is taken to be two or three dimensional periodic domain in this work, and $t$ is time. Here $\nu$ is a small parameter, occasionally we denote $\varepsilon = \sqrt{\nu}$ to represent an average distance over which phases mix. The energy term $f(u)$ is often chosen to be

$$f(u) = F'(u) = u^3 - u \ , \ F(u) = \frac{1}{4}(u^2-1)^2.$$

It is well known that, as $\varepsilon \to 0$, the limiting problem of (AC) is given by a mean curvature flow while the limiting problem of (CH) becomes Mullins-Sekerka problem; we refer to [22] for AC and [29], [2] for CH, where asymptotic and rigorous analysis are provided. Although the limiting behavior of AC and CH are well known, there are related materials science models that are studied only numerically and this current work presents an idea about how to approach these models in an appropriate way numerically.

In this paper, we consider the spatial domain $\Omega$ to be the two dimensional $2\pi$-periodic torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ and three dimensional $2\pi$-periodic torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$. In fact, our proof can be applied to more general settings such as Dirichlet and Neumann boundary conditions in a bounded domain. However, considering the periodic domain allows the use of efficient and accurate Fourier-spectral numerical methods; moreover, periodic domain is often appropriate for application questions, which involve the formation of micro-structure away from physical boundaries.

As a standard result, the mass of the smooth solution of Cahn-Hilliard (CH) is conserved, i.e. $\frac{d}{dt}M(t) \equiv 0$ , $M(t) = \int_\Omega u(x,t)\ dx$. This represents the conservation of the two phases in the mixture. In particular, $M(t) \equiv 0$ if $M(0) = 0$ and hence oftentimes zero-mean initial data (equal amounts of both phases) will be considered as a simpler but representative case. The associated energy functional of CH equation is given by

$$E(u) = \int_\Omega \left( \frac{1}{2}\nu|\nabla u|^2 + F(u) \right)\ dx.$$

Assuming $u(x,t)$ is a smooth solution with zero mean, one can deduce

$$\frac{d}{dt}E(u(t)) + \int_\Omega |\nabla(-\nu\Delta u + f(u))|^2\ dx = 0,$$

which implies energy decay: $\frac{d}{dt}E(u(t)) \leq 0$, and hence contributes to the existence of global solutions to Cahn-Hilliard equation as it provides an a-priori $H^1$-norm bound. In this sense, the energy decay property is an important index for whether a numerical scheme is “stable” or not.

In comparison, Allen-Cahn equation does not share the mass conservation property; however, it still follows the energy decay property with the same energy functional. Moreover, the solution to the fractional Cahn-Hilliard equation (FCH) satisfies both mass conservation and energy properties, see [1, 5] for example. The fractional Cahn-Hilliard equation (FCH) is defined as the following:

$$\begin{cases} \partial_t u = \nu\Delta\left((-\Delta)^\alpha u + (-\Delta)^{\alpha-1} f(u)\right)\ ,\ 0 < \alpha \leq 1 \\ u(x,0) = u_0\ . \end{cases} \tag{FCH}$$

The difficulty in dealing with this FCH model arises from the non-local behavior of fractional laplacian, where fractional laplacian on the torus is given from the Fourier side: for $x \in \mathbb{T}^d$, $(-\Delta)^\alpha f(x) = \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d} |k|^{2\alpha}\hat{f}(k)e^{-ik\cdot x}$. The convention of Fourier series is given in the next section.

Various approaches have been developed to study numerical simulations on Cahn-Hilliard and related field models, [14, 19, 21, 35, 32, 18, 17, 9, 36, 4, 30, 11, 13, 12] as examples, in which different approaches are applied to the time stepping including fully explicit (forward Euler) scheme, fully implicit (backward Euler) scheme, finite element scheme and convex splitting scheme; and different schemes are used for the spatial discretization including the Fourier-spectral method. These numerical approximations should give accurate results to the values and qualitative features of the solution; a key feature is energy decay.

From the analysis point of view, Feng and Prohl [17] introduced a semi-discrete in time and fully spatially discrete finite element method for Cahn-Hilliard equation(CH) where they obtained an error bound of size of powers of $1/\nu$.

Explicit time-stepping schemes require strict time-step restrictions and do not obey energy decay in general. To guarantee the energy decay property and increase the time step, a good alternative is to use semi-implicit schemes in which the linear term is implicit (such as backward time differentiation) and the nonlinear term is treated explicitly. Having only a linear implicit at every time step has computational advantages, as suggested in [9], Chen and Shen considered a semi-implicit Fourier-spectral scheme for (CH).

On the other hand, semi-implicit schemes can lose stability for large time steps and thus smaller time steps are needed in practice. To resolve this problem, semi-implicit methods with better stability have been introduced, e.g. [20, 33, 21, 35, 36, 32, 31, 34]. Specifically, [21, 35, 36, 32] give different semi-implicit Fourier-spectral schemes, which involved different stabilizing terms of different “size”, that preserve the energy decay property (we say these schemes are “energy stable”). However, those works either require a strong Lipschitz condition on the nonlinear source term, or require certain $L^\infty$ bounds on the numerical solutions.

In the seminal works [25, 26, 27, 28], Li et al. developed a large time-stepping semi-implicit Fourier-spectral scheme for Cahn-Hilliard equation and proved that it preserves energy decay with no a-priori assumptions (unconditional stability). The proof uses tools from harmonic analysis in [6, 7], and introduces a novel energy bootstrap scheme in order to obtain a $L^\infty$-bound of the numerical solution.

The scheme for (CH) has the form:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = -\nu\Delta^2 u^{n+1} + A\Delta(u^{n+1}-u^n) + \Delta(f(u^n)) \ , \ n \geq 0 \\ u^0 = u_0 \ . \end{cases} \tag{1.1}$$

Here $\tau$ is the time step and $A$ is a large coefficient for the $O(\tau)$ stabilizing term. As a result of their work, the energy decay can still be satisfied with a well-chosen large number $A$, with at least a size of $O(1/\nu|\log(\nu)|^2)$, or $c/\nu|\log(\nu)|^2$ for some positive constant $c$ that depends on the initial conditions.

However, their arguments cannot apply to Allen Cahn equation (AC) directly: this is due to lack of mass conservation. Our work extends their first order semi-implicit scheme to the related Allen-Cahn equation (AC). Following the same path the fractional Cahn-Hilliard equation (FCH) can be studied as well. We will also consider a 3D case by extending Li et al. work in [24], moreover we will study the stability of second order schemes introduced by Li et al. in [23], for the AC model.

**Remark 1.1.** *As a remark, in fractional CH case, as $\alpha \to 0$, (FCH) becomes the zero-mass projected Allen-Cahn equation and for $\alpha = 1$, it coincides the original Cahn-Hilliard equation. Roughly speaking, the fractional Cahn-Hilliard equation is an interpolation of the zero-mass projected Allen-Cahn and Cahn-Hilliard equations. Here the zero-mass projected Allen-Cahn equation is defined as:*

$$\begin{cases} \partial_t u = \Pi_0\left(\nu\Delta u - f(u)\right) \\ u(x,0) = u_0 \ , \end{cases} \tag{1.2}$$

*where $\Pi_0$ is the zero mass projector, i.e. $\Pi_0(g) = g - \int_\Omega g \ dx$, or $= \frac{1}{(2\pi)^d}\sum_{|k|\geq 1}\hat{g}(k)e^{ik\cdot x}$ from the Fourier side. The difference between* (AC) *and the zero-mass projected Allen-Cahn equation* (1.2) *results from mass conservation.*

**Remark 1.2.** *More general cases can be discussed. Roughly speaking we can define a general "gradient" operator $\mathcal{G}$ and rewrite the equation as :*

$$\begin{cases} \partial_t u = \mathcal{G}\left(\nu\Delta u - f(u)\right) \\ u(x,0) = u_0 \end{cases} . \tag{1.3}$$

*When $\mathcal{G} = id$, the identity map, (1.3) becomes the Allen-Cahn equation; when $\mathcal{G} = (-\Delta)^\alpha$, (1.3) becomes the fractional Cahn-Hilliard equation as discussed above. And the corresponding semi-implicit scheme is*

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = \mathcal{G}\left(\nu\Delta u^{n+1} - f(u^n)\right) - A\mathcal{G}(u^{n+1}-u^n) \ , \ n \geq 0 \\ u^0 = u_0 \end{cases} . \tag{1.4}$$

The main result of this paper states that for any fixed time step $\tau$, we can always define a large constant $A$ independent of $\tau$ in (1.4), such that the numerical solution will be stable in the sense of satisfying the energy-decay condition for "gradient" cases of AC and fractional CH (2D and 3D cases). For completeness, error estimates will be discussed. We refer to [10] for full details.

## 2. Notation and preliminaries

Throughout this paper we denote $L^p(\Omega)$ for $1 \leq p < \infty$ as the space consists of all complex-valued measurable functions that satisfy $\int_\Omega |f(x)|^p \ dx < \infty$ and $\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \ dx\right)^{1/p}$.

Similarly, we use the weak derivative in the following sense: For $u, \ v \in L^1_{loc}(\Omega)$, (i.e they are locally integrable); $\forall \phi \in C_0^\infty(\Omega)$, i.e $\phi$ is infinitely differentiable (smooth) and compactly supported; and

$$\int_\Omega u(x) \ \partial^\alpha \phi(x) \ dx = (-1)^{\alpha_1+\cdots+\alpha_n}\int_\Omega v(x) \ \phi(x) \ dx,$$

then $v$ is defined to be the weak partial derivative of $u$, denoted by $\partial^\alpha u$. Suppose $u \in L^p(\Omega)$ and all weak derivatives $\partial^\alpha u$ exist for $|\alpha| = \alpha_1 + \cdots + \alpha_n \le k$ , such that $\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \le k$, then we denote $u \in W^{k,p}(\Omega)$ to be the standard Sobolev space. The corresponding norm of $W^{k,p}(\Omega)$ is :

$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u|^p \, dx \right)^{\frac{1}{p}} .$$

For $p = 2$ case, we use the convention $H^k(\Omega)$ to denote the space $W^{k,2}(\Omega)$. For more details, we refer to section 5 of [16]. We often use $D^m u$ to denote any differential operator $D^\alpha u$ for any $|\alpha| = m$: $D^2$ denotes $\partial^2_{x_i x_j} u$ for $1 \le i,j \le d$, as an example.

In this paper we use the following convention for Fourier expansion on $\mathbb{T}^d$:

$$f(x) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{ik\cdot x} \ , \ \hat{f}(k) = \int_\Omega f(x) e^{-ik\cdot x} \, dx \ .$$

By taking advantage of Fourier expansion, we use the equivalent $H^s$-norm and $\dot{H}^s$-semi-norm of function $f$ by

$$\|f\|_{H^s} = \frac{1}{(2\pi)^{d/2}} \left( \sum_{k \in \mathbb{Z}^d} (1 + |k|^{2s}) |\hat{f}(k)|^2 \right)^{\frac{1}{2}} \ , \ \|f\|_{\dot{H}^s} = \frac{1}{(2\pi)^{d/2}} \left( \sum_{k \in \mathbb{Z}^d} |k|^{2s} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} .$$

The equivalence of two norms are well known.

**Lemma 2.1** (Sobolev inequality on $\mathbb{T}^d$). *Let $0 < s < d$ and $f \in L^q(\mathbb{T}^d)$ for any $\frac{d}{d-s} < p < \infty$, then*

$$\| \langle \nabla \rangle^{-s} f \|_{L^p(\mathbb{T}^d)} \lesssim_{s,p,d} \|f\|_{L^q(\mathbb{T}^d)} \ , \ \textit{where } \frac{1}{q} = \frac{1}{p} + \frac{s}{d} \ ,$$

*where $\langle \nabla \rangle^{-s}$ denotes $(1-\Delta)^{-\frac{s}{2}}$ and $A \lesssim_{s,p,d} B$ is defined as $A \le C_{s,p,d} \, B$ where $C_{s,p,d}$ is a constant dependent on $s, p$ and $d$.*

**Remark 2.2.** *Note that the this Sobolev inequality is slightly different from the standard version. On the Fourier side the symbol of $\langle \nabla \rangle^{-s}$ is given by $(1 + |k|^2)^{-\frac{s}{2}}$. This inequality can be proved by Littlewood-Paley decomposition and one can refer to [25] for a detailed proof. In particular, $\|f\|_{\infty(\mathbb{T}^d)} \lesssim \|f\|_{H^2(\mathbb{T}^d)}$, known as Morrey's inequality in section 5 in [16].*

**Lemma 2.3** (Discrete Grönwall's inequality). *Let $\tau > 0$ and $y_n \ge 0$, $\alpha_n \ge 0$, $\beta_n \ge 0$ for $n = 1, 2, 3 \cdots$. Suppose*

$$\frac{y_{n+1} - y_n}{\tau} \le \alpha_n y_n + \beta_n \ , \ \forall \ n \ge 0 \ .$$

*Then for any $m \ge 1$, we have*

$$y_m \le \exp\left( \tau \sum_{n=0}^{m-1} \alpha_n \right) \left( y_0 + \sum_{k=0}^{m-1} \beta_k \right) .$$

**Remark 2.4.** *We sketch the proof here. By the assumption, it follows that for $n \ge 0$,*

$$y_{n+1} \le (1 + \alpha_n \tau) y_n + \tau \beta_n \le e^{\tau \alpha_n} y_n + \tau \beta_n;$$

*therefore, we obtain*

$$\exp\left( -\tau \sum_{j=0}^{n} \alpha_j \right) y_{n+1} \le \exp\left( -\tau \sum_{j=0}^{n-1} \alpha_j \right) y_n + \exp\left( -\tau \sum_{j=0}^{n} \alpha_j \right) \beta_n.$$

*Conducting a telescoping summation, we derived the desired result. We refer to [25] for details of the proof.*

## 3. Stability of a first order semi-implicit scheme on the 2D Allen-Cahn equation

Recall that the Allen-Cahn equation (AC) is as follows:

$$\begin{cases} \partial_t u = \nu \Delta u - f(u) \\ u(x,0) = u_0 \end{cases} .$$

Here $f(u) = u^3 - u$, and the spatial domain $\Omega$ is taken to be the two dimensional $2\pi-$periodic torus $\mathbb{T}^2$. The corresponding energy is defined by $E(u) = \int_\Omega (\frac{\nu}{2}|\nabla u|^2 + F(u))\,dx$ , where $F(u) = \frac{1}{4}(u^2-1)^2$, the anti-derivative of $f(u)$.

As is well known, the energy satisfies $E(u(t)) \le E(u(s))$ ,$\forall\ t \ge s$, which gives a priori bound. Now we consider a stabilized semi-implicit scheme introduced in [25]. The form is the following:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = \nu \Delta u^{n+1} - A(u^{n+1} - u^n) - \Pi_N f(u^n) \\ u^0 = \Pi_N u_0 \end{cases} . \tag{3.1}$$

where $\tau$ is the time step and $A > 0$ is the coefficient for the $O(\tau)$ regularization term. For $N \ge 2$, define

$$X_N = \text{span}\left\{\cos(k\cdot x)\ , \sin(k\cdot x) : \quad k = (k_1,k_2) \in \mathbb{Z}^2\ ,\ |k|_\infty = \max\{|k_1|,|k_2|\} \le N\right\} .$$

Define the $L^2$ projection operator $\Pi_N :\ L^2(\Omega) \to X_N$ by $(\Pi_N u - u, \phi) = 0\quad \forall \phi \in X_N$, where $(\cdot,\cdot)$ denotes the $L^2$ inner product on $\Omega$. In other words, the projection operator $\Pi_N$ is just the truncation of Fourier modes $|k|_\infty \le N$. $\Pi_N u_0 \in X_N$ and by induction, we have $u^n \in X_N$ ,$\forall n \ge 0$.

**Theorem 3.1** (Unconditional energy stability for AC)**.** *Consider (3.1) with $\nu > 0$ and assume $u_0 \in H^2(\mathbb{T}^2)$. Then there exists a constant $\beta_0$ depending only on the initial energy $E_0 = E(u_0)$ such that if*

$$A \ge \beta \cdot \left(\|u_0\|_{H^2}^2 + \nu^{-1}|\log \nu| + 1\right)\ ,\ \beta \ge \beta_0$$

*then $E(u^{n+1}) \le E(u^n)$, $\forall n \ge 0$, where $E$ is defined above.*

**Remark 3.2.** *Similar to [25], the stability result is valid for any time step $\tau$. Our choice of $A$ is independent of $\tau$ as long as it has size of $O\left(1/\nu|\log(\nu)|\right)$. Note that the choice of $A$ may not be optimal and further work can be done.*

**Remark 3.3.** *No mean zero assumption is needed for $u_0$ due to the lack of mass conservation.*

To prove this we need a log-type interpolation inequality:

**Lemma 3.4** (Log-type interpolation)**.** *For all $f \in H^s(\mathbb{T}^2)$ , $s > 1$, then*

$$\|f\|_\infty \le C_s \cdot \left(\|f\|_{\dot{H}^1}\sqrt{\log(\|f\|_{\dot{H}^s} + 3)} + |\hat{f}(0)| + 1\right) .$$

*Here $C_s$ is a constant which only depends on $s$.*

*Proof.* To prove the lemma, we write $f(x) = \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2} \hat{f}(k)\ e^{ik\cdot x}$, i.e. the Fourier series of $f$, which is convergent pointwisely to $f$. We then estimate as follows:

$$\begin{aligned}
\|f\|_\infty &\le \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2}|\hat{f}(k)| \\
&\le \frac{1}{(2\pi)^2}\left(|\hat{f}(0)| + \sum_{0<|k|\le N}|\hat{f}(k)| + \sum_{|k|>N}|\hat{f}(k)|\right) \\
&\lesssim |\hat{f}(0)| + \sum_{0<|k|\le N}(|\hat{f}(k)||k|\cdot|k|^{-1}) + \sum_{|k|>N}(|\hat{f}(k)||k|^s\cdot|k|^{-s}) \\
&\lesssim |\hat{f}(0)| + \left(\sum_{0<|k|\le N}|\hat{f}(k)|^2|k|^2\right)^{\frac12}\cdot\left(\sum_{0<|k|\le N}|k|^{-2}\right)^{\frac12} + \left(\sum_{|k|>N}|\hat{f}(k)|^2|k|^{2s}\right)^{\frac12}\cdot(\sum_{|k|>N}|k|^{-2s})^{\frac12} \\
&\lesssim |\hat{f}(0)| + \frac{1}{N^{s-1}}\left(\sum_{|k|>N}|\hat{f}(k)|^2|k|^{2s}\right)^{\frac12} + \left(\sum_{0<|k|\le N}|\hat{f}(k)|^2|k|^2\right)^{\frac12}\cdot\sqrt{\log(N+3)} \\
&\lesssim |\hat{f}(0)| + \frac{1}{N^{s-1}}\|f\|_{\dot{H}^s} + \sqrt{\log(N+3)}\|f\|_{\dot{H}^1}\ .
\end{aligned}$$

If $\|f\|_{\dot{H}^s} \le 3$, we can simply take $N = 1$; otherwise take $N^{s-1}$ close to $\|f\|_{\dot{H}^s}$ . As a remark, this lemma is a variation of the log-type Bernstein inequality in [25] and [26]. $\square$

We will prove Theorem 3.1 by induction. To start with, let us recall the numerical scheme (3.1):

$$\frac{u^{n+1}-u^n}{\tau} = \nu\Delta u^{n+1} - A(u^{n+1}-u^n) - \Pi_N f(u^n).$$

Here $\Pi_N$ is truncation of Fourier modes of $L^2$ functions to $|k|_\infty \le N$. Multiply the equation by $(u^{n+1}-u^n)$ and integrate over $\Omega$, one has

$$\frac{1}{\tau}\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 = \nu\int_{\mathbb{T}^2}\Delta u^{n+1}(u^{n+1}-u^n) - A\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 - \left(\Pi_N f(u^n), u^{n+1}-u^n\right).$$

Because $u^n$ is periodic, (as $u^n \in X_N$), hence by integration by parts, we have

$$\left(\frac{1}{\tau}+A\right)\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 + \nu\int_{\mathbb{T}^2}\nabla u^{n+1}\nabla(u^{n+1}-u^n) = -\left(\Pi_N f(u^n), u^{n+1}-u^n\right)\ .$$

Note $\nabla u^{n+1}\nabla(u^{n+1}-u^n) = \frac12\left(|\nabla u^{n+1}|^2 - |\nabla u^n|^2 + |\nabla(u^{n+1}-u^n)|^2\right)$, we have

$$\left(\frac{1}{\tau}+A\right)\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 + \frac{\nu}{2}\int_{\mathbb{T}^2}|\nabla u^{n+1}|^2 - |\nabla u^n|^2 + |\nabla(u^{n+1}-u^n)|^2 = -\left(\Pi_N f(u^n), u^{n+1}-u^n\right)\ .$$

Moreover, every $u^n \in X_N$, we have

$$\left(\frac{1}{\tau}+A\right)\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 + \frac{\nu}{2}\int_{\mathbb{T}^2}|\nabla u^{n+1}|^2 - |\nabla u^n|^2 + |\nabla(u^{n+1}-u^n)|^2 = -\left(f(u^n), u^{n+1}-u^n\right).$$

Now, by fundamental theorem of calculus and integration by parts,

$$\begin{aligned}
F(u^{n+1}) - F(u^n) &= f(u^n)(u^{n+1}-u^n) + \int_{u^n}^{u^{n+1}} f'(s)(u^{n+1}-s)\ ds \\
&= f(u^n)(u^{n+1}-u^n) + \int_{u^n}^{u^{n+1}} (3s^2-1)(u^{n+1}-s)\ ds \\
&= f(u^n)(u^{n+1}-u^n) + \frac14(u^{n+1}-u^n)^2\left(3(u^n)^2 + (u^{n+1})^2 + 2u^nu^{n+1} - 2\right)\ .
\end{aligned}$$

Combine previous two equations, and denote $E(u^n)$ by $E^n$ we have

$$
\begin{aligned}
&\left(\frac{1}{\tau}+A\right)\|u^{n+1}-u^n\|_{L^2}^2+\frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2+\frac{\nu}{2}\|\nabla u^{n+1}\|_{L^2}^2-\frac{\nu}{2}\|\nabla u^n\|_{L^2}^2\\
&+\int_{\mathbb{T}^2}F(u^{n+1})-F(u^n)=\frac{1}{4}\left((u^{n+1}-u^n)^2,3(u^n)^2+(u^{n+1})^2+2u^nu^{n+1}-2\right)\\
\text{Note }&\frac{\nu}{2}\|\nabla u^{n+1}\|_{L^2}^2+\int_{\mathbb{T}^2}F(u^{n+1})=E(u^{n+1})=E^{n+1}\\
\Longrightarrow\ &(\frac{1}{\tau}+A+\frac{1}{2})\|u^{n+1}-u^n\|_{L^2}^2+\frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2+E^{n+1}-E^n\\
&=\frac{1}{4}\left((u^{n+1}-u^n)^2,3(u^n)^2+(u^{n+1})^2+2u^nu^{n+1}\right)\\
&\le\|u^{n+1}-u^n\|_{L^2}^2\left(\|u^n\|_\infty^2+\frac{1}{2}\|u^{n+1}\|_\infty^2\right)\ .
\end{aligned}
$$

To show $E^{n+1}\le E^n$, clearly it suffices to show

$$
\frac{1}{\tau}+A+\frac{1}{2}\ge\frac{3}{2}\max\left\{\|u^n\|_\infty^2\ ,\ \|u^{n+1}\|_\infty^2\right\}\ . \tag{3.2}
$$

Note that $E^0=E(\Pi_Nu_0)$ while $E_0=E(u_0)$ and in general $E_0\neq E^0$. We claim that

**Proposition 3.1.** *Suppose $E^0=E(\Pi_Nu_0)$ and $E_0=E(u_0)$ as defined above, the following inequality holds:*

$$
\sup_N\ E(\Pi_Nu_0)\lesssim 1+E_0,\textit{where } u_0\in H^1(\mathbb{T}^2)\ .
$$

*Proof.* We rewrite $\Pi_Nu_0$ as $\frac{1}{(2\pi)^2}\sum_{|k|\le N}\widehat{u_0}(k)e^{ik\cdot x}$ , namely the Dirichlet partial sum of $u_0$.

$$
\|\nabla\left(\Pi_Nu_0\right)\|_{L^2(\mathbb{T}^2)}^2=\frac{1}{(2\pi)^2}\sum_{|k|\le N}|k|^2|\widehat{u_0}(k)|^2\le\frac{1}{(2\pi)^2}\sum_{|k|\in\mathbb{Z}^2}|k|^2|\widehat{u_0}(k)|^2=\|\nabla\left(u_0\right)\|_{L^2(\mathbb{T}^2)}^2\ .
$$

On the potential energy part, by the Sobolev inequality Lemma 2.1, $\|u_0\|_{L^4(\mathbb{T}^2)}\lesssim\|u_0\|_{H^1(\mathbb{T}^2)}$, this shows $u_0\in L^4(\mathbb{T}^2)$ and hence the Dirichlet partial sum $\Pi_Nu_0$ converges to $u_0$ in $L^4(\mathbb{T}^2)$. Then $\|\Pi_Nu_0\|_{L^4(\mathbb{T}^2)}\to\|u_0\|_{L^4(\mathbb{T}^2)}$, which leads to $\sup_N\|\Pi_Nu_0\|_{L^4(\mathbb{T}^2)}<\infty$. By the Uniform Boundedness Principle, we derive $\sup_N\|\Pi_N\|<\infty$, i.e. $\sup_N\|\Pi_Nu_0\|_{L^4(\mathbb{T}^2)}\le c\|u_0\|_{L^4(\mathbb{T}^2)}$ for an absolute constant $c$. Combining the two estimates above we conclude the proof for the claim. See [24] for an alternate proof; this claim holds for 3D case as well with a similar proof. □

We rewrite the numerical scheme (3.1) as follows:

$$
u^{n+1}=\frac{1+A\tau}{1+A\tau-\nu\tau\Delta}u^n-\frac{\tau}{1+A\tau-\nu\tau\Delta}\Pi_N[f(u^n)]\ . \tag{3.3}
$$

By the interpolation lemma (Lemma 3.4), to control $\|u^{n+1}\|_\infty$ and $\|u^n\|_\infty$, we may consider $\dot{H}^1$-norm and $\dot{H}^{\frac{3}{2}}$-norm together with 0th-mode $|\widehat{u^{n+1}}(0)|$. We start by estimating $|\widehat{u^{n+1}}(0)|$,

$$
\begin{aligned}
|\widehat{u^{n+1}}(0)|&\le|\widehat{u^n}(0)|+\frac{\tau}{1+A\tau}|\widehat{f(u^n)}(0)|\\
&\le|\widehat{u^n}(0)|+\frac{1}{A}|\widehat{f(u^n)}(0)|\\
&\le\left|\int_{\mathbb{T}^2}u^n\ dx\right|+\left|\int_{\mathbb{T}^2}u^n-(u^n)^3\ dx\right|\\
&\lesssim 1+\left|\int_{\mathbb{T}^2}(u^n)^2\ dx\right|^{\frac{1}{2}}+\left|\int_{\mathbb{T}^2}(1-(u^n)^2)^2\ dx\right|^{\frac{1}{2}}\\
&\lesssim 1+\sqrt{E^n}\ .
\end{aligned}
$$

**Lemma 3.5.** *There is an absolute constant $c_1 > 0$ such that for any $n \geq 0$*

$$\begin{cases} \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \leq c_1 \cdot \left(\dfrac{A+1}{\nu} + \dfrac{1}{\nu\tau}\right) \cdot (E^n + 1) \\ \|u^{n+1}\|_{\dot{H}^1(\mathbb{T}^2)} \leq \left(1 + \dfrac{1}{A} + \dfrac{3}{A}\|u^n\|_\infty^2\right) \cdot \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} \ . \end{cases}$$

*Proof.* As $0th$-mode will not contribute to $\dot{H}^1$ norm and $\dot{H}^{\frac{3}{2}}$ norm, we can just consider Fourier modes $|k| \geq 1$ from the Fourier side.

