# SECOND ORDER ENERGY STABLE SEMI-IMPLICIT SCHEMES FOR THE 2D ALLEN-CAHN EQUATION

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ABSTRACT. In this work we are interested in a class of second order numerical schemes for certain phase field models. It is known that phase field models are gradient flows therefore the energy dissipates in time. As a matter of fact, numerical simulations with unconditional energy stability (energy decays in time regardless of the size of the time step) indicate good stability. In recent work [8], several first order semi-implicit schemes for the Allen-Cahn and fractional Cahn-Hilliard equations were developed satisfying the energy-decay property, in this paper we extend the analysis to second order schemes for the two dimensional Allen-Cahn equation with a rigorous proof of energy stability.

## 1. INTRODUCTION

Partial differential equations (PDE) often describe mathematical models of physical, biological phenomena. Among PDEs, phase field equations are models of essential importance in the study of material sciences. Particularly, in this work we consider a classic phase field models, the Allen-Cahn (AC equation. The (AC) model was developed in [1] by Allen and Cahn to study the competition of crystal grain orientations in an annealing process separation of different metals in a binary alloy. More specifically, the Allen-Cahn equation takes the form as follows:

$$\begin{cases} \partial_t u = \nu \Delta u - f(u), \quad (x,t) \in \Omega \times (0,\infty) \\ u(x,0) = u_0 \end{cases}, \tag{AC}$$

AC

where $u(x,t)$ is a real valued function and the values of $u$ are in $(-1,1)$ representing a mixture of the two phases. $-1$ represents the pure state of one phase and $+1$ indicates the pure state of the other phase. In this paper we take the spatial domain $\Omega$ to be the two dimensional periodic domain $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. Here $\nu$ is a small parameter, occasionally we denote $\varepsilon = \sqrt{\nu}$ to represent an average distance over which phases mix. The nonlinearity $f(u)$ is often chosen to be

$$f(u) = F'(u) = u^3 - u \ , \ F(u) = \frac{1}{4}(u^2-1)^2.$$

It is well known that, as $\varepsilon \to 0$, the limiting problem of (AC) is driven by a mean curvature flow and we refer the readers to [18] for the scalar AC and a recent work [14] for the matrix-valued AC, where asymptotic and rigorous analysis are provided. Unlike the limiting behavior of AC, there are many other related materials science models that are studied only numerically and one of the modest goals of this current work is to present an idea on approaching these models in an appropriate way numerically.

As mentioned earlier, in this note we take the spatial domain $\Omega$ to be the two dimensional $2\pi$-periodic torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. It is worth mentioning that our proof can be applied to more general settings such as Dirichlet and Neumann boundary conditions in a 2D bounded domain. However, considering the periodic domain allows us to apply the efficient and accurate Fourier-spectral numerical methods. Nevertheless, periodic domain is very natural in practical problems, which usually involve the formation of micro-structure away from physical boundaries. As is well-known that the (AC) behaves as a gradient, therefore its energy dissipates in time. Here the associated energy functional of (AC) is given by

$$E(u) = \int_\Omega \left( \frac{1}{2}\nu |\nabla u|^2 + F(u) \right) \, dx. \tag{1.1}$$

?energy?

Assume that $u(x,t)$ is a smooth solution, it is clear that

$$\frac{d}{dt}E(u(t)) + \int_\Omega |\nu\Delta u - f(u)|^2 \ dx = 0,$$

which implies the decay of the energy: $\frac{d}{dt}E(u(t)) \leq 0$. This thus provides an *a priori* $H^1$-norm bound and since the scaling-critical space for (AC) is $L^2$ in 2D (and $H^{\frac{1}{2}}$ in 3D), the global well-posedness follow from standard energy estimates. Therefore from the analysis point of view, the energy dissipation property is an important index for whether a numerical scheme is "stable" or not.

Various approaches have been developed to study numerical simulations on Allen-Cahn and other related phase field models, cf. [12, 16, 17, 30, 15, 7, 31, 3, 13, 22, 23, 24, 2, 19]. Among which different time stepping approaches are applied including the fully explicit (forward Euler) scheme, fully implicit (backward Euler) scheme, finite element scheme and convex splitting scheme; and different schemes are used for the spatial discretization including the Fourier-spectral method and finite difference schemes. To guarantee the accuracy and stability, numerical approximations usually need to obey certain qualitative behaviors and features. One of the key features is the energy dissipation or conservation as mentioned earlier.

Starting from this point, we briefly go through the main related results in the literature. To start with, Feng and Prohl [15] introduced a semi-discrete in time and fully spatially discrete finite element method for the Cahn-Hilliard equation (CH) where they obtained an error bound of size of powers of $1/\nu$. Here CH is another classic phase field model introduced in [6] by Cahn and Hilliard to describe the process of phase separation of different metals in a binary alloy. However, explicit time-stepping schemes usually require strict time-step restrictions and do not obey energy decay in general. To guarantee the energy dissipation with bigger time steps, a good alternative is to use semi-implicit schemes in which the linear term is implicit (such as backward time differentiation) and the nonlinear term is treated explicitly. Having only a linear implicit at every time step has computational advantages, as suggested in [7], Chen and Shen considered a semi-implicit Fourier-spectral scheme for CH. On the other hand, semi-implicit schemes can lose stability for large time steps and thus smaller time steps are needed in practice. To resolve this problem, semi-implicit methods with better stability have been introduced, cf. [17, 27, 30, 31, 29, 20, 21]. To be more specific, the work [17, 30, 31, 28] study different semi-implicit Fourier-spectral schemes, which involved different stabilizing terms of different size, that preserve the energy decay property (we say these schemes are "energy stable"). However, those works either require a strong Lipschitz condition on the nonlinear source term, or require certain $L^\infty$ bounds on the numerical solutions.

In the seminal works [20, 21, 25], Li et al. developed a large time-stepping semi-implicit Fourier-spectral scheme for Cahn-Hilliard equation and proved that it preserves energy decay with no *a priori* assumptions (unconditional stability). The proof uses harmonic analysis tools developed in [4, 5], and introduces a novel energy bootstrap scheme in order to obtain a $L^\infty$-bound of the numerical solution. One of their schemes for CH takes the following form:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = -\nu\Delta^2 u^{n+1} + A\Delta(u^{n+1}-u^n) + \Delta(f(u^n)) \ , \ n \geq 0 \\ u^0 = u_0 \ . \end{cases} \tag{1.2}$$ {?}

Here $\tau$ is the time step and $A$ is a large coefficient for the $O(\tau)$ stabilizing term. As a result of their work, the energy decay is satisfied with a well-chosen large number $A$, with at least a size of $O(1/\nu|\log(\nu)|^2)$. However, their arguments cannot be applied to the Allen Cahn equation (AC) directly: this is due to the lack of the mass conservation, i.e. $\frac{d}{dt}M(t) = 0$, where $M(t) = \int_{\mathbb{T}^2} u \ dx$. The work [8] then extends their first order semi-implicit scheme to the related Allen-Cahn equation (AC) and the more general non-local fractional Cahn-Hilliard equation. Following the same idea in [8], in this work we show the arguments can be applied to second-order semi-implicit schemes of the Allen-Cahn equation in $\mathbb{T}^2$. In particular, we consider two second-order schemes below.

We first consider the following second order semi-implicit Fourier spectral scheme I:

$$\frac{3u^{n+1}-4u^n+u^{n-1}}{2\tau} = \nu\Delta u^{n+1} - A\tau(u^{n+1}-u^n) - \Pi_N\left(2f(u^n) - f(u^{n-1})\right) \ \ , \ n \geq 1 \ , \tag{1.3}$$ 2ndSchemeI

where $\tau > 0$ is the time step and this scheme applies second order backward derivative in time with a second order extrapolation for the nonlinear term. To start the iteration, we need to derive $u^1$ according to the following first order scheme:

$$\begin{cases} \dfrac{u^1 - u^0}{\tau_1} = \nu \Delta u^1 - \Pi_N f(u^0) \ , \\ u^0 = \Pi_N u_0 \ , \end{cases} \tag{1.4}$$ 2ndSchemeI_1storc

where $\tau_1 = \min\{\tau^{\frac{4}{3}} \ , \ 1\}$. Such choice of $\tau_1$ is due to the error analysis and will be shown later. Afterwards, we consider the second order scheme II:

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} = \nu \Delta u^{n+1} - A(u^{n+1} - 2u^n + u^{n-1}) - \Pi_N \left(2f(u^n) - f(u^{n-1})\right), \tag{1.5}$$ 2nd_SchemeII

where $\tau > 0$ is the time step and $n \geq 1$. We again need to derive $u^1$ according to the following first order scheme:

$$\begin{cases} \dfrac{u^1 - u^0}{\tau_1} = \nu \Delta u^1 - \Pi_N f(u^0) \ , \\ u^0 = \Pi_N u_0 \ , \end{cases} \tag{1.6}$$ 2nd_SchemeII_1st

where $\tau_1 = \min\{\tau^{\frac{4}{3}} \ , \ 1 \ , \frac{1}{\sqrt{A+1}}\}$. The choice of such $\tau$ is to guarantee the error estimate and to ensure that the new modified energy function can be controlled by the initial data. In this work we will show the energy stability of Scheme I (1.3)-(1.4) and Scheme II (1.5)-(1.6). Our main results state below:

⟨2ndThm_I⟩ **Theorem 1.1** (Unconditional stability of Scheme I)**.** *Consider the scheme (1.3)-(1.4) with $\nu > 0$, $\tau > 0$ and $N \geq 2$. Assume $u_0 \in H^2(\mathbb{T}^2)$. The initial energy is denoted by $E_0 = E(u_0)$. If there exists a constant $\beta_c > 0$ depending only on $E_0$ and $\|u_0\|_{H^2}$, such that*

$$A \geq \beta \cdot (\nu^2 + \nu^{-10} |\log \nu|^4) \ , \ \beta \geq \beta_c \ ,$$

*then*

$$\tilde{E}(u^{n+1}) \leq \tilde{E}(u^n) \ , \ n \geq 1 \ ,$$

*where $\tilde{E}(u^n)$ for $n \geq 1$ is a modified energy functional and is defined as*

$$\tilde{E}(u^n) := E(u^n) + \frac{\nu}{4}\|u^n - u^{n-1}\|_2^2 + \frac{1}{4\tau}\|u^n - u^{n-1}\|_2^2 \ .$$

⟨Thmerror⟩ **Theorem 1.2** ($L^2$ error estimate of Scheme I)**.** *Let $\nu > 0$ and $u_0 \in H^s$, $s \geq 8$. Let $0 < \tau \leq M$ for some $M > 0$. Let $u(t)$ be the continuous solution to the 2D Allen-Cahn equation with initial data $u_0$. Let $u^1$ be defined according to (1.4) with initial data $u^0 = \Pi_N u_0$. Let $u^m$, $m \geq 2$ be defined in (1.3) with initial data $u^0$ and $u^1$. Assume $A$ satisfies the same condition in Theorem 1.1. Define $t_0 = 0$, $t_1 = \tau_1$ and $t_m = \tau_1 + (m-1)\tau$ for $m \geq 2$. Then for any $m \geq 1$,*

$$\|u(t_m) - u^m\|_2 \leq C_1 \cdot e^{C_2 t_m} \cdot (N^{-s} + \tau^2) \ ,$$

*where $C_1$ , $C_2 > 0$ are constants depending only on $(u_0, \ \nu, \ s, \ A, \ M)$.*

**Remark 1.3.** *Here we require that $\tau$ is not arbitrarily large. This is a result of loss of the mass conservation as preserved by Cahn-Hilliard equation. However, in practice it is not a big issue as we always use small time steps.*

⟨2ndThm_cond⟩ **Theorem 1.4** (Conditional stability of Scheme II)**.** *Consider the scheme (1.5) - (1.6) with $\nu > 0$, $\tau > 0$ and $N \geq 2$. Assume $u_0 \in H^2(\mathbb{T}^2)$. The initial energy is denoted by $E_0 = E(u_0)$. There exist constants $C_i > 0$, $i = 1, \ 2, \ 3, \ 4$ depending only on $E_0$ and $\|u_0\|_{H^2}$, such that the following holds:*
`Case 1:` $A = 0$. *If*

$$\tau \leq \begin{cases} C_1 \dfrac{\nu^4}{1 + |\log \nu|^2}, & \textit{when } 0 < \nu < 1 \ ; \\ C_2 \dfrac{\nu^{-2}}{1 + |\log \nu|^2}, & \textit{when } \nu \geq 1 \ . \end{cases}$$

*then*

$$\mathring{E}(u^{n+1}) \le \mathring{E}(u^n) \ .$$

*Case 2:* $A = constant \cdot (\nu^4 + \nu)$. *If*

$$\tau \le \begin{cases} C_3 \dfrac{\nu^2}{1+|\log \nu|}, & when\ 0 < \nu < 1 \ ; \\ C_4 \dfrac{\nu^{-1}}{1+|\log \nu|}, & when\ \nu \ge 1 \ . \end{cases}$$

*then*

$$\mathring{E}(u^{n+1}) \le \mathring{E}(u^n) \ .$$

*Here* $\mathring{E}(u^n)$ *for* $n \ge 1$ *is a modified energy functional and is defined as*

$$\mathring{E}(u^n) := E(u^n) + \frac{A+1}{2} \|u^n - u^{n-1}\|_2^2 + \frac{1}{4\tau} \|u^n - u^{n-1}\|_2^2 \ .$$

**Remark 1.5.** *The* $L^2$ *error estimate of Scheme II can be obtained via very similar arguments as in Theorem 1.2 and we skip the details.*

The presenting paper is organized as follows. In Section 2 we list the notation and preliminaries including several useful lemmas. The energy stability of the second order semi-implicit scheme I of the 2D Allen-Cahn will be shown in Section 3 and the error estimate is given therein. The second order semi-implicit scheme II will be discussed in Section 4.

## 2. Notation and preliminaries

Throughout this paper, for any two (non-negative in particular) quantities $X$ and $Y$, we denote $X \lesssim Y$ if $X \le CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant $C$ on other parameters or constants are usually clear from the context and we will often suppress this dependence. We shall denote $X \lesssim_{Z_1, Z_2, \cdots, Z_k} Y$ if $X \le CY$ and the constant $C$ depends on the quantities $Z_1, \cdots, Z_k$.

