ENERGY STABLE SEMI-IMPLICIT SCHEMES FOR THE 2D ALLEN-CAHN AND FRACTIONAL CAHN-HILLIARD EQUATIONS

XINYU CHENG

Abstract. In this work we are interested in a class of numerical schemes for certain phase field models. It is well known that unconditional energy stability (energy decays in time regardless of the size of the time step) provides a fidelity check in practical numerical simulations. In recent work [24], a type of semi-implicit schemes for the Cahn-Hilliard equation was developed satisfying the energy-decay property, in this paper we extend such semi-implicit schemes to the Allen-Cahn equation and the fractional Cahn-Hilliard equation with a rigorous proof of similar energy stability. Models in 2 spatial dimensions are discussed.

1. Introduction

1.1. Introduction to the models and historical review

In this work we consider two classic phase field models: Allen-Cahn (AC) and Cahn-Hilliard (CH) equations. The (AC) model was developed in [3] by Allen and Cahn to study the competition of crystal grain orientations in an annealing process separation of different metals in a binary alloy; while the (CH) was introduced in [9] by Cahn and Hilliard to describe the process of phase separation of different metals in a binary alloy. These equations are presented as:

$$\begin{cases} \partial_t u = \nu\Delta u - f(u), \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0, \end{cases} \tag{AC} \boxed{\texttt{AC}}$$

and

$$\begin{cases} \partial_t u = \Delta(-\nu\Delta u + f(u)), \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0, \end{cases} \tag{CH} \boxed{\texttt{CH}}$$

where $u(x,t)$ is a real valued function and values of u in $(-1,1)$ represent a mixture of the two phases, with -1 representing the pure state of one phase and $+1$ representing the pure state of the other phase. Vector position x is in the spatial domain Ω, which is oftentimes taken to be two or three dimensional periodic domain and t is the time variable. Here ν is a small parameter, occasionally we denote $\varepsilon = \sqrt{\nu}$ to represent an average distance over which phases mix. The energy term $f(u)$ is often chosen to be

$$f(u) = F'(u) = u^3 - u \ , \ F(u) = \frac{1}{4}(u^2-1)^2.$$

It is well known that, as $\varepsilon \to 0$, the limiting problem of (AC) is given by a mean curvature flow while the limiting problem of (CH) becomes Mullins-Sekerka problem; we refer to [22] for AC and [31], [2] for CH and a recent work for matrix-valued AC [18]. Both asymptotic and rigorous analysis are well-studied. Although the limiting behavior of AC and CH are well known, there are related materials science models that are studied only numerically and this current work presents an idea about how to approach these models in an appropriate way numerically.

In this paper, we consider the spatial domain Ω to be the two dimensional 2π-periodic torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. In fact, our proof can be applied to more general settings such as Dirichlet and Neumann boundary conditions in a bounded domain. However, considering the periodic domain allows the use of efficient and accurate Fourier-spectral numerical methods; moreover, periodic domain is often appropriate for application questions, which involve the formation of micro-structure away from physical boundaries.

As is well known, the mass of the smooth solution of Cahn-Hilliard (CH) is conserved, i.e. $\frac{d}{dt}M(t) \equiv 0$, $M(t) = \int_\Omega u(x,t)\ dx$. This represents the conservation of the two phases in the mixture. In particular, $M(t) \equiv 0$ if $M(0) = 0$ and hence oftentimes zero-mean initial data (equal amounts of both phases) will be considered as a simpler but representative case. The associated energy functional of (CH) is given by

$$E(u) = \int_\Omega \left(\frac{1}{2}\nu |\nabla u|^2 + F(u) \right) \, dx. \tag{1.1}$$ energy

Assume that $u(x,t)$ is a smooth solution with zero mean, one can deduce

$$\frac{d}{dt}E(u(t)) + \int_\Omega |\nabla(-\nu\Delta u + f(u))|^2 \, dx = 0,$$

which implies the decay of the energy: $\frac{d}{dt}E(u(t)) \le 0$. This thus provides an *a priori* H^1-norm bound and since the scaling-critical space for (CH) is L^2 in 2D and $H^{\frac{1}{2}}$ in 3D, the global well-posedness and regularity of solutions follow from standard arguments. In this sense, the energy decay property is an important index for whether a numerical scheme is "stable" or not. In comparison, Allen-Cahn equation does not share the mass conservation property; however, it still follows the energy decay property with the same energy functional. Moreover, the solution to the fractional Cahn-Hilliard equation (FCH) satisfies both mass conservation and energy properties, see [1, 6] for example. The fractional Cahn-Hilliard equation (FCH) is defined as the following:

$$\begin{cases} \partial_t u = \nu\Delta\left((-\Delta)^\alpha u + (-\Delta)^{\alpha-1} f(u)\right) \ , \ 0 < \alpha \le 1 \\ u(x,0) = u_0 \ . \end{cases} \tag{FCH}$$ frac_eq

The difficulty in dealing with this FCH model arises from the non-local behavior of the fractional laplacian, where the fractional laplacian on the torus is given from the Fourier side: for $x \in \mathbb{T}^d$, $(-\Delta)^\alpha f(x) = \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d} |k|^{2\alpha}\hat{f}(k)e^{-ik\cdot x}$. The convention of Fourier series is given in the next section.