Use the symbol $f \lesssim g$ to denote $f \leq c \cdot g$ with $c$ being a constant. We then obtain that

$$\begin{cases} \dfrac{(1+A\tau)|k|^{\frac{3}{2}}}{1+A\tau+\nu\tau|k|^2} \lesssim \dfrac{1+A\tau}{\nu\tau} \\ \dfrac{\tau|k|^{\frac{3}{2}}}{1+A\tau+\nu\tau|k|^2} \lesssim \dfrac{\tau}{\tau\nu}|k|^{-\frac{1}{2}} = \dfrac{1}{\nu}|k|^{-\frac{1}{2}} \ . \end{cases}$$

Hence

$$\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \lesssim \left(\frac{1+A\tau}{\nu\tau}\right)\|u^n\|_{L^2(\mathbb{T}^2)} + \frac{1}{\nu}\|\langle\nabla\rangle^{-\frac{1}{2}} f(u^n)\|_{L^2(\mathbb{T}^2)}. \tag{3.4}$$

Here the notaion $\langle\nabla\rangle^s = (1-\Delta)^{\frac{s}{2}}$ , corresponds to the Fourier side $(1+|k|^2)^{s/2}$. Note that

$$\|u^n\|_{L^2(\mathbb{T}^2)} \lesssim \int_{\mathbb{T}^2} \frac{1}{4}(u^4 - 2u^2 + 1)\ dx + 1 \lesssim E^n + 1$$

by Cauchy-Schwarz inequality. On the other hand, by the Sobolev inequality

$$\begin{aligned} \|\langle\nabla\rangle^{-\frac{1}{2}} f(u^n)\|_{L^2(\mathbb{T}^2)} &\lesssim \|f(u^n)\|_{L^{\frac{4}{3}}(\mathbb{T}^2)} = \|(u^n)^3 - u^n\|_{L^{\frac{4}{3}}(\mathbb{T}^2)} \\ &= \left(\int_{\mathbb{T}^2} ((u^n)^3 - u^n)^{\frac{4}{3}}\ dx\right)^{\frac{3}{4}} \\ &\lesssim E^n + 1\ . \end{aligned}$$

Therefore (3.4) becomes

$$\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \lesssim \left(\frac{1+A\tau}{\nu\tau} + \frac{1}{\nu}\right)(E^n + 1).$$

Similarly, we get

$$\begin{cases} \dfrac{(1+A\tau)|k|}{1+A\tau+\nu\tau|k|^2} \lesssim |k| \\ \dfrac{\tau|k|}{1+A\tau+\nu\tau|k|^2} \lesssim \dfrac{\tau}{\tau A}|k| = \dfrac{1}{A}|k|\ . \end{cases}$$

This implies

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^1(\mathbb{T}^2)} &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \frac{1}{A}\|f(u^n)\|_{\dot{H}^1(\mathbb{T}^2)} \\ &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \frac{1}{A}\|\nabla(f(u^n))\|_{L^2(\mathbb{T}^2)} \\ &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \frac{1}{A}\|(3(u^n)^2 - 1)\cdot(\nabla u^n)\|_{L^2(\mathbb{T}^2)} \\ &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + (\frac{1}{A} + \frac{3\|u\|_\infty^2}{A})\|u^n\|_{\dot{H}^1(\mathbb{T}^2)} \\ &\lesssim \left(1 + \frac{1}{A} + \frac{3\|u\|_\infty^2}{A}\right)\|u^n\|_{\dot{H}^1(\mathbb{T}^2)}\ . \end{aligned}$$

□

Now we will complete the proof for **Theorem** 3.1 by induction:
**Step 1:** The induction $n \to n+1$ step. Assuming $E^n \le E^{n-1} \le \cdots \le E^0$ and $E^n \le \sup_N E(\Pi_N u_0)$, we will show $E^{n+1} \le E^n$. This implies $\|u^n\|_{\dot{H}^1}^2 = \|\nabla u^n\|_{L^2}^2 \le \frac{2E^n}{\nu} \le \frac{2E^0}{\nu}$.

By Lemma 3.4, use the notation $f \lesssim_{E^0} g$ to denote that $f \le C(E^0)\cdot g$ for some constant $C(E^0)$ depending only on $E^0$, we have

$$\begin{aligned} \|u^n\|_\infty^2 &\lesssim \|u^n\|_{\dot{H}^1}^2 \left(\sqrt{\log(3 + c_1\left(\frac{1}{\nu\tau} + \frac{A+1}{\nu}\right)(E^n+1))}\right)^2 + E^n + 1 \\ &\lesssim \frac{2E^0}{\nu}\left(1 + \log(A) + \log(\frac{1}{\nu}) + (\log(1+\frac{1}{\tau}))\right) + E^0 + 1 \\ &\lesssim_{E^0} \nu^{-1}\left(1 + \log(A) + \log(\frac{1}{\nu})\right) + \nu^{-1}|\log(\tau)| + 1 \ . \end{aligned} \tag{3.5}$$

Define $m_0 := \nu^{-1}(1 + \log(A) + |\log(\nu)|)$, and note that $E^0 \le \sup_N E(\Pi_N u_0) \lesssim E_0 + 1$, the inequality above (3.5) is then estimated as follows:

$$\|u^n\|_\infty^2 \lesssim_{E_0} m_0 + \nu^{-1}|\log(\tau)| + 1.$$

On the other hand by Lemma 3.5,

$$\begin{aligned} \|u^{n+1}\|_\infty &\lesssim 1 + \|u^{n+1}\|_{\dot{H}^1}\sqrt{\log(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}})} \\ &\lesssim 1 + (\frac{1+\|u^n\|_\infty^2}{A})\|u^n\|_{\dot{H}^1}\sqrt{\log(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}})} \\ &\lesssim_{E_0} 1 + (1 + \frac{m_0 + \nu^{-1}|\log(\tau)|}{A})(\sqrt{\frac{1}{\nu}}\sqrt{\log(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}})} \\ &\lesssim_{E_0} 1 + (1 + \frac{m_0 + \nu^{-1}|\log(\tau)|}{A})(\sqrt{m_0 + \nu^{-1}|\log(\tau)|}) \\ &\lesssim_{E_0} 1 + \sqrt{m_0 + \nu^{-1}|\log(\tau)|} + \frac{(\sqrt{m_0 + \nu^{-1}|\log(\tau)|})^3}{A} \\ &\lesssim_{E_0} \sqrt{1 + \frac{m_0^3}{A^2} + m_0 + \nu^{-3}|\log(\tau)|^3} \ . \end{aligned} \tag{3.6}$$

The sufficient condition (3.2) thus becomes

$$\begin{cases} A + \frac{1}{2} + \frac{1}{\tau} \ge C(E_0)\left(m_0 + 1 + \frac{m_0^3}{A^2} + \nu^{-3}|\log(\tau)|^3\right) \\ m_0 = \nu^{-1}(1 + \log(A) + |\log(\nu)|) \ . \end{cases}$$

We now discuss two cases.
`Case 1:`$\frac{1}{\tau} \ge C(E_0)\nu^{-3}|\log(\tau)|^3$. In this case, we need to choose $A$ such that

$$A \gg_{E_0} m_0 = \nu^{-1}(1 + \log(A) + |\log(\nu)|) \ ,$$

where $B \gg_{E_0} D$ means there exists a large constant depending only on $E_0$. In fact, for $\nu \gtrsim 1$ , we can take $A \gg_{E_0} 1$; if $0 < \nu \ll 1$, we will choose $A = C_{E_0}\cdot\nu^{-1}|\log\nu|$ , where $C_{E_0}$ is a large constant depending only on $E_0$. Therefore it suffices to choose

$$A = C_{E_0}\cdot\max\left\{\nu^{-1}|\log(\nu)| \ , \ 1\right\}. \tag{3.7}$$

`Case 2:` $\frac{1}{\tau} \le C(E_0)\nu^{-3}|\log(\tau)|^3$. This implies $|\log(\tau)| \lesssim_{E_0} 1 + |\log(\nu)|$. Going back to equations (3.5), we have

$$\|u^n\|_\infty^2 \lesssim_{E_0} m_0,$$

as $\nu^{-1}|\log(\tau)|$ will be absorbed by $m_0$, where $m_0 = \nu^{-1}(1 + \log(A) + |\log(\nu)|)$. Hence substituting this new bound into (3.6), we get

$$\begin{aligned}
\|u^{n+1}\|_\infty &\lesssim 1 + (\frac{1+\|u^n\|_\infty^2}{A})\|u^n\|_{\dot H^1}\sqrt{\log(3+\|u^{n+1}\|_{\dot H^{\frac32}})} \\
&\lesssim_{E_0} 1 + (1+\frac{m_0}{A})\sqrt{\frac{1}{\nu}}\sqrt{\log(3+\|u^{n+1}\|_{\dot H^{\frac32}})} \\
&\lesssim_{E_0} 1 + (1+\frac{m_0}{A})\sqrt{m_0} \\
&\lesssim_{E_0} \sqrt{1+\frac{m_0^3}{A^2}+m_0}\ .
\end{aligned}$$

This shows it suffices to take

$$A \geq C_{E_0} m_0,$$

for a large enough constant $C_{E_0}$ depending only on $E_0$. The same choice of $A$ in `Case 1` (with a larger $C_{E_0}$ if necessary) will still work.

**Step 2:** We check the induction base step $n = 1$. Clearly we only need to check

$$A + \frac{1}{2} + \frac{1}{\tau} \geq \|\Pi_N u_0\|_\infty^2 + \frac{1}{2}\|u^1\|_\infty^2.$$

By Lemma 3.5,

$$\begin{aligned}
\|u^1\|_{\dot H^1} &\leq \left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\cdot\|u_0\|_{\dot H^1} \\
&\leq \left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\cdot\sqrt{\frac{2E^0}{\nu}}\ .
\end{aligned}$$

As a result,

$$\begin{aligned}
\|u^1\|_\infty &\lesssim 1 + |\widehat{u^1}(0)| + \|u^1\|_{\dot H^1}\sqrt{\log(3+\|u^1\|_{\dot H^{\frac32}})} \\
&\lesssim 1+\sqrt{E^0} + \left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\sqrt{\frac{2E^0}{\nu}}\sqrt{\log\left(3+c_1\left(\frac{A+1}{\nu}+\frac{1}{\nu\tau}\right)(E_0+1)\right)} \\
&\lesssim_{E^0} 1 + \left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\cdot\nu^{-\frac12}\cdot\sqrt{1+\log(A)+|\log(\nu)|+|\log(\tau)|} \\
&\lesssim_{E_0} \left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\cdot\nu^{-\frac12}\cdot\sqrt{1+\log(A)+|\log(\nu)|+|\log(\tau)|}\ .
\end{aligned}$$

Thus we need to choose $A$ such that

$$\begin{aligned}
A+\frac{1}{2}+\frac{1}{\tau} \geq \|\Pi_N u_0\|_\infty^2 + C_{E_0}\cdot\left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)^2\cdot\nu^{-1} \\
\cdot\left(1+\log(A)+|\log(\nu)|+|\log(\tau)|\right)\ ,
\end{aligned}$$

where $C_{E_0}$ is a large constant depending only on $E_0$. Note that by Morrey's inequality,

$$\|\Pi_N u_0\|_{L^\infty(\mathbb{T}^2)} \lesssim \|\Pi_N u_0\|_{H^2(\mathbb{T}^2)} \lesssim \|u_0\|_{H^2(\mathbb{T}^2)}.$$

Then it suffices to take $A$ such that

$$A \gg_{E_0} \|u_0\|_{H^2}^2 + \nu^{-1}|\log(\nu)| + 1\ . \tag{3.8}$$

This completes the induction and hence proves the theorem.

## 4. $L^2$ error estimate of the first order semi-implicit scheme on the 2D Allen-Cahn equation

In this section, we will like to study the $L^2$ error between the semi-implicit numerical solution and the exact PDE solution in the domain $\mathbb{T}^2$. To start with, we consider the auxiliary $L^2$ error estimate for near solutions.

4.1. **Auxiliary $L^2$ error estimate for near solutions**

Consider the following auxiliary system $u^n$ and $v^n$ for the first order scheme:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = \nu\Delta u^{n+1} - \Pi_N f(u^n) - A(u^{n+1}-u^n) + G_n^1 \\ \dfrac{v^{n+1}-v^n}{\tau} = \nu\Delta v^{n+1} - \Pi_N f(v^n) - A(v^{n+1}-v^n) + G_n^2 \\ u^0 = u_0 \ , \ v^0 = v_0 \ . \end{cases} \tag{4.1}$$

We define that $G_n = G_n^1 - G_n^2$.

**Proposition 4.1.** *For solutions of (4.1), assume for some* $N_1 > 0$,

$$\sup_{n\geq 0} \|u^n\|_\infty + \sup_{n\geq 0} \|v^n\|_\infty \leq N_1 \ .$$

*Then for any* $m \geq 1$,

$$\begin{aligned} &\|u^m - v^m\|_{L^2}^2 = \|e^m\|_{L^2}^2 \\ &\leq \exp\left(m\tau \cdot \left\{ C\left(\frac{(1+N_1^2)N_1}{\nu} + N_1^2 + \nu(1+N_1^2)N_1\right) + \frac{B}{\nu}\right\}\right) \\ &\cdot \left((1+A\tau)\|u_0 - v_0\|_{L^2}^2 + B\tau\nu \sum_{n=0}^{m-1} \|G_n\|_{L^2}^2\right) \end{aligned}$$

*where* $B$ , $C > 0$ *are absolute constants.*

*Proof.* Write $e^n = u^n - v^n$. Then

$$\frac{e^{n+1}-e^n}{\tau} = \nu\Delta e^{n+1} - A(e^{n+1}-e^n) - \Pi_N\left(f(u^n) - f(v^n)\right) + G_n \ .$$

Taking $L^2$-inner product with $e^{n+1}$ on both sides and recalling similar computations in previous section, we have

$$\begin{aligned} &\frac{1}{2\tau}\left(\|e^{n+1}\|_{L^2}^2 - \|e^n\|_{L^2}^2 + \|e^{n+1}-e^n\|_{L^2}^2\right) + \nu\|\nabla e^{n+1}\|_{L^2}^2 + \frac{A}{2}(\|e^{n+1}\|_{L^2}^2 - \|e^n\|_{L^2}^2 + \\ &\|e^{n+1}-e^n\|_{L^2}^2) = \left(G_n, \ e^{n+1}\right) + \left(f(u^n) - f(v^n), \ \Pi_N e^{n+1}\right) \end{aligned}$$

where $(. \ , \ .)$ denotes the $L^2$ inner product and the last term is because $\Pi_N$ is a self-adjoint operator $(\Pi_N f, \ g) = (f, \ \Pi_N g)$, since it is just an $N$-th Fourier mode truncation. By Hölder's inequality, we obtain that

$$|\left(G_n, \ e^{n+1}\right)| \leq \|e^{n+1}\|_{L^2}\|G_n\|_{L^2} \leq 2B\left(\nu\|G_n\|_{L^2}^2 + \frac{\|e^{n+1}\|_{L^2}^2}{\nu}\right).$$

Next, by the fundamental theorem of calculus, we have

$$\begin{aligned} f(u^n) - f(v^n) &= \int_0^1 f'(v^n + se^n)ds \ e^n \\ &= (a_1 + a_2(v^n)^2)e^n + a_3 v^n (e^n)^2 + a_4(e^n)^3 \ , \end{aligned}$$

where $a_i$ are constants can be computed. Note that we will denote $C$ to be an absolute constant whose value may vary in different lines:

$$\begin{aligned} |\left((a_1 + a_2(v^n)^2)e^n \ , \ e^{n+1}\right)| &\leq C(1 + \|v^n\|_\infty^2)\|e^{n+1}\|_{L^2}\|e^n\|_{L^2} \\ &\leq C(1+N_1^2)N_1\left(\frac{\|e^{n+1}\|_{L^2}^2}{\nu} + \nu\|e^n\|_{L^2}^2\right) \\ &\leq \frac{C(1+N_1^2)N_1}{\nu}\|e^{n+1}\|_{L^2}^2 + \nu \cdot C(1+N_1^2)N_1\|e^n\|_{L^2}^2 \ , \end{aligned}$$

moreover, the other two terms can be estimated similarly:

$$\begin{aligned} |\left(a_3 v^n (e^n)^2 \ , \ e^{n+1}\right)| &\leq C\|v^n\|_\infty\|e^{n+1}\|_\infty\|e^n\|_{L^2}^2 \\ &\leq CN_1^2\|e^n\|_{L^2}^2 \ , \end{aligned}$$

$$
\begin{aligned}
|\left(a_4(e^n)^3\ ,\ e^{n+1}\right)| &\le C\|e^{n+1}\|_\infty\|e^n\|_\infty\|e^n\|_{L^2}^2\\
&\le CN_1^2\|e^n\|_{L^2}^2\ .
\end{aligned}
$$

To simplify the formula, we use the notation $\|u\|_2$ to denote the $L^2$ norm. Collecting all estimates, we get

$$
\begin{aligned}
&\frac{\|e^{n+1}\|_2^2-\|e^n\|_2^2}{\tau}+A(\|e^{n+1}\|_2^2-\|e^n\|_2^2)\le B\nu\|G_n\|_2^2+\frac{B}{\nu}\|e^{n+1}\|_2^2\\
&+C\left(\nu(1+N_1^2)N_1+N_1^2\right)\|e^n\|_2^2+\frac{C(1+N_1^2)N_1}{\nu}\|e^{n+1}\|_2^2
\end{aligned}
$$

where $B$ and $C$ are two absolute constants that can be computed exactly. Recalling that $A$ is chosen larger than $O(\nu^{-1}|\log\nu|)$ for $\nu$ small, we derive that

$$
\begin{aligned}
&\frac{\|e^{n+1}\|_2^2-\|e^n\|_2^2}{\tau}+\left(A-\frac{C(1+N_1^2)N_1}{\nu}-\frac{B}{\nu}\right)(\|e^{n+1}\|_2^2-\|e^n\|_2^2)\le B\nu\|G_n\|_2^2+\\
&\left\{C\left(\frac{(1+N_1^2)N_1}{\nu}+N_1^2+\nu(1+N_1^2)N_1\right)+\frac{B}{\nu}\right\}\|e^n\|_2^2.
\end{aligned}
$$

Define

$$
\begin{aligned}
y_n&=\left(1+\left(A-\frac{C(1+N_1^2)N_1}{\nu}-\frac{B}{\nu}\right)\tau\right)\|e^n\|_2^2\ ,\\
\alpha&=C\left(\frac{(1+N_1^2)N_1}{\nu}+N_1^2+\nu(1+N_1^2)N_1\right)+\frac{B}{\nu}\ ,\\
\beta_n&=B\nu\|G_n\|_2^2\ .
\end{aligned}
$$

Then for $\nu$ small, we have

$$
\frac{y_{n+1}-y_n}{\tau}\le\alpha y_n+\beta_n\ .
$$

Applying discrete Grönwall's inequality in Lemma 2.3, we have

$$
\begin{aligned}
\|u^m-v^m\|_2^2=&\|e^m\|_2^2\le y_m\\
\le&\exp\left(m\tau\cdot\left\{C\left(\frac{(1+N_1^2)N_1}{\nu}+N_1^2+\nu(1+N_1^2)N_1\right)+\frac{B}{\nu}\right\}\right)\\
&\cdot\left(\left(1+\left(A-\frac{C(1+N_1^2)N_1}{\nu}-\frac{B}{\nu}\right)\tau\right)\|u_0-v_0\|_2^2+B\tau\nu\sum_{n=0}^{m-1}\|G_n\|_2^2\right)\\
\le&\exp\left(m\tau\cdot\left\{C\left(\frac{(1+N_1^2)N_1}{\nu}+N_1^2+\nu(1+N_1^2)N_1\right)+\frac{B}{\nu}\right\}\right)\\
&\cdot\left((1+A\tau)\|u_0-v_0\|_2^2+B\tau\nu\sum_{n=0}^{m-1}\|G_n\|_2^2\right)\ .
\end{aligned}
\tag{4.2}
$$

□

## 4.2. $L^2$ error estimate of 2D Allen-Cahn equation

In this section, to simplify the notation, we will write $x\lesssim y$ if $x\le C(\nu,u_0)\ y$ for a constant $C$ depending on $\nu$ and $u_0$. We consider the system

$$
\begin{cases}
\dfrac{u^{n+1}-u^n}{\tau}=\nu\Delta u^{n+1}-\Pi_N f(u^n)-A(u^{n+1}-u^n)\\
\partial_t u=\nu\Delta u-f(u)\\
u^0=\Pi_N u_0\ ,\ u(0)=u_0\ .
\end{cases}
\tag{4.3}
$$

**Theorem 4.1.** *Let $\nu > 0$. Let $u_0 \in H^s$, $s \geq 4$ and $u(t)$ be the solution to the Allen-Cahn equation with initial data $u_0$. Let $u^n$ be the numerical solution with initial data $\Pi_N u_0$. Assume $A$ satisfies the same condition in the stability theorem. Define $t_m = m\tau$, $m \geq 1$. Then*

$$\|u^m - u(t_m)\|_2 \leq A \cdot e^{C_1 t_m} \cdot C_2 \cdot \left(N^{-s} + \tau\right) ,$$

*where $C_1 > 0$ depends only on $(u_0, \nu)$ and $C_2$ depends on $(u_0, \nu, s)$.*

In order to prove this theorem, it is clear that we shall estimate $G_n$ in the previous proposition. Note that for a one-variable function $h(t)$, one has the formula:

$$\begin{cases} \frac{1}{\tau}\int_{t_n}^{t_{n+1}} h(t) = h(t_n) + \frac{1}{\tau}\int_{t_n}^{t_{n+1}} h'(t)\cdot(t_{n+1}-t)\ dt \\ \frac{1}{\tau}\int_{t_n}^{t_{n+1}} h(t) = h(t_{n+1}) + \frac{1}{\tau}\int_{t_n}^{t_{n+1}} h'(t)\cdot(t_n-t)\ dt\ . \end{cases}$$

Using the formula above and integrating the Allen-Cahn equation on the time interval $[t_n, t_{n+1}]$, we get

$$\begin{aligned} \frac{u(t_{n+1}) - u(t_n)}{\tau} =& \nu\Delta u(t_{n+1}) - A\left(u(t_{n+1}) - u(t_n)\right) \\ & - \Pi_N f(u(t_n)) - \Pi_{>N} f(u(t_n)) + G_n\ , \end{aligned} \tag{4.4}$$

where $\Pi_{>N} = id - \Pi_N$, the large mode truncation operator, and

$$G_n = \frac{\nu}{\tau}\int_{t_n}^{t_{n+1}} \partial_t\Delta u\cdot(t_n - t)\ dt - \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \partial_t(f(u))(t_{n+1}-t)\ dt + A\int_{t_n}^{t_{n+1}} \partial_t u\ dt\ . \tag{4.5}$$

To bound $\|G_n\|_2$, we introduce some useful lemmas.