For any two quantities $X$ and $Y$, we shall denote $X \ll Y$ if $X \le cY$ for some sufficiently small constant $c$. The smallness of the constant $c$ is usually clear from the context. The notation $X \gg Y$ is similarly defined. Note that our use of $\ll$ and $\gg$ here is *different* from the usual Vinogradov notation in number theory or asymptotic analysis.

For a real-valued function $u : \Omega \to \mathbb{R}$ we denote its usual Lebesgue $L^p$-norm by

$$\|u\|_p = \|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u|^p \ dx\right)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \operatorname{esssup}_{x \in \Omega} |u(x)|, & p = \infty. \end{cases} \tag{2.1}$$ {?}

Similarly, we use the weak derivative in the following sense: For $u$, $v \in L^1_{loc}(\Omega)$, (i.e they are locally integrable); $\forall \phi \in C_0^\infty(\Omega)$, i.e $\phi$ is infinitely differentiable (smooth) and compactly supported; and

$$\int_\Omega u(x) \ \partial^\alpha \phi(x) \ dx = (-1)^{\alpha_1 + \cdots + \alpha_n} \int_\Omega v(x) \ \phi(x) \ dx,$$

then $v$ is defined to be the weak partial derivative of $u$, denoted by $\partial^\alpha u$. Suppose $u \in L^p(\Omega)$ and all weak derivatives $\partial^\alpha u$ exist for $|\alpha| = \alpha_1 + \cdots + \alpha_n \le k$ , such that $\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \le k$, then we denote $u \in W^{k,p}(\Omega)$ to be the standard Sobolev space. The corresponding norm of $W^{k,p}(\Omega)$ is :

$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u|^p \ dx \right)^{\frac{1}{p}} .$$

For $p = 2$ case, we use the convention $H^k(\Omega)$ to denote the space $W^{k,2}(\Omega)$. We often use $D^m u$ to denote any differential operator $D^\alpha u$ for any $|\alpha| = m$: $D^2$ denotes $\partial^2_{x_i x_j} u$ for $1 \le i, j \le d$, as an example.

In this paper we use the following convention for Fourier expansion on $\mathbb{T}^d$:

$$f(x) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{ik \cdot x} \ , \ \hat{f}(k) = \int_\Omega f(x) e^{-ik \cdot x} \ dx \ .$$

Taking advantage of the Fourier expansion, we use the well-known equivalent $H^s$-norm and $\dot{H}^s$-semi-norm of function $f$ by

$$\|f\|_{H^s} = \frac{1}{(2\pi)^{d/2}}\left(\sum_{k\in\mathbb{Z}^d}(1+|k|^{2s})|\hat{f}(k)|^2\right)^{\frac{1}{2}}\ ,\ \|f\|_{\dot{H}^s} = \frac{1}{(2\pi)^{d/2}}\left(\sum_{k\in\mathbb{Z}^d}|k|^{2s}|\hat{f}(k)|^2\right)^{\frac{1}{2}}\ .$$

⟨Sobolevineq⟩ **Lemma 2.1** (Sobolev's inequalities on $\mathbb{T}^d$). *Let $0<s<d$ and $f\in L^q(\mathbb{T}^d)$ for any $\frac{d}{d-s}<p<\infty$, then*

$$\|\langle\nabla\rangle^{-s} f\|_{L^p(\mathbb{T}^d)} \lesssim_{s,p,d} \|f\|_{L^q(\mathbb{T}^d)}\ ,\ \textit{where}\ \frac{1}{q}=\frac{1}{p}+\frac{s}{d}\ ,$$

*where $\langle\nabla\rangle^{-s}$ denotes $(1-\Delta)^{-\frac{s}{2}}$ and $A\lesssim_{s,p,d} B$ is defined as $A\le C_{s,p,d}\,B$ where $C_{s,p,d}$ is a constant dependent on $s,p$ and $d$.*

**Remark 2.2.** *Note that the this Sobolev inequality is a variety of the standard version. Note that on the Fourier side the symbol of $\langle\nabla\rangle^{-s}$ is given by $(1+|k|^2)^{-\frac{s}{2}}$. In particular, $\|f\|_{\infty(\mathbb{T}^d)}\lesssim\|f\|_{H^2(\mathbb{T}^d)}$, known as Morrey's inequality.*

⟨DiscreteGronwall⟩ **Lemma 2.3** (Discrete Grönwall's inequality). *Let $\tau>0$ and $y_n\ge 0$, $\alpha_n\ge 0$, $\beta_n\ge 0$ for $n=1,2,3\cdots$. Suppose*

$$\frac{y_{n+1}-y_n}{\tau}\le \alpha_n y_n+\beta_n\ ,\ \forall\ n\ge 0\ .$$

*Then for any $m\ge 1$, we have*

$$y_m\le \exp\left(\tau\sum_{n=0}^{m-1}\alpha_n\right)\left(y_0+\sum_{k=0}^{m-1}\beta_k\right)\ .$$

*Proof.* We refer the readers to [8] and [9] for the proof and skip the details here. □

⟨MP_ac⟩ **Lemma 2.4** (Maximum principle for smooth solutions to the Allen-Cahn equation). *Let $T>0$, $d\le 3$ and assume $u\in C_x^2C_t^1(\mathbb{T}^d\times[0,T])$ is a classical solution to Allen-Cahn equation with initial data $u_0$. Then*

$$\|u(.\ ,\ t)\|_\infty\le\max\{\|u_0\|_\infty\ ,\ 1\}\ ,\ \forall\ 0\le t\le T\ .$$

*Proof.* We refer the readers to Lemma 4.2 in [8] for the proof. □

⟨H^kregularity_AC⟩ **Lemma 2.5** ($H^k$ boundedness of the exact solution to the Allen-Cahn equation). *Assume $u(x,t)$ is a smooth solution to the Allen-Cahn equation in $\mathbb{T}^d$ with $d\le 3$ and the initial data $u_0\in H^k(\mathbb{T}^d)$ for $k\ge 2$. Then,*

$$\sup_{t\ge 0}\|u(t)\|_{H^k(\mathbb{T}^d)}\lesssim_k 1 \tag{2.2}$$ {?}

*where we omit the dependence on $\nu$ and $u_0$.*

*Proof.* All cases $d=1,2,3$ have been proved and we refer the readers to Lemma 4.4 in [8] for the proof. □

⟨Loginterpolation⟩ **Lemma 2.6** (Log-type interpolation). *For all $f\in H^s(\mathbb{T}^2)$ , $s>1$, then*

$$\|f\|_\infty\le C_s\cdot\left(\|f\|_{\dot{H}^1}\sqrt{\log(\|f\|_{\dot{H}^s}+3)}+|\hat{f}(0)|+1\right)\ .$$

*Here $C_s$ is a constant which only depends on $s$.*

*Proof.* The proof of Lemma 2.6 is given in [8]. For the sake of completeness we sketch the proof here. We first consider the Fourier series of $f$: $f(x)=\frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2}\hat{f}(k)\,e^{ik\cdot x}$, which converge pointwisely to $f$. It then follows that

$$
\begin{aligned}
\|f\|_\infty &\le \frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2}|\hat f(k)| \\
&\le \frac{1}{(2\pi)^2}\left(|\hat f(0)| + \sum_{0<|k|\le N}|\hat f(k)| + \sum_{|k|>N}|\hat f(k)|\right) \\
&\lesssim |\hat f(0)| + \sum_{0<|k|\le N}(|\hat f(k)||k|\cdot|k|^{-1}) + \sum_{|k|>N}(|\hat f(k)||k|^s\cdot|k|^{-s}) \\
&\lesssim |\hat f(0)| + \left(\sum_{0<|k|\le N}|\hat f(k)|^2|k|^2\right)^{\frac12}\cdot\left(\sum_{0<|k|\le N}|k|^{-2}\right)^{\frac12} + \left(\sum_{|k|>N}|\hat f(k)|^2|k|^{2s}\right)^{\frac12}\cdot(\sum_{|k|>N}|k|^{-2s})^{\frac12} \\
&\lesssim |\hat f(0)| + \frac{1}{N^{s-1}}\left(\sum_{|k|>N}|\hat f(k)|^2|k|^{2s}\right)^{\frac12} + \left(\sum_{0<|k|\le N}|\hat f(k)|^2|k|^2\right)^{\frac12}\cdot\sqrt{\log(N+3)} \\
&\lesssim |\hat f(0)| + \frac{1}{N^{s-1}}\|f\|_{\dot H^s} + \sqrt{\log(N+3)}\|f\|_{\dot H^1}\ .
\end{aligned}
$$

If $\|f\|_{\dot H^s}\le 3$, we can simply take $N=1$; otherwise take $N^{s-1}$ close to $\|f\|_{\dot H^s}$. As a remark, this lemma can be viewed as a variation of the well-known log-type Bernstein's inequality. □

## 3. Second order semi-implicit scheme I

Recall that the second order semi-implicit Fourier spectral scheme I is given by:

$$
\frac{3u^{n+1}-4u^n+u^{n-1}}{2\tau} = \nu\Delta u^{n+1} - A\tau(u^{n+1}-u^n) - \Pi_N\left(2f(u^n)-f(u^{n-1})\right)\quad,\ n\ge 1\ , \tag{3.1}
$$
{?}

where $\tau>0$ is the time step and this scheme applies second order backward derivative in time with a second order extrapolation for the nonlinear term. Recall that we need to derive $u^1$ according to the following first order scheme:

$$
\begin{cases}
\dfrac{u^1-u^0}{\tau_1} = \nu\Delta u^1 - \Pi_N f(u^0)\quad ,\\
u^0 = \Pi_N u_0\ ,
\end{cases} \tag{3.2}
$$
{?}

where $\tau_1=\min\{\tau^{\frac43}\ ,\ 1\}$. The choice of $\tau_1$ is due to the error analysis which will be shown later. Roughly speaking,

$$\|u^1-u(\tau_1)\|_2 \lesssim N^{-s}+\tau_1^{\frac32}\ ,$$

where $u(\tau_1)$ denotes the exact PDE solution at $\tau_1$. As expected in $L^2$ error analysis of the second order scheme, we require that $\tau_1^{\frac32}\lesssim\tau^2$ or $\tau_1\lesssim\tau^{\frac43}$.

### 3.1. Estimate of the first order scheme (1.4)

In this section we will estimate some bounds for $u^1$ which will be used to prove the stability and $L^2$ error estimate of the second order scheme I.

⟨2ndSchemem_I_lem1⟩ **Lemma 3.1.** *Consider the scheme (1.4). Assume $u_0\in H^2(\mathbb{T}^2)$, then*

$$\|u^1\|_\infty + \frac{\|u^1-u^0\|_2^2}{\tau_1} + \frac{\nu}{2}\|\nabla u^1\|_2^2 \lesssim_{E(u_0)\ ,\ \|u_0\|_{H^2}} 1\ .$$

*Proof.* Firstly, we consider $\|u^1\|_\infty$. We write

$$u^1 = \frac{1}{1-\tau_1\nu\Delta}u^0 - \frac{\tau_1\Pi_N}{1-\tau_1\nu\Delta}f(u^0)\ .$$

Note that

$$\frac{1}{1+\tau_1\nu|k|^2}\le 1\ ,\ \tau_1\le 1\ ,$$

thus we have

$$\begin{aligned}\|u^1\|_\infty \lesssim \|u^1\|_{H^2} &\lesssim \|u^0\|_{H^2} + \|f(u^0)\|_{H^2}\\ &\lesssim \|u^0\|_{H^2} + \|(u^0)^3\|_{H^2}\\ &\lesssim_{\|u_0\|_{H^2}} 1\ ,\end{aligned}$$

as $\|u^0\|_\infty \lesssim 1$ by Morrey's inequality. Secondly, we take $L^2$ inner product with $u^1 - u^0$ on both sides of (1.4) and it then follows that

$$\begin{aligned}&\frac{\|u^1-u^0\|_2^2}{\tau_1} + \frac{\nu}{2}\left(\|\nabla u^1\|_2^2 - \|\nabla u^0\|_2^2 + \|\nabla(u^1-u^0)\|_2^2\right)\\ &= -(f(u^0)\ ,\ u^1-u^0)\\ &\leq \|f(u^0)\|_{\frac{4}{3}}\|u^1-u^0\|_4\\ &\lesssim_{E(u^0)} 1\ ,\end{aligned}$$

by $\|u_0\|_\infty\ ,\ \|u_1\|_\infty \lesssim 1$. As a result, $\|u^1\|_\infty + \frac{\|u^1-u^0\|_2^2}{\tau_1} + \frac{\nu}{2}\|\nabla u^1\|_2^2 \lesssim_{E(u_0)\ ,\ \|u_0\|_{H^2}} 1$.