Various approaches have been developed to study numerical simulations on Cahn-Hilliard and related models, [15, 20, 21, 35, 19, 10, 36, 5, 17, 26, 27, 28, 4, 23, 13, 14] as examples, in which different approaches are applied to the time stepping including fully explicit (forward Euler) scheme, fully implicit (backward Euler) scheme, finite element scheme, convex splitting scheme and operator splitting schemes. Different strategies are adopted for the spatial discretization including the Fourier-spectral method. All these numerical approximations give accurate results to the values and qualitative features of the solution. One of the key features is the energy dissipation.

We hereby give a list of work in the historical review. From the analysis point of view, Feng and Prohl [19] introduced a semi-discrete in time and fully spatially discrete finite element method for Cahn-Hilliard equation(CH) where they obtained an error bound of size of powers of $1/\nu$. Explicit time-stepping schemes require strict time-step restrictions and do not obey energy decay in general. To guarantee the energy decay property and increase the time step, a good alternative is to use semi-implicit schemes in which the linear term is implicit (such as backward time differentiation) and the nonlinear term is treated explicitly. Having only a linear implicit at every time step has computational advantages, as suggested in [10], Chen and Shen considered a semi-implicit Fourier-spectral scheme for (CH). On the other hand, semi-implicit schemes can lose stability for large time steps and thus smaller time steps are needed in practice. To resolve this problem, semi-implicit methods with better stability have been introduced, e.g. [21, 32, 35, 36, 34, 24, 25]. Specifically speaking, [21, 35, 36, 33] give different semi-implicit Fourier-spectral schemes, which involved different stabilizing terms of different "size", that preserve the energy decay property (we say these schemes are "energy stable"). However, those works either require a strong Lipschitz condition on the nonlinear source term, or require certain L^∞ bounds on the numerical solutions.

In the seminal works [24, 25, 29], Li et al. developed a large time-stepping semi-implicit Fourier-spectral scheme for Cahn-Hilliard equation and proved that it preserves energy decay with no *a priori* assumptions (unconditional stability). The proof uses tools from harmonic analysis in [7, 8], and introduces a novel energy bootstrap scheme in order to obtain a L^∞-bound of the numerical solution.

Their scheme for (CH) has the form:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = -\nu\Delta^2 u^{n+1} + A\Delta(u^{n+1}-u^n) + \Delta(f(u^n)) \ , \ n \geq 0 \\ u^0 = u_0 \ . \end{cases} \tag{1.2}$$ {?}

Here τ is the time step and A is a large coefficient for the $O(\tau)$ stabilizing term. As a result of their work, the energy decay can still be satisfied with a well-chosen large number A, with at least a size of $O(1/\nu|\log(\nu)|^2)$, or $c/\nu|\log(\nu)|^2$ for some positive constant c that depends on the initial conditions.

However, their arguments cannot be applied to the Allen Cahn equation (AC) directly: this is due to the lack of mass conservation. Our work extends their first order semi-implicit scheme to the related Allen-Cahn equation (AC). Following the same path the fractional Cahn-Hilliard equation (FCH) can be studied as well. To be more specific, we consider the following stabilized semi-implicit scheme for (AC):

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = \nu\Delta u^{n+1} - A(u^{n+1}-u^n) - \Pi_N f(u^n) \\ u^0 = \Pi_N u_0, \end{cases} \tag{1.3}$$ 1stScheme

where τ is the time step and $A > 0$ is the coefficient for the $O(\tau)$ regularization term. For $N \geq 2$, we define

$$X_N = \operatorname{span}\left\{\cos(k\cdot x)\ ,\sin(k\cdot x):\quad k=(k_1,k_2)\in\mathbb{Z}^2\ ,\ |k|_\infty = \max\{|k_1|,|k_2|\}\leq N\right\}\ .$$

Define the L^2 projection operator $\Pi_N :\ L^2(\Omega)\to X_N$ by $(\Pi_N u - u, \phi) = 0\ \ \forall\phi\in X_N$, where $(\cdot,\cdot)$ denotes the L^2 inner product on Ω. In other words, the projection operator Π_N is just the truncation of Fourier modes $|k|_\infty \leq N$. $\Pi_N u_0 \in X_N$ and by induction, we have $u^n \in X_N\ ,\forall n\geq 0$. Similarly, the semi-implicit scheme for (FCH) is given by the following:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = -\nu(-\Delta)^{\alpha+1}u^{n+1} - (-\Delta)^\alpha A(u^{n+1}-u^n) - (-\Delta)^\alpha \Pi_N f(u^n) \\ u^0 = \Pi_N u_0. \end{cases} \tag{1.4}$$ 1stFracscheme

We will show the numerical solutions in (1.3) and (1.4) are unconditionally energy stable and prove a L^2 error estimate.