### 4.3. Bounds on the Allen-Cahn exact solution and numerical solution

**Lemma 4.2** (Maximum principle for smooth solutions to the Allen-Cahn equation)**.** *Let $T > 0$, $d \leq 3$ and assume $u \in C_x^2 C_t^1(\mathbb{T}^d \times [0,T])$ is a classical solution to Allen-Cahn equation with initial data $u_0$. Then*

$$\|u(.\ ,\ t)\|_\infty \leq \max\{\|u_0\|_\infty\ ,\ 1\}\ ,\ \forall 0 \leq t \leq T\ .$$

**Remark 4.3.** *As proved in [15], there exists a global $H_x^4 C_t^1$ solution to Allen-Cahn equation. In fact as pointed out by Li et al. in [26], the regularity will be higher due to the smoothing effect. Therefore we assume a smooth solution here.*

*Proof.* We define $f(x,t) = u(x,t)^2$ and $f^\epsilon(x,t) = f(x,t) - \epsilon t$. Since $f^\epsilon$ is a continuous function on the compact domain $\mathbb{T}^d \times [0,T]$, it achieves maximum at some point $(x_*, t_*)$, i.e.

$$\max_{\substack{0\leq t\leq T\ ,\\ x\in\mathbb{T}^d}} f^\epsilon(x,t) = f^\epsilon(x_*, t_*) := M_\epsilon\ .$$

We discuss several cases.

`Case 1:` $0 < t_* \leq T$ and $M_\epsilon > 1$. This shows $\nabla f^\epsilon(x_*, t_*) = 0\ ,\ \Delta f^\epsilon(x_*, t_*) \leq 0$. Note that

$$\nabla f^\epsilon = 2u\nabla u\ ,\ \Delta f^\epsilon = 2|\nabla u|^2 + 2u\Delta u\ ,$$

this shows $\nabla u(x_*, t_*) = 0$, $u\Delta u(x_*, t_*) < 0$. However, we also have

$$\begin{aligned} \partial_t f^\epsilon(x_*, t_*) &= 2u(x_*, t_*)\partial_t u(x_*, t_*) - \epsilon \\ &= 2u(x_*, t_*)(\nu\Delta u(x_*, t_*) - u^3(x_*, t_*) + u(x_*, t_*)) - \epsilon \\ &< -2u^4(x_*, t_*) + 2u^2(x_*, t_*) - \epsilon \\ &< -2(u^2(x_*, t_*) - \frac{1}{2})^2 + \frac{1}{2} - \epsilon \\ &< -\epsilon < 0 \end{aligned}$$

as $u^2(x_*,t_*) > 1$ by assumption. This contradicts the hypothesis that $f^\epsilon$ achieves its maximum at $(x_*,t_*)$.
`Case 2:` $0 < t_* \le T$ and $M_\epsilon \le 1$. In this case we obtain

$$\max_{0\le t\le T\ ,\ x\in\mathbb{T}^d} f(x,t) \le 1+\epsilon T\ ,$$

letting $\epsilon \to 0$, we obtain $f(x,t)\le 1$.
`Case 3:` $t_* = 0$, then

$$\max_{0\le t\le T\ ,\ x\in\mathbb{T}^d} f(x,t) \le \max_{x\in\mathbb{T}^d} f(x,0)+\epsilon T\ ,$$

sending $\epsilon$ to 0, we obtain $f(x,t)\le f(x,0)$.

This concludes $\|u\|_\infty \le \max\{\|u_0\|_\infty\ ,\ 1\}$. □

**Lemma 4.4** ($H^k$ boundedness of the exact solution)**.** *Assume $u(x,t)$ is a smooth solution to the Allen-Cahn equation in $\mathbb{T}^d$ with $d=1,2,3$ and the initial data $u_0\in H^k(\mathbb{T}^d)$ for $k\ge 2$. Then,*

$$\sup_{t\ge 0}\|u(t)\|_{H^k(\mathbb{T}^d)} \lesssim_k 1 \tag{4.6}$$

*where we omit the dependence on $\nu$ and $u_0$.*

*Proof.* We write the solution $u$ in the mild form

$$u(t) = e^{\nu t\Delta}u_0 + \int_0^t e^{\nu(t-s)\Delta}(u-u^3)\ ds\ .$$

We will prove this argument inductively. By previous lemma 4.2, we have $\|u\|_2 \lesssim 1$ as $\|u\|_\infty \lesssim 1$ and we will show $\|u\|_{H^1}\lesssim 1$ for any $t\ge 1$. Then by taking the spatial derivative and $L^2$ norm in the formula above, we derive

$$\|Du\|_2 \le \|De^{\nu t\Delta}u_0\|_2 + \int_0^t \|De^{\nu(t-s)\Delta}(u-u^3)\|_2\ ds$$

where $D^m u$ denotes any differential operator $D^\alpha u$ for any $|\alpha| = m$.

First, we consider the nonlinear part.

$$\|De^{\nu(t-s)\Delta}(u-u^3)\|_2 \lesssim \|De^{\nu(t-s)\Delta}(u-u^3)\|_\infty \lesssim |K_1*(u-u^3)|\ ,$$

where $K_1$ is the kernel corresponding to $De^{\nu(t-s)\Delta}$. Therefore we estimate that

$$\begin{aligned}|K_1*(u-u^3)| &\le \|K_1\|_2\cdot\|u-u^3\|_2\\ &\lesssim \|K_1\|_2\cdot\|u\|_2\end{aligned}$$

by the boundedness of $\|u\|_\infty$. Note that

$$\begin{aligned}\|K_1\|_2 &\lesssim \left(\sum_{k\in\mathbb{Z}^d}|k|^2e^{-2\nu(t-s)|k|^2}\right)^{\frac12}\\ &= \left(\sum_{|k|\ge 1}|k|^2e^{-2\nu(t-s)|k|^2}\right)^{\frac12}\\ &\lesssim \left(\int_1^\infty e^{-2\nu(t-s)r^2}r^{d+1}\ dr\right)^{\frac12}\ .\end{aligned}$$

The estimates for different dimensions are different. Now we will assume $t\ge 1$ because the other case $t<1$ is much easier.
`Case 1:` $d=1$. $\int_1^\infty e^{-2\nu(t-s)r^2}r^2\ dr \lesssim \frac{e^{-2\nu(t-s)}}{t-s} + \frac{\mathrm{erf}(\sqrt{2\nu(t-s)})}{(t-s)^{3/2}}$, where $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\ dt$, the complementary error function. Letting $\gamma = t-s$,

$$\int_0^t \|De^{\nu(t-s)\Delta}u\|_2\ ds \lesssim \left(\int_0^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}} + \frac{(\mathrm{erf}(\sqrt{\nu\gamma}))^{1/2}}{\gamma^{3/4}}\ d\gamma\right)\cdot\|u\|_2\ .$$

For $\gamma$ small enough, $\frac{(\mathrm{erf}(\sqrt{\nu\gamma}))^{1/2}}{\gamma^{3/4}}$ will dominate the estimate and for $\gamma$ away from 0, $\frac{e^{-\nu\gamma}}{\gamma^{1/2}}$ shall dominate the estimate. Then we split the integral as below (recall that $t \geq 1$):

$$\begin{aligned}\int_0^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}} + \frac{(\mathrm{erf}(\sqrt{\nu\gamma}))^{1/2}}{\gamma^{3/4}}\, d\gamma &\lesssim \int_0^1 \frac{1}{\gamma^{3/4}}\, d\gamma + \int_1^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\, d\gamma \\ &\lesssim 1 + \int_0^\infty \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\, d\gamma \\ &\lesssim 1\ .\end{aligned}$$

`Case 2:` $d = 2$. $\int_1^\infty e^{-2\nu(t-s)r^2} r^3\, dr \lesssim \frac{e^{-2\nu(t-s)}}{(t-s)^2} + \frac{e^{-2\nu(t-s)}}{t-s}$. Similar to Case 1, we will split the integral as well. Letting $\gamma = t - s$, we have

$$\begin{aligned}\int_1^t \frac{e^{-\nu\gamma}}{\gamma} + \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\, d\gamma &\lesssim \int_1^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\, d\gamma \\ &\lesssim \int_0^\infty \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\, d\gamma \\ &\lesssim 1\ .\end{aligned}$$

However, the estimate in Case 1 does not work for $\gamma \leq 1$. Now we estimate $\|K_1 * (u - u^3)\|_{L^2(\mathbb{T}^d)}$ differently. We compute from the Fourier side:

$$\begin{aligned}\|K_1 * (u - u^3)\|_{L^2(\mathbb{T}^d)}^2 &= \sum_{|k|\geq 1} |k|^2 e^{-2\nu(t-s)|k|^2} |\widehat{u - u^3}(k)|^2 \\ &\leq \max_{|k|\geq 1} \left\{ |k|^2 e^{-2\nu(t-s)|k|^2} \right\} \cdot \sum_{|k|\geq 1} |\widehat{u - u^3}(k)|^2 \\ &\lesssim \max_{|k|\geq 1} \left\{ |k|^2 e^{-2\nu(t-s)|k|^2} \right\} \cdot \|u\|_{L^2(\mathbb{T}^d)}^2\ .\end{aligned}$$

Define $g(x) = x^2 e^{-2\nu\gamma x^2}$, where $x \geq 0$. Then,

$$g'(x) = 2x e^{-2\nu\gamma x^2}\left(1 - 2\nu\gamma x^2\right)\ ,$$

which shows the maximum is achieved at $x = \frac{1}{\sqrt{2\nu\gamma}}$ and hence

$$g(x) \leq g\left(\frac{1}{\sqrt{2\nu\gamma}}\right) \lesssim \frac{1}{\gamma}\ .$$

Therefore

$$\|De^{\nu(t-s)\Delta}(u - u^3)\|_{L^2(\mathbb{T}^d)} \lesssim \frac{1}{\sqrt{t-s}} \|u\|_{L^2(\mathbb{T}^d)}\ .$$

Note that this proof works for any dimension. As a result,

$$\int_0^1 \|De^{\nu\gamma\Delta} u\|_2\, d\gamma \lesssim \int_0^1 \frac{1}{\sqrt{\gamma}}\, d\gamma \cdot \|u\|_2 \lesssim 1\ .$$

This shows $\int_0^t \|De^{\nu(t-s)\Delta} u\|_2\, ds \lesssim 1$.

`Case 3:` $d = 3$. As proved in previous case, we will only need to check the case $\gamma \geq 1$. Note that $\int_1^\infty e^{-2\nu\gamma r^2} r^4\, dr \lesssim \frac{e^{-2\nu\gamma}}{\gamma}$ for $\gamma \geq 1$. This shows that

$$\begin{aligned}\int_1^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\, d\gamma &\lesssim \int_0^\infty \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\, d\gamma \\ &\lesssim 1\ .\end{aligned}$$

For the case where $t \leq 1$, it is easier because we do not need to split the integral and all integrals from 0 to $t$ can be bounded by the integral from 0 to 1.

Now for the linear part, by Duhamel's Principle, $e^{\nu t\Delta} u_0$ denotes the solution to the heat equation. As is well known, every spatial derivative of the solution $e^{\nu t\Delta} u_0$ solves the heat equation, hence by the energy decay property, we have $\|e^{\nu t\Delta} u_0\|_{H^m} \lesssim \|u_0\|_{H^m}$ for any $1 \leq m \leq k$. Combining the nonlinear and linear parts, we obtain that $\|u\|_{H^1} \lesssim 1$ independent of $t \geq 0$ and hence $\sup_{t\geq 0} \|u\|_{H^1} \lesssim 1$.

Assume that we have $\sup_{t\geq 0}\|u\|_{H^{m-1}}\lesssim 1$, then the estimate follows by repeating the process above:

$$\begin{aligned}\|D(D^{m-1}u)\|_2 &\leq \|De^{\nu t\Delta}D^{m-1}u_0\|_2+\int_0^t\|De^{\nu(t-s)\Delta}D^{m-1}u\|_2\,ds\\ &\lesssim \|u_0\|_{H^m}+\int_0^1\|De^{\nu\gamma\Delta}D^{m-1}u\|_2\,d\gamma+\int_1^t\|De^{\nu\gamma\Delta}D^{m-1}u\|_2\\ &\lesssim 1+\int_0^1\frac{1}{\sqrt{\gamma}}\,d\gamma\cdot\|D^{m-1}u\|_2+\int_0^\infty\frac{e^{-\nu\gamma}}{\sqrt{\gamma}}\,d\gamma\cdot\|D^{m-1}u\|_2\\ &\lesssim 1\ ,\end{aligned}$$

We finally obtain that

$$\sup_{t\geq 0}\|u\|_{H^k(\mathbb{T}^d)}\lesssim_k 1\ . \tag{4.7}$$

□

**Lemma 4.5** (Discrete version $H^k$ boundedness)**.** *Suppose $u_0\in H^k(\mathbb{T}^d)$ with $d\leq 3$ and $k\geq 2$. Then, suppose $u^n$ is the numerical solution that satisfies*

$$\begin{cases}\dfrac{u^{n+1}-u^n}{\tau}=\nu\Delta u^{n+1}-A(u^{n+1}-u^n)-\Pi_N f(u^n)\\ u^0=\Pi_N u_0\ ,\end{cases}$$

*then*

$$\sup_{n\geq 0}\|u^n\|_{H^k(\mathbb{T}^d)}\lesssim_{A,k} 1\ .$$

**Remark 4.6.** *The bound on $u^n$ is independent of time step $\tau$ and truncation number $N$.*

**Remark 4.7.** *The proof uses the energy decay property of the numerical scheme, so we will assume this property for now, as the proof for the energy decay in 3D case will be given in section 6.*

*Proof.* To simplify the notation, we will use "$\lesssim$" instead of "$\lesssim_{\nu,u_0,A,k}$" only in this lemma. We will use a similar method to the one provided in [24].

We can write the scheme as follows:

$$\begin{aligned}u^{n+1}&=\underbrace{\frac{1+A\tau}{1+A\tau-\nu\tau\Delta}}_{:=L_1}u^n+\underbrace{\frac{-\tau\Pi_N}{1+A\tau-\nu\tau\Delta}}_{:=L_2}f(u^n)\\ &=L_1\left(L_1u^{n-1}+L_2f(u^{n-1})\right)+L_2f(u^n)\\ &=L_1^{m_0+1}u^{n-m_0}+\sum_{l=0}^{m_0}L_1^lL_2f(u^{n-l})\ ,\end{aligned} \tag{4.8}$$

where $m_0$ will be chosen later.

Similar to the continuous version, we prove inductively. To demonstrate the idea, we show

$$\sup_{n\geq 0}\|u^n\|_{H^3(\mathbb{T}^d)}\lesssim 1\ .$$

Recall $\sup_{n\geq 0}\|u^n\|_{H^1}\lesssim 1$ and $\sup_{n\geq 0}\|f(u^n)\|_2\lesssim 1$ by the energy decay property, then we only need to consider the $\dot{H}^3$ semi norm.

We discuss 2 cases:

`Case 1:` $A\tau\geq\frac{1}{10}$. Then for $0\neq k\in\mathbb{Z}^d$:

$$\begin{aligned}\left|\widehat{L_1}(k)\right|&=\frac{1+A\tau}{1+A\tau+\nu\tau|k|^2}\\ &\leq\frac{11A\tau}{A\tau+\nu\tau|k|^2}\\ &\lesssim\frac{1}{1+|k|^2}\ ,\end{aligned}$$

and

$$\left|\widehat{L_2}(k)\right| = \frac{\tau}{1 + A\tau + \nu\tau|k|^2} \lesssim \frac{1}{1+|k|^2} \ .$$

To conclude the result for this case, we prove the following bootstrap argument:

**Claim.** *Let $L_1$ and $L_2$ be defined as above. Suppose given some $s > 0$,*

$$\sup_{0\le j\le n} \|\widehat{f^j}(k)|k|^s\|_{l^\infty} \le \alpha < \infty,$$

*where $f^j := f(u^j)$. Define*

$$v = \sum_{j=1}^{n} L_1^j L_2 f^j.$$

*We have*

$$\|\hat{v}(k)|k|^{s+2}\|_{l^\infty} \le \frac{\alpha}{\nu},$$

*moreover for $d \le 3$,*

$$\|v\|_{H^{s+0.4}(\mathbb{T}^d)} \le \beta < \infty,$$

*where $\beta > 0$ depends only on $\alpha$ and $\nu$.*

**Proof of claim.** *Notice that for each $k \neq 0$,*

$$|\widehat{L_1^j}(k)\widehat{L_2}(k)| = \left(\frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2}\right)^j \cdot \frac{\tau}{1 + A\tau + \nu\tau|k|^2};$$

*therefore for each $k \neq 0$,*

$$\begin{aligned} |\hat{v}(k)| \le& \alpha|k|^{-s} \sum_{j=1}^{n} \left(\frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2}\right)^j \cdot \frac{\tau}{1 + A\tau + \nu\tau|k|^2} \\ \le& \alpha|k|^{-s} \cdot \frac{1}{\nu|k|^2}. \end{aligned}$$

*This concludes that $\|\hat{v}(k)|k|^{s+2}\|_{l^\infty} \le \frac{\alpha}{\nu}$ and therefore*

$$\|v\|_{H^{s+0.4}(\mathbb{T}^d)} \le \beta$$

*for some $\beta$ depending on $\alpha$ and $\nu$ only.*$\square$

Recall that $\sup_{n\ge 0} \|u^n\|_{H^1} \lesssim 1$, we have

$$\||k|\widehat{f^j}(k)\|_{l^\infty} \lesssim 1,$$

therefore applying this argument, we get $\||k|^3 u^{n+1}\|_{l^\infty} \lesssim 1$. It then follows that, by applying the argument again, $\|u^{n+1}\|_{H^3} \lesssim 1$.

`Case 2:` $A\tau < \frac{1}{10}$. Take $m_0$ to be one integer such that $\frac{1}{2} \le m_0\tau < 1$ and thus $m_0 \ge 5$.

$$\begin{aligned} \left|\widehat{L_1^{m_0+1}}(k)\right| &\le \left(\frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2}\right)^{m_0+1} \\ &\le \left(\frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2}\right)^{m_0} \\ &= \left(1 + \frac{\nu\tau|k|^2}{1 + A\tau}\right)^{-m_0} . \end{aligned}$$

Recall $A\tau < \frac{1}{10} < 1$, then

$$\left(1 + \frac{\nu\tau|k|^2}{1 + A\tau}\right)^{-m_0} \le \left(1 + \frac{\nu\tau|k|^2}{2}\right)^{-m_0} ,$$

define $t_0 := m_0\tau$ and we derive

$$\left|\widehat{L_1^{m_0+1}}(k)\right| \le \left(1 + \frac{1}{2}\nu|k|^2 \frac{t_0}{m_0}\right)^{-m_0} .$$

For any $a > 0$, we consider the function $h(x) = -x\log\left(1+\frac{a}{x}\right)$, $x > 0$. Then

$$h'(x) = -\log\left(1+\frac{a}{x}\right) + \frac{a}{a+x}$$
$$h''(x) = \frac{a}{x+a}\left(\frac{1}{x} - \frac{1}{x+a}\right) > 0 \ .$$

By direct computation, $h(x)$ decreases on $(0,\infty)$. Therefore, recalling $m_0 \geq 5$,

$$\left|\widehat{L_1^{m_0+1}}(k)\right| \leq \left(1+\frac{1}{2}\nu|k|^2\frac{t_0}{m_0}\right)^{-m_0} \leq \left(1+\frac{1}{2}\nu|k|^2\cdot\frac{t_0}{5}\right)^{-5} \ .$$

As a direct result, we have

$$\begin{aligned}\left|\widehat{L_2}(k)\right|\cdot\sum_{l=0}^{m_0}\left|\widehat{L_1}(k)\right|^l &\leq \left|\widehat{L_2}(k)\right|\cdot\frac{1}{1-\left|\widehat{L_1}(k)\right|}\\ &= \frac{\tau}{1+A\tau+\nu\tau|k|^2}\cdot\frac{1}{1-\frac{1+A\tau}{1+A\tau+\nu\tau|k|^2}}\\ &= \frac{1}{\nu|k|^2}\\ &\lesssim \frac{1}{|k|^2} \ .\end{aligned}$$

Therefore for $n \geq m_0$,

$$\|u^{n+1}\|_{\dot{H}^2} \lesssim \|u^{n-m_0}\|_2 + \sup_{0\leq l\leq m_0}\|f(u^{n-l})\|_2 \lesssim 1 \ .$$

For $1 \leq n \leq m_0+1$, we apply

$$u^n = L_1^n u^0 + \sum_{l=0}^{n-1} L_1^l L_2 f(u^{n-1-l}) \ .$$

Hence we get

$$\|u^n\|_{\dot{H}^2} \lesssim \|u^0\|_{\dot{H}^2} + \sup_{0\leq l\leq n-1}\|f(u^{n-l-1})\|_2 \lesssim 1 \ .$$

Then by the bootstrap lemma, we can conclude

$$\|u^{n+1}\|_{H^3(\mathbb{T}^d)} \lesssim 1 \ .$$

By the energy decay property, the constant depends only on $\nu, u_0$ and $A$, we can conclude that

$$\sup_{n\geq 0}\|u^n\|_{H^3(\mathbb{T}^d)} \lesssim 1 \ . \tag{4.9}$$

To obtain a different $H^k$ norm control, we can repeat the bootstrap lemma above and derive the desired result.