□

⟨2ndSchemem_I_lem2⟩ **Lemma 3.2** (Error estimate for $u^1$)**.** *Consider the system for first time step $u^1$:*

$$\begin{cases}\dfrac{u^1-u^0}{\tau_1} = \nu\Delta u^1 - \Pi_N f(u^0)\\ \partial_t u = \nu\Delta u - f(u)\\ u^0 = \Pi_N u_0\ ,\ u(0) = u_0\ .\end{cases}$$

*Let $u_0 \in H^s$, $s \geq 6$. There exists a constant $D_1 > 0$ depending only on $(u_0, \nu, s)$, such that $\|u(\tau_1) - u^1\|_2 \leq D_1 \cdot (N^{-s} + \tau_1^{\frac{3}{2}})$.*

*Proof.* We start the proof in three steps:

**Step 1:** We discretize the exact solution $u$ in time. Write the continuous time PDE in time interval $[0, \tau_1]$. Note that for a one-variable function $h(s)$,

$$\begin{aligned}h(0) &= h(\tau_1) + \int_{\tau_1}^0 h'(s)\ ds\\ &= h(\tau_1) - h'(\tau_1)\tau_1 + \int_0^{\tau_1} h''(s)\cdot s\ ds\ .\end{aligned}$$

By applying this formula, we have

$$\begin{aligned}\frac{u(\tau_1)-u(0)}{\tau_1} &= \partial_t u(\tau_1) - \frac{1}{\tau_1}\int_0^{\tau_1}(\partial_{tt}u)\cdot s\ ds\\ &= \nu\Delta u(\tau_1) - f(u(\tau_1)) - \frac{1}{\tau_1}\int_0^{\tau_1}(\partial_{tt}u)\cdot s\ ds\\ &= \nu\Delta u(\tau_1) - \Pi_N f(u(0)) - \Pi_{>N} f(u(0)) - [f(u(\tau_1) - f(u(0))]\\ &\quad - \frac{1}{\tau_1}\int_0^{\tau_1}(\partial_{tt}u)\cdot s\ ds\ ,\end{aligned}$$

where $\Pi_{>N} = id - \Pi_N$. Therefore, we get

$$\frac{u(\tau_1)-u(0)}{\tau_1} = \nu\Delta u(\tau_1) - \Pi_N f(u(0)) + G^0\ ,$$

where

$$\begin{aligned}G^0 &= -\Pi_{>N} f(u(0)) - [f(u(\tau_1) - f(u(0))] - \frac{1}{\tau_1}\int_0^{\tau_1}(\partial_{tt}u)\cdot s\ ds\\ &= -\Pi_{>N} f(u(0)) - [f(u(\tau_1) - f(u(0))] - \frac{1}{\tau_1}\int_0^{\tau_1}(\nu\Delta\partial_t u - f'(u)\partial_t u)\cdot s\ ds\end{aligned}$$

**Step 2:** Estimate $\|u(\tau_1) - u^1\|_2$. We consider

$$\begin{cases} \dfrac{u(\tau_1)-u(0)}{\tau_1} = \nu\Delta u(\tau_1) - \Pi_N f(u(0)) + G^0 \\ \dfrac{u^1-u^0}{\tau_1} = \nu\Delta u^1 - \Pi_N f(u^0) \\ u^0 = \Pi_N u_0 \ , \ u(0) = u_0 \ . \end{cases}$$

Define $e^1 = u(\tau_1) - u^1$ and $e^0 = u(0) - u^0$. Then we get

$$\frac{e^1 - e^0}{\tau_1} = \nu\Delta e^1 - \Pi_N \left(f(u(0)) - f(u^0)\right) + G^0 \ .$$

Taking the $L^2$ inner product with $e^1$ on both sides, we derive

$$\begin{aligned} &\frac{1}{2\tau_1}\left(\|e^1\|_2^2 - \|e^0\|_2^2 + \|e^1 - e^0\|_2^2\right) + \nu\|\nabla e^1\|_2^2 \\ &\leq \|f(u(0)) - f(u^0)\|_2 \cdot \|e^1\|_2 + \|G^0\|_2 \cdot \|e^1\|_2 \\ &\lesssim \left(\|e^0\|_2 + \|G^0\|_2\right)\|e^1\|_2 \\ &\lesssim \left(\|e^0\|_2^2 + \|G^0\|_2^2\right) + \frac{1}{4}\|e^1\|_2^2 \ . \end{aligned}$$

As a result, we have

$$\left(1 - \frac{\tau_1}{2}\right)\|e^1\|_2^2 \leq 2\tau_1\left(\|e^0\|_2^2 + \|G^0\|_2^2\right) + \|e^0\|_2^2 \ .$$

Note that $\tau_1 \leq 1$, so $1 - \frac{\tau_1}{2} \geq \frac{1}{2}$ and

$$\|e^1\|_2^2 \lesssim (1+\tau_1)\|e^0\|_2^2 + \tau_1\|G^0\|_2^2 \ .$$

**Step 3:** Estimate $\|e^0\|_2^2$ and $\|G^0\|_2^2$. Note that $\|e^0\|_2^2 = \|u(0) - u^0\|_2^2 = \|u_0 - \Pi_N u_0\|_2^2 = \|\Pi_{>N} u_0\|_2^2$. It is clear that (cf.[9])

$$\|e^0\|_2^2 = \|\Pi_{>N} u_0\|_2^2 \lesssim N^{-2s} \ .$$

For $\|G^0\|_2$, note that $\|\Pi_{>N} f(u(0))\|_2 \lesssim N^{-s}$, by the maximum principle (Lemma 2.4). On the other hand, by the mean value theorem,

$$f(u(\tau_1)) - f(u(0)) = f'(\xi)(u(\tau_1) - u(0)) \ ,$$

where $\xi$ is a number between $u(\tau_1)$ and $u(0)$. Again by the maximum principle (Lemma 2.4),

$$\|f(u(\tau_1)) - f(u(0))\|_2 \lesssim \|u(\tau_1) - u(0)\|_2 \lesssim \tau_1\|\partial_t u\|_{L_t^\infty L_x^2([0\ ,\ \tau_1]\times\mathbb{T}^2)} \lesssim \tau_1 \ ,$$

by the Sobolev bound of the exact solution Lemma 2.5. Finally, we have

$$\begin{aligned} &\left\|\frac{1}{\tau_1}\int_0^{\tau_1} (\nu\Delta\partial_t u - f'(u)\partial_t u)\cdot s \ ds\right\|_2 \\ &\lesssim \left\|\int_0^{\tau_1} \nu\Delta\partial_t u - f'(u)\partial_t u \ ds\right\|_2 \\ &\lesssim \int_0^{\tau_1} \|\nu\Delta\partial_t u\|_2 \ ds + \int_0^{\tau_1} \|f'(u)\partial_t u\|_2 \ ds \\ &\lesssim \tau_1 \ . \end{aligned}$$

This implies $\|G^0\|_2^2 \lesssim N^{-2s} + \tau_1^2$. Therefore we get

$$\|e^1\|_2^2 \lesssim (1+\tau_1)N^{-2s} + \tau_1(N^{-2s} + \tau_1^2) \lesssim N^{-2s} + \tau_1^3 \ .$$

As a result, we obtain that

$$\|e^1\|_2 \lesssim N^{-s} + \tau_1^{\frac{3}{2}} \ . \tag{3.3}$$ ?2ndorderSchI:e1?

This completes the proof. □

3.2. **Unconditional stability of the second order scheme I (1.3) & (1.4)**

In this section we will prove Theorem 1.1 for the second order scheme (1.3) combining (1.4). Before proving this stability theorem, we begin with several lemmas.

⟨2ndScheme_I_lem3⟩ **Lemma 3.3.** *Consider (1.3) for $n \ge 1$. Suppose $E(u^n) \le B$ and $E(u^{n-1}) \le B$ for some $B > 0$. Then*

$$\|u^{n+1}\|_\infty \le \alpha_B \cdot \left\{ (1+\nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1})} + \tau + 1 \right\} ,$$

*for some $\alpha_B > 0$ only depending on $B$.*

*Proof.* For simplicity we write $\lesssim$ instead of $\lesssim_B$. Recall that (1.3)

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} = \nu\Delta u^{n+1} - A\tau(u^{n+1} - u^n) - \Pi_N \left(2f(u^n) - f(u^{n-1})\right) .$$

We rewrite (1.3) as

$$\begin{aligned} u^{n+1} =& \frac{4 + 2A\tau^2}{3 - 2\nu\tau\Delta + 2A\tau^2} u^n - \frac{1}{3 - 2\nu\tau\Delta + 2A\tau^2} \\ & - \frac{2\tau\Pi_N}{3 - 2\nu\tau\Delta + 2A\tau^2} \left(2f(u^n) - f(u^{n-1})\right) . \end{aligned}$$

For the case when $k = 0$, we have

$$\begin{cases} \dfrac{4 + 2A\tau^2}{3 + 2A\tau^2} \lesssim 1 \\ \dfrac{1}{3 + 2A\tau^2} \lesssim 1 \\ \dfrac{2}{3 + 2A\tau^2} \lesssim \tau . \end{cases}$$

We thus have

$$|\widehat{u^{n+1}}(0)| \lesssim \tau + 1 .$$

Note that for the case when $|k| \ge 1$,

$$\begin{cases} \dfrac{4 + 2A\tau^2}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim 1 \\ \dfrac{1}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim 1 \\ \dfrac{2\tau|k|}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim \dfrac{\tau|k|}{\nu\tau|k|^2} \lesssim \dfrac{1}{\nu} \cdot |k|^{-1} . \end{cases}$$

Therefore we get

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^1} &\lesssim \|u^n\|_{\dot{H}^1} + \|u^{n-1}\|_{\dot{H}^1} + \frac{1}{\nu} \|\langle\nabla\rangle^{-1} \left(2f(u^n) - f(u^{n-1})\right)\|_2 \\ &\lesssim \nu^{-\frac{1}{2}} + \nu^{-1}(\|(u^n)^3\|_{4/3} + \|(u^{n-1})^3\|_{4/3} + \|u^n\|_2 + \|u^{n-1}\|_2) \\ &\lesssim \nu^{-\frac{1}{2}} + \nu^{-1} , \end{aligned}$$

here we apply Sobolev's inequality Lemma 2.1 and apply the energy bound. Similarly, we can derive that

$$\begin{cases} \dfrac{|k|^2(4 + 2A\tau^2)}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim \dfrac{|k|^2(1 + A\tau^2)}{\nu\tau|k|^2} \lesssim \dfrac{1}{\nu\tau} + \dfrac{A\tau}{\nu} \\ \dfrac{|k|^2}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim \dfrac{1}{\nu\tau} \\ \dfrac{2\tau|k|^2}{3 + 2\nu\tau|k|^2 + 2A\tau^2} \lesssim \dfrac{\tau|k|^2}{\nu\tau|k|^2} \lesssim \dfrac{1}{\nu} . \end{cases}$$

This implies

$$\begin{aligned}
\|u^{n+1}\|_{\dot{H}^2} &\lesssim \left(\frac{1}{\nu\tau}+\frac{A\tau}{\nu}\right)\|u^n\|_2+\frac{1}{\nu\tau}\|u^{n-1}\|_2+\frac{1}{\nu}\|2f(u^n)-f(u^{n-1})\|_2\\
&\lesssim \frac{1}{\nu\tau}+\frac{A\tau}{\nu}+\frac{1}{\nu}\left(\|u^n\|_6^3+\|u^{n-1}\|_6^3+\|u^n\|_2+\|u^{n-1}\|_2\right)\\
&\lesssim \frac{1}{\nu\tau}+\frac{A\tau}{\nu}+\frac{1}{\nu}\left(\|u^n\|_{H^1}^3+\|u^{n-1}\|_{H^1}^3+1\right)\\
&\lesssim \frac{1}{\nu\tau}+\frac{A\tau}{\nu}+\frac{1}{\nu}(\nu^{-\frac{3}{2}}+1)\ .
\end{aligned}$$

Finally, by applying the log-interpolation lemma (Lemma 2.6), we can get

$$\begin{aligned}
\|u^{n+1}\|_\infty &\lesssim (1+\|u^{n+1}\|_{\dot{H}^1})\cdot\sqrt{\log(3+\|u^{n+1}\|_{\dot{H}^2})}+|\widehat{u^{n+1}(0)}|\\
&\lesssim (1+\nu^{-1})\cdot\sqrt{\log(3+\frac{A\tau}{\nu}+\frac{1}{\nu\tau}+\nu^{-\frac{5}{2}}+\nu^{-1})}+\tau+1\ ,
\end{aligned}$$

where $\nu^{-\frac{1}{2}}$ is bounded by $\nu^{-1}+1$.

□

### 3.3. **Proof of the unconditional stability**

In this section we show Theorem 1.1. To start with we introduce some notation. We denote $\delta u^{n+1} := u^{n+1}-u^n$ and $\delta^2 u^{n+1} := u^{n+1}-2u^n+u^{n-1}$. Clearly,

$$\begin{cases}
3u^{n+1}-4u^n+u^{n-1}=2\delta u^{n+1}+\delta^2 u^{n+1}\\
\delta^2 u^{n+1}-\delta u^{n+1}=-\delta u^n\\
\delta u^n\cdot u^n=(u^n-u^{n-1})u^n=\frac{1}{2}\left(|u^n|^2-|u^{n-1}|^2+|\delta u^n|^2\right)\ .
\end{cases}$$

As a result, we have

$$\begin{aligned}
&\left(3u^{n+1}-4u^n+u^{n-1}\ ,\ u^{n+1}-u^n\right)\\
&=\left(2\delta u^{n+1}+\delta^2 u^{n+1}\ ,\ \delta u^{n+1}\right)\\
&=2\|\delta u^{n+1}\|_2^2+\left(\delta u^{n+1}-\delta u^n\ ,\ \delta u^{n+1}\right)\\
&=2\|\delta u^{n+1}\|_2^2+\frac{1}{2}\left(\|\delta u^{n+1}\|_2^2-\|\delta u^n\|_2^2+\|\delta^2 u^{n+1}\|_2^2\right)\ .
\end{aligned}$$

Now recall the scheme (1.3)

$$\frac{3u^{n+1}-4u^n+u^{n-1}}{2\tau}=\nu\Delta u^{n+1}-A\tau(u^{n+1}-u^n)-\Pi_N\left(2f(u^n)-f(u^{n-1})\right)\ .$$

Taking the $L^2$ inner product with $\delta u^{n+1}=u^{n+1}-u^n$ on both sides of (1.3), we have

$$\begin{aligned}
&\frac{1}{\tau}\|\delta u^{n+1}\|_2^2+\frac{1}{4\tau}\left(\|\delta u^{n+1}\|_2^2-\|\delta u^n\|_2^2+\|\delta^2 u^{n+1}\|_2^2\right)\\
&\qquad+\frac{\nu}{2}\left(\|\nabla u^{n+1}\|_2^2-\|\nabla u^n\|_2^2+\|\delta\nabla u^{n+1}\|_2^2\right)\\
&+A\tau\|\delta u^{n+1}\|_2^2=-\left(\Pi_N(2f(u^n)-f(u^{n-1}))\ ,\ \delta u^{n+1}\right)\ .
\end{aligned}$$