$\square$

**Remark 4.8.** *The proof for the exact solution and the numerical solution is similar in the sense that we develop bootstrap process and split the time interval.*

### 4.4. Proof of $L^2$ error estimate of 2D Allen-Cahn equation

By Lemma 4.5, $\sup_{n\geq 0}\|u^n\|_\infty \lesssim 1$ using Morrey's inequality. Thus the assumptions of Proposition 4.1 (auxiliary $L^2$ error estimate proposition) are satisfied. Recall that

$$G_n = \frac{\nu}{\tau}\int_{t_n}^{t_{n+1}}\partial_t\Delta u\cdot(t_n-t)\ dt - \frac{1}{\tau}\int_{t_n}^{t_{n+1}}\partial_t(f(u))(t_{n+1}-t)\ dt + A\int_{t_n}^{t_{n+1}}\partial_t u\ dt \ .$$

Then we can estimate that

$$
\begin{aligned}
\|G_n\|_2 &\lesssim \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \, dt + \int_{t_n}^{t_{n+1}} \|\partial_t (f(u))\|_2 \, dt + A \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2 \, dt \\
&\lesssim \underbrace{\int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \, dt}_{I_1} + \underbrace{\int_{t_n}^{t_{n+1}} \|\partial_t u\|_2 \, dt \cdot \left(A + \|f'(u)\|_{L_t^\infty L_x^\infty}\right)}_{I_2} \; .
\end{aligned}
\tag{4.10}
$$

Note that $\partial_t u = \nu \Delta u - u + u^3$ and hence by Lemma 4.4,

$$
\|\partial_t u\|_2 \lesssim 1 \; , \quad \|f'(u)\|_\infty \lesssim 1 \; .
$$

Recall the energy decay property:

$$
\frac{dE}{dt} = -\|\partial_t u\|_2^2 \; .
$$

This shows

$$
\int_0^\infty \|\partial_t u\|_2^2 \, dt \lesssim 1 \; .
$$

Note that by the Gagliardo-Nirenberg interpolation inequality, we have

$$
\|\partial_t \Delta u\|_2 \lesssim \|\langle \nabla \rangle^3 \partial_t u\|_2^{\frac{2}{3}} \cdot \|\partial_t u\|_2^{\frac{1}{3}} \lesssim \|\partial_t u\|_2^{\frac{1}{3}} \; .
$$

This implies

$$
\begin{aligned}
&\int_0^\infty \|\partial_t \Delta u\|_2^6 \, dt \lesssim 1 \; , \\
\Longrightarrow &\int_0^T \|\partial_t \Delta u\|_2^2 \, dt \lesssim \left( \int_0^T \|\partial_t \Delta u\|_2^6 \, dt \right)^{\frac{1}{3}} \cdot \left( \int_0^T 1 \, dt \right)^{\frac{2}{3}} \lesssim 1 + T^{\frac{2}{3}} \; .
\end{aligned}
$$

Moreover, we can estimate (4.10) now

$$
I_1 = \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \, dt \lesssim \left( \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2^2 \, dt \right)^{\frac{1}{2}} \cdot \sqrt{\tau} \; .
$$

Similarly for $I_2$, we obtain that

$$
I_2 \lesssim (1+A) \cdot \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2 \, dt \lesssim (1+A) \cdot \left( \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2^2 \, dt \right)^{\frac{1}{2}} \cdot \sqrt{\tau} \; .
$$

Hence for $t_m \geq 1$,

$$
\begin{aligned}
\sum_{n=0}^{m-1} \|G_n\|_2^2 &\lesssim \sum_{n=0}^{m-1} \left( (I_1)^2 + (I_2)^2 \right) \\
&\lesssim \sum_{n=0}^{m-1} \left( \tau \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2^2 \, dt + (1+A)^2 \tau \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2^2 \, dt \right) \\
&\lesssim \tau \int_0^{t_m} \|\partial_t \Delta u\|_2^2 \, dt + (1+A)^2 \tau \int_0^{t_m} \|\partial_t u\|_2^2 \, dt \\
&\lesssim \tau (1 + t_m) + (1+A)^2 \tau \\
&\lesssim (1+A)^2 \tau \cdot (1 + t_m) \; .
\end{aligned}
\tag{4.11}
$$

On the other hand, by the high Sobolev bound lemma (Lemma 4.4) $\sup_{t \geq 0} \|u(t)\|_{H^s} \lesssim_s 1$, we have $\sup_{n \geq 0} \|f(u(t_n))\|_{H^s} \lesssim_s 1$. We can then derive that

$$
\begin{aligned}
\|\Pi_{>N} f(u(t_n))\|_2^2 &= \sum_{|k|>N} \left| \widehat{f(u(t_n))}(k) \right|^2 \\
&\leq \sum_{|k|>N} |k|^{2s} \left| \widehat{f(u(t_n))}(k) \right|^2 \cdot |k|^{-2s} \\
&\lesssim N^{-2s} \cdot \sum_{|k|>N} |k|^{2s} \left| \widehat{f(u(t_n))}(k) \right|^2 \\
&\lesssim N^{-2s} \cdot \|f(u(t_n))\|_{H^s}^2 \\
&\lesssim N^{-2s} \ ,
\end{aligned}
$$

thus

$$
\sum_{n=0}^{m-1} \|\Pi_{>N} f(u(t_n))\|_2^2 \lesssim_s m \cdot N^{-2s} \lesssim \frac{t_m N^{-2s}}{\tau} \ .
$$

Therefore,

$$
\tau \sum_{n=0}^{m-1} \left( \|G_n\|_2^2 + \|\Pi_{>N} f(u(t_n))\|_2^2 \right) \lesssim_s (1+t_m)(\tau^2 + N^{-2s})(1+A)^2 \ .
$$

Similarly, we have

$$
\|u^0 - u(0)\|_2^2 = \|\Pi_N u_0 - u_0\|_2^2 \lesssim N^{-2s} \ .
$$

Applying the auxiliary solutions estimate in Proposition 4.1 and noting that $t_m = m\tau$, we can get

$$
\|u^m - u(t_m)\|_2^2 \lesssim_s (1+A)^2 e^{Ct_m} \left( N^{-2s} + \tau \cdot N^{-2s} + (1+t_m)(\tau^2 + N^{-2s}) \right) \ .
$$

Note that

$$
\begin{cases}
\tau \cdot N^{-2s} \lesssim \tau^2 + N^{-4s} \lesssim \tau^2 + N^{-2s} \\
1 + t_m \lesssim e^{C' t_m} \ ,
\end{cases}
$$

which leads to

$$
\|u^m - u(t_m)\|_2^2 \lesssim_s (1+A)^2 e^{Ct_m} \left( N^{-2s} + \tau^2 \right) \ .
$$

Thus

$$
\|u^m - u(t_m)\|_2 \leq (1+A) \cdot C_2 \cdot e^{C_1 t_m} \left( N^{-s} + \tau \right) \ , \tag{4.12}
$$

where $C_1 > 0$ is a constant depending on $\nu$ , $u_0$; $C_2 > 0$ is a constant depending on $s$ , $\nu$ and $u_0$. This completes the proof of $L^2$ error estimate.

## 5. Stability of a first order semi-implicit scheme on the 2D fractional Cahn-Hilliard equation

As mentioned in the introduction, the fractional Cahn-Hilliard equation is an "interpolation" between Allen-Cahn equation and original Cahn-Hilliard equation.

$$
\begin{cases}
\partial_t u = \nu \Delta \left( (-\Delta)^\alpha u + (-\Delta)^{\alpha-1} f(u) \right) \ , \ 0 < \alpha \leq 1 \\
u(x,0) = u_0
\end{cases} .
$$

In this section, we stick to the same region, two dimensional $2\pi$-periodic torus $\mathbb{T}^2 = \mathbb{R}^2 / 2\pi\mathbb{Z}^2$. $f(u) = u^3 - u$ and the energy $E(u) = \int_{\mathbb{T}^2} \left( \frac{\nu}{2} |\nabla u|^2 + F(u) \right) \, dx$ , with $F(u) = \frac{1}{4}(u^2 - 1)^2$. Similarly, the semi-implicit scheme is given by the following:

$$
\begin{cases}
\dfrac{u^{n+1} - u^n}{\tau} = -\nu (-\Delta)^{\alpha+1} u^{n+1} - (-\Delta)^\alpha A (u^{n+1} - u^n) - (-\Delta)^\alpha \Pi_N f(u^n) \\
u^0 = \Pi_N u_0
\end{cases} . \tag{5.1}
$$

**Theorem 5.1** (Unconditional energy stability for FCH)**.** *Consider* (5.1) *with* $\nu > 0$ *and assume* $u_0 \in H^2(\mathbb{T}^2)$ *and obeys the zero-mean condition. Then there exists a constant* $\beta_0$ *depending only on the initial energy* $E_0 = E(u_0)$ *such that if*

$$
A \geq \beta \cdot \left( \|u_0\|_{H^2}^2 + \nu^{-1} |\log \nu| + 1 \right) \ , \ \beta \geq \beta_0
$$

*then* $E(u^{n+1}) \le E(u^n)$, $\forall n \ge 0$, *where* $E$ *is defined above.*

**Remark 5.2.** *Here we require a zero-mean assumption on* $u_0$ *which implies* $u^n$ *has mean zero for each* $n$*. This assumption will guarantee that negative fractional Laplacian is well defined. Here we use the notation* $|\nabla|^{-\alpha} = (-\Delta)^{-\frac{\alpha}{2}}$ *to denote the fractional Laplacian.*

*Proof.* The proof uses a similar computation given in previous section. We recall the scheme (5.1):

$$\frac{u^{n+1}-u^n}{\tau} = -\nu(-\Delta)^{\alpha+1}u^{n+1} - (-\Delta)^{\alpha}A(u^{n+1}-u^n) - (-\Delta)^{\alpha}\Pi_N f(u^n) \ .$$

Now we multiply the equation by $(-\Delta)^{-\alpha}(u^{n+1}-u^n)$ and apply the fundamental theorem of calculus as in section 3. We then obtain

$$\begin{aligned}&\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2}^2 + \frac{\nu}{2}\left(\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \|\nabla u^{n+1}\|_{L^2}^2 - \|\nabla u^n\|_{L^2}^2\right)\\ &+A\|u^{n+1}-u^n\|_{L^2}^2 = -\left(f(u^n), u^{n+1}-u^n\right) \ .\end{aligned}$$

This then implies that

$$\begin{aligned}&\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2}^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \left(A+\frac{1}{2}\right)\|u^{n+1}-u^n\|_{L^2}^2 + E^{n+1} - E^n\\ &\le \|u^{n+1}-u^n\|_{L^2}^2\left(\|u^n\|_\infty^2 + \frac{1}{2}\|u^{n+1}\|_\infty^2\right) \ .\end{aligned} \tag{5.2}$$

It is clear that the first two norms $\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2}^2$ and $\frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2$ will be hard to control as we will expect more help from $\|u^{n+1}-u^n\|_{L^2}^2$.

**Lemma 5.3.** *There exists a constant* $C_{\alpha\nu\tau}$ *that is determined by* $\alpha$*,* $\nu$ *and* $\tau$*, such that*

$$\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2 \ge C_{\alpha\nu\tau}\|u^{n+1}-u^n\|_{L^2(\mathbb{T}^2)}^2 \ .$$

*Proof.* It is natural to examine the above norms $\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2$ and $\frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2$ on the Fourier side. Then we obtain that

$$\begin{aligned}&\frac{1}{\tau}\sum_{k\neq 0}|k|^{-2\alpha}|\widehat{u^{n+1}}(k) - \widehat{u^n}(k)|^2 + \frac{\nu}{2}\sum_{k\neq 0}|\widehat{u^{n+1}}(k) - \widehat{u^n}(k)|^2\\ &= \sum_{k\neq 0}|\widehat{u^{n+1}}(k) - \widehat{u^n}(k)|^2 \cdot \left(\frac{|k|^{-2\alpha}}{\tau} + \frac{\nu|k|^2}{2}\right) \ .\end{aligned}$$

We apply standard Young's inequality for product to estimate: $ab \le \frac{a^\gamma}{\gamma} + \frac{b^\beta}{\beta}$ , with $\frac{1}{\gamma} + \frac{1}{\beta} = 1$. We then take $a = |k|^p$ , $b = |k|^q$, where $p + q = 0$. To fulfill the condition, we choose $\gamma, \beta, p$ and $q$ as follows:

$$\begin{cases} p = \dfrac{-2\alpha}{\alpha+1} \\ q = \dfrac{2\alpha}{\alpha+1} \\ \gamma = \alpha+1 \\ \beta = \dfrac{\alpha+1}{\alpha} \ . \end{cases} \implies \begin{cases} -2\alpha = p\gamma \\ 2 = q\beta \end{cases} .$$

Therefore, we have

$$\begin{cases} a^\gamma = |k|^{p\gamma} = |k|^{-2\alpha} \\ b^\beta = |k|^{q\beta} = |k|^2 \end{cases} .$$

As a result, we obtain that

$$\begin{aligned}
&\sum_{k\neq 0}|\widehat{u^{n+1}}(k)-\widehat{u^n}(k)|^2\cdot\left(\frac{|k|^{-2\alpha}}{\tau}+\frac{\nu|k|^2}{2}\right)\\
=&\sum_{k\neq 0}|\widehat{u^{n+1}}(k)-\widehat{u^n}(k)|^2\cdot\left[\frac{\alpha+1}{\tau}\cdot\left(\frac{|k|^{-2\alpha}}{\alpha+1}\right)+\frac{\nu(\alpha+1)}{2\alpha}\cdot\left(\frac{|k|^2}{\frac{\alpha+1}{\alpha}}\right)\right]\\
\geq&\sum_{k\neq 0}|\widehat{u^{n+1}}(k)-\widehat{u^n}(k)|^2\cdot\left(\frac{\alpha+1}{\tau}\right)^{\frac{1}{\alpha+1}}\cdot\left(\frac{\nu(\alpha+1)}{2\alpha}\right)^{\frac{\alpha+1}{\alpha}}\ .
\end{aligned}$$

Clearly it suffices to take $C_{\alpha\tau\nu}=\left(\frac{\alpha+1}{\tau}\right)^{\frac{1}{\alpha+1}}\cdot\left(\frac{\nu(\alpha+1)}{2\alpha}\right)^{\frac{\alpha+1}{\alpha}}$. □

**Remark 5.4.** *In the proof above, $C_{\alpha\tau\nu}\to\infty$ as $\alpha\to 0$. As a result, the method above will not work for the* (AC) *case.*

Back to the proof of Theorem 5.1, (5.2) leads to

$$(A+\frac{1}{2}+C_{\alpha\tau\nu})\|u^{n+1}-u^n\|_{L^2}^2+E^{n+1}-E^n\leq\|u^{n+1}-u^n\|_{L^2}^2\left(\|u^n\|_\infty^2+\frac{1}{2}\|u^{n+1}\|_\infty^2\right)\ .$$

To prove $E^{n+1}\leq E^n$, it suffices to show $A+\frac{1}{2}+C_{\alpha\tau\nu}\geq\frac{3}{2}\max\left\{\|u^{n+1}\|_\infty^2\ ,\ \|u^n\|_\infty^2\right\}$. We rewrite the scheme (5.1) as

$$u^{n+1}=\frac{1+A\tau(-\Delta)^\alpha}{1+\tau\nu(-\Delta)^{\alpha+1}+A\tau(-\Delta)^\alpha}u^n-\frac{\tau(-\Delta)^\alpha}{1+\tau\nu(-\Delta)^{\alpha+1}+A\tau(-\Delta)^\alpha}\Pi_N[f(u^n)]\ .$$

Similarly, we can still apply Lemma 3.4 under the assumption $u_0$ satisfies zero-mean condition. Recall that

$$\|u^{n+1}\|_\infty\lesssim\|u^{n+1}\|_{\dot H^1}\sqrt{\log(\|u^{n+1}\|_{\dot H^{\frac{3}{2}}}+3)}\ .$$

We will estimate $\|u^{n+1}\|_{\dot H^1}$ and $\|u^{n+1}\|_{\dot H^{\frac{3}{2}}}$. As we did in section 3,

$$\begin{cases}\dfrac{1+A\tau|k|^{2\alpha}}{1+A\tau|k|^{2\alpha}+\nu\tau|k|^{2+2\alpha}}\cdot|k|\lesssim|k|\\[2ex]\dfrac{\tau|k|^{2\alpha}}{1+A\tau|k|^{2\alpha}+\nu\tau|k|^{2+2\alpha}}\cdot|k|\lesssim\dfrac{\tau}{\tau A}|k|=\dfrac{1}{A}|k|\end{cases}.$$

Hence we derive

$$\|u^{n+1}\|_{\dot H^1(\mathbb{T}^2)}\lesssim\left(1+\frac{1}{A}+\frac{3\|u\|_\infty^2}{A}\right)\|u^n\|_{\dot H^1(\mathbb{T}^2)}\ ,$$

which is the same argument as before. Similarly, we can derive

$$\|u^{n+1}\|_{\dot H^{\frac{3}{2}}(\mathbb{T}^2)}\lesssim\left(\frac{1+A\tau}{\nu\tau}+\frac{1}{\nu}\right)(E^n+1)\ .$$

We then prove by induction again.

**Step 1:** The induction $n\to n+1$ step. Assume $E^n\leq E^{n-1}\leq\cdots\leq E^0$ and $E^n\leq\sup_N\ E(\Pi_N u_0)$, we will show $E^{n+1}\leq E^n$. This implies $\|u^n\|_{\dot H^1}^2=\|\nabla u^n\|_{L^2}^2\leq\frac{2E^n}{\nu}\leq\frac{2E^0}{\nu}$. By applying the main lemma carefully and $E^0\lesssim E_0+1$,

$$\|u^n\|_\infty^2\lesssim_{E_0}\nu^{-1}\left(1+\log(A)+|\log(\nu)|\right)+\nu^{-1}|\log(\tau)|+1\quad .$$

Define $m_0:=\nu^{-1}\left(1+\log(A)+|\log(\nu)|\right)$, then the inequality above can be written as

$$\|u^n\|_\infty^2\lesssim_{E_0}m_0+\nu^{-1}|\log(\tau)|+1\ . \tag{5.3}$$

Similarly,

$$\|u^{n+1}\|_\infty^2\lesssim_{E_0}1+\frac{m_0^3}{A^2}+m_0+\nu^{-3}|\log(\tau)|^3. \tag{5.4}$$

Therefore we require the following condition:

$$\begin{cases} A + \frac{1}{2} + (\frac{\alpha+1}{\tau})^{\frac{1}{\alpha+1}} \cdot (\frac{\nu(\alpha+1)}{2\alpha})^{\frac{\alpha+1}{\alpha}} \geq C(E_0)\left(m_0 + 1 + \frac{m_0^3}{A^2} + \nu^{-3}|\log(\tau)|^3\right) \\ m_0 = \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right) \ . \end{cases}$$

Now we discuss 2 cases again:

`Case 1:` $(\frac{\alpha+1}{\tau})^{\frac{1}{\alpha+1}} \cdot (\frac{\nu(\alpha+1)}{2\alpha})^{\frac{\alpha+1}{\alpha}} \geq C(E_0)\nu^{-3}|\log(\tau)|^3$. In this case, it suffices to choose $A$ such that

$$A \gg_{E_0} m_0 = \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right) \ .$$

In fact, for $\nu \gtrsim 1$ , we can take $A \gg_{E_0} 1$; if $0 < \nu \ll 1$, we will choose $A = C_{E_0} \cdot \nu^{-1}|\log \nu|$ , where $C_{E_0}$ is a large constant depending only on $E_0$. Therefore in both cases it suffices to choose

$$A = C_{E_0} \cdot \max\left\{\nu^{-1}|\log(\nu)| \ , \ 1\right\} .$$

`Case 2:` $(\frac{\alpha+1}{\tau})^{\frac{1}{\alpha+1}} \cdot (\frac{\nu(\alpha+1)}{2\alpha})^{\frac{\alpha+1}{\alpha}} \leq C(E_0)\nu^{-3}|\log(\tau)|^3$. This implies $(\frac{1}{\tau})^{\frac{1}{\alpha+1}} \lesssim (\frac{1}{\nu})^{-4-\frac{1}{\alpha}}$ , hence $|\log(\tau)| \lesssim_{E_0} 1 + |\log(\nu)|$ for fixed $0 < \alpha \leq 1$. Now going back to equations (5.3), we have

$$\|u^n\|_\infty^2 \lesssim_{E_0} m_0$$

as $\nu^{-1}|\log(\tau)|$ will be dominated by $m_0$, recall that $m_0 = \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right)$. Substituting this new bound to (5.4), we derive that

$$\begin{aligned} \|u^{n+1}\|_\infty^2 &\lesssim \left(1 + (\frac{1 + \|u^n\|_\infty^2}{A})\|u^n\|_{\dot{H}^1}\sqrt{\log(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}})}\right)^2 \\ &\lesssim_{E_0} \left(1 + (1 + \frac{m_0}{A})(\sqrt{\frac{1}{\nu}}\sqrt{\log(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}})}\right)^2 \\ &\lesssim_{E_0} \left(1 + (1 + \frac{m_0}{A})\sqrt{m_0}\right)^2 \\ &\lesssim_{E_0} 1 + \frac{m_0^3}{A^2} + m_0 \ . \end{aligned}$$

Thus it suffices to take

$$A \geq C_{E_0} m_0 \ . \tag{5.5}$$

For the induction base **Step 2**, the proof is exactly the same as in section 3 and this shows stability of the semi-implicit scheme in the fractional Cahn-Hilliard case. □

## 6. Stability of a first order semi-implicit scheme on the 3D Allen-Cahn equation

In this section, we explore a bit more the three dimensional case. What makes the difference is that the log-interpolation lemma 3.4 does not hold. To clarify, the $\dot{H}^1$-norm should be replaced by the $\dot{H}^{\frac{3}{2}}$-norm in Lemma 3.4, as a result of scaling invariance. However, the $\dot{H}^{\frac{3}{2}}$-norm will not help to prove the 3D theorem as there is no a-priori energy bound for the $\dot{H}^{\frac{3}{2}}$-norm. To solve this issue, we will try an alternate interpolation inequality. For simplicity, we only consider Allen-Cahn equation in 3D periodic domain $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ in this section as other Cahn-Hilliard type equations can be handled similarly. To begin with, we recall the numerical scheme (3.1) for Allen-Cahn equation.

$$\begin{cases} \frac{u^{n+1} - u^n}{\tau} = \nu\Delta u^{n+1} - A(u^{n+1} - u^n) - \Pi_N f(u^n) \\ u^0 = \Pi_N u_0 \end{cases} . \tag{6.1}$$

where $\tau$ is the time step and $A > 0$ is the coefficient for the $O(\tau)$ regularization term. As usual, for $N \geq 2$, define

$$X_N = \operatorname{span}\left\{\cos(k \cdot x) \ , \sin(k \cdot x) : \quad k = (k_1, k_2, k_3) \in \mathbb{Z}^3 \ , \ |k|_\infty = \max\{|k_1|, |k_2|, |k_3|\} \leq N\right\} \ .$$

6.1. **Unconditional stability theorem**

**Theorem 6.1** (3D energy stability for AC)**.** *Consider (6.1) with $\nu > 0$ and assume $u_0 \in H^2(\mathbb{T}^3)$. Then there exists a constant $\beta_0$ depending only on the initial energy $E_0 = E(u_0)$ such that if*

$$A \geq \beta \cdot \left(\|u_0\|_{H^2}^2 + \nu^{-3} + 1\right) \ , \ \beta \geq \beta_0$$

*then $E(u^{n+1}) \leq E(u^n)$, $\forall n \geq 0$, where $E$ is defined before.*

**Remark 6.2.** *Unlike in section 3, our choice of $A$ is independent of $\tau$ as long as it has size of $O\left(\nu^{-3}\right)$ at least, which is much larger than $O\left(\nu^{-1}|\log(\nu)|^2\right)$. This results from the loss of log type control for the $L^\infty$ bound.*

Before proving Theorem 6.1, we will prove a new interpolation lemma here.

**Lemma 6.3.** *For all $f \in H^2(\mathbb{T}^3)$ , one has*

$$\|f\|_\infty \lesssim \|f\|_{\dot{H}^1}^{\frac{1}{2}} \|f\|_{\dot{H}^2}^{\frac{1}{2}} + |\hat{f}(0)| \ .$$

*Proof.* The proof is given in [24]. First we write $f(x) = \frac{1}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3} \hat{f}(k)\ e^{ik\cdot x}$, the Fourier series of $f$ in $\mathbb{T}^3$. So,

$$\begin{aligned}
\|f\|_\infty &\leq \frac{1}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3} |\hat{f}(k)| \\
&\leq \frac{1}{(2\pi)^3}|\hat{f}(0)| + \frac{1}{(2\pi)^3}\left(\sum_{0<|k|\leq N} |\hat{f}(k)| + \sum_{|k|>N} |\hat{f}(k)|\right) \\
&\lesssim |\hat{f}(0)| + \sum_{0<|k|\leq N} (|\hat{f}(k)||k| \cdot |k|^{-1}) + \sum_{|k|>N} (|\hat{f}(k)||k|^2 \cdot |k|^{-2}) \\
&\lesssim |\hat{f}(0)| + \left(\sum_{0<|k|\leq N} |\hat{f}(k)|^2|k|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{0<|k|\leq N} |k|^{-2}\right)^{\frac{1}{2}} + \left(\sum_{|k|>N} |\hat{f}(k)|^2|k|^4\right)^{\frac{1}{2}} \cdot (\sum_{|k|>N} |k|^{-4})^{\frac{1}{2}} \\
&\lesssim |\hat{f}(0)| + \left(\sum_{|k|>N} |\hat{f}(k)|^2|k|^4\right)^{\frac{1}{2}} \cdot \left(\int_N^\infty \frac{\pi r^2}{r^4}\ dr\right)^{\frac{1}{2}} + \left(\sum_{0<|k|\leq N} |\hat{f}(k)|^2|k|^2\right)^{\frac{1}{2}} \cdot \left(\int_1^N \frac{\pi r^2}{r^2}\ dr\right)^{\frac{1}{2}} \\
&\lesssim |\hat{f}(0)| + \|f\|_{\dot{H}^2} \cdot N^{-\frac{1}{2}} + \|f\|_{\dot{H}^1} \cdot N^{\frac{1}{2}} \ .
\end{aligned}$$

We optimize $N$ and hence derive

$$\|f\|_\infty \lesssim |\hat{f}(0)| + \|f\|_{\dot{H}^1}^{\frac{1}{2}} \|f\|_{\dot{H}^2}^{\frac{1}{2}} \ .$$

□

6.2. **Proof of the 3D stability theorem**

By the same argument in section 3 with notation $E^n = E(u^n)$,

$$\begin{aligned}
&(\frac{1}{\tau} + A + \frac{1}{2})\|u^{n+1} - u^n\|_{L^2}^2 + \frac{\nu}{2}\|\nabla(u^{n+1} - u^n)\|_{L^2}^2 + E^{n+1} - E^n \\
&\leq \|u^{n+1} - u^n\|_{L^2}^2 \left(\|u^n\|_\infty^2 + \frac{1}{2}\|u^{n+1}\|_\infty^2\right) \ .
\end{aligned}$$

Clearly, in order to show $E^{n+1} \leq E^n$, it suffices to show

$$\frac{1}{\tau} + A + \frac{1}{2} \geq \frac{3}{2} \max\left\{\|u^n\|_\infty^2 \ , \ \|u^{n+1}\|_\infty^2\right\} \ . \tag{6.2}$$

Now we rewrite the scheme (3.1) as the following:

$$u^{n+1} = \frac{1 + A\tau}{1 + A\tau - \nu\tau\Delta} u^n - \frac{\tau}{1 + A\tau - \nu\tau\Delta} \Pi_N[f(u^n)].$$

Recall that

$$\|u^{n+1}\|_\infty \lesssim |\widehat{u^{n+1}}(0)| + \|u^{n+1}\|_{\dot{H}^1}^{\frac{1}{2}} \|u^{n+1}\|_{\dot{H}^2}^{\frac{1}{2}} \ .$$

Clearly, we need to estimate $|\widehat{u^{n+1}}(0)|$, $\|u^{n+1}\|_{\dot{H}^1}$ and $\|u^{n+1}\|_{\dot{H}^2}$. By the same argument,

$$|\widehat{u^{n+1}}(0)| \lesssim 1 + \sqrt{E^n} \ .$$

Note that

$$\begin{cases} \dfrac{(1+A\tau)|k|}{1+A\tau+\nu\tau|k|^2} \le |k| \\ \dfrac{\tau|k|}{1+A\tau+\nu\tau|k|^2} \le \dfrac{\tau|k|}{2\tau\sqrt{A\nu}|k|} \lesssim \dfrac{1}{\sqrt{A\nu}} \ . \end{cases}$$

Hence we have

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^1} &\lesssim \|u^n\|_{\dot{H}^1} + \frac{1}{\sqrt{A\nu}} \|f(u^n)\|_{L^2} \\ &\lesssim \|u^n\|_{\dot{H}^1} + \frac{1}{\sqrt{A\nu}} (\|(u^n)^3\|_{L^2} + 1) \ . \end{aligned}$$

Similarly, we have

$$\begin{cases} \dfrac{(1+A\tau)|k|^2}{1+A\tau+\nu\tau|k|^2} \lesssim \left( \dfrac{1}{\tau\sqrt{A\nu}} + \sqrt{\dfrac{A}{\nu}} \right) |k| \\ \dfrac{\tau|k|^2}{1+A\tau+\nu\tau|k|^2} \le \dfrac{1}{\nu} \ . \end{cases}$$

This implies

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^2} &\lesssim \left( \frac{1}{\tau\sqrt{A\nu}} + \sqrt{\frac{A}{\nu}} \right) \|u^n\|_{\dot{H}^1} + \frac{1}{\nu} \|f(u^n)\|_{L^2} \\ &\lesssim \left( \frac{1}{\tau\sqrt{A\nu}} + \sqrt{\frac{A}{\nu}} \right) \|u^n\|_{\dot{H}^1} + \frac{1}{\nu} \left( \|(u^n)^3\|_{L^2} + 1 \right) \ . \end{aligned}$$

Note that by a standard Sobolev inequality,

$$\|(u^n)^3\|_{L^2} = \|u^n\|_{L^6}^3 \lesssim \|u^n\|_{H^1}^3 \lesssim \|\nabla u^n\|_{L^2}^3 + \|u^n\|_{L^2}^3 \lesssim \|u^n\|_{\dot{H}^1}^3 + 1 + (E^n)^{\frac{3}{2}} \ .$$

As a result, we get

$$\begin{cases} \|u^{n+1}\|_{\dot{H}^1} \lesssim \|u^n\|_{\dot{H}^1} + \dfrac{1}{\sqrt{A\nu}} \left( \|(u^n)\|_{\dot{H}^1}^3 + 1 + (E^n)^{\frac{3}{2}} \right) \\ \|u^{n+1}\|_{\dot{H}^2} \lesssim \left( \dfrac{1}{\tau\sqrt{A\nu}} + \sqrt{\dfrac{A}{\nu}} \right) \|u^n\|_{\dot{H}^1} + \dfrac{1}{\nu} \left( \|(u^n)\|_{\dot{H}^1}^3 + 1 + (E^n)^{\frac{3}{2}} \right) \end{cases} . \tag{6.3}$$

We prove the 3D stability theorem inductively as in the 2D case.

**Step 1:** The induction $n \to n+1$ step. Assume $E^n \le E^{n-1} \le \cdots \le E^0$ and $E^n \le \sup_N E(\Pi_N u_0)$, we show $E^{n+1} \le E^n$. This implies $\|u^n\|_{\dot{H}^1}^2 = \|\nabla u^n\|_{L^2}^2 \le \frac{2E^n}{\nu} \le \frac{2E^0}{\nu}$. Recall $\sup_N E(\Pi_N u_0) \lesssim E_0 + 1$ as well. Hence we derive from (6.3) that

$$\begin{cases} \|u^{n+1}\|_{\dot{H}^1} \lesssim_{E_0} \nu^{-\frac{1}{2}} + A^{-\frac{1}{2}} \nu^{-\frac{1}{2}} \left( \nu^{-\frac{3}{2}} + 1 \right) \lesssim_{E_0} \nu^{-\frac{1}{2}} + A^{-\frac{1}{2}} \nu^{-2} \\ \|u^{n+1}\|_{\dot{H}^2} \lesssim_{E_0} A^{\frac{1}{2}} \nu^{-1} + \nu^{-\frac{5}{2}} + \tau^{-1} A^{-\frac{1}{2}} \nu^{-1} \end{cases} .$$

Applying Lemma 6.3, we get

$$\begin{aligned} \|u^{n+1}\|_\infty^2 &\lesssim_{E_0} \left( \nu^{-\frac{1}{2}} + A^{-\frac{1}{2}} \nu^{-2} \right) \cdot \left( A^{\frac{1}{2}} \nu^{-1} + \nu^{-\frac{5}{2}} + \tau^{-1} A^{-\frac{1}{2}} \nu^{-1} \right) + 1 \\ &\lesssim_{E_0} A^{\frac{1}{2}} \nu^{-\frac{3}{2}} + \nu^{-3} + A^{-\frac{1}{2}} \nu^{-\frac{9}{2}} + \tau^{-1} A^{-\frac{1}{2}} \nu^{-\frac{3}{2}} + \tau^{-1} A^{-1} \nu^{-3} + 1 \ . \end{aligned}$$

To satisfy the sufficient condition (6.2),

$$A^{\frac{1}{2}} \nu^{-\frac{3}{2}} + \nu^{-3} + A^{-\frac{1}{2}} \nu^{-\frac{9}{2}} + \tau^{-1} A^{-\frac{1}{2}} \nu^{-\frac{3}{2}} + \tau^{-1} A^{-1} \nu^{-3} \lesssim_{E_0} A + \frac{1}{\tau} \ ,$$

it suffices to take

$$A \geq C_{E_0} \nu^{-3} \ , \tag{6.4}$$

for a large enough constant $C_{E_0}$ depending only on $E_0$.

**Step 2:** Check the induction base step $n = 1$. It is clear that we only need to check

$$A + \frac{1}{2} + \frac{1}{\tau} \geq \frac{3}{2}\|\Pi_N u_0\|_\infty^2 + \frac{3}{2}\|u^1\|_\infty^2.$$

By standard Sobolev inequality in $\mathbb{T}^3$, we get

$$\|\Pi_N u_0\|_\infty^2 \lesssim \|\Pi_N u_0\|_{H^2}^2 \lesssim \|u_0\|_{H^2}^2 \ .$$

On the other hand, by Lemma 6.3 it suffices to take

$$A + \frac{1}{\tau} \geq c_1 \|u_0\|_{H^2}^2 + \alpha_{E_0} \left( A^{\frac{1}{2}} \nu^{-\frac{3}{2}} + \nu^{-3} + A^{-\frac{1}{2}} \nu^{-\frac{9}{2}} + \tau^{-1} A^{-\frac{1}{2}} \nu^{-\frac{3}{2}} + \tau^{-1} A^{-1} \nu^{-3} \right) \ ,$$

where $c_1$ is an absolute constant and $\alpha_{E_0}$ is a constant only depending on $E_0$. Hence it suffices to take

$$A \geq C_{E_0} \left( \|u_0\|_{\dot{H}^2}^2 + \nu^{-3} + 1 \right) \ , \tag{6.5}$$

for a large constant $C_{E_0}$ only depending on $E_0$. This completes the proof. By using this new main lemma, the 3D fractional Cahn-Hilliard can be handled similarly.

It remains to estimate the $L^2$ error as in section 4.

### 6.3. $L^2$ error estimate of 3D Allen-Cahn equation

**Theorem 6.4.** *Let $\nu > 0$. Let $u_0 \in H^s$, $s \geq 4$ and $u(t)$ be the solution to Allen-Cahn equation with initial data $u_0$. Let $u^n$ be the numerical solution with initial data $\Pi_N u_0$. Assume $A$ satisfies the same condition in the stability theorem. Define $t_m = m\tau$, $m \geq 1$. Then*

$$\|u^m - u(t_m)\|_2 \leq A \cdot e^{C_1 t_m} \cdot C_2 \cdot \left( N^{-s} + \tau \right) \ ,$$

*where $C_1 > 0$ depends only on $(u_0, \nu)$ and $C_2$ depends on $(u_0, \nu, s)$.*

*Proof.* Recall the semi-implicit scheme together with exact solution

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = \nu \Delta u^{n+1} - \Pi_N f(u^n) - A(u^{n+1} - u^n) \\ \partial_t u = \nu \Delta u - f(u) \\ u^0 = \Pi_N u_0 \ , \ u(0) = u_0 \ . \end{cases} \tag{6.6}$$

As we proved in section 4, the auxiliary $L^2$ estimate lemma and all boundedness lemma work for 3D case. The only difference is the estimate for $\|\partial_t \Delta u\|_2$ using Gagliardo-Nirenberg interpolation inequality,

$$\|\partial_t \Delta u\|_2 \lesssim \|\langle \nabla \rangle^3 \partial_t u\|_2^{\frac{2}{3}} \cdot \|\partial_t u\|_2^{\frac{1}{3}} \lesssim \|\partial_t u\|_2^{\frac{1}{3}} \ ,$$

which works as well for the same power. This leads to the conclusion of Theorem 6.4 by the same argument in section 4.

□

## 7. Second order semi-implicit schemes on the 2D Allen-Cahn equation

In previous sections we introduce a first order semi-implicit scheme for the Allen-Cahn equation and the fractional Cahn-Hilliard equation in both two dimensional periodic domain and three dimensional periodic domain. For completeness, we will like to study some second order schemes. As a representative case, we only consider the 2D Allen-Cahn equation here and other cases are analyzed similarly. We introduce two second order schemes and prove the unconditional stability for Scheme I and conditional stability for Scheme II. We obtain these results by extending the study of [23], where Li et al. studied two second order schemes for the 2D periodic Cahn-Hilliard equation.

7.1. **Unconditional stability of scheme I:**

As introduced in [23], the second order semi-implicit Fourier spectral scheme I is given by:

$$\frac{3u^{n+1}-4u^n+u^{n-1}}{2\tau}=\nu\Delta u^{n+1}-A\tau(u^{n+1}-u^n)-\Pi_N\left(2f(u^n)-f(u^{n-1})\right)\quad,\ n\geq 1\ , \tag{7.1}$$

where $\tau>0$ is the time step and this scheme applies second order backward derivative in time with a second order extrapolation for the nonlinear term.

To start the iteration, we need to derive $u^1$ according to the following first order scheme:

$$\begin{cases}\dfrac{u^1-u^0}{\tau_1}=\nu\Delta u^1-\Pi_N f(u^0)\quad,\\ u^0=\Pi_N u_0\ ,\end{cases} \tag{7.2}$$

where $\tau_1=\min\{\tau^{\frac{4}{3}}\ ,\ 1\}$. The choice of $\tau$ is due to the error analysis which will be shown later. Roughly speaking,

$$\|u^1-u(\tau_1)\|_2\lesssim N^{-s}+\tau_1^{\frac{3}{2}}\ ,$$

where $u(\tau_1)$ denotes the exact PDE solution at $\tau_1$. As expected in $L^2$ error analysis of the second order scheme, we require that $\tau_1^{\frac{3}{2}}\lesssim\tau^2$ or $\tau_1\lesssim\tau^{\frac{4}{3}}$.

7.1.1. **Estimate of the first order scheme (7.2)**

In this section we will estimate some bounds for $u^1$ which will be used to prove the stability and $L^2$ error estimate of the second order scheme.

**Lemma 7.1.** *Consider the scheme (7.2). Assume $u_0\in H^2(\mathbb{T}^2)$, then*

$$\|u^1\|_\infty+\frac{\|u^1-u^0\|_2^2}{\tau_1}+\frac{\nu}{2}\|\nabla u^1\|_2^2\lesssim_{E(u_0)\ ,\ \|u_0\|_{H^2}}1\ .$$

*Proof.* Firstly, we consider $\|u^1\|_\infty$. We write

$$u^1=\frac{1}{1-\tau_1\nu\Delta}u^0-\frac{\tau_1\Pi_N}{1-\tau_1\nu\Delta}f(u^0)\ .$$

Note that

$$\frac{1}{1+\tau_1\nu|k|^2}\leq 1\ ,\ \tau_1\leq 1\ ,$$

thus we have

$$\begin{aligned}\|u^1\|_\infty\lesssim\|u^1\|_{H^2}&\lesssim\|u^0\|_{H^2}+\|f(u^0)\|_{H^2}\\ &\lesssim\|u^0\|_{H^2}+\|(u^0)^3\|_{H^2}\\ &\lesssim_{\|u_0\|_{H^2}}1\ ,\end{aligned}$$

as $\|u^0\|_\infty\lesssim 1$ by Morrey's inequality.

Secondly, we take $L^2$ inner product with $u^1-u^0$ on both sides of (7.2).

$$\begin{aligned}&\frac{\|u^1-u^0\|_2^2}{\tau_1}+\frac{\nu}{2}\left(\|\nabla u^1\|_2^2-\|\nabla u^0\|_2^2+\|\nabla(u^1-u^0)\|_2^2\right)\\ &=-(f(u^0)\ ,\ u^1-u^0)\\ &\leq\|f(u^0)\|_{\frac{4}{3}}\|u^1-u^0\|_4\\ &\lesssim_{E(u^0)}1\ ,\end{aligned}$$

by $\|u_0\|_\infty\ ,\ \|u_1\|_\infty\lesssim 1$. As a result, $\|u^1\|_\infty+\frac{\|u^1-u^0\|_2^2}{\tau_1}+\frac{\nu}{2}\|\nabla u^1\|_2^2\lesssim_{E(u_0)\ ,\ \|u_0\|_{H^2}}1$.

∎

**Lemma 7.2** (Error estimate for $u^1$)**.** *Consider the system for first time step $u^1$:*

$$\begin{cases} \dfrac{u^1 - u^0}{\tau_1} = \nu\Delta u^1 - \Pi_N f(u^0) \\ \partial_t u = \nu\Delta u - f(u) \\ u^0 = \Pi_N u_0 \ , \ u(0) = u_0 \ . \end{cases}$$

*Let $u_0 \in H^s$, $s \geq 6$. There exists a constant $D_1 > 0$ depending only on $(u_0, \nu, s)$, such that $\|u(\tau_1) - u^1\|_2 \leq D_1 \cdot (N^{-s} + \tau_1^{\frac{3}{2}})$.*

*Proof.* We start the proof in three steps:

**Step 1:** We discretize the exact solution $u$ in time. Write the continuous time PDE in time interval $[0, \tau_1]$. Note that for a one-variable function $h(s)$,

$$\begin{aligned} h(0) &= h(\tau_1) + \int_{\tau_1}^{0} h'(s)\ ds \\ &= h(\tau_1) - h'(\tau_1)\tau_1 + \int_0^{\tau_1} h''(s) \cdot s\ ds\ . \end{aligned}$$

By applying this formula, we have

$$\begin{aligned} \frac{u(\tau_1) - u(0)}{\tau_1} &= \partial_t u(\tau_1) - \frac{1}{\tau_1}\int_0^{\tau_1} (\partial_{tt} u) \cdot s\ ds \\ &= \nu\Delta u(\tau_1) - f(u(\tau_1)) - \frac{1}{\tau_1}\int_0^{\tau_1} (\partial_{tt} u) \cdot s\ ds \\ &= \nu\Delta u(\tau_1) - \Pi_N f(u(0)) - \Pi_{>N} f(u(0)) - [f(u(\tau_1) - f(u(0))] \\ &- \frac{1}{\tau_1}\int_0^{\tau_1} (\partial_{tt} u) \cdot s\ ds\ , \end{aligned}$$

where $\Pi_{>N} = id - \Pi_N$ as in section 4. Therefore, we get

$$\frac{u(\tau_1) - u(0)}{\tau_1} = \nu\Delta u(\tau_1) - \Pi_N f(u(0)) + G^0\ ,$$

where

$$\begin{aligned} G^0 &= -\Pi_{>N} f(u(0)) - [f(u(\tau_1) - f(u(0))] - \frac{1}{\tau_1}\int_0^{\tau_1} (\partial_{tt} u) \cdot s\ ds \\ &= -\Pi_{>N} f(u(0)) - [f(u(\tau_1) - f(u(0))] - \frac{1}{\tau_1}\int_0^{\tau_1} (\nu\Delta\partial_t u - f'(u)\partial_t u) \cdot s\ ds \end{aligned}$$

**Step 2:** Estimate $\|u(\tau_1) - u^1\|_2$. We consider

$$\begin{cases} \dfrac{u(\tau_1) - u(0)}{\tau_1} = \nu\Delta u(\tau_1) - \Pi_N f(u(0)) + G^0 \\ \dfrac{u^1 - u^0}{\tau_1} = \nu\Delta u^1 - \Pi_N f(u^0) \\ u^0 = \Pi_N u_0 \ , \ u(0) = u_0 \ . \end{cases}$$

Define $e^1 = u(\tau_1) - u^1$ and $e^0 = u(0) - u^0$. Then we get

$$\frac{e^1 - e^0}{\tau_1} = \nu\Delta e^1 - \Pi_N \left(f(u(0)) - f(u^0)\right) + G^0\ .$$

Taking the $L^2$ inner product with $e^1$ on both sides, we derive

$$
\begin{aligned}
&\frac{1}{2\tau_1}\left(\|e^1\|_2^2 - \|e^0\|_2^2 + \|e^1 - e^0\|_2^2\right) + \nu\|\nabla e^1\|_2^2 \\
&\le \|f(u(0)) - f(u^0)\|_2 \cdot \|e^1\|_2 + \|G^0\|_2 \cdot \|e^1\|_2 \\
&\lesssim \left(\|e^0\|_2 + \|G^0\|_2\right)\|e^1\|_2 \\
&\lesssim \left(\|e^0\|_2^2 + \|G^0\|_2^2\right) + \frac{1}{4}\|e^1\|_2^2 \ .
\end{aligned}
$$

As a result, we have

$$
\left(1 - \frac{\tau_1}{2}\right)\|e^1\|_2^2 \le 2\tau_1\left(\|e^0\|_2^2 + \|G^0\|_2^2\right) + \|e^0\|_2^2 \ .
$$

Note that $\tau_1 \le 1$, so $1 - \frac{\tau_1}{2} \ge \frac{1}{2}$ and

$$
\|e^1\|_2^2 \lesssim (1+\tau_1)\|e^0\|_2^2 + \tau_1\|G^0\|_2^2 \ .
$$

**Step 3:** Estimate $\|e^0\|_2^2$ and $\|G^0\|_2^2$. Note that $\|e^0\|_2^2 = \|u(0) - u^0\|_2^2 = \|u_0 - \Pi_N u_0\|_2^2 = \|\Pi_{>N} u_0\|_2^2$. As proved in section 4,

$$
\|e^0\|_2^2 = \|\Pi_{>N} u_0\|_2^2 \lesssim N^{-2s} \ .
$$

For $\|G^0\|_2$, note that $\|\Pi_{>N} f(u(0))\|_2 \lesssim N^{-s}$, by the maximum principle (Lemma 4.2) proved in section 4. On the other hand, by the mean value theorem,

$$
f(u(\tau_1)) - f(u(0)) = f'(\xi)(u(\tau_1) - u(0)) \ ,
$$

where $\xi$ is a number between $u(\tau_1)$ and $u(0)$. Again by the maximum principle (Lemma 4.2),

$$
\|f(u(\tau_1)) - f(u(0))\|_2 \lesssim \|u(\tau_1) - u(0)\|_2 \lesssim \tau_1\|\partial_t u\|_{L_t^\infty L_x^2([0\ ,\ \tau_1]\times\mathbb{T}^2)} \lesssim \tau_1 \ ,
$$

by the Sobolev bound of the exact solution proved in section 4.

Finally, we have

$$
\begin{aligned}
&\left\|\frac{1}{\tau_1}\int_0^{\tau_1}(\nu\Delta\partial_t u - f'(u)\partial_t u)\cdot s\ ds\right\|_2 \\
&\lesssim \left\|\int_0^{\tau_1}\nu\Delta\partial_t u - f'(u)\partial_t u\ ds\right\|_2 \\
&\lesssim \int_0^{\tau_1}\|\nu\Delta\partial_t u\|_2\ ds + \int_0^{\tau_1}\|f'(u)\partial_t u\|_2\ ds \\
&\lesssim \tau_1 \ .
\end{aligned}
$$

This implies $\|G^0\|_2^2 \lesssim N^{-2s} + \tau_1^2$. Therefore we get

$$
\|e^1\|_2^2 \lesssim (1+\tau_1)N^{-2s} + \tau_1(N^{-2s} + \tau_1^2) \lesssim N^{-2s} + \tau_1^3 \ .
$$

As a result, we obtain that

$$
\|e^1\|_2 \lesssim N^{-s} + \tau_1^{\frac{3}{2}} \ . \tag{7.3}
$$

This completes the proof. □

7.1.2. **Unconditional stability of the second order scheme I (7.1) & (7.2)**

In this section we will prove a unconditional stability theorem for the second order scheme (7.1) combining (7.2). To get started, we state the theorem first.

**Theorem 7.3** (Unconditional stability)**.** *Consider the scheme (7.1)-(7.2) with $\nu > 0$, $\tau > 0$ and $N \ge 2$. Assume $u_0 \in H^2(\mathbb{T}^2)$. The initial energy is denoted by $E_0 = E(u_0)$. If there exists a constant $\beta_c > 0$ depending only on $E_0$ and $\|u_0\|_{H^2}$, such that*

$$
A \ge \beta\cdot(\nu^2 + \nu^{-10}|\log\nu|^4) \ ,\ \beta \ge \beta_c \ ,
$$

*then*

$$
\tilde{E}(u^{n+1}) \le \tilde{E}(u^n) \ ,\ n \ge 1 \ ,
$$

*where* $\tilde{E}(u^n)$ *for* $n \geq 1$ *is a modified energy functional and is defined as*

$$\tilde{E}(u^n) := E(u^n) + \frac{\nu}{4}\|u^n - u^{n-1}\|_2^2 + \frac{1}{4\tau}\|u^n - u^{n-1}\|_2^2 \ .$$

Before proving this stability theorem, we begin with several lemmas.