To analyze $\left(2f(u^n)-f(u^{n-1})\ ,\ \delta u^{n+1}\right)$, we consider

$$2f(u^n)-f(u^{n-1})=f(u^n)+\left(f(u^n)-f(u^{n-1})\right).$$

Note that $F'=f$, hence by the fundamental theorem of calculus,

$$\begin{aligned}
&F(u^{n+1})-F(u^n)\\
=&f(u^n)\delta u^{n+1}+\int_0^1 f'(u^n+s\delta u^{n+1})(1-s)\ ds\cdot(\delta u^{n+1})^2\\
=&f(u^n)\delta u^{n+1}+\int_0^1 \tilde{f}(u^n+s\delta u^{n+1})(1-s)\ ds\cdot(\delta u^{n+1})^2-\frac{1}{2}(\delta u^{n+1})^2\ ,
\end{aligned}$$

where $\tilde{f}(x) = 3x^2$, as $f'(x) = 3x^2 - 1$. Therefore, we can get

$$f(u^n)\delta u^{n+1} \geq F(u^{n+1}) - F(u^n) + \frac{1}{2}(\delta u^{n+1})^2 - \frac{3}{2}\left(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2\right) \cdot (\delta u^{n+1})^2 \ .$$

On the other hand by the mean value theorem,

$$f(u^n) - f(u^{n-1}) = f'(\xi)\delta u^n \ ,$$

and hence we have

$$\begin{aligned}
&\int_{\mathbb{T}^2} (f(u^n) - f(u^{n-1})) \cdot \delta u^{n+1} \geq -\left(3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2 + 1\right) \cdot \|\delta u^n\|_2 \cdot \|\delta u^{n+1}\|_2 \\
&\geq -\frac{\left(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2\right)^2}{\nu} \cdot \|\delta u^{n+1}\|_2^2 - \frac{\nu}{4}\|\delta u^n\|_2^2 \ .
\end{aligned}$$

Then the estimate of the nonlinear term is as following:

$$\begin{aligned}
&-\left(\Pi_N(2f(u^n) - f(u^{n-1}))\ ,\ \delta u^{n+1}\right) \\
=&-\left(2f(u^n) - f(u^{n-1})\ ,\ \delta u^{n+1}\right) \\
\leq& -\int_{\mathbb{T}^2} F(u^{n+1})\ dx + \int_{\mathbb{T}^2} F(u^n)\ dx - \frac{1}{2}\|\delta u^{n+1}\|_2^2 \\
&+ \frac{3}{2}\left(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2\right) \cdot \|\delta u^{n+1}\|_2^2 \\
&+ \frac{\left(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2\right)^2}{\nu} \cdot \|\delta u^{n+1}\|_2^2 + \frac{\nu}{4}\|\delta u^n\|_2^2 \ .
\end{aligned}$$

Combining all estimates we get

$$\begin{aligned}
&\frac{1}{\tau}\|\delta u^{n+1}\|_2^2 + \frac{1}{4\tau}\left(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2\right) \\
&+\frac{\nu}{2}\left(\|\nabla u^{n+1}\|_2^2 - \|\nabla u^n\|_2^2 + \|\delta\nabla u^{n+1}\|_2^2\right) \\
&+A\tau\|\delta u^{n+1}\|_2^2 \\
&\leq -\int_{\mathbb{T}^2} F(u^{n+1})\ dx + \int_{\mathbb{T}^2} F(u^n)\ dx - \frac{1}{2}\|\delta u^{n+1}\|_2^2 \\
&+\frac{3}{2}\left(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2\right) \cdot \|\delta u^{n+1}\|_2^2 \\
&+\frac{\left(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2\right)^2}{\nu} \cdot \|\delta u^{n+1}\|_2^2 + \frac{\nu}{4}\|\delta u^n\|_2^2 \ .
\end{aligned}$$

After simplification, we obtain

$$\begin{aligned}
&\left(\frac{1}{\tau} + A\tau - \frac{\nu}{4} + \frac{1}{2}\right) \cdot \|\delta u^{n+1}\|_2^2 + \tilde{E}(u^{n+1}) \\
\leq& \left\{\frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{\left(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2\right)^2}{\nu}\right\} \cdot \|\delta u^{n+1}\|_2^2 + \tilde{E}(u^n) \ .
\end{aligned}$$

Clearly for $\tilde{E}(u^{n+1}) \leq \tilde{E}(u^n)$, it suffices to show

$$\begin{aligned}
&\frac{1}{\tau} + A\tau - \frac{\nu}{4} + \frac{1}{2} \geq \\
&\frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{\left(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2\right)^2}{\nu} \ .
\end{aligned} \tag{3.4}$$

2nd_condtionI

Now we prove this sufficient condition inductively. Set

$$B = \max\left\{\tilde{E}(u^1)\ ,\ E(u^0)\right\} \ ,$$

by Lemma 3.1 in previous section, $B \lesssim 1$. We shall prove for every $m \geq 2$,

$$\begin{cases} \tilde{E}(u^m) \leq B \ , \ \tilde{E}(u^m) \leq \tilde{E}(u^{m-1}) \ , \\ \|u^m\|_\infty \leq \alpha_B \cdot \left[ (1+\nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1})} + \tau + 1 \right] \ , \end{cases}$$

where $\alpha_B > 0$ is the same constant in Lemma 3.3.

We first check the base case when $m = 2$. Note that $E(u^1) \leq \tilde{E}(u^1) \leq B$ and $E(u^0) \leq B$, then we can apply Lemma 3.3, and hence obtain

$$\|u^2\|_\infty \leq \alpha_B \cdot \left\{ (1+\nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1})} + \tau + 1 \right\} \ .$$

It then suffices to check $\tilde{E}(u^2) \leq \tilde{E}(u^1)$. By the sufficient condition (3.4), we only need to check the inequality

$$\frac{1}{\tau} + A\tau - \frac{\nu}{4} + \frac{1}{2} \geq \frac{3}{2}(\|u^1\|_\infty^2 + \|u^2\|_\infty^2) + \frac{\left(1 + 3\|u^1\|_\infty^2 + 3\|u^0\|_\infty^2\right)^2}{\nu} \ .$$

By Lemma 3.1, $\|u^0\|_\infty \ , \ \|u^1\|_\infty \lesssim 1$, it suffices to choose $A$ such that

$$\frac{1}{\tau} + A\tau - \frac{\nu}{4} + \frac{1}{2} \geq C \cdot (1+\nu^{-2}) \cdot \log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1}) + C\nu^{-1} + C + C\tau \ .$$

We discuss two case and denote $X = A\tau + \frac{1}{\tau}$.

`Case 1:` $0 < \nu \leq 1/2$. In this case we need

$$X + \frac{1}{2} \geq C\nu^{-2} \cdot (|\log \nu| + |\log X|) \ .$$

Hence we need

$$X \geq C \cdot \nu^{-2} |\log \nu| \ .$$

`Case 2:` $\nu > 1/2$. Then we need

$$X \geq C \cdot (|\log X| + 1 + \nu) \ ,$$

namely,

$$X \geq C \cdot (1 + \nu) \ .$$

In conclusion, as $X \geq 2\sqrt{A}$,

$$A \geq C \cdot (1 + \nu^2 + \nu^{-4}|\log \nu|^2) \geq C \cdot (\nu^2 + \nu^{-4}|\log \nu|^2) \ .$$

Now we check the induction step. Assume the induction hypopaper hold for $2 \leq m \leq n$, then for $m = n+1$,

$$\|u^{n+1}\|_\infty \leq \alpha_B \cdot \left\{ (1+\nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1})} + \tau \right\} \ ,$$

by Lemma 3.3. It remains to show $\tilde{E}(u^{n+1}) \leq \tilde{E}(u^n)$. It suffices to choose $A$ such that

$$\begin{aligned} \frac{1}{\tau} + A\tau + \frac{1}{2} \geq & \frac{\nu}{4} + C \cdot (1+\nu^{-2}) \cdot \log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1}) \\ & + \frac{C(1+\nu^{-4})}{\nu} \left( (\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-\frac{5}{2}} + \nu^{-1}))^2 + \tau \right) \ . \end{aligned}$$

In terms of $X = A\tau + \frac{1}{\tau}$ again, we need to discuss two cases as well.

`Case 1:` $0 < \nu \leq 1/2$. Then

$$X \geq C \cdot \nu^{-5}(|\log \nu|^2 + |\log X|^2) \ .$$

As a result, we have

$$X \geq C \cdot \nu^{-5} |\log \nu|^2 \ .$$

`Case 2:` $\nu > 1/2$. Then we need

$$X \geq C\nu + C \cdot (\log X + (\log X)^2 \nu^{-1}) \ ,$$

hence $X \geq C \cdot (\nu + 1)$.

To conclude these two cases, we require that

$$A \geq C \cdot (\nu^2 + 1 + \nu^{-10} |\log \nu|^4) \geq C \cdot (\nu^2 + \nu^{-10} |\log \nu|^4) \ .$$

This completes the induction. Combining the estimate, we can take

$$A \geq C \cdot (\nu^2 + \nu^{-10} |\log \nu|^4) \ , \tag{3.5}$$ {?}

such that $\tilde{E}(u^{n+1}) \leq \tilde{E}(u^n)$, for $n \geq 1$.

### 3.4. $L^2$ error estimate of the second order scheme I

⟨subsection:error2nd⟩ It remains to estimate the $L^2$ error of this second order scheme. We will study the auxiliary error estimate behavior and time discretization behavior of Allen-Cahn equation before proving the theorem.

#### 3.4.1. Auxiliary $L^2$ error estimate for near solutions

Consider for $n \geq 1$,

$$\begin{cases} \dfrac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} = \nu \Delta u^{n+1} - A\tau(u^{n+1} - u^n) - \Pi_N \left(2f(u^n) - f(u^{n-1})\right) + G^n \\ \dfrac{3v^{n+1} - 4v^n + v^{n-1}}{2\tau} = \nu \Delta v^{n+1} - A\tau(v^{n+1} - v^n) - \Pi_N \left(2f(v^n) - f(v^{n-1})\right) \quad , \end{cases} \tag{3.6}$$ 2nd_I_aux

where $(u^1,\ u^0,\ v^1,\ v^0)$ are given.

⟨2ndpropI⟩ **Proposition 3.1.** *For solutions to (3.6), assume for some $N_1 > 0$,*

$$\sup_{n \geq 0} \|u^n\|_\infty + \sup_{n \geq 0} \|v^n\|_\infty \leq N_1 \ ,$$

*Then for any $m \geq 2$,*

$$\begin{aligned} \|u^m - v^m\|_2^2 \leq & \ C \cdot \exp\left((m-1)\tau \cdot \frac{C(1+N_1^4)}{\eta}\right) \\ & \cdot \left((1 + A\tau^2)\|u^1 - v^1\|_2^2 + \|u^0 - v^0\|_2^2 + \frac{\tau}{\eta} \sum_{n=1}^{m-1} \|G^n\|_2^2\right) \ , \end{aligned}$$

*where $C > 0$ is a absolute constant that can be computed and $0 < \eta < \frac{1}{100M}$ is a constant depending only on $M$, that is the upper bound for $\tau$.*

*Proof.* We still denote the constant by $C$ whose value may vary in different lines. Denote $e^n = u^n - v^n$, then

$$\begin{aligned} & \frac{3e^{n+1} - 4e^n + e^{n-1}}{2\tau} - \nu \Delta e^{n+1} + A\tau(e^{n+1} - e^n) \\ = & - \Pi_N \left(2f(u^n) - 2f(v^n)\right) + \Pi_N \left(f(u^{n-1}) - f(v^{n-1})\right) + G^n \end{aligned}$$

Taking the $L^2$ inner product with $e^{n+1}$ on both sides, we derive that

$$\begin{aligned} & \frac{1}{2\tau}(3e^{n+1} - 4e^n + e^{n-1}, e^{n+1}) + \nu \|\nabla e^{n+1}\|_2^2 + \frac{A\tau}{2}\left(\|e^{n+1}\|_2^2 - \|e^n\|_2^2 + \|e^{n+1} - e^n\|_2^2\right) \\ & = -2(f(u^n) - f(v^n),\ e^{n+1}) + (f(u^{n-1}) - f(v^{n-1}),\ e^{n+1}) + (G^n,\ e^{n+1}) \ . \end{aligned} \tag{3.7}$$ 2nd_I_Ineq1

To estimate the right hand side of (3.7), first we observe that

$$|(f(u^n)-f(v^n),e^{n+1})|\le \|f(u^n)-f(v^n)\|_2\|e^{n+1}\|_2\le \frac{\|f(u^n)-f(v^n)\|_2^2}{\eta}+\eta\|e^{n+1}\|_2^2,$$

where $\eta<\frac{1}{100M}$ is a small number only depending on $M$. Moreover, note that

$$\begin{aligned} f(u^n)-f(v^n) &= f(u^n)-f(u^n-e^n)\\ &=(u^n)^3-(u^n-e^n)^3-e^n\\ &=-(e^n)^3-e^n-3u^n(e^n)^2+3(u^n)^2e^n\ . \end{aligned}$$

By assumption we then have

$$\begin{aligned} \|f(u^n)-f(v^n)\|_2^2 \lesssim&\|e^n\|_\infty^4\|e^n\|_2^2+\|e^n\|_2^2+\|u^n\|_\infty^2\|e^n\|_2^2+\|u^n\|_\infty^4\|e^n\|_2^2\\ \lesssim&(1+N_1^4)\|e^n\|_2^2\ . \end{aligned}$$

Similarly, we have

$$\|f(u^{n-1})-f(v^{n-1})\|_2^2\lesssim (1+N_1^4)\|e^{n-1}\|_2^2\ .$$

As a result, we obtain that

$$\text{RHS of (3.7)}\le \frac{C(1+N_1^4)}{\eta}\left(\|e^n\|_2^2+\|e^{n-1}\|_2^2\right)+\frac{1}{\eta}\|G^n\|_2^2+\eta\|e^{n+1}\|_2^2\ .$$

On the other hand, we have

$$\begin{aligned} (3e^{n+1}-4e^n+e^{n-1},e^{n+1}-e^n) &= (2\delta e^{n+1}+\delta^2e^{n+1},\ \delta e^{n+1})\\ &=2\|\delta e^{n+1}\|_2^2+\frac12\left(\|\delta e^{n+1}\|_2^2-\|\delta e^n\|_2^2+\|\delta^2e^{n+1}\|_2^2\right)\ . \end{aligned}$$