**Lemma 7.4.** *Consider (7.1) for* $n \geq 1$*. Suppose* $E(u^n) \leq B$ *and* $E(u^{n-1}) \leq B$ *for some* $B > 0$*. Then*

$$\|u^{n+1}\|_\infty \leq \alpha_B \cdot \left\{ (1+\nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1})} + \tau + 1 \right\} \ ,$$

*for some* $\alpha_B > 0$ *only depending on* $B$*.*

*Proof.* For simplicity we write $\lesssim$ instead of $\lesssim_B$. Recall that (7.1)

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} = \nu\Delta u^{n+1} - A\tau(u^{n+1} - u^n) - \Pi_N\left(2f(u^n) - f(u^{n-1})\right) \ .$$

We rewrite (7.1) as

$$\begin{aligned} u^{n+1} = & \frac{4 + 2A\tau^2}{3 - 2\nu\tau\Delta + 2A\tau^2} u^n - \frac{1}{3 - 2\nu\tau\Delta + 2A\tau^2} \\ & - \frac{2\tau\Pi_N}{3 - 2\nu\tau\Delta + 2A\tau^2}\left(2f(u^n) - f(u^{n-1})\right) \ . \end{aligned}$$

For the case when $k = 0$, we have

$$\begin{cases} \frac{4 + 2A\tau^2}{3 + 2A\tau^2} \lesssim 1 \\ \frac{1}{3 + 2A\tau^2} \lesssim 1 \\ \frac{2}{3 + 2A\tau^2} \lesssim \tau \ . \end{cases}$$

We thus have

$$|\widehat{u^{n+1}}(0)| \lesssim \tau + 1 \ .$$

Note that for the case when $|k| \geq 1$,

$$\begin{cases} \frac{4 + 2A\tau^2}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim 1 \\ \frac{1}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim 1 \\ \frac{2\tau|k|}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim \frac{\tau|k|}{\nu\tau|k|^2} \lesssim \frac{1}{\nu} \cdot |k|^{-1} \ . \end{cases}$$

Therefore we get

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^1} & \lesssim \|u^n\|_{\dot{H}^1} + \|u^{n-1}\|_{\dot{H}^1} + \frac{1}{\nu}\|\langle\nabla\rangle^{-1}\left(2f(u^n) - f(u^{n-1})\right)\|_2 \\ & \lesssim \nu^{-\frac{1}{2}} + \nu^{-1}(\|(u^n)^3\|_{4/3} + \|(u^{n-1})^3\|_{4/3} + \|u^n\|_2 + \|u^{n-1}\|_2) \\ & \lesssim \nu^{-\frac{1}{2}} + \nu^{-1} \ , \end{aligned}$$

here we apply Sobolev's inequality and apply the energy bound as proved in section 3. Similarly, we can derive that

$$\begin{cases} \frac{|k|^2(4 + 2A\tau^2)}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim \frac{|k|^2(1 + A\tau^2)}{\nu\tau|k|^2} \lesssim \frac{1}{\nu\tau} + \frac{A\tau}{\nu} \\ \frac{|k|^2}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim \frac{1}{\nu\tau} \\ \frac{2\tau|k|^2}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim \frac{\tau|k|^2}{\nu\tau|k|^2} \lesssim \frac{1}{\nu} \ . \end{cases}$$

This implies

$$
\begin{aligned}
\|u^{n+1}\|_{\dot H^2} &\lesssim \left(\frac{1}{\nu\tau}+\frac{A\tau}{\nu}\right)\|u^n\|_2+\frac{1}{\nu\tau}\|u^{n-1}\|_2+\frac{1}{\nu}\|2f(u^n)-f(u^{n-1})\|_2\\
&\lesssim \frac{1}{\nu\tau}+\frac{A\tau}{\nu}+\frac{1}{\nu}\left(\|u^n\|_6^3+\|u^{n-1}\|_6^3+\|u^n\|_2+\|u^{n-1}\|_2\right)\\
&\lesssim \frac{1}{\nu\tau}+\frac{A\tau}{\nu}+\frac{1}{\nu}\left(\|u^n\|_{H^1}^3+\|u^{n-1}\|_{H^1}^3+1\right)\\
&\lesssim \frac{1}{\nu\tau}+\frac{A\tau}{\nu}+\frac{1}{\nu}(\nu^{-\frac{3}{2}}+1)\ .
\end{aligned}
$$

Finally, by applying the log-interpolation lemma (Lemma 3.4), we can get

$$
\begin{aligned}
\|u^{n+1}\|_\infty &\lesssim (1+\|u^{n+1}\|_{\dot H^1})\cdot\sqrt{\log(3+\|u^{n+1}\|_{\dot H^2})}+|\widehat{u^{n+1}(0)}|\\
&\lesssim (1+\nu^{-1})\cdot\sqrt{\log(3+\frac{A\tau}{\nu}+\frac{1}{\nu\tau}+\nu^{-\frac{5}{2}}+\nu^{-1})}+\tau+1\ ,
\end{aligned}
$$

where $\nu^{-\frac{1}{2}}$ is bounded by $\nu^{-1}+1$.

□

### 7.1.3. **Proof of unconditional stability Theorem 7.3**

Before proving Theorem 7.3, we introduce some notation. We denote $\delta u^{n+1} := u^{n+1}-u^n$ and $\delta^2 u^{n+1} := u^{n+1}-2u^n+u^{n-1}$. Clearly,

$$
\begin{cases}
3u^{n+1}-4u^n+u^{n-1} = 2\delta u^{n+1}+\delta^2 u^{n+1}\\
\delta^2 u^{n+1}-\delta u^{n+1} = -\delta u^n\\
\delta u^n\cdot u^n = (u^n-u^{n-1})u^n = \frac{1}{2}\left(|u^n|^2-|u^{n-1}|^2+|\delta u^n|^2\right)\ .
\end{cases}
$$

As a result, we have

$$
\begin{aligned}
&\left(3u^{n+1}-4u^n+u^{n-1}\ ,\ u^{n+1}-u^n\right)\\
&=\left(2\delta u^{n+1}+\delta^2 u^{n+1}\ ,\ \delta u^{n+1}\right)\\
&=2\|\delta u^{n+1}\|_2^2+\left(\delta u^{n+1}-\delta u^n\ ,\ \delta u^{n+1}\right)\\
&=2\|\delta u^{n+1}\|_2^2+\frac{1}{2}\left(\|\delta u^{n+1}\|_2^2-\|\delta u^n\|_2^2+\|\delta^2 u^{n+1}\|_2^2\right)\ .
\end{aligned}
$$

Now recall the scheme (7.1)

$$
\frac{3u^{n+1}-4u^n+u^{n-1}}{2\tau} = \nu\Delta u^{n+1}-A\tau(u^{n+1}-u^n)-\Pi_N\left(2f(u^n)-f(u^{n-1})\right)\ .
$$

Taking the $L^2$ inner product with $\delta u^{n+1}=u^{n+1}-u^n$ on both sides of (7.1), we have

$$
\begin{aligned}
&\frac{1}{\tau}\|\delta u^{n+1}\|_2^2+\frac{1}{4\tau}\left(\|\delta u^{n+1}\|_2^2-\|\delta u^n\|_2^2+\|\delta^2 u^{n+1}\|_2^2\right)\\
&\qquad+\frac{\nu}{2}\left(\|\nabla u^{n+1}\|_2^2-\|\nabla u^n\|_2^2+\|\delta\nabla u^{n+1}\|_2^2\right)\\
&+A\tau\|\delta u^{n+1}\|_2^2 = -\left(\Pi_N(2f(u^n)-f(u^{n-1}))\ ,\ \delta u^{n+1}\right)\ .
\end{aligned}
$$

To analyze $\left(2f(u^n)-f(u^{n-1})\ ,\ \delta u^{n+1}\right)$, we consider

$$
2f(u^n)-f(u^{n-1}) = f(u^n)+\left(f(u^n)-f(u^{n-1}\right).
$$

Note that $F'=f$, hence by the fundamental theorem of calculus,

$$
\begin{aligned}
&F(u^{n+1})-F(u^n)\\
=&f(u^n)\delta u^{n+1}+\int_0^1 f'(u^n+s\delta u^{n+1})(1-s)\ ds\cdot(\delta u^{n+1})^2\\
=&f(u^n)\delta u^{n+1}+\int_0^1 \tilde f(u^n+s\delta u^{n+1})(1-s)\ ds\cdot(\delta u^{n+1})^2-\frac{1}{2}(\delta u^{n+1})^2\ ,
\end{aligned}
$$

where $\tilde{f}(x) = 3x^2$, as $f'(x) = 3x^2 - 1$. Therefore, we can get

$$f(u^n)\delta u^{n+1} \geq F(u^{n+1}) - F(u^n) + \frac{1}{2}(\delta u^{n+1})^2 - \frac{3}{2}\left(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2\right) \cdot (\delta u^{n+1})^2 \ .$$

On the other hand by the mean value theorem,

$$f(u^n) - f(u^{n-1}) = f'(\xi)\delta u^n \ ,$$

and hence we have

$$\begin{aligned}
&(f(u^n) - f(u^{n-1})) \cdot \delta u^{n+1} \geq -\left(3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2 + 1\right) \cdot |\delta u^n| \cdot |\delta u^{n+1}| \\
&\geq -\frac{\left(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2\right)^2}{\nu} \cdot \|\delta u^n\|_2^2 - \frac{\nu}{4}\|\delta u^{n+1}\|_2^2 \ .
\end{aligned}$$

Then the estimate of the nonlinear term is as following:

$$\begin{aligned}
&-\left(\Pi_N(2f(u^n) - f(u^{n-1}))\ ,\ \delta u^{n+1}\right) \\
=&-\left(2f(u^n) - f(u^{n-1})\ ,\ \delta u^{n+1}\right) \\
\leq&-\int_{\mathbb{T}^2} F(u^{n+1})\ dx + \int_{\mathbb{T}^2} F(u^n)\ dx - \frac{1}{2}\|\delta u^{n+1}\|_2^2 \\
&+\frac{3}{2}\left(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2\right) \cdot \|\delta u^{n+1}\|_2^2 \\
&+\frac{\left(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2\right)^2}{\nu} \cdot \|\delta u^n\|_2^2 + \frac{\nu}{4}\|\delta u^{n+1}\|_2^2 \ .
\end{aligned}$$

Combining all estimates we get

$$\begin{aligned}
&\frac{1}{\tau}\|\delta u^{n+1}\|_2^2 + \frac{1}{4\tau}\left(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2\right) \\
&+\frac{\nu}{2}\left(\|\nabla u^{n+1}\|_2^2 - \|\nabla u^n\|_2^2 + \|\delta\nabla u^{n+1}\|_2^2\right) \\
&+A\tau\|\delta u^{n+1}\|_2^2 \\
&\leq -\int_{\mathbb{T}^2} F(u^{n+1})\ dx + \int_{\mathbb{T}^2} F(u^n)\ dx - \frac{1}{2}\|\delta u^{n+1}\|_2^2 \\
&+\frac{3}{2}\left(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2\right) \cdot \|\delta u^{n+1}\|_2^2 \\
&+\frac{\left(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2\right)^2}{\nu} \cdot \|\delta u^n\|_2^2 + \frac{\nu}{4}\|\delta u^{n+1}\|_2^2 \ .
\end{aligned}$$

After simplification, we obtain

$$\begin{aligned}
&\left(\frac{1}{\tau} + A\tau - \frac{\nu}{4} + \frac{1}{2}\right) \cdot \|\delta u^{n+1}\|_2^2 + \tilde{E}(u^{n+1}) \\
\leq&\left\{\frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{\left(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2\right)^2}{\nu}\right\} \cdot \|\delta u^{n+1}\|_2^2 + \tilde{E}(u^n) \ .
\end{aligned}$$

Clearly for $\tilde{E}(u^{n+1}) \leq \tilde{E}(u^n)$, it suffices to show

$$\begin{aligned}
&\frac{1}{\tau} + A\tau - \frac{\nu}{4} + \frac{1}{2} \geq \\
&\frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{\left(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2\right)^2}{\nu} \ .
\end{aligned} \tag{7.4}$$

Now we prove this sufficient condition inductively. Set

$$B = \max\left\{\tilde{E}(u^1)\ ,\ E(u^0)\right\} \ ,$$

by Lemma 7.1 in previous section, $B \lesssim 1$. We shall prove for every $m \geq 2$,

$$\begin{cases} \tilde{E}(u^m) \leq B \ , \ \tilde{E}(u^m) \leq \tilde{E}(u^{m-1}) \ , \\ \|u^m\|_\infty \leq \alpha_B \cdot \left[ (1+\nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1})} + \tau + 1 \right] \ , \end{cases}$$

where $\alpha_B > 0$ is the same constant in Lemma 7.4.

We first check the base case when $m = 2$. Note that $E(u^1) \leq \tilde{E}(u^1) \leq B$ and $E(u^0) \leq B$, then we can apply Lemma 7.4, and hence obtain

$$\|u^2\|_\infty \leq \alpha_B \cdot \left\{ (1+\nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1})} + \tau + 1 \right\} \ .$$

It then suffices to check $\tilde{E}(u^2) \leq \tilde{E}(u^1)$. By the sufficient condition (7.4), we only need to check the inequality

$$\frac{1}{\tau} + A\tau - \frac{\nu}{4} + \frac{1}{2} \geq \frac{3}{2}(\|u^1\|_\infty^2 + \|u^2\|_\infty^2) + \frac{\left(1 + 3\|u^1\|_\infty^2 + 3\|u^0\|_\infty^2\right)^2}{\nu} \ .$$

By Lemma 7.1, $\|u^0\|_\infty \ , \ \|u^1\|_\infty \lesssim 1$, it suffices to choose $A$ such that

$$\frac{1}{\tau} + A\tau - \frac{\nu}{4} + \frac{1}{2} \geq C \cdot (1+\nu^{-2}) \cdot \log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1}) + C\nu^{-1} + C + C\tau \ .$$

We discuss two case and denote $X = A\tau + \frac{1}{\tau}$.

`Case 1:` $0 < \nu \leq 1/2$. In this case we need

$$X + \frac{1}{2} \geq C\nu^{-2} \cdot (|\log \nu| + |\log X|) \ .$$

Hence we need

$$X \geq C \cdot \nu^{-2} |\log \nu| \ .$$

`Case 2:` $\nu > 1/2$. Then we need

$$X \geq C \cdot (|\log X| + 1 + \nu) \ ,$$

namely,

$$X \geq C \cdot (1+\nu) \ .$$

In conclusion, as $X \geq 2\sqrt{A}$,

$$A \geq C \cdot (1 + \nu^2 + \nu^{-4}|\log \nu|^2) \geq C \cdot (\nu^2 + \nu^{-4}|\log \nu|^2) \ .$$

Now we check the induction step. Assume the induction hypopaper hold for $2 \leq m \leq n$, then for $m = n+1$,

$$\|u^{n+1}\|_\infty \leq \alpha_B \cdot \left\{ (1+\nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1})} + \tau \right\} \ ,$$

by Lemma 7.4. It remains to show $\tilde{E}(u^{n+1}) \leq \tilde{E}(u^n)$. It suffices to choose $A$ such that

$$\begin{aligned} \frac{1}{\tau} + A\tau + \frac{1}{2} \geq & \frac{\nu}{4} + C \cdot (1+\nu^{-2}) \cdot \log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1}) \\ & + \frac{C(1+\nu^{-4})}{\nu} \left( (\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1}))^2 + \tau \right) \ . \end{aligned}$$

In terms of $X = A\tau + \frac{1}{\tau}$ again, we need to discuss two cases as well.

`Case 1:` $0 < \nu \leq 1/2$. Then

$$X \geq C \cdot \nu^{-5} (|\log \nu|^2 + |\log X|^2) \ .$$

As a result, we have

$$X \geq C \cdot \nu^{-5} |\log \nu|^2 \ .$$

`Case 2:` $\nu > 1/2$. Then we need

$$X \geq C\nu + C \cdot (\log X + (\log X)^2 \nu^{-1}) \ ,$$

hence $X \geq C \cdot (\nu + 1)$.

To conclude these two cases, we require that

$$A \geq C \cdot (\nu^2 + 1 + \nu^{-10} |\log \nu|^4) \geq C \cdot (\nu^2 + \nu^{-10} |\log \nu|^4) \ .$$

This completes the induction. Combining the estimate, we can take

$$A \geq C \cdot (\nu^2 + \nu^{-10} |\log \nu|^4) \ , \tag{7.5}$$

such that $\tilde{E}(u^{n+1}) \leq \tilde{E}(u^n)$, for $n \geq 1$.

### 7.2. $L^2$ error estimate of the second order scheme I

It remains to estimate the $L^2$ error of this second order scheme.

**Theorem 7.5** ($L^2$ error estimate)**.** *Let $\nu > 0$ and $u_0 \in H^s$, $s \geq 8$. Let $0 < \tau \leq M$ for some $M > 0$. Let $u(t)$ be the continuous solution to the 2D Allen-Cahn equation with initial data $u_0$. Let $u^1$ be defined according to (7.2) with initial data $u^0 = \Pi_N u_0$. Let $u^m$, $m \geq 2$ be defined in (7.1) with initial data $u^0$ and $u^1$. Assume $A$ satisfies the same condition in Theorem 7.3. Define $t_0 = 0$, $t_1 = \tau_1$ and $t_m = \tau_1 + (m-1)\tau$ for $m \geq 2$. Then for any $m \geq 1$,*

$$\|u(t_m) - u^m\|_2 \leq C_1 \cdot e^{C_2 t_m} \cdot (N^{-s} + \tau^2) \ ,$$

*where $C_1$ , $C_2 > 0$ are constants depending only on $(u_0, \ \nu, \ s, \ A, \ M)$.*

**Remark 7.6.** *Here we require that $\tau$ is not arbitrarily large. This is a result of loss of mass conservation as preserved by Cahn-Hilliard equation. However, in practice it is not a big issue as we always use small time steps.*

Similar to section 4, we will study the auxiliary error estimate behavior and time discretization behavior of Allen-Cahn equation before proving the theorem.

#### 7.2.1. **Auxiliary $L^2$ error estimate for near solutions**

Consider for $n \geq 1$,

$$\begin{cases} \dfrac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} = \nu\Delta u^{n+1} - A\tau(u^{n+1} - u^n) - \Pi_N \left(2f(u^n) - f(u^{n-1})\right) + G^n \\ \dfrac{3v^{n+1} - 4v^n + v^{n-1}}{2\tau} = \nu\Delta v^{n+1} - A\tau(v^{n+1} - v^n) - \Pi_N \left(2f(v^n) - f(v^{n-1})\right) \end{cases} , \tag{7.6}$$

where $(u^1, \ u^0, \ v^1, \ v^0)$ are given.

**Proposition 7.1.** *For solutions to (7.6), assume for some $N_1 > 0$,*

$$\sup_{n \geq 0} \|u^n\|_\infty + \sup_{n \geq 0} \|v^n\|_\infty \leq N_1 \ ,$$

*Then for any $m \geq 2$,*

$$\|u^m - v^m\|_2^2 \leq C \cdot \exp\left((m-1)\tau \cdot \frac{C(1+N_1^4)}{\eta}\right) \cdot \left((1 + A\tau^2)\|u^1 - v^1\|_2^2 + \|u^0 - v^0\|_2^2 + \frac{\tau}{\eta} \sum_{n=1}^{m-1} \|G^n\|_2^2\right) \ ,$$

*where $C > 0$ is a absolute constant that can be computed and $0 < \eta < \frac{1}{100M}$ is a constant depending only on $M$, that is the upper bound for $\tau$.*

*Proof.* We still denote the constant by $C$ whose value may vary in different lines. Denote $e^n = u^n - v^n$, then

$$\begin{aligned} &\frac{3e^{n+1} - 4e^n + e^{n-1}}{2\tau} - \nu\Delta e^{n+1} + A\tau(e^{n+1} - e^n) \\ =& - \Pi_N\left(2f(u^n) - 2f(v^n)\right) + \Pi_N\left(f(u^{n-1}) - f(v^{n-1})\right) + G^n \end{aligned}$$

Taking the $L^2$ inner product with $e^{n+1}$ on both sides, we derive that

$$\begin{aligned} &\frac{1}{2\tau}(3e^{n+1} - 4e^n + e^{n-1}, e^{n+1}) + \nu\|\nabla e^{n+1}\|_2^2 + \frac{A\tau}{2}\left(\|e^{n+1}\|_2^2 - \|e^n\|_2^2 + \|e^{n+1} - e^n\|_2^2\right) \\ &= -2(f(u^n) - f(v^n),\ e^{n+1}) + (f(u^{n-1}) - f(v^{n-1}),\ e^{n+1}) + (G^n,\ e^{n+1})\ . \end{aligned} \tag{7.7}$$

To estimate the right hand side of (7.7), first we observe that

$$|(f(u^n) - f(v^n), e^{n+1})| \le \|f(u^n) - f(v^n)\|_2\|e^{n+1}\|_2 \le \frac{\|f(u^n) - f(v^n)\|_2^2}{\eta} + \eta\|e^{n+1}\|_2^2,$$

where $\eta < \frac{1}{100M}$ is a small number only depending on $M$. Moreover, as proved in section 4,

$$\begin{aligned} f(u^n) - f(v^n) &= f(u^n) - f(u^n - e^n) \\ &= (u^n)^3 - (u^n - e^n)^3 - e^n \\ &= -(e^n)^3 - e^n - 3u^n(e^n)^2 + 3(u^n)^2e^n\ . \end{aligned}$$

By assumption we then have

$$\begin{aligned} \|f(u^n) - f(v^n)\|_2^2 \lesssim& \|e^n\|_\infty^4\|e^n\|_2^2 + \|e^n\|_2^2 + \|u^n\|_\infty^2\|e^n\|_2^2 + \|u^n\|_\infty^4\|e^n\|_2^2 \\ \lesssim&(1 + N_1^4)\|e^n\|_2^2\ . \end{aligned}$$

Similarly, we have

$$\|f(u^{n-1}) - f(v^{n-1})\|_2^2 \lesssim (1 + N_1^4)\|e^{n-1}\|_2^2\ .$$

As a result, we obtain that

$$\text{RHS of (7.7)} \le \frac{C(1 + N_1^4)}{\eta}\left(\|e^n\|_2^2 + \|e^{n-1}\|_2^2\right) + \frac{1}{\eta}\|G^n\|_2^2 + \eta\|e^{n+1}\|_2^2\ .$$

On the other hand, we have

$$\begin{aligned} (3e^{n+1} - 4e^n + e^{n-1}, e^{n+1} - e^n) &= (2\delta e^{n+1} + \delta^2 e^{n+1},\ \delta e^{n+1}) \\ &= 2\|\delta e^{n+1}\|_2^2 + \frac{1}{2}\left(\|\delta e^{n+1}\|_2^2 - \|\delta e^n\|_2^2 + \|\delta^2 e^{n+1}\|_2^2\right)\ . \end{aligned}$$

We also have that

$$\begin{aligned} &(3e^{n+1} - 4e^n + e^{n-1},\ e^n) = 3(\delta e^{n+1},\ e^n) - (\delta e^n,\ e^n) \\ =&\frac{3}{2}\left(\|e^{n+1}\|_2^2 - \|e^n\|_2^2 - \|e^{n+1} - e^n\|_2^2\right) - \frac{1}{2}\left(\|e^n\|_2^2 - \|e^{n-1}\|_2^2 + \|e^n - e^{n-1}\|_2^2\right)\ . \end{aligned}$$

These two equations give that

$$\begin{aligned} &(3e^{n+1} - 4e^n + e^{n-1},\ e^{n+1}) \\ =&\frac{3}{2}(\|e^{n+1}\|_2^2 - \|e^n\|_2^2) - \frac{1}{2}(\|e^n\|_2^2 - \|e^{n-1}\|_2^2) + \|\delta e^{n+1}\|_2^2 - \|\delta e^n\|_2^2 \\ &+ \frac{1}{2}\|\delta^2 e^{n+1}\|_2^2\ . \end{aligned}$$

Collecting all estimates, we bound (7.7) as

$$\begin{aligned}
&\frac{1}{2\tau}\left(\frac{3}{2}\|e^{n+1}\|_2^2-\frac{1}{2}\|e^n\|_2^2+\|e^{n+1}-e^n\|_2^2\right)+\frac{A\tau}{2}\|e^{n+1}\|_2^2\\
\le&\frac{1}{2\tau}\left(\frac{3}{2}\|e^n\|_2^2-\frac{1}{2}\|e^{n-1}\|_2^2+\|e^n-e^{n-1}\|_2^2\right)+\frac{A\tau}{2}\|e^n\|_2^2\\
+&\frac{C(1+N_1^4)}{\eta}\left(\|e^n\|_2^2+\|e^{n-1}\|_2^2\right)+\frac{1}{\eta}\|G^n\|_2^2+\eta\|e^{n+1}\|_2^2\ .
\end{aligned}\tag{7.8}$$

Define $X^{n+1}:=\frac{3}{2}\|e^{n+1}\|_2^2-\frac{1}{2}\|e^n\|_2^2+\|e^{n+1}-e^n\|_2^2$. We observe that

$$X^{n+1}=\begin{cases}\frac{1}{2}\|e^{n+1}\|_2^2+\frac{1}{2}\|2e^{n+1}-e^n\|_2^2\\ \frac{1}{10}\|e^n\|_2^2+\frac{5}{2}\|e^{n+1}-\frac{2}{5}e^n\|_2^2\ .\end{cases}$$

This shows

$$X^{n+1}\ge\frac{1}{10}\max\left\{\|e^{n+1}\|_2^2,\ \|e^n\|_2^2\right\}\ .$$

Making use of $X^{n+1}$, (7.8) becomes

$$\begin{aligned}
&\frac{\left(X^{n+1}+A\tau^2\|e^{n+1}\|_2^2\right)-\left(X^n+A\tau^2\|e^n\|_2^2\right)}{2\tau}\\
\le&\frac{C(1+N_1^4)}{\eta}\left(\|e^n\|_2^2+\|e^{n-1}\|_2^2\right)+\frac{1}{\eta}\|G^n\|_2^2+\eta\|e^{n+1}\|_2^2\ .
\end{aligned}$$

This leads to

$$\begin{aligned}
&\frac{\left(X^{n+1}-2\eta\tau\|e^{n+1}\|_2^2+A\tau^2\|e^{n+1}\|_2^2\right)-\left(X^n-2\eta\tau\|e^n\|_2^2+A\tau^2\|e^n\|_2^2\right)}{2\tau}\\
\le&\frac{C(1+N_1^4)}{\eta}\left(\|e^n\|_2^2+\|e^{n-1}\|_2^2\right)+\frac{1}{\eta}\|G^n\|_2^2+\eta\|e^n\|_2^2\\
\le&\left(\frac{C(1+N_1^4)}{\eta}+C\eta\right)\cdot\left(X^n-2\eta\tau\|e^n\|_2^2\right)+\frac{1}{\eta}\|G^n\|_2^2\ .
\end{aligned}$$

Define that

$$\begin{aligned}
&y_n=X^n-2\eta\tau\|e^n\|_2^2+A\tau^2\|e^n\|_2^2\ ,\\
&\alpha=\frac{C(1+N_1^4)}{\eta}+C\eta\ ,\\
&\beta_n=\frac{\|G_n\|_2^2}{\eta}\ .
\end{aligned}$$

Then for $\nu$ small, we get

$$\frac{y_{n+1}-y_n}{\tau}\le\alpha y_n+\beta_n\ .$$

Applying the discrete Gronwall's inequality (Lemma 2.3), we have for $m\ge2$,

$$\|e^m\|_2^2\le C\left(X^m-2\eta\tau\|e^m\|_2^2\right)\le Ce^{(m-1)\tau\cdot\frac{C(1+N_1^4)}{\eta}}\left(X^1+A\tau^2\|e^1\|_2^2+\frac{\tau}{\eta}\sum_{n=1}^{m-1}\|G^n\|_2^2\right),$$

which gives

$$
\begin{aligned}
&\|u^m - v^m\|_2^2 \\
\le& C \cdot \exp\left((m-1)\tau \cdot \frac{C(1+N_1^4)}{\eta}\right) \cdot \left(\frac{3}{2}\|e^1\|_2^2 - \frac{1}{2}\|e^0\|_2^2 + \|e^1 - e^0\|_2^2\right. \\
&\left. + A\tau^2 \|e^1\|_2^2 + \frac{\tau}{\eta}\sum_{n=1}^{m-1}\|G^n\|_2^2\right) \\
\le& C \cdot \exp\left((m-1)\tau \cdot \frac{C(1+N_1^4)}{\eta}\right) \cdot \left((1+A\tau^2)\|u^1 - v^1\|_2^2 + \|u^0 - v^0\|_2^2\right. \\
&\left. + \frac{\tau}{\eta}\sum_{n=1}^{m-1}\|G^n\|_2^2\right) .
\end{aligned}
$$

□

7.2.2. **Time discretization of the Allen-Cahn equation**

We first rewrite the AC equation in terms of the second order scheme.

**Lemma 7.7** (Time discrete Allen-Cahn equation)**.** *Let $u(t)$ be the exact solution to Allen-Cahn equation with initial data $u_0 \in H^s$, $s \ge 8$. Define $t_0 = 0$, $t_1 = \tau_1$ and $t_m = \tau_1 + (m-1)\tau$ for $m \ge 2$. Then for any $n \ge 1$,*

$$
\begin{aligned}
&\frac{3u(t_{n+1}) - 4u(t_n) + u(t_{n-1})}{2\tau} \\
&= \nu\Delta u(t_{n+1}) - A\tau\left(u(t_{n+1}) - u(t_n)\right) - \Pi_N\left[2f(u(t_n)) - f(u(t_{n-1}))\right] + G^n .
\end{aligned}
$$

*For any $m \ge 2$,*

$$
\tau \sum_{n=1}^{m-1}\|G^n\|_2^2 \lesssim (1+t_m)\cdot\left(\tau^4 + N^{-2s}\right) .
$$

*Proof.* The proof will be proceeded in several steps and we write $\lesssim$ instead of $\lesssim_{A,\ \nu,\ u_0}$ for simplicity.

**Step 1:** We write the PDE in the discrete form in time. Recall that

$$
\partial_t u = \nu\Delta u - f(u) .
$$

For a one variable function $h(t)$, the following equation holds:

$$
h(t) = h(t_0) + h'(t_0)(t-t_0) + \frac{1}{2}h''(t_0)(t-t_0)^2 + \frac{1}{2}\int_{t_0}^{t} h'''(s)(s-t)^2\, ds .
$$

We then apply this to AC,

$$
\begin{cases}
u(t_n) = u(t_{n+1}) - \partial_t u(t_{n+1})\cdot\tau + \dfrac{1}{2}\partial_{tt}u(t_{n+1})\tau^2 + \dfrac{1}{2}\displaystyle\int_{t_{n+1}}^{t_n}\partial_{ttt}u(s)(s-t_n)^2\, ds \\
u(t_{n-1}) = u(t_{n+1}) - \partial_t u(t_{n+1})\cdot 2\tau + 2\partial_{tt}u(t_{n+1})\tau^2 + \dfrac{1}{2}\displaystyle\int_{t_{n+1}}^{t_{n-1}}\partial_{ttt}u(s)(s-t_{n-1})^2\, ds.
\end{cases}
$$

The second equation minus 4 times the first equation results in

$$
\begin{aligned}
&\frac{3u(t_{n+1})-4u(t_n)+u(t_{n-1})}{2\tau}\\
&=\frac{1}{2\tau}\Bigg(2\tau\cdot\partial_t u(t_{n+1})-2\int_{t_{n+1}}^{t_n}\partial_{ttt}u(s)(s-t_n)^2\ ds\\
&+\frac{1}{2}\int_{t_{n+1}}^{t_{n-1}}\partial_{ttt}u(s)(s-t_{n-1})^2\ ds\Bigg)\\
&=\partial_t u(t_{n+1})+\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\partial_{ttt}u(s)(s-t_n)^2\ ds-\frac{1}{4\tau}\int_{t_{n-1}}^{t_{n+1}}\partial_{ttt}u(s)(s-t_{n-1})^2\ ds\\
&=\nu\Delta u(t_{n+1})-A\tau\left(u(t_{n+1})-u(t_n)\right)-\Pi_N\left[2f(u(t_n))-f(u(t_{n-1}))\right]\\
&+A\tau\left(u(t_{n+1})-u(t_n)\right)-\Pi_{>N}\left[2f(u(t_n))-f(u(t_{n-1}))\right]\\
&+2f(u(t_n))-f(u(t_{n-1}))-f(u(t_{n+1}))\\
&+\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\partial_{ttt}u(s)(s-t_n)^2\ ds-\frac{1}{4\tau}\int_{t_{n-1}}^{t_{n+1}}\partial_{ttt}u(s)(s-t_{n-1})^2\ ds\ .
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
G^n=&\underbrace{A\tau\left(u(t_{n+1})-u(t_n)\right)}_{I_1}-\underbrace{\Pi_{>N}\left[2f(u(t_n))-f(u(t_{n-1}))\right]}_{I_2}\\
&+\underbrace{2f(u(t_n))-f(u(t_{n-1}))-f(u(t_{n+1}))}_{I_3}\\
&+\underbrace{\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\partial_{ttt}u(s)(s-t_n)^2\ ds}_{I_4}-\underbrace{\frac{1}{4\tau}\int_{t_{n-1}}^{t_{n+1}}\partial_{ttt}u(s)(s-t_{n-1})^2\ ds}_{I_5}\ .
\end{aligned}
$$

**Step 2:** We will estimate $\|I_1\|_2\sim\|I_5\|_2$.

$I_1$: Apply the fundamental theorem of calculus,

$$
\begin{aligned}
\|I_1\|_2^2=&\|A\tau\left(u(t_{n+1})-u(t_n)\right)\|_2^2\\
\lesssim&\tau^2\|u(t_{n+1})-u(t_n)\|_2^2\\
\lesssim&\tau^2\|\int_{t_n}^{t_{n+1}}\partial_t u(s)\ ds\|_2^2\\
\lesssim&\tau^2\int_{\mathbb{T}^2}\left(\int_{t_n}^{t_{n+1}}\partial_t u(s)\ ds\right)^2\\
\lesssim&\tau^2\int_{\mathbb{T}^2}\left((\int_{t_n}^{t_{n+1}}|\partial_t u(s)|^2\ ds)^{1/2}\cdot\sqrt{\tau}\right)^2\\
\lesssim&\tau^2\cdot\tau\cdot\int_{t_n}^{t_{n+1}}\|\partial_t u(s)\|_2^2\ ds\\
\lesssim&\tau^3\int_{t_n}^{t_{n+1}}\|\partial_t u(s)\|_2^2\ ds\ .
\end{aligned}
$$

$I_2$: By the maximum principle proved in section 4 and $u\in L_t^\infty H_x^s$,

$$
\begin{aligned}
\|I_2\|_2\lesssim&N^{-s}\cdot\left(\|f(u(t_n))\|_{H^s}+\|f(u(t_{n-1}))\|_{H^s}\right)\\
\lesssim&N^{-s}\ .
\end{aligned}
$$

$I_3$: To bound $\|I_3\|_2$, we recall that for a one-variable function $h(t)$,

$$
h(t)=h(t_0)+h'(t_0)(t-t_0)-\int_{t_0}^{t}h''(s)\cdot(s-t)\ ds\ .
$$

Then we apply to $f(u(t_n))$,

$$\begin{cases} f(u(t_n)) = f(u(t_{n+1})) - \partial_t(f(u))(t_{n+1}) \cdot \tau + \int_{t_n}^{t_{n+1}} \partial_{tt}(f(u)) \cdot (s - t_n)\ ds \\ f(u(t_{n-1})) = f(u(t_{n+1})) - \partial_t(f(u))(t_{n+1}) \cdot 2\tau + \int_{t_{n-1}}^{t_{n+1}} \partial_{tt}(f(u)) \cdot (s - t_{n-1})\ ds\ . \end{cases}$$

Then we subtract the second equation above by 2 times the first equation and derive:

$$\begin{aligned} & f(u(t_{n+1})) - 2f(u(t_n)) + f(u(t_{n-1})) \\ =& -2\int_{t_n}^{t_{n+1}} \partial_{tt}(f(u)) \cdot (s - t_n)\ ds + \int_{t_{n-1}}^{t_{n+1}} \partial_{tt}(f(u)) \cdot (s - t_{n-1})\ ds\ . \end{aligned}$$

As a result, we get

$$\begin{aligned} \|I_3\|_2^2 =& \|f(u(t_{n+1})) - 2f(u(t_n)) + f(u(t_{n-1}))\|_2^2 \\ \lesssim& \|\int_{t_n}^{t_{n+1}} \partial_{tt}(f(u)) \cdot (s - t_n)\ ds\|_2^2 + \|\int_{t_{n-1}}^{t_{n+1}} \partial_{tt}(f(u)) \cdot (s - t_{n-1})\ ds\|_2^2 \\ \lesssim& \tau^2 \cdot \|\int_{t_n}^{t_{n+1}} \partial_{tt}(f(u))\ ds\|_2^2 + \tau^2 \cdot \|\int_{t_{n-1}}^{t_{n+1}} \partial_{tt}(f(u))\ ds\|_2^2 \\ \lesssim& \tau^3 \int_{t_{n-1}}^{t_{n+1}} \|\partial_{tt}(f(u))\|_2^2\ ds\ , \end{aligned}$$

by a similar estimate in $I_1$.

$I_4$&$I_5$:

$$\begin{aligned} & \|I_4\|_2^2 + \|I_5\|_2^2 \\ \lesssim& \left\| \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \partial_{ttt} u(s)(s - t_n)^2\ ds \right\|_2^2 + \left\| \frac{1}{\tau} \int_{t_{n-1}}^{t_{n+1}} \partial_{ttt} u(s)(s - t_{n-1})^2\ ds \right\|_2^2 \\ \lesssim& \left\| \frac{1}{\tau} \int_{t_{n-1}}^{t_{n+1}} \partial_{ttt} u(s) \cdot \tau^2\ ds \right\|_2^2 \\ \lesssim& \tau^2 \cdot \left\| \int_{t_{n-1}}^{t_{n+1}} \partial_{ttt} u(s)\ ds \right\|_2^2 \\ \lesssim& \tau^3 \int_{t_{n-1}}^{t_{n+1}} \|\partial_{ttt} u(s)\|_2^2\ ds\ . \end{aligned}$$

**Step 3:** Estimate $\tau \cdot \sum_{n=1}^{m-1} \|G^n\|_2^2$.

Collecting estimates above, we have

$$\begin{aligned} \tau \cdot \sum_{n=1}^{m-1} \|G^n\|_2^2 =& \tau \cdot \sum_{n=1}^{m-1} (\|I_1\|_2^2 + \|I_2\|_2^2 + \|I_3\|_2^2 + \|I_4\|_2^2 + \|I_5\|_2^2) \\ & \lesssim m\tau \cdot N^{-2s} + \tau^4 \cdot \int_0^{t_m} \|\partial_t u\|_2^2 + \|\partial_{tt}(f(u))\|_2^2 + \|\partial_{ttt} u\|_2^2\ d\tilde{s}\ . \end{aligned}$$

Note that by differentiating the original AC equation, we have

$$\begin{cases} \partial_{tt} u = \nu \partial_t \Delta u - \partial_t(f(u)) \\ \partial_{ttt} u = \nu \partial_{tt} \Delta u - \partial_{tt}(f(u)) \\ \partial_t(f(u)) = f'(u)\partial_t u \\ \partial_{tt}(f(u)) = f'(u)\partial_{tt} u + f''(u)(\partial_t u)^2\ , \end{cases}$$

hence together with maximum principle and higher Sobolev bounds proved in section 4, one has

$$\begin{aligned}
\tau\cdot\sum_{n=1}^{m-1}\|G^n\|_2^2 &\lesssim m\tau\cdot N^{-2s}+\tau^4\cdot\int_0^{t_m}\|\partial_t u\|_2^2+\|\partial_{tt}(f(u))\|_2^2+\|\partial_{ttt}u\|_2^2\ d\tilde{s}\\
&\lesssim t_m\cdot N^{-2s}+\tau^4\cdot\int_0^{t_m}\|u\|_{H^s}^2\ d\tilde{s}\\
&\lesssim t_m\cdot N^{-2s}+\tau^4\cdot(1+t_m)\\
&\lesssim (1+t_m)\cdot(\tau^4+N^{-2s})\ .
\end{aligned}$$

This completes the proof of Lemma 7.7. □

7.2.3. **Proof of $L^2$ error estimate of second order scheme I (7.1)**

Note that the assumptions in Proposition 7.1 are satisfied by the unconditional Theorem 7.3 and the maximum principle of the Allen-Cahn equation. Thus we apply the auxiliary estimate Proposition 7.1. Then

$$\|u(t_m)-u^m\|_2^2\lesssim e^{Cm\tau}\cdot\left((1+A\tau^2)\|u^1-v^1\|_2^2+\|u^0-v^0\|_2^2+\tau\sum_{n=1}^{m-1}\|G^n\|_2^2\right)\ .$$

By Lemma 7.2 and Lemma 7.7,

$$\begin{aligned}
\|u(t_m)-u^m\|_2^2 &\lesssim e^{Cm\tau}\cdot\left((1+A\tau^2)\|u^1-v^1\|_2^2+\|u^0-v^0\|_2^2+\tau\sum_{n=1}^{m-1}\|G^n\|_2^2\right)\\
&\lesssim e^{Ct_m}\cdot\left((1+A\tau^2)(N^{-2s}+\tau^4)+N^{-2s}+(1+t_m)\cdot(\tau^4+N^{-2s})\right)\\
&\lesssim e^{Ct_m}\cdot(N^{-2s}+\tau^4)\ .
\end{aligned}$$

Thus for $m\geq 2$,

$$\|u(t_m)-u^m\|_2\lesssim e^{Ct_m}\cdot(N^{-s}+\tau^2)\ .$$

**Remark 7.8.** *For the error estimate, we actually do not need high regularity of the initial data because of a smoothing effect of Allen-Cahn equation.*

7.3. **Conditional stability of scheme II**

In this section, we introduce another second order scheme:

$$\frac{3u^{n+1}-4u^n+u^{n-1}}{2\tau}=\nu\Delta u^{n+1}-A(u^{n+1}-2u^n+u^{n-1})-\Pi_N\left(2f(u^n)-f(u^{n-1})\right), \tag{7.9}$$

where $\tau>0$ is the time step and $n\geq 1$.

To start the iteration, we again need to derive $u^1$ according to the following first order scheme:

$$\begin{cases}\dfrac{u^1-u^0}{\tau_1}=\nu\Delta u^1-\Pi_N f(u^0)\ \ ,\\ u^0=\Pi_N u_0\ ,\end{cases}\tag{7.10}$$

where $\tau_1=\min\{\tau^{\frac{4}{3}}\ ,\ 1\ ,\ \frac{1}{\sqrt{A+1}}\}$. The choice of such $\tau$ is to guarantee the error estimate as proved in section 7.2, and to ensure that the new modified energy function can be controlled by the initial data.

7.3.1. **Estimate of the First Order Scheme (7.10)**

In this section we will still estimate some bounds of $u^1$ which will be used to prove the stability of the second order scheme and it will be slightly different from scheme (7.2).

**Lemma 7.9.** *Consider the scheme (7.10).*

$$\begin{cases} \dfrac{u^1 - u^0}{\tau_1} = \nu \Delta u^1 - \Pi_N f(u^0) \quad , \\ u^0 = \Pi_N u_0 \ , \end{cases}$$

*where* $\tau_1 = \min\{\tau^{\frac{4}{3}} \ , \ 1 \ , \frac{1}{\sqrt{A+1}}\}$. *Assume* $u_0 \in H^2(\mathbb{T}^2)$, *then*

$$\begin{cases} E(u^1) + \dfrac{\|u^1 - u^0\|_2^2}{\tau_1} \lesssim_{E(u_0), \|u_0\|_{H^2}} 1 \\ (1+A)\|u^1 - u^0\|_2^2 \lesssim_{\|u_0\|_{H^2}} (1+\nu)^2 \ . \end{cases}$$

*Proof.* The first inequality shares the same proof as in previous section since the scheme (7.10) is a refined version of (7.2).

For the second inequality, recall that $\|u^1\|_{H^2} \lesssim_{\|u_0\|_{H^2}}$. We get

$$\frac{1}{\tau_1}\|u^1 - u^0\|_2 \leq \nu \|u^1\|_{H^2} + \|f(u^0)\|_2 \lesssim_{\|u_0\|_{H^2}} 1 + \nu \ .$$

This implies

$$(A+1)\|u^1 - u^0\|_2^2 \lesssim_{\|u_0\|_{H^2}} (1+\nu)^2 \ .$$

□

7.3.2. **Conditional stability of the second order scheme II (7.9) & (7.10)**

In this section we will prove a conditional stability theorem for the second order scheme (7.9) & (7.10).

**Theorem 7.10** (Conditional stability)**.** *Consider the scheme (7.9) - (7.10) with* $\nu > 0$, $\tau > 0$ *and* $N \geq 2$. *Assume* $u_0 \in H^2(\mathbb{T}^2)$. *The initial energy is denoted by* $E_0 = E(u_0)$. *There exist constants* $C_i > 0$, $i = 1, \ 2, \ 3, \ 4$ *depending only on* $E_0$ *and* $\|u_0\|_{H^2}$, *such that the following holds:*
`Case 1:` $A = 0$. *If*

$$\tau \leq \begin{cases} C_1 \dfrac{\nu^4}{1 + |\log \nu|^2}, & \text{when } 0 < \nu < 1 \ ; \\ C_2 \dfrac{\nu^{-2}}{1 + |\log \nu|^2}, & \text{when } \nu \geq 1 \ . \end{cases}$$

*then*

$$\mathring{E}(u^{n+1}) \leq \mathring{E}(u^n) \ .$$

`Case 2:` $A = \textit{constant} \cdot (\nu^4 + \nu)$. *If*

$$\tau \leq \begin{cases} C_3 \dfrac{\nu^2}{1 + |\log \nu|}, & \text{when } 0 < \nu < 1 \ ; \\ C_4 \dfrac{\nu^{-1}}{1 + |\log \nu|}, & \text{when } \nu \geq 1 \ . \end{cases}$$

*then*

$$\mathring{E}(u^{n+1}) \leq \mathring{E}(u^n) \ .$$

*Here* $\mathring{E}(u^n)$ *for* $n \geq 1$ *is a modified energy functional and is defined as*

$$\mathring{E}(u^n) := E(u^n) + \frac{A+1}{2}\|u^n - u^{n-1}\|_2^2 + \frac{1}{4\tau}\|u^n - u^{n-1}\|_2^2 \ .$$

Before proving this stability theorem, we begin with several lemmas.

**Lemma 7.11.** *Consider (7.9) for* $n \geq 1$. *Suppose* $E(u^n) \leq B \cdot (1+\nu)^2$ *and* $E(u^{n-1}) \leq B \cdot (1+\nu)^2$ *for some* $B > 0$. *Then*

$$\|u^{n+1}\|_\infty \leq \alpha_B \cdot \left\{ (\nu^{\frac{1}{2}} + \nu^{-1}) \cdot \sqrt{1 + \log(A+1) + |\log \nu| + |\log \tau|} + 1 \right\} \ ,$$

*for some* $\alpha_B > 0$ *only depending on* $B$.