We also have that

$$\begin{aligned} &(3e^{n+1}-4e^n+e^{n-1},\ e^n)=3(\delta e^{n+1},\ e^n)-(\delta e^n,\ e^n)\\ =&\frac32\left(\|e^{n+1}\|_2^2-\|e^n\|_2^2-\|e^{n+1}-e^n\|_2^2\right)-\frac12\left(\|e^n\|_2^2-\|e^{n-1}\|_2^2+\|e^n-e^{n-1}\|_2^2\right)\ . \end{aligned}$$

These two equations give that

$$\begin{aligned} &(3e^{n+1}-4e^n+e^{n-1},\ e^{n+1})\\ =&\frac32(\|e^{n+1}\|_2^2-\|e^n\|_2^2)-\frac12(\|e^n\|_2^2-\|e^{n-1}\|_2^2)+\|\delta e^{n+1}\|_2^2-\|\delta e^n\|_2^2\\ &+\frac12\|\delta^2e^{n+1}\|_2^2\ . \end{aligned}$$

Collecting all estimates, we bound (3.7) as

$$\begin{aligned} &\frac{1}{2\tau}\left(\frac32\|e^{n+1}\|_2^2-\frac12\|e^n\|_2^2+\|e^{n+1}-e^n\|_2^2\right)+\frac{A\tau}{2}\|e^{n+1}\|_2^2\\ \le&\frac{1}{2\tau}\left(\frac32\|e^n\|_2^2-\frac12\|e^{n-1}\|_2^2+\|e^n-e^{n-1}\|_2^2\right)+\frac{A\tau}{2}\|e^n\|_2^2\\ &+\frac{C(1+N_1^4)}{\eta}\left(\|e^n\|_2^2+\|e^{n-1}\|_2^2\right)+\frac{1}{\eta}\|G^n\|_2^2+\eta\|e^{n+1}\|_2^2\ . \end{aligned} \tag{3.8}$$

2nd_I_Ineq2

Define $X^{n+1}:=\frac32\|e^{n+1}\|_2^2-\frac12\|e^n\|_2^2+\|e^{n+1}-e^n\|_2^2$. We observe that

$$X^{n+1}=\begin{cases}\frac12\|e^{n+1}\|_2^2+\frac12\|2e^{n+1}-e^n\|_2^2\\ \frac{1}{10}\|e^n\|_2^2+\frac52\|e^{n+1}-\frac25e^n\|_2^2\ .\end{cases}$$

This shows

$$X^{n+1}\ge\frac{1}{10}\max\left\{\|e^{n+1}\|_2^2,\ \|e^n\|_2^2\right\}\ .$$

Making use of $X^{n+1}$, (3.8) becomes

$$\begin{aligned}&\frac{\left(X^{n+1}+A\tau^2\|e^{n+1}\|_2^2\right)-\left(X^n+A\tau^2\|e^n\|_2^2\right)}{2\tau}\\ \le&\frac{C(1+N_1^4)}{\eta}\left(\|e^n\|_2^2+\|e^{n-1}\|_2^2\right)+\frac{1}{\eta}\|G^n\|_2^2+\eta\|e^{n+1}\|_2^2\ .\end{aligned}$$

This leads to

$$\begin{aligned}&\frac{\left(X^{n+1}-2\eta\tau\|e^{n+1}\|_2^2+A\tau^2\|e^{n+1}\|_2^2\right)-\left(X^n-2\eta\tau\|e^n\|_2^2+A\tau^2\|e^n\|_2^2\right)}{2\tau}\\ \le&\frac{C(1+N_1^4)}{\eta}\left(\|e^n\|_2^2+\|e^{n-1}\|_2^2\right)+\frac{1}{\eta}\|G^n\|_2^2+\eta\|e^n\|_2^2\\ \le&\left(\frac{C(1+N_1^4)}{\eta}+C\eta\right)\cdot\left(X^n-2\eta\tau\|e^n\|_2^2\right)+\frac{1}{\eta}\|G^n\|_2^2\ .\end{aligned}$$

Define that

$$\begin{aligned}y_n&=X^n-2\eta\tau\|e^n\|_2^2+A\tau^2\|e^n\|_2^2\ ,\\ \alpha&=\frac{C(1+N_1^4)}{\eta}+C\eta\ ,\\ \beta_n&=\frac{\|G_n\|_2^2}{\eta}\ .\end{aligned}$$

Then for $\nu$ small, we get

$$\frac{y_{n+1}-y_n}{\tau}\le\alpha y_n+\beta_n\ .$$

Applying the discrete Gronwall's inequality (Lemma 2.3), we have for $m\ge 2$,

$$\|e^m\|_2^2\le C\left(X^m-2\eta\tau\|e^m\|_2^2\right)\le Ce^{(m-1)\tau\cdot\frac{C(1+N_1^4)}{\eta}}\left(X^1+A\tau^2\|e^1\|_2^2+\frac{\tau}{\eta}\sum_{n=1}^{m-1}\|G^n\|_2^2\right),$$

which gives

$$\begin{aligned}&\|u^m-v^m\|_2^2\\ \le&C\cdot\exp\left((m-1)\tau\cdot\frac{C(1+N_1^4)}{\eta}\right)\cdot\left(\frac{3}{2}\|e^1\|_2^2-\frac{1}{2}\|e^0\|_2^2+\|e^1-e^0\|_2^2\right.\\ &\left.+A\tau^2\|e^1\|_2^2+\frac{\tau}{\eta}\sum_{n=1}^{m-1}\|G^n\|_2^2\right)\\ \le&C\cdot\exp\left((m-1)\tau\cdot\frac{C(1+N_1^4)}{\eta}\right)\cdot\left((1+A\tau^2)\|u^1-v^1\|_2^2+\|u^0-v^0\|_2^2\right.\\ &\left.+\frac{\tau}{\eta}\sum_{n=1}^{m-1}\|G^n\|_2^2\right)\ .\end{aligned}$$

□

3.4.2. **Time discretization of the Allen-Cahn equation**

We first rewrite the AC equation in terms of the second order scheme.

⟨2ndSchemem_I_lem4⟩ **Lemma 3.4** (Time discrete Allen-Cahn equation)**.** *Let $u(t)$ be the exact solution to Allen-Cahn equation with initial data $u_0\in H^s(\mathbb{T}^2)$, $s\ge 8$. Define $t_0=0$, $t_1=\tau_1$ and $t_m=\tau_1+(m-1)\tau$ for $m\ge 2$. Then for any $n\ge 1$,*

$$\begin{aligned}&\frac{3u(t_{n+1})-4u(t_n)+u(t_{n-1})}{2\tau}\\ =&\nu\Delta u(t_{n+1})-A\tau\left(u(t_{n+1})-u(t_n)\right)-\Pi_N\left[2f(u(t_n))-f(u(t_{n-1}))\right]+G^n\ .\end{aligned}$$

*For any $m \geq 2$,*

$$\tau \sum_{n=1}^{m-1} \|G^n\|_2^2 \lesssim (1+t_m) \cdot \left(\tau^4 + N^{-2s}\right) \ .$$

*Proof.* The proof will be proceeded in several steps and we write $\lesssim$ instead of $\lesssim_{A,\ \nu,\ u_0}$ for simplicity.

**Step 1:** We write the PDE in the discrete form in time. Recall that

$$\partial_t u = \nu \Delta u - f(u) \ .$$

For a one variable function $h(t)$, the following equation holds:

$$h(t) = h(t_0) + h'(t_0)(t-t_0) + \frac{1}{2} h''(t_0)(t-t_0)^2 + \frac{1}{2} \int_{t_0}^{t} h'''(s)(s-t)^2 \ ds \ .$$

We then apply this to AC,

$$\begin{cases} u(t_n) = u(t_{n+1}) - \partial_t u(t_{n+1}) \cdot \tau + \dfrac{1}{2} \partial_{tt} u(t_{n+1}) \tau^2 + \dfrac{1}{2} \displaystyle\int_{t_{n+1}}^{t_n} \partial_{ttt} u(s)(s-t_n)^2 \ ds \\ u(t_{n-1}) = u(t_{n+1}) - \partial_t u(t_{n+1}) \cdot 2\tau + 2\partial_{tt} u(t_{n+1}) \tau^2 + \dfrac{1}{2} \displaystyle\int_{t_{n+1}}^{t_{n-1}} \partial_{ttt} u(s)(s-t_{n-1})^2 \ ds. \end{cases}$$

The second equation minus 4 times the first equation results in

$$\begin{aligned} &\frac{3u(t_{n+1}) - 4u(t_n) + u(t_{n-1})}{2\tau} \\ &= \frac{1}{2\tau} \left( 2\tau \cdot \partial_t u(t_{n+1}) - 2 \int_{t_{n+1}}^{t_n} \partial_{ttt} u(s)(s-t_n)^2 \ ds \right. \\ &\left. + \frac{1}{2} \int_{t_{n+1}}^{t_{n-1}} \partial_{ttt} u(s)(s-t_{n-1})^2 \ ds \right) \\ &= \partial_t u(t_{n+1}) + \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \partial_{ttt} u(s)(s-t_n)^2 \ ds - \frac{1}{4\tau} \int_{t_{n-1}}^{t_{n+1}} \partial_{ttt} u(s)(s-t_{n-1})^2 \ ds \\ &= \nu \Delta u(t_{n+1}) - A\tau \left(u(t_{n+1}) - u(t_n)\right) - \Pi_N \left[2f(u(t_n)) - f(u(t_{n-1}))\right] \\ &+ A\tau \left(u(t_{n+1}) - u(t_n)\right) - \Pi_{>N} \left[2f(u(t_n)) - f(u(t_{n-1}))\right] \\ &+ 2f(u(t_n)) - f(u(t_{n-1})) - f(u(t_{n+1})) \\ &+ \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \partial_{ttt} u(s)(s-t_n)^2 \ ds - \frac{1}{4\tau} \int_{t_{n-1}}^{t_{n+1}} \partial_{ttt} u(s)(s-t_{n-1})^2 \ ds \ . \end{aligned}$$

Clearly we arrive at

$$\begin{aligned} G^n =& A\tau \left(u(t_{n+1}) - u(t_n)\right) - \Pi_{>N} \left[2f(u(t_n)) - f(u(t_{n-1}))\right] \\ &+ 2f(u(t_n)) - f(u(t_{n-1})) - f(u(t_{n+1})) \\ &+ \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \partial_{ttt} u(s)(s-t_n)^2 \ ds - \frac{1}{4\tau} \int_{t_{n-1}}^{t_{n+1}} \partial_{ttt} u(s)(s-t_{n-1})^2 \ ds \\ :=& I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

**Step 2:** We hereby estimate $\|I_1\|_2 \sim \|I_5\|_2$.

$I_1$: Apply the fundamental theorem of calculus,

$$\begin{aligned}\|I_1\|_2^2 =& \|A\tau\,(u(t_{n+1})-u(t_n))\,\|_2^2\\ \lesssim& \tau^2\|u(t_{n+1})-u(t_n)\|_2^2\\ \lesssim& \tau^2\|\int_{t_n}^{t_{n+1}}\partial_t u(s)\;ds\|_2^2\\ \lesssim& \tau^2\int_{\mathbb{T}^2}\left(\int_{t_n}^{t_{n+1}}\partial_t u(s)\;ds\right)^2\\ \lesssim& \tau^2\int_{\mathbb{T}^2}\left((\int_{t_n}^{t_{n+1}}|\partial_t u(s)|^2\;ds)^{1/2}\cdot\sqrt{\tau}\right)^2\\ \lesssim& \tau^2\cdot\tau\cdot\int_{t_n}^{t_{n+1}}\|\partial_t u(s)\|_2^2\;ds\\ \lesssim& \tau^3\int_{t_n}^{t_{n+1}}\|\partial_t u(s)\|_2^2\;ds\;.\end{aligned}$$

$I_2$: By the maximum principle Lemma 2.4 and $u\in L_t^\infty H_x^s$,

$$\begin{aligned}\|I_2\|_2 \lesssim& N^{-s}\cdot(\|f(u(t_n))\|_{H^s}+\|f(u(t_{n-1}))\|_{H^s})\\ \lesssim& N^{-s}\;.\end{aligned}$$

$I_3$: To bound $\|I_3\|_2$, we recall that for a one-variable function $h(t)$,

$$h(t)=h(t_0)+h'(t_0)(t-t_0)-\int_{t_0}^{t}h''(s)\cdot(s-t)\;ds\;.$$

Then we apply to $f(u(t_n))$,

$$\begin{cases} f(u(t_n))=f(u(t_{n+1}))-\partial_t(f(u))(t_{n+1})\cdot\tau+\displaystyle\int_{t_n}^{t_{n+1}}\partial_{tt}(f(u))\cdot(s-t_n)\;ds\\ f(u(t_{n-1}))=f(u(t_{n+1}))-\partial_t(f(u))(t_{n+1})\cdot 2\tau+\displaystyle\int_{t_{n-1}}^{t_{n+1}}\partial_{tt}(f(u))\cdot(s-t_{n-1})\;ds\;.\end{cases}$$

Then we subtract the second equation above by 2 times the first equation and derive:

$$\begin{aligned}&f(u(t_{n+1}))-2f(u(t_n))+f(u(t_{n-1}))\\ =&-2\int_{t_n}^{t_{n+1}}\partial_{tt}(f(u))\cdot(s-t_n)\;ds+\int_{t_{n-1}}^{t_{n+1}}\partial_{tt}(f(u))\cdot(s-t_{n-1})\;ds\;.\end{aligned}$$

As a result, we get

$$\begin{aligned}\|I_3\|_2^2 =& \|f(u(t_{n+1}))-2f(u(t_n))+f(u(t_{n-1}))\|_2^2\\ \lesssim& \|\int_{t_n}^{t_{n+1}}\partial_{tt}(f(u))\cdot(s-t_n)\;ds\|_2^2+\|\int_{t_{n-1}}^{t_{n+1}}\partial_{tt}(f(u))\cdot(s-t_{n-1})\;ds\|_2^2\\ \lesssim& \tau^2\cdot\|\int_{t_n}^{t_{n+1}}\partial_{tt}(f(u))\;ds\|_2^2+\tau^2\cdot\|\int_{t_{n-1}}^{t_{n+1}}\partial_{tt}(f(u))\;ds\|_2^2\\ \lesssim& \tau^3\int_{t_{n-1}}^{t_{n+1}}\|\partial_{tt}(f(u))\|_2^2\;ds\;,\end{aligned}$$

by a similar estimate in $I_1$.

$I_4$&$I_5$:

$$
\begin{aligned}
&\|I_4\|_2^2+\|I_5\|_2^2\\
\lesssim&\left\|\frac{1}{\tau}\int_{t_n}^{t_{n+1}}\partial_{ttt}u(s)(s-t_n)^2\ ds\right\|_2^2+\left\|\frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}}\partial_{ttt}u(s)(s-t_{n-1})^2\ ds\right\|_2^2\\
\lesssim&\left\|\frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}}\partial_{ttt}u(s)\cdot\tau^2\ ds\right\|_2^2\\
\lesssim&\tau^2\cdot\left\|\int_{t_{n-1}}^{t_{n+1}}\partial_{ttt}u(s)\ ds\right\|_2^2\\
\lesssim&\tau^3\int_{t_{n-1}}^{t_{n+1}}\|\partial_{ttt}u(s)\|_2^2\ ds\ .
\end{aligned}
$$

**Step 3:** Estimate $\tau\cdot\sum_{n=1}^{m-1}\|G^n\|_2^2$.
Collecting estimates above, we have

$$
\begin{aligned}
\tau\cdot\sum_{n=1}^{m-1}\|G^n\|_2^2=&\tau\cdot\sum_{n=1}^{m-1}(\|I_1\|_2^2+\|I_2\|_2^2+\|I_3\|_2^2+\|I_4\|_2^2+\|I_5\|_2^2)\\
&\lesssim m\tau\cdot N^{-2s}+\tau^4\cdot\int_0^{t_m}\|\partial_t u\|_2^2+\|\partial_{tt}(f(u))\|_2^2+\|\partial_{ttt}u\|_2^2\ d\tilde{s}\ .
\end{aligned}
$$

Note that by differentiating the original AC equation, we have

$$
\begin{cases}
\partial_{tt}u=\nu\partial_t\Delta u-\partial_t(f(u))\\
\partial_{ttt}u=\nu\partial_{tt}\Delta u-\partial_{tt}(f(u))\\
\partial_t(f(u))=f'(u)\partial_t u\\
\partial_{tt}(f(u))=f'(u)\partial_{tt}u+f''(u)(\partial_t u)^2\ ,
\end{cases}
$$

hence together with maximum principle and higher Sobolev bounds Lemma 2.5 one has

$$
\begin{aligned}
\tau\cdot\sum_{n=1}^{m-1}\|G^n\|_2^2&\lesssim m\tau\cdot N^{-2s}+\tau^4\cdot\int_0^{t_m}\|\partial_t u\|_2^2+\|\partial_{tt}(f(u))\|_2^2+\|\partial_{ttt}u\|_2^2\ d\tilde{s}\\
&\lesssim t_m\cdot N^{-2s}+\tau^4\cdot\int_0^{t_m}\|u\|_{H^s}^2\ d\tilde{s}\\
&\lesssim t_m\cdot N^{-2s}+\tau^4\cdot(1+t_m)\\
&\lesssim(1+t_m)\cdot(\tau^4+N^{-2s})\ .
\end{aligned}
$$

This completes the proof of Lemma 3.4. □

3.4.3. **Proof of $L^2$ error estimate of second order scheme I (1.3)**

Note that the assumptions in Proposition 3.1 are satisfied by the unconditional Theorem 1.1 and the maximum principle of the Allen-Cahn equation. Thus we apply the auxiliary estimate Proposition 3.1. Then

$$
\|u(t_m)-u^m\|_2^2\lesssim e^{Cm\tau}\cdot\left((1+A\tau^2)\|u^1-v^1\|_2^2+\|u^0-v^0\|_2^2+\tau\sum_{n=1}^{m-1}\|G^n\|_2^2\right)\ .
$$

By Lemma 3.2 and Lemma 3.4,

$$
\begin{aligned}
\|u(t_m)-u^m\|_2^2&\lesssim e^{Cm\tau}\cdot\left((1+A\tau^2)\|u^1-v^1\|_2^2+\|u^0-v^0\|_2^2+\tau\sum_{n=1}^{m-1}\|G^n\|_2^2\right)\\
&\lesssim e^{Ct_m}\cdot\left((1+A\tau^2)(N^{-2s}+\tau^4)+N^{-2s}+(1+t_m)\cdot(\tau^4+N^{-2s})\right)\\
&\lesssim e^{Ct_m}\cdot(N^{-2s}+\tau^4)\ .
\end{aligned}
$$

Thus for $m \geq 2$,

$$\|u(t_m) - u^m\|_2 \lesssim e^{Ct_m} \cdot (N^{-s} + \tau^2) \ .$$

**Remark 3.5.** *For the error estimate, we actually do not need high regularity of the initial data because of a smoothing effect of Allen-Cahn equation.*

## 4. Second order semi-implicit scheme II

In this section, we recall the second order scheme II:

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} = \nu\Delta u^{n+1} - A(u^{n+1} - 2u^n + u^{n-1}) - \Pi_N\left(2f(u^n) - f(u^{n-1})\right), \quad (4.1)\ \{?\}$$

where $\tau > 0$ is the time step and $n \geq 1$. We again need to derive $u^1$ according to the following first order scheme:

$$\begin{cases} \dfrac{u^1 - u^0}{\tau_1} = \nu\Delta u^1 - \Pi_N f(u^0) \ , \\ u^0 = \Pi_N u_0 \ , \end{cases} \quad (4.2)\ \{?\}$$

where $\tau_1 = \min\{\tau^{\frac{4}{3}}\ ,\ 1\ ,\ \frac{1}{\sqrt{A+1}}\}$. The choice of such $\tau$ is to guarantee the error estimate as mentioned in Section 3.4, and to ensure that the new modified energy function can be controlled by the initial data.

### 4.1. **Estimate of the First Order Scheme (1.6)**

In this section we will still estimate some bounds of $u^1$ which will be used to prove the stability of the second order scheme and it will be slightly different from scheme (1.4).

⟨2ndSchemem_II_lem1⟩ **Lemma 4.1.** *Consider the scheme (1.6).*

$$\begin{cases} \dfrac{u^1 - u^0}{\tau_1} = \nu\Delta u^1 - \Pi_N f(u^0) \ , \\ u^0 = \Pi_N u_0 \ , \end{cases}$$

*where* $\tau_1 = \min\{\tau^{\frac{4}{3}}\ ,\ 1\ ,\ \frac{1}{\sqrt{A+1}}\}$. *Assume* $u_0 \in H^2(\mathbb{T}^2)$, *then*

$$\begin{cases} E(u^1) + \dfrac{\|u^1 - u^0\|_2^2}{\tau_1} \lesssim_{E(u_0), \|u_0\|_{H^2}} 1 \\ (1+A)\|u^1 - u^0\|_2^2 \lesssim_{\|u_0\|_{H^2}} (1+\nu)^2 \ . \end{cases}$$

*Proof.* The first inequality shares the same proof as in previous section since the scheme (1.6) is a refined version of (1.4).

For the second inequality, recall that $\|u^1\|_{H^2} \lesssim_{\|u_0\|_{H^2}}$. We get

$$\frac{1}{\tau_1}\|u^1 - u^0\|_2 \leq \nu\|u^1\|_{H^2} + \|f(u^0)\|_2 \lesssim_{\|u_0\|_{H^2}} 1 + \nu \ .$$

This implies

$$(A+1)\|u^1 - u^0\|_2^2 \lesssim_{\|u_0\|_{H^2}} (1+\nu)^2 \ .$$

□

### 4.2. **Conditional stability of the second order scheme II (1.5) & (1.6)**

In this section we will prove Theorem 1.4 for the second order scheme (1.5) & (1.6). Before proving this stability theorem, we begin with several lemmas.

⟨2ndSchemem_II_lem2⟩ **Lemma 4.2.** *Consider (1.5) for $n \geq 1$. Suppose $E(u^n) \leq B \cdot (1+\nu)^2$ and $E(u^{n-1}) \leq B \cdot (1+\nu)^2$ for some $B > 0$. Then*

$$\|u^{n+1}\|_\infty \leq \alpha_B \cdot \left\{(\nu^{\frac{1}{2}} + \nu^{-1}) \cdot \sqrt{1 + \log(A+1) + |\log\nu| + |\log\tau|} + 1\right\} \ ,$$

*for some $\alpha_B > 0$ only depending on $B$.*

*Proof.* For simplicity we write $\lesssim$ instead of $\lesssim_B$. Note that by the energy estimates,

$$\|\nabla u^{n-1}\|_2 + \|\nabla u^n\|_2 \lesssim \nu^{-\frac{1}{2}}(1+\nu)\ ,\ \|u^{n-1}\|_4 + \|u^n\|_4 \lesssim (1+\nu)^{\frac{1}{2}}\ .$$

We rewrite the scheme (1.5) as

$$\begin{aligned} u^{n+1} =& \frac{4+4A\tau}{3+2A\tau-2\nu\tau\Delta}u^n - \frac{1+2A\tau}{3+2A\tau-2\nu\tau\Delta}u^{n-1} \\ &- \frac{2\tau\Pi_N}{3+2A\tau-2\nu\tau\Delta}\left(2f(u^n)-f(u^{n-1})\right)\ . \end{aligned}$$

For Fourier mode $k=0$, we have

$$\begin{cases} \dfrac{4+4A\tau}{3+2A\tau} \lesssim 1 \\ \dfrac{1+2A\tau}{3+2A\tau} \lesssim 1 \\ \dfrac{2\tau}{3+2A\tau} \lesssim \dfrac{1}{A} \lesssim 1\ . \end{cases}$$

Thus

$$|\widehat{u^{n+1}}(0)| \lesssim 1\ .$$

For $|k| \geq 1$,

$$\begin{cases} \dfrac{4+4A\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim 1 \\ \dfrac{1+2A\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim 1 \\ \dfrac{2\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim \dfrac{1}{\nu|k|^2}\ . \end{cases}$$

This implies

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^1} \lesssim& \|u^n\|_{\dot{H}^1} + \|u^{n-1}\|_{\dot{H}^1} + \frac{1}{\nu}\|\langle\nabla\rangle^{-1}(2f(u^n)-f(u^{n-1}))\|_2 \\ \lesssim& \nu^{-\frac{1}{2}}(1+\nu) + \nu^{-1}\cdot\left(\|(u^n)^3\|_{4/3} + \|(u^{n-1})^3\|_{4/3} + \|u^n\|_2 + \|u^{n-1}\|_2\right) \\ \lesssim& \nu^{-\frac{1}{2}}(1+\nu) + \nu^{-1}\cdot\left((1+\nu)^{\frac{3}{2}} + (1+\nu)^{\frac{1}{2}}\right) \\ \lesssim& \nu^{-1} + \nu^{\frac{1}{2}}\ . \end{aligned}$$

Similarly, we can derive that

$$\begin{cases} \dfrac{4+4A\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim \left(\dfrac{1}{\nu\tau} + \dfrac{A}{\nu}\right)\cdot\dfrac{1}{|k|^2} \\ \dfrac{1+2A\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim \left(\dfrac{1}{\nu\tau} + \dfrac{A}{\nu}\right)\cdot\dfrac{1}{|k|^2} \\ \dfrac{2\tau}{3+2A\tau+2\nu\tau|k|^2} \lesssim \dfrac{1}{\nu|k|^2}\ . \end{cases}$$

Thus by the standard Sobolev inequality

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^2} \lesssim& \left(\frac{1}{\nu\tau} + \frac{A}{\nu}\right)\cdot\left(\|u^n\|_2 + \|u^{n-1}\|_2\right) + \frac{1}{\nu}\|(2f(u^n)-f(u^{n-1}))\|_2 \\ \lesssim& \left(\frac{1}{\nu\tau} + \frac{A}{\nu}\right)\cdot(1+\nu)^{\frac{1}{2}} + \nu^{-1}\left(\|u^n\|_6^3 + \|u^{n-1}\|_6^3 + \|u^n\|_2 + \|u^{n-1}\|_2\right) \\ \lesssim& \left(\frac{1}{\nu\tau} + \frac{A}{\nu}\right)\cdot(1+\nu)^{\frac{1}{2}} + \nu^{-1}(\nu^{-\frac{3}{2}}(1+\nu)^3 + (1+\nu)^{\frac{1}{2}}) \\ \lesssim& \frac{1}{\nu\tau} + \frac{A}{\nu} + \frac{1}{\nu^{\frac{1}{2}}\tau} + \frac{A+1}{\nu^{\frac{1}{2}}} + \nu^{-\frac{5}{2}} + \nu^{\frac{1}{2}}\ . \end{aligned}$$

As a result, by the log interpolation Lemma 2.6 again, we get

$$\|u^{n+1}\|_\infty \lesssim (\nu^{\frac{1}{2}} + \nu^{-1}) \cdot \sqrt{1 + \log(A+1) + |\log \nu| + |\log \tau|} + 1 \ .$$

□

### 4.3. **Proof of the conditional stability**

To prove Theorem 1.4, we first recall that

$$\begin{aligned} \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} &- \nu \Delta u^{n+1} + A(u^{n+1} - 2u^n + u^{n-1}) \\ &= -\Pi_N \left(2f(u^n) - f(u^{n-1})\right) . \end{aligned} \tag{4.3}$$ eq:2ndL2est

We apply the $L^2$ inner product with $\delta u^{n+1} = u^{n+1} - u^n$ on both sides of (4.3). Recall that

$$\begin{aligned} &\left(3u^{n+1} - 4u^n + u^{n-1} \ , \ u^{n+1} - u^n\right) \\ &= 2\|\delta u^{n+1}\|_2^2 + \frac{1}{2}\left(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2\right) \ . \end{aligned}$$