*Proof.* For simplicity we write $\lesssim$ instead of $\lesssim_B$. Note that by the energy estimates,

$$\|\nabla u^{n-1}\|_2 + \|\nabla u^n\|_2 \lesssim \nu^{-\frac{1}{2}}(1+\nu)\ ,\ \|u^{n-1}\|_4 + \|u^n\|_4 \lesssim (1+\nu)^{\frac{1}{2}}\ .$$

We rewrite the scheme (7.9) as

$$\begin{aligned} u^{n+1} =& \frac{4+4A\tau}{3+2A\tau-2\nu\tau\Delta} u^n - \frac{1+2A\tau}{3+2A\tau-2\nu\tau\Delta} u^{n-1} \\ &- \frac{2\tau\Pi_N}{3+2A\tau-2\nu\tau\Delta}\left(2f(u^n) - f(u^{n-1})\right)\ . \end{aligned}$$

For Fourier mode $k=0$, we have

$$\begin{cases} \dfrac{4+4A\tau}{3+2A\tau} \lesssim 1 \\ \dfrac{1+2A\tau}{3+2A\tau} \lesssim 1 \\ \dfrac{2\tau}{3+2A\tau} \lesssim \dfrac{1}{A} \lesssim 1\ . \end{cases}$$

Thus

$$|\widehat{u^{n+1}}(0)| \lesssim 1\ .$$

For $|k| \geq 1$,

$$\begin{cases} \dfrac{4+4A\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim 1 \\ \dfrac{1+2A\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim 1 \\ \dfrac{2\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim \dfrac{1}{\nu|k|^2}\ . \end{cases}$$

This implies

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^1} \lesssim& \|u^n\|_{\dot{H}^1} + \|u^{n-1}\|_{\dot{H}^1} + \frac{1}{\nu}\|\langle\nabla\rangle^{-1}(2f(u^n) - f(u^{n-1}))\|_2 \\ \lesssim& \nu^{-\frac{1}{2}}(1+\nu) + \nu^{-1}\cdot\left(\|(u^n)^3\|_{4/3} + \|(u^{n-1})^3\|_{4/3} + \|u^n\|_2 + \|u^{n-1}\|_2\right) \\ \lesssim& \nu^{-\frac{1}{2}}(1+\nu) + \nu^{-1}\cdot\left((1+\nu)^{\frac{3}{2}} + (1+\nu)^{\frac{1}{2}}\right) \\ \lesssim& \nu^{-1} + \nu^{\frac{1}{2}}\ . \end{aligned}$$

Similarly, we can derive that

$$\begin{cases} \dfrac{4+4A\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim \left(\dfrac{1}{\nu\tau} + \dfrac{A}{\nu}\right)\cdot\dfrac{1}{|k|^2} \\ \dfrac{1+2A\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim \left(\dfrac{1}{\nu\tau} + \dfrac{A}{\nu}\right)\cdot\dfrac{1}{|k|^2} \\ \dfrac{2\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim \dfrac{1}{\nu|k|^2}\ . \end{cases}$$

Thus by the standard Sobolev inequality

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^2} \lesssim& \left(\frac{1}{\nu\tau} + \frac{A}{\nu}\right)\cdot\left(\|u^n\|_2 + \|u^{n-1}\|_2\right) + \frac{1}{\nu}\|(2f(u^n) - f(u^{n-1}))\|_2 \\ \lesssim& \left(\frac{1}{\nu\tau} + \frac{A}{\nu}\right)\cdot(1+\nu)^{\frac{1}{2}} + \nu^{-1}\left(\|u^n\|_6^3 + \|u^{n-1}\|_6^3 + \|u^n\|_2 + \|u^{n-1}\|_2\right) \\ \lesssim& \left(\frac{1}{\nu\tau} + \frac{A}{\nu}\right)\cdot(1+\nu)^{\frac{1}{2}} + \nu^{-1}(\nu^{-\frac{3}{2}}(1+\nu)^3 + (1+\nu)^{\frac{1}{2}}) \\ \lesssim& \frac{1}{\nu\tau} + \frac{A}{\nu} + \frac{1}{\nu^{\frac{1}{2}}\tau} + \frac{A+1}{\nu^{\frac{1}{2}}} + \nu^{-\frac{5}{2}} + \nu^{\frac{1}{2}}\ . \end{aligned}$$

As a result, by the log interpolation Lemma 3.4 again, we get

$$\|u^{n+1}\|_\infty \lesssim (\nu^{\frac{1}{2}} + \nu^{-1}) \cdot \sqrt{1 + \log(A+1) + |\log \nu| + |\log \tau|} + 1 \ .$$

$\square$

### 7.4. **Proof of conditional stability Theorem 7.10**

To prove Theorem 7.10, we first recall that

$$\begin{aligned} \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} &- \nu \Delta u^{n+1} + A(u^{n+1} - 2u^n + u^{n-1}) \\ &= -\Pi_N \left(2f(u^n) - f(u^{n-1})\right) . \end{aligned} \tag{7.11}$$

We apply the $L^2$ inner product with $\delta u^{n+1} = u^{n+1} - u^n$ on both sides of (7.11). Recall that

$$\begin{aligned} &\left(3u^{n+1} - 4u^n + u^{n-1} \ , \ u^{n+1} - u^n\right) \\ &= 2\|\delta u^{n+1}\|_2^2 + \frac{1}{2}\left(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2\right) \ . \end{aligned}$$

Applying this equation, we derive the estimate of the left hand side of (7.11) after taking inner products:

$$\begin{aligned} \text{LHS} =& \frac{1}{\tau}\|\delta u^{n+1}\|_2^2 + \frac{1}{4\tau}\left(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2\right) \\ &+ \frac{\nu}{2}\left(\|\nabla u^{n+1}\|_2^2 - \|\nabla u^n\|_2^2 + \|\delta \nabla u^{n+1}\|_2^2\right) \\ &+ \frac{A}{2}\left(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2\right) \\ \geq& \left[\frac{\nu}{2}\|\nabla u^{n+1}\|_2^2 + \frac{1}{4\tau}\|\delta u^{n+1}\|_2^2 + \frac{A}{2}\|\delta u^{n+1}\|_2^2\right] \\ &- \left[\frac{\nu}{2}\|\nabla u^n\|_2^2 + \frac{1}{4\tau}\|\delta u^n\|_2^2 + \frac{A}{2}\|\delta u^n\|_2^2\right] \\ &+ \frac{1}{\tau}\|\delta u^{n+1}\|_2^2 + \frac{A}{2}\|\delta^2 u^{n+1}\|_2^2 \ . \end{aligned}$$

Now it remains to estimate the right hand side of (7.11) after taking inner products:

$$\begin{aligned} \text{RHS} =& -\left(2f(u^n) - f(u^{n-1}) \ , \ \delta u^{n+1}\right) \\ =& \underbrace{\left(2u^n - u^{n-1} \ , \ \delta u^{n+1}\right)}_{I_1} + \underbrace{\left((u^{n-1})^3 - 2(u^n)^3 \ , \ \delta u^{n+1}\right)}_{I_2} \ . \end{aligned}$$

To estimate $I_1$:

$$\begin{aligned} I_1 &= \left(-\delta^2 u^{n+1} \ , \ \delta u^{n+1}\right) + \left(u^{n+1} \ , \ \delta u^{n+1}\right) \\ &= -\frac{1}{2}\left(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2\right) \\ &\quad + \frac{1}{2}\left(\|u^{n+1}\|_2^2 - \|u^n\|_2^2 + \|\delta u^{n+1}\|_2^2\right) \ . \end{aligned}$$

For $I_2$, we use the identity $u^{n-1} = \delta^2 u^{n+1} + 2u^n - u^{n+1}$, then

$$\begin{aligned} (u^{n-1})^3 - 2(u^n)^3 =& (\delta^2 u^{n+1} + 2u^n - u^{n+1})^3 - 2(u^n)^3 \\ =& (\delta^2 u^{n+1})^3 + 3(\delta^2 u^{n+1})^2(2u^n - u^{n+1}) + 3\delta^2 u^{n+1}(2u^n - u^{n+1})^2 \\ &+ (2u^n - u^{n+1})^3 - 2(u^n)^3 \ . \end{aligned}$$

Note that

$$\begin{aligned} 3\delta^2 u^{n+1}(2u^n - u^{n+1})^2 &= 3\delta^2 u^{n+1}(\delta^2 u^{n+1} - u^{n-1})^2 \\ &= 3(\delta^2 u^{n+1})^3 - 6(\delta^2 u^{n+1})^2 u^{n-1} + 3\delta^2 u^{n+1}(u^{n-1})^2 \ . \end{aligned}$$

As a result, we get

$$\begin{aligned}
&(u^{n-1})^3 - 2(u^n)^3\\
=&4(\delta^2 u^{n+1})^3 + (\delta^2 u^{n+1})^2(6u^n - 3u^{n+1} - 6u^{n-1})\\
&+ 3\delta^2 u^{n+1}(u^{n-1})^2 + (2u^n - u^{n+1})^3 - 2(u^n)^3\\
=&(\delta^2 u^{n+1})^2 \cdot \left[4(u^{n+1} - 2u^n + u^{n-1}) + 6u^n - 3u^{n+1} - 6u^{n-1}\right]\\
&+ 3\delta^2 u^{n+1}(u^{n-1})^2 + 6(u^n)^3 - 12(u^n)^2 u^{n+1} + 6u^n(u^{n+1})^2 - (u^{n+1})^3\\
=&(\delta^2 u^{n+1})^2 \cdot (u^{n+1} - 2u^n - 2u^{n-1}) + 3\delta^2 u^{n+1}(u^{n-1})^2\\
&+ 6u^n(u^{n+1} - u^n)^2 - (u^{n+1})^3 \ .
\end{aligned}$$

Therefore,

$$\begin{aligned}
|I_2| \leq& \|\delta^2 u^{n+1}\|_\infty \cdot \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2\\
&\cdot \left(\|u^{n+1}\|_\infty + 2\|u^n\|_\infty + 2\|u^{n-1}\|_\infty\right)\\
&+ \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \cdot 3\|u^{n-1}\|_\infty^2\\
&+ \left((\delta u^{n+1})^2 \ , \ 6u^n(u^{n+1} - u^n)\right) - \left((u^{n+1})^3 \ , \ \delta u^{n+1}\right) \ .
\end{aligned}$$

Now note that

$$\begin{aligned}
&\frac{(u^n)^4}{4}\\
=&\frac{1}{4}(u^{n+1} - \delta u^{n+1})^4\\
=&\frac{1}{4}\left[(u^{n+1})^4 - 4(u^{n+1})^3\delta u^{n+1} + 6(u^{n+1})^2(\delta u^{n+1})^2 - 4u^{n+1}(\delta u^{n+1})^3 + (\delta u^{n+1})^4\right]\\
=&\frac{(u^{n+1})^4}{4} - (u^{n+1})^3\delta u^{n+1} + \frac{1}{4}(\delta u^{n+1})^2\left[6(u^{n+1})^2 - 4u^{n+1}(u^{n+1} - u^n) + (u^{n+1} - u^n)^2\right]\\
=&\frac{(u^{n+1})^4}{4} - (u^{n+1})^3\delta u^{n+1} + \frac{(\delta u^{n+1})^2}{4}\left[(u^n)^2 + 2u^n u^{n+1} + 3(u^{n+1})^2\right] \ .
\end{aligned}$$

Applying this identity we have

$$\begin{aligned}
&\left((\delta u^{n+1})^2 \ , \ 6u^n(u^{n+1} - u^n)\right) - \left((u^{n+1})^3 \ , \ \delta u^{n+1}\right)\\
=&\int_{\mathbb{T}^2} \frac{(u^n)^4}{4} - \int_{\mathbb{T}^2} \frac{(u^{n+1})^4}{4} - \left((\delta u^{n+1})^2 \ , \ \frac{25}{4}(u^n)^2 - \frac{11}{2}u^n u^{n+1} + \frac{3}{4}(u^{n+1})^2\right)\\
=&\int_{\mathbb{T}^2} \frac{(u^n)^4}{4} - \int_{\mathbb{T}^2} \frac{(u^{n+1})^4}{4} - \left((\delta u^{n+1})^2 \ , \ \frac{25}{4}(u^n - \frac{11}{25}u^{n+1})^2\right)\\
&+ \frac{23}{50}\left((\delta u^{n+1})^2 \ , \ (u^{n+1})^2\right) \ .
\end{aligned}$$

Observe that

$$\|\delta^2 u^{n+1}\|_\infty \leq 4\max\left\{\|u^{n-1}\|_\infty, \ \|u^n\|_\infty, \ \|u^{n+1}\|_\infty\right\},$$

$$\begin{aligned}
&\text{RHS}\\
\leq& -\frac{1}{4}\|u^{n+1}\|_4^4 + \frac{1}{2}\|u^{n+1}\|_2^2 - \frac{1}{2}\|\delta u^{n+1}\|_2^2\\
&+ \frac{1}{4}\|u^n\|_4^4 - \frac{1}{2}\|u^n\|_2^2 + \frac{1}{2}\|\delta u^n\|_2^2\\
&- \frac{1}{2}\|\delta^2 u^{n+1}\|_2^2 + \|\delta u^{n+1}\|_2^2 \cdot \left(\frac{1}{2} + \frac{23}{50}\|u^{n+1}\|_\infty^2\right)\\
&+ \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \cdot 23\max\left\{\|u^{n-1}\|_\infty^2, \ \|u^n\|_\infty^2, \ \|u^{n+1}\|_\infty^2\right\} \ .
\end{aligned}$$

Recall that

$$E(u) = \frac{\nu}{2}\|\nabla u\|_2^2 + \frac{1}{4}\|u\|_4^4 - \frac{1}{2}\|u\|_2^2 + \frac{1}{4}\cdot\mu(\mathbb{T}^2) \ ,$$

where $\mu(\mathbb{T}^2)$ is the measure of $\mathbb{T}^2$. Hence by comparing the LHS and RHS, we get

$$
\begin{aligned}
&E(u^{n+1}) + \frac{1}{4\tau}\|\delta u^{n+1}\|_2^2 + \frac{A+1}{2}\|\delta u^{n+1}\|_2^2 \\
\leq &E(u^n) + \frac{1}{4\tau}\|\delta u^n\|_2^2 + \frac{A+1}{2}\|\delta u^n\|_2^2 \\
&+ \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \cdot 23\max\left\{\|u^{n-1}\|_\infty^2,\ \|u^n\|_\infty^2,\ \|u^{n+1}\|_\infty^2\right\} \\
&- \left\{\frac{A+1}{2}\|\delta^2 u^{n+1}\|_2^2 + \left(\frac{1}{\tau} - \frac{1}{2} - \frac{23}{50}\|u^{n+1}\|_\infty^2\right)\cdot\|\delta u^{n+1}\|_2^2\right\} .
\end{aligned}
$$

To show the desired energy decay, it suffices to require

$$
\begin{aligned}
&2(A+1)\left(\frac{1}{\tau} - \frac{1}{2} - \frac{23}{50}\|u^{n+1}\|_\infty^2\right) \\
&\qquad \geq 529\max\left\{\|u^{n-1}\|_\infty^4,\ \|u^n\|_\infty^4,\ \|u^{n+1}\|_\infty^4\right\} .
\end{aligned}
\tag{7.12}
$$

We will prove it inductively as in the previous section. Set

$$
B = \max\left\{\mathring{E}(u^1)\ ,\ E(u^0)\right\}\ ,
$$

by Lemma 7.9 in previous section, $B \lesssim 1$. We shall prove for every $m \geq 2$,

$$
\begin{cases}
\mathring{E}(u^m) \leq B\cdot(1+\nu)^2\ ,\ \mathring{E}(u^m) \leq \mathring{E}(u^{m-1})\ , \\
\|u^m\|_\infty \leq \alpha_B \cdot \left[(\nu^{\frac{1}{2}} + \nu^{-1})\cdot\sqrt{1+\log(A+1)+|\log\nu|+|\log\tau|} + 1\right]\ ,
\end{cases}
$$

where $\alpha_B > 0$ is the same constant in Lemma 7.11. Then it suffices to verify the main inequality (7.12):

$$
\begin{aligned}
&2(A+1)\left(\frac{1}{\tau} - \frac{1}{2} - C_1 - C_1\cdot(\nu^{-2}+\nu)\cdot(1+\log(A+1)+|\log\nu|+|\log\tau|)\right) \\
&> C_2(\nu^{-4}+\nu^2)\cdot\left(1+|\log(A+1)|^2+|\log\nu|^2+|\log\tau|^2\right) + C_2\ .
\end{aligned}
$$

We consider two cases.

`Case 1:` $A=0$. Then we need

$$
\frac{1}{\tau} \gg (\nu^{-4}+\nu^2)\cdot\left(1+|\log(A+1)|^2+|\log\nu|^2+|\log\tau|^2\right) + 1\ .
$$

If $0<\nu<1$, we require that

$$
\tau \ll \frac{\nu^4}{1+|\log\nu|^2}\ ;
$$

If $\nu \geq 1$, we then require that

$$
\tau \ll \frac{\nu^{-2}}{1+|\log\nu|^2}\ .
$$

`Case 2:` $A = const\cdot(\nu^2+\nu^{-4})$. In this case,

$$
\frac{1}{\tau} \gg (\nu^{-2}+\nu)\cdot(1+|\log(A+1)|+|\log\nu|+|\log\tau|) + |\log\nu|^2 + |\log\tau|^2 + 1\ .
$$

If $0<\nu<1$, we need

$$
\tau \ll \frac{\nu^2}{1+|\log\nu|}\ ;
$$

If $\nu \geq 1$, then we need

$$
\tau \ll \frac{\nu^{-1}}{1+|\log\nu|}\ .
$$

This completes the proof.

## 8. Concluding remark

Throughout this paper, we discussed first order and second order semi-implicit Fourier spectral methods on the Allen-Cahn equation and the fractional Cahn-Hilliard equation in both two dimensional and three dimensional cases. We proved the stability (energy decay) of the first order numerical scheme by adding a stabilizing term $A(u^{n+1}-u^n)$ and $(-\Delta)^\alpha A(u^{n+1}-u^n)$ with a large constant $A$ at least of size $O(\nu^{-1}|\log(\nu)|)$ for the 2D case and $O(\nu^{-3})$ for the 3D case. Note that this stability is preserved independent of time step $\tau$. We also proved a $L^2$ error estimate between numerical solutions from the semi-implicit scheme and exact solutions. We proved stability results and $L^2$ error estimates for two second order schemes as well.

Future work can be done in other gradient cases (as mentioned in Remark 1.2) such as general nonlocal Allen-Cahn and Cahn-Hilliard equations, MBE equations, and other equations describing phenomena of interest in materials science.

## Acknowledgement

The author is supported by the International Doctoral Fellowship (IDF) provided by the University of British Columbia, Vancouver, BC, Canada.

## References

[1] M. Ainsworth and Z. Mao. Analysis and Approximation of a Fractional Cahn–Hilliard Equation . *SIAM Journal on Numerical Analysis.*, 55(4):1689–1718, 2017.

[2] N. Alikakos, P. Bates, and X. Chen. Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Archive for Rational Mechanics and Analysis.*, 128(2):165–205, 1994.

[3] S. M. Allen and J. W. Cahn. Ground state structures in ordered binary alloys with second neighbor interactions. *Acta Metallurgica.*, 20(3):423–433, 1972.

[4] A. Bertozzi, N. Ju, and H.-W. Lu. A biharmonic-modified forward time stepping method for fourth order nonlinear diffusion equations. *Discrete and Continuous Dynamical Systems.*, 29(4):1367–1391, 2011.

[5] J. Bosch and M. Stoll. A fractional inpainting model based on the vector-valued Cahn-Hilliard equation. *SIAM Journal on Imaging Sciences.*, 8(4):2352–2382, 2015.

[6] J. Bourgain and D. Li. Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces. *Inventiones Mathematicae.*, 201(1):97–157, 2015.

[7] J. Bourgain and D. Li. Strong illposedness of the incompressible Euler equation in integer $C^m$ spaces. *Geometric and Functional Analysis.*, 25(1):1–86, 2015.

[8] J. W. Cahn and J. E. Hilliard. Free Energy of a Nonuniform System. I. Interfacial Free Energy. *The Journal of Chemical Physics.*, 28(2):258–267, 1958.

[9] L.-Q. Chen and J. Shen. Applications of semi-implicit Fourier-spectral method to phase field equations. *Computer Physics Communications.*, 108(2–3):147–158, 1998.

[10] X. Cheng. On the stability of a semi-implicit scheme of cahn-hilliard type equations. Master's thesis, University of British Columbia, Vancouver, 2017.

[11] X. Cheng, D. Li, K. Promsilow, and B. Wetton. Asymptotic Behaviour of Time Stepping Methods for Phase Field Models. *Journal of Scientific Computing.*, 86(3):1–34, 2021.

[12] X. Cheng, D. Li, C. Quan, and W. Yang. On a sinc-type mbe model. ArXiv:2106.16193.

[13] X. Cheng, D. Li, C. Quan, and W. Yang. On a parabolic Sine-Gordon model. *Numerical Mathematics: Theory, Methods and Applications.*, 14(4):1068–1084, 2021.

[14] A. Christlieb, J. Jones, K. Promislow, B. Wetton, and M. Willoughby. High accuracy solutions to energy gradient flows from material science models. *Journal of computational physics.*, 257:193–215, 2014.

[15] C. M. Elliott and S. Zheng. On the Cahn-Hilliard equation. *Archive for Rational Mechanics and Analysis.*, 96(4):339–357, 1986.

[16] L. C. Evans. *Partial Differential Equations: Second Edition.* American Mathematical Society Providence., Rhode Island, 2010.

[17] X. Feng and A. Prohl. Error analysis of a mixed finite element method for the Cahn-Hilliard equation. *Numerische Mathematik.*, 99(1):47–84, 2004.

[18] X. Feng, T. Tang, and J. Yang. Stabilized Crank-Nicolson/Adams-Bashforth Schemes for Phase Field Models. *East Asian Journal on Applied Mathematics.*, 3(1):59–80, 2013.

[19] G. Gavish, J. Jones, Z. X. Xu, A. Christlieb, and K. Promislow. Variational Models of Network Formation and Ion Transport: Applications to Perfluorosulfonate Ionomer Membranes. *Polymers.*, 4(1):630–655, 2012.

[20] Y. He, Y. Liu, and T. Tang. On large time-stepping methods for the Cahn–Hilliard equation. *Applied numerical mathematics.*, 57(5–7):616–628, 2007.

[21] Y. He, Y. Liu, and T. Tang. On large time-stepping methods for the Cahn–Hilliard equation. *Applied Numerical Mathematics.*, 57(5–7):616–628, 2007.

[22] T. Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *Journal of Differential Geometry.*, 38:417–461, 1993.

[23] D. Li and Z. Qiao. On second order semi-implicit fourier spectral methods for 2d Cahn-Hilliard equations. *Journal of Scientific Computing.*, 70(1):1–41, 2017.

[24] D. Li and Z. Qiao. On the stabilization size of semi-implicit Fourier-spectral methods for 3D Cahn-Hilliard equations. *Communications in Mathematical Sciences.*, 15(6):1489–1506, 2017.

[25] D. Li, Z. Qiao, and T. Tang. Characterizing the stabilization size for semi-implicit Fourier-spectral method to phase field equations. *SIAM Journal on Numerical Analysis.*, 54(3):1653–1681, 2016.

[26] D. Li, Z. Qiao, and T. Tang. Gradient bounds for a thin film epitaxy equation. *Journal of Differential Equations.*, 262(3):1720–1746, 2017.

[27] D. Li and T. Tang. Stability analysis for the Implicit-Explicit discretization of the Cahn-Hilliard equation. ArXiv:2008.03701.

[28] D. Li and T. Tang. Stability of semi-implicit methods for the Cahn-Hilliard equation with logarithmic potentials. *Ann. Appl. Math..*, 37(1):31–60, 2021.

[29] R. Pego. Front migration in the nonlinear Cahn-Hilliard equation. *Proc. Roy. Soc. London.*, A(422):261–278, 1989.

[30] J. Shen, C. Wang, X. Wang, and S. Wise. Second-order convex splitting schemes for gradient flows with Ehrlich-Schwoebel type energy: application to thin film epitaxy. *SIAM Journal on Numerical Analysis.*, 50(1):105–125, 2012.

[31] J. Shen and X. Yang. Energy stable schemes for Cahn-Hilliard phase-field model of two-phase incompressible flows. *Chinese Annals of Mathematics.*, 31(5):743–758, 2010.

[32] J. Shen and X. Yang. Numerical approximations of Allen-Cahn and Cahn-Hilliard equations. *Discrete and Continuous Dynamical Systems.*, 28(4):1669–1691, 2010.

[33] J. Shen and X. Yang. Numerical approximations of Allen–Cahn and Cahn–Hilliard equations. *Discrete and Continuous Dynamical Systems.*, 28(4):1669–1691, 2010.

[34] T. Tang and J. Yang. Implicit-explicit scheme for the Allen-Cahn equation preserves the maximum Principle. *Journal of Computational Mathematics.*, 34(5):471–481, 2016.

[35] C. Xu and T. Tang. Stability analysis of large time-stepping methods for epitaxial growth models. *SIAM Journal on Numerical Analysis.*, 44(4):1759–1779, 2006.

[36] J. Zhu, L.-Q. Chen, J. Shen, and V. Tikare. Coarsening kinetics from a variable-mobility Cahn-Hilliard equation: Application of a semi-implicit Fourier spectral method. *Phys. Rev. E.*, 60(4):3564–3572, 1999.

Xinyu Cheng
School of Mathematical Sciences, Fudan University
xycheng@fudan.edu.cn