Applying this equation, we derive the estimate of the left hand side of (4.3) after taking inner products:

$$\begin{aligned} \text{LHS} =& \frac{1}{\tau}\|\delta u^{n+1}\|_2^2 + \frac{1}{4\tau}\left(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2\right) \\ &+ \frac{\nu}{2}\left(\|\nabla u^{n+1}\|_2^2 - \|\nabla u^n\|_2^2 + \|\delta \nabla u^{n+1}\|_2^2\right) \\ &+ \frac{A}{2}\left(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2\right) \\ \geq& \left[\frac{\nu}{2}\|\nabla u^{n+1}\|_2^2 + \frac{1}{4\tau}\|\delta u^{n+1}\|_2^2 + \frac{A}{2}\|\delta u^{n+1}\|_2^2\right] \\ &- \left[\frac{\nu}{2}\|\nabla u^n\|_2^2 + \frac{1}{4\tau}\|\delta u^n\|_2^2 + \frac{A}{2}\|\delta u^n\|_2^2\right] \\ &+ \frac{1}{\tau}\|\delta u^{n+1}\|_2^2 + \frac{A}{2}\|\delta^2 u^{n+1}\|_2^2 \ . \end{aligned}$$

Now it remains to estimate the right hand side of (4.3) after taking inner products:

$$\begin{aligned} \text{RHS} =& -\left(2f(u^n) - f(u^{n-1}) \ , \ \delta u^{n+1}\right) \\ =& \left(2u^n - u^{n-1} \ , \ \delta u^{n+1}\right) + \left((u^{n-1})^3 - 2(u^n)^3 \ , \ \delta u^{n+1}\right) \\ :=& I_1 + I_2 \ . \end{aligned}$$

To estimate $I_1$:

$$\begin{aligned} I_1 =& \left(-\delta^2 u^{n+1} \ , \ \delta u^{n+1}\right) + \left(u^{n+1} \ , \ \delta u^{n+1}\right) \\ =& -\frac{1}{2}\left(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2\right) \\ &+ \frac{1}{2}\left(\|u^{n+1}\|_2^2 - \|u^n\|_2^2 + \|\delta u^{n+1}\|_2^2\right) \ . \end{aligned}$$

For $I_2$, we use the identity $u^{n-1} = \delta^2 u^{n+1} + 2u^n - u^{n+1}$, then

$$\begin{aligned} (u^{n-1})^3 - 2(u^n)^3 =& (\delta^2 u^{n+1} + 2u^n - u^{n+1})^3 - 2(u^n)^3 \\ =& (\delta^2 u^{n+1})^3 + 3(\delta^2 u^{n+1})^2(2u^n - u^{n+1}) + 3\delta^2 u^{n+1}(2u^n - u^{n+1})^2 \\ &+ (2u^n - u^{n+1})^3 - 2(u^n)^3 \ . \end{aligned}$$

Note that

$$\begin{aligned} 3\delta^2 u^{n+1}(2u^n - u^{n+1})^2 &= 3\delta^2 u^{n+1}(\delta^2 u^{n+1} - u^{n-1})^2 \\ &= 3(\delta^2 u^{n+1})^3 - 6(\delta^2 u^{n+1})^2 u^{n-1} + 3\delta^2 u^{n+1}(u^{n-1})^2 \ . \end{aligned}$$

As a result, we get

$$\begin{aligned}
&(u^{n-1})^3 - 2(u^n)^3 \\
=&4(\delta^2 u^{n+1})^3 + (\delta^2 u^{n+1})^2 (6u^n - 3u^{n+1} - 6u^{n-1}) \\
&+ 3\delta^2 u^{n+1}(u^{n-1})^2 + (2u^n - u^{n+1})^3 - 2(u^n)^3 \\
=&(\delta^2 u^{n+1})^2 \cdot \left[4(u^{n+1} - 2u^n + u^{n-1}) + 6u^n - 3u^{n+1} - 6u^{n-1}\right] \\
&+ 3\delta^2 u^{n+1}(u^{n-1})^2 + 6(u^n)^3 - 12(u^n)^2 u^{n+1} + 6u^n (u^{n+1})^2 - (u^{n+1})^3 \\
=&(\delta^2 u^{n+1})^2 \cdot (u^{n+1} - 2u^n - 2u^{n-1}) + 3\delta^2 u^{n+1}(u^{n-1})^2 \\
&+ 6u^n (u^{n+1} - u^n)^2 - (u^{n+1})^3 \ .
\end{aligned}$$

Therefore,

$$\begin{aligned}
|I_2| \leq &\|\delta^2 u^{n+1}\|_\infty \cdot \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \\
&\cdot \left(\|u^{n+1}\|_\infty + 2\|u^n\|_\infty + 2\|u^{n-1}\|_\infty\right) \\
&+ \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \cdot 3\|u^{n-1}\|_\infty^2 \\
&+ \left((\delta u^{n+1})^2 \ , \ 6u^n (u^{n+1} - u^n)\right) - \left((u^{n+1})^3 \ , \ \delta u^{n+1}\right) \ .
\end{aligned}$$

Now note that

$$\begin{aligned}
&\frac{(u^n)^4}{4} \\
=&\frac{1}{4}(u^{n+1} - \delta u^{n+1})^4 \\
=&\frac{1}{4}\left[(u^{n+1})^4 - 4(u^{n+1})^3 \delta u^{n+1} + 6(u^{n+1})^2 (\delta u^{n+1})^2 - 4u^{n+1}(\delta u^{n+1})^3 + (\delta u^{n+1})^4\right] \\
=&\frac{(u^{n+1})^4}{4} - (u^{n+1})^3 \delta u^{n+1} + \frac{1}{4}(\delta u^{n+1})^2 \left[6(u^{n+1})^2 - 4u^{n+1}(u^{n+1} - u^n) + (u^{n+1} - u^n)^2\right] \\
=&\frac{(u^{n+1})^4}{4} - (u^{n+1})^3 \delta u^{n+1} + \frac{(\delta u^{n+1})^2}{4}\left[(u^n)^2 + 2u^n u^{n+1} + 3(u^{n+1})^2\right] \ .
\end{aligned}$$

Applying this identity we have

$$\begin{aligned}
&\left((\delta u^{n+1})^2 \ , \ 6u^n (u^{n+1} - u^n)\right) - \left((u^{n+1})^3 \ , \ \delta u^{n+1}\right) \\
=&\int_{\mathbb{T}^2} \frac{(u^n)^4}{4} - \int_{\mathbb{T}^2} \frac{(u^{n+1})^4}{4} - \left((\delta u^{n+1})^2 \ , \ \frac{25}{4}(u^n)^2 - \frac{11}{2}u^n u^{n+1} + \frac{3}{4}(u^{n+1})^2\right) \\
=&\int_{\mathbb{T}^2} \frac{(u^n)^4}{4} - \int_{\mathbb{T}^2} \frac{(u^{n+1})^4}{4} - \left((\delta u^{n+1})^2 \ , \ \frac{25}{4}(u^n - \frac{11}{25}u^{n+1})^2\right) \\
&+ \frac{23}{50}\left((\delta u^{n+1})^2 \ , \ (u^{n+1})^2\right) \ .
\end{aligned}$$

Observe that

$$\|\delta^2 u^{n+1}\|_\infty \leq 4\max\left\{\|u^{n-1}\|_\infty, \ \|u^n\|_\infty, \ \|u^{n+1}\|_\infty\right\},$$

$$\begin{aligned}
&\text{RHS} \\
&\leq -\frac{1}{4}\|u^{n+1}\|_4^4 + \frac{1}{2}\|u^{n+1}\|_2^2 - \frac{1}{2}\|\delta u^{n+1}\|_2^2 \\
&\quad + \frac{1}{4}\|u^n\|_4^4 - \frac{1}{2}\|u^n\|_2^2 + \frac{1}{2}\|\delta u^n\|_2^2 \\
&\quad - \frac{1}{2}\|\delta^2 u^{n+1}\|_2^2 + \|\delta u^{n+1}\|_2^2 \cdot \left(\frac{1}{2} + \frac{23}{50}\|u^{n+1}\|_\infty^2\right) \\
&\quad + \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \cdot 23\max\left\{\|u^{n-1}\|_\infty^2, \ \|u^n\|_\infty^2, \ \|u^{n+1}\|_\infty^2\right\} \ .
\end{aligned}$$

Recall that

$$E(u) = \frac{\nu}{2}\|\nabla u\|_2^2 + \frac{1}{4}\|u\|_4^4 - \frac{1}{2}\|u\|_2^2 + \frac{1}{4}\cdot\mu(\mathbb{T}^2) \ ,$$

where $\mu(\mathbb{T}^2)$ is the measure of $\mathbb{T}^2$. Hence by comparing the LHS and RHS, we get

$$
\begin{aligned}
&E(u^{n+1}) + \frac{1}{4\tau}\|\delta u^{n+1}\|_2^2 + \frac{A+1}{2}\|\delta u^{n+1}\|_2^2 \\
\le & E(u^n) + \frac{1}{4\tau}\|\delta u^n\|_2^2 + \frac{A+1}{2}\|\delta u^n\|_2^2 \\
&+ \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \cdot 23 \max\left\{\|u^{n-1}\|_\infty^2,\ \|u^n\|_\infty^2,\ \|u^{n+1}\|_\infty^2\right\} \\
&- \left\{\frac{A+1}{2}\|\delta^2 u^{n+1}\|_2^2 + \left(\frac{1}{\tau} - \frac{1}{2} - \frac{23}{50}\|u^{n+1}\|_\infty^2\right) \cdot \|\delta u^{n+1}\|_2^2\right\} .
\end{aligned}
$$

To show the desired energy decay, it suffices to require

$$
\begin{aligned}
2(A+1)&\left(\frac{1}{\tau} - \frac{1}{2} - \frac{23}{50}\|u^{n+1}\|_\infty^2\right) \\
&\ge 529 \max\left\{\|u^{n-1}\|_\infty^4,\ \|u^n\|_\infty^4,\ \|u^{n+1}\|_\infty^4\right\} .
\end{aligned}
\tag{4.4}
$$

2ndScheme_II_cond

We will prove it inductively as in the previous section. Set

$$
B = \max\left\{\mathring{E}(u^1)\ ,\ E(u^0)\right\} ,
$$

by Lemma 4.1 in the previous section, $B \lesssim 1$. We shall prove for every $m \ge 2$,

$$
\begin{cases}
\mathring{E}(u^m) \le B \cdot (1+\nu)^2\ ,\ \mathring{E}(u^m) \le \mathring{E}(u^{m-1})\ , \\
\|u^m\|_\infty \le \alpha_B \cdot \left[(\nu^{\frac{1}{2}} + \nu^{-1}) \cdot \sqrt{1 + \log(A+1) + |\log \nu| + |\log \tau|} + 1\right] ,
\end{cases}
$$

where $\alpha_B > 0$ is the same constant in Lemma 4.2. Then it suffices to verify the main inequality (4.4):

$$
\begin{aligned}
2(A+1)&\left(\frac{1}{\tau} - \frac{1}{2} - C_1 - C_1 \cdot (\nu^{-2} + \nu) \cdot (1 + \log(A+1) + |\log \nu| + |\log \tau|)\right) \\
&> C_2(\nu^{-4} + \nu^2) \cdot \left(1 + |\log(A+1)|^2 + |\log \nu|^2 + |\log \tau|^2\right) + C_2 .
\end{aligned}
$$

We consider two cases.

`Case 1:` $A = 0$. Then we need

$$
\frac{1}{\tau} \gg (\nu^{-4} + \nu^2) \cdot \left(1 + |\log(A+1)|^2 + |\log \nu|^2 + |\log \tau|^2\right) + 1 .
$$

If $0 < \nu < 1$, we require that

$$
\tau \ll \frac{\nu^4}{1 + |\log \nu|^2} ;
$$

If $\nu \ge 1$, we then require that

$$
\tau \ll \frac{\nu^{-2}}{1 + |\log \nu|^2} .
$$

`Case 2:` $A = const \cdot (\nu^2 + \nu^{-4})$. In this case,

$$
\frac{1}{\tau} \gg (\nu^{-2} + \nu) \cdot (1 + |\log(A+1)| + |\log \nu| + |\log \tau|) + |\log \nu|^2 + |\log \tau|^2 + 1 .
$$

If $0 < \nu < 1$, we need

$$
\tau \ll \frac{\nu^2}{1 + |\log \nu|} ;
$$

If $\nu \ge 1$, then we need

$$
\tau \ll \frac{\nu^{-1}}{1 + |\log \nu|} .
$$

This completes the proof.

## 5. Concluding remark

Throughout this paper, we discussed two second order semi-implicit Fourier spectral methods on the Allen-Cahn equation in the two dimensional torus. We proved the stability (energy decay) of the schemes by adding stabilizing terms with a large constant $A$. We also proved the $L^2$ error estimate between numerical solutions from the semi-implicit scheme and the exact solutions. Future work can be done in other gradient cases such as general non-local Cahn-Hilliard equations, MBE equations, and other equations describing interesting phenomena in material sciences.

## 6. Numerical results

Several numerical results are given below. For the Scheme I, we perform the following numerical simulation. In Figure 1, we choose $\nu = 0.1$, A=1, $u_0 = \sin(x)\sin(y)$, $\tau = 0.01$, $N_x = N_y = 256$.

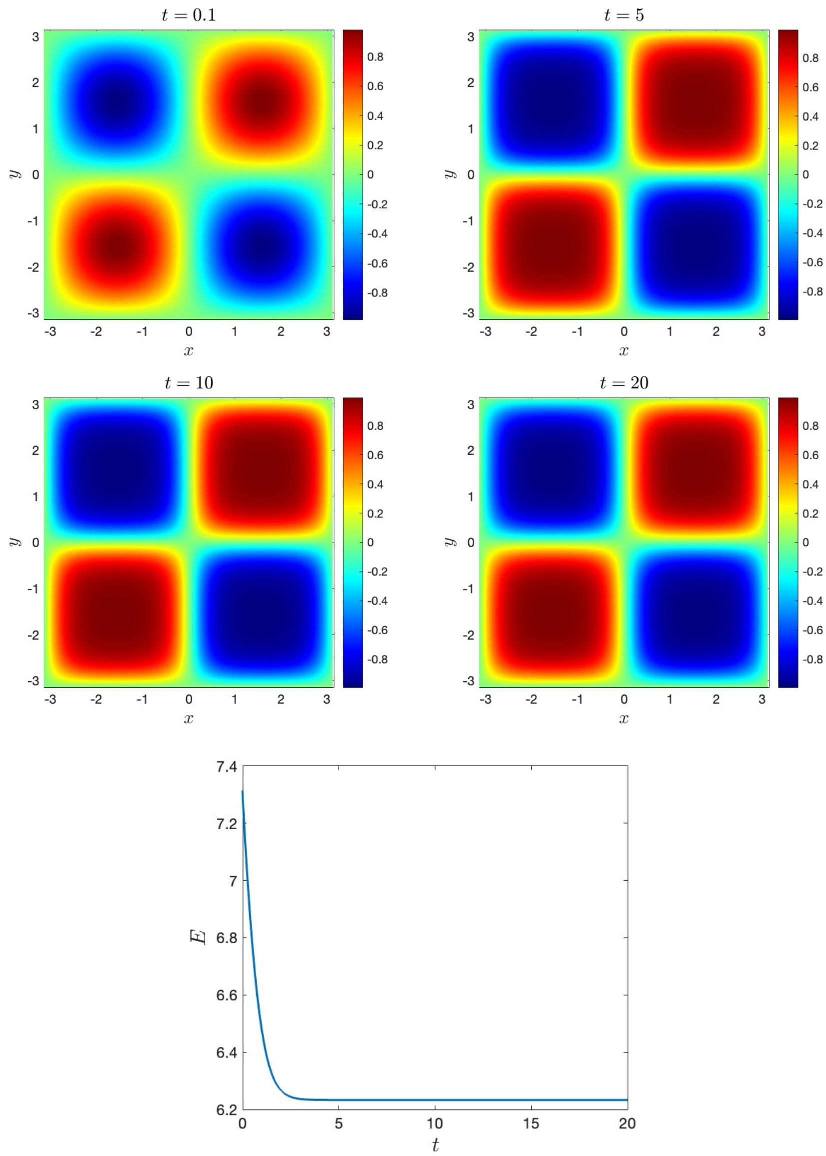


Figure 1. Dynamics of 2D Allen-Cahn equation using semi-implicit scheme I where $\nu = 0.1$, A=1, $u_0 = \sin(x)\sin(y)$, $\tau = 0.01$, $N_x = N_y = 256$.

⟨fig:2do1AC⟩

The next numerical simulation is shown in Figure 2. In this simulation we choose $\nu = 0.01$, A=1, $\tau = 0.01$, $N_x = N_y = 256$ and the initial data $u_0$ is given basically supported in several circles as below:

$$u_0(x,y) = -1 + \sum_{i=1}^{7} f_0\left(\sqrt{(x-x_i)^2+(y-y_i)^2} - r_i\right), \tag{6.1}$$ 3.6

where

$$f_0(s) = \begin{cases} 2e^{-\frac{\nu}{s^2}}, & \text{if } s<0; \\ 0, & \text{otherwise.} \end{cases}$$

The centers and radii of the chosen circles are given in the table 1 below.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x_i$ | $-\frac{\pi}{2}$ | $-\frac{3\pi}{4}$ | $-\frac{\pi}{2}$ | $0$ | $\frac{\pi}{2}$ | $0$ | $\frac{\pi}{2}$ |
| $y_i$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{4}$ | $\frac{\pi}{4}$ | $-\frac{3\pi}{4}$ | $-\frac{3\pi}{4}$ | $0$ | $\frac{\pi}{2}$ |
| $r_i$ | $\frac{\pi}{5}$ | $\frac{2\pi}{15}$ | $\frac{2\pi}{15}$ | $\frac{\pi}{10}$ | $\frac{\pi}{10}$ | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ |

TABLE 1. Choice of centers and radii

⟨table:1⟩

For Scheme II, an numerical result is given in Figure 3 where we choose $\nu = 0.1$, A=1, $u_0 = \sin(x)\sin(y)$, $\tau = 0.01$, $N_x = N_y = 256$.

We also provide a similar simulation using the same $u_0$ in (6.1) as a comparison to Scheme I. The numerical simulation can be seen below in Figure 4 where we also choose $\nu = 0.01$, A=1, $\tau = 0.01$, $N_x = N_y = 256$.

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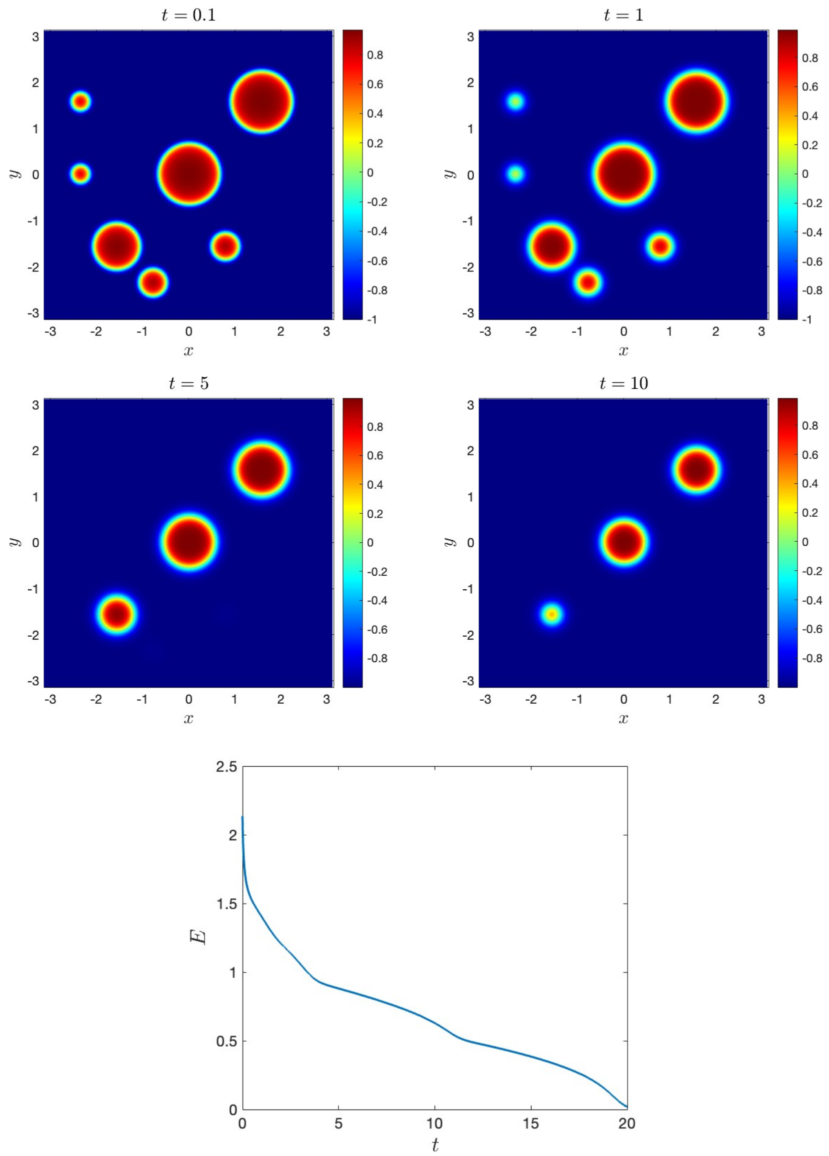


FIGURE 2. Dynamics of 2D Allen-Cahn equation using semi-implicit scheme I where $\nu = 0.01$, A=1, $\tau = 0.01$, $N_x = N_y = 256$ and the initial data $u_0$ is given in (6.1).

⟨fig:ACIcircle⟩

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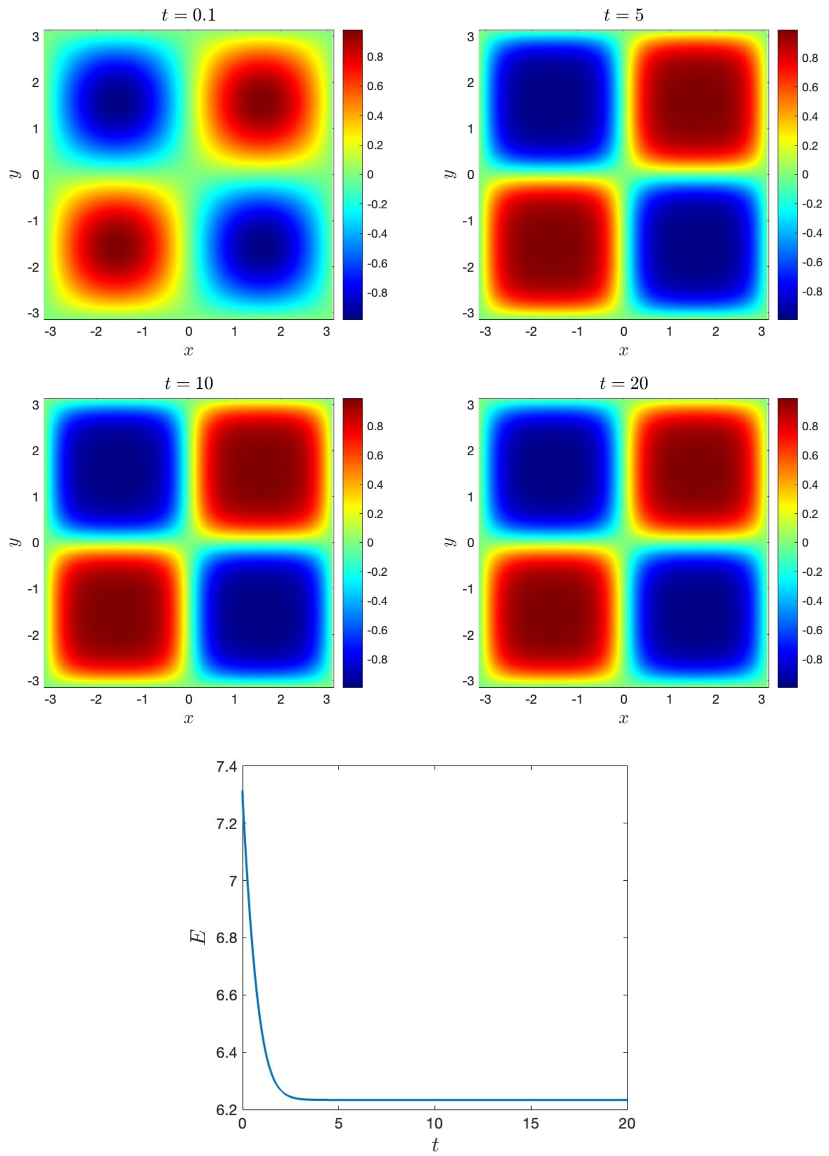


FIGURE 3. Dynamics of 2D Allen-Cahn equation using semi-implicit scheme II where $\nu = 0.1$, A=1, $u_0 = \sin(x)\sin(y)$, $\tau = 0.01$, $N_x = N_y = 256$.

⟨fig:2do2AC⟩


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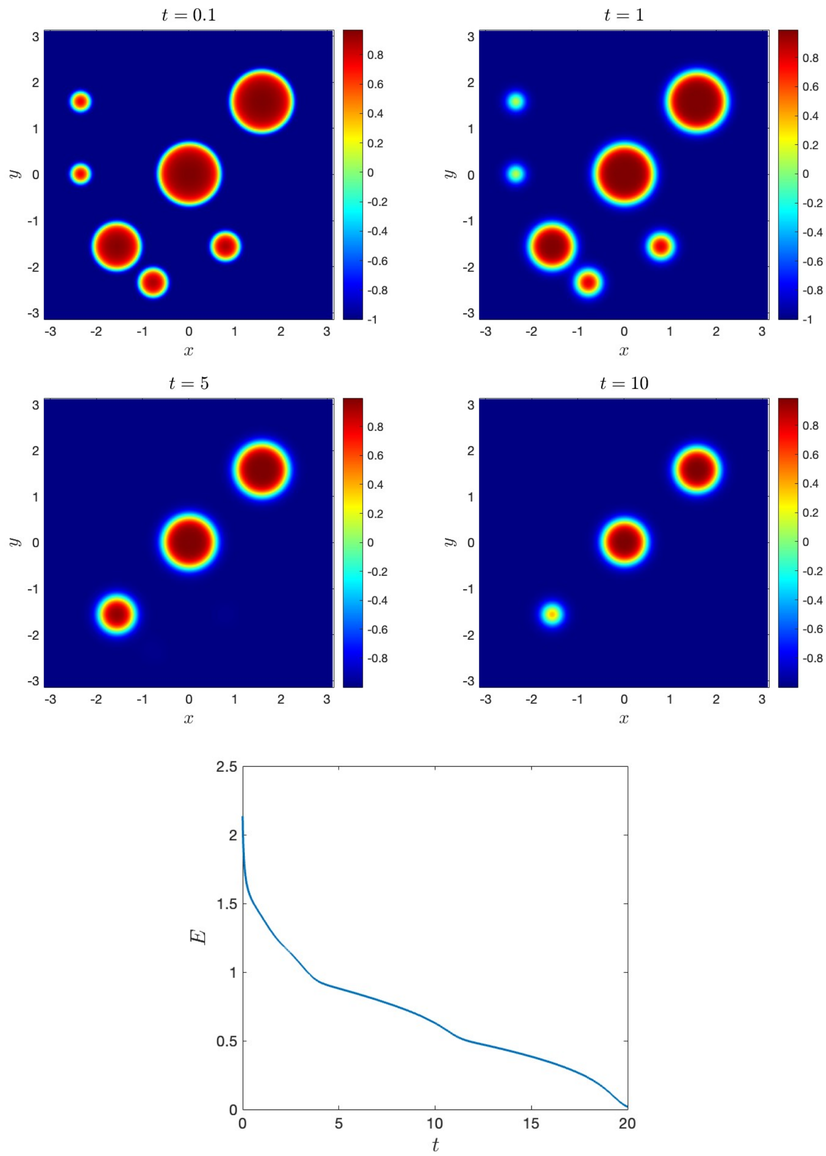


FIGURE 4. Dynamics of 2D Allen-Cahn equation using semi-implicit scheme II where $\nu = 0.01$, A=1, $\tau = 0.01$, $N_x = N_y = 256$ and the initial data $u_0$ is given in (6.1).

⟨fig:ACIIcircle⟩

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