1.2. **Main results**

Our main results state below:

⟨Thm_1stab⟩ **Theorem 1.1** (Unconditional energy stability for AC)**.** Consider (1.3) with $\nu > 0$ and assume $u_0 \in H^2(\mathbb{T}^2)$. Then there exists a constant β_0 depending only on the initial energy $E_0 = E(u_0)$ such that if

$$A \geq \beta\cdot\left(\|u_0\|_{H^2}^2 + \nu^{-1}|\log\nu| + 1\right)\ ,\ \beta\geq\beta_0$$

then $E(u^{n+1}) \leq E(u^n)$, $\forall n \geq 0$ and for any choice of the time step τ, where E is defined in (1.1).

Remark 1.2. Note that here in Theorem 1.1 no mean zero assumption is needed for u_0 due to the lack of mass conservation.

⟨1st_error⟩ **Theorem 1.3.** Let $\nu > 0$. Let $u_0 \in\ H^s$, $s \geq 4$ and $u(t)$ be the solution to the Allen-Cahn equation (AC) with initial data u_0. Let u^n be the numerical solution with initial data $\Pi_N u_0$ in (1.3). Assume A satisfies the same condition in Theorem 1.1. Define $t_m = m\tau$, $m \geq 1$. Then

$$\|u^m - u(t_m)\|_2 \leq A\cdot e^{C_1 t_m}\cdot C_2\cdot\left(N^{-s}+\tau\right)\ ,$$

where $C_1 > 0$ depends only on (u_0,ν) and C_2 depends on (u_0,ν,s).

⟨1stFracThm⟩ **Theorem 1.4** (Unconditional energy stability for FCH)**.** Consider (1.4) with $\nu > 0$ and assume $u_0 \in H^2(\mathbb{T}^2)$ and obeys the zero-mean condition. Then there exists a constant β_0 depending only on the initial energy $E_0 = E(u_0)$ such that if

$$A \geq \beta\cdot\left(\|u_0\|_{H^2}^2 + \nu^{-1}|\log\nu| + 1\right)\ ,\ \beta\geq\beta_0$$

then $E(u^{n+1}) \leq E(u^n)$, $\forall n \geq 0$ and for any time step τ. Here E is defined above in (1.1).

Remark 1.5. Here in Theorem 1.4 we require a zero-mean assumption on u_0 which implies u^n has mean zero for each n. This assumption will guarantee that negative fractional Laplacian is well defined. Here we use the notation $|\nabla|^{-\alpha} = (-\Delta)^{-\frac{\alpha}{2}}$ to denote the fractional Laplacian.

Remark 1.6. It is worth mentioning that the stability results above in Theorem 1.1 and Theorem 1.4 are valid for any time step τ. Our choice of A is independent of τ as long as it has size of $O\left(1/\nu|\log(\nu)|\right)$. Note that the choice of A may not be optimal and further work can be done in this direction.

Remark 1.7. As a remark, in the fractional CH case, as $\alpha \to 0$, (FCH) becomes the zero-mass projected Allen-Cahn equation and for $\alpha = 1$, it coincides the original Cahn-Hilliard equation. Roughly speaking, the fractional Cahn-Hilliard equation is an interpolation of the zero-mass projected Allen-Cahn and Cahn-Hilliard equations. Here the zero-mass projected Allen-Cahn equation is defined as:

$$\begin{cases} \partial_t u = \Pi_0\left(\nu\Delta u - f(u)\right) \\ u(x,0) = u_0\ , \end{cases} \tag{1.5}$$

0mAC

where Π_0 is the zero mass projector, i.e. $\Pi_0(g) = g - \fint_\Omega g\ dx$, or $= \frac{1}{(2\pi)^d}\sum_{|k|\geq 1}\hat{g}(k)e^{ik\cdot x}$ from the Fourier side. The difference between (AC) and the zero-mass projected Allen-Cahn equation (1.5) results from the loss of mass conservation.

⟨rmk_gen_grad⟩ *Remark* 1.8. More general cases can be discussed. To be more specific by defining a general "gradient" operator $\mathcal{G}$, we can rewrite the equation as :

$$\begin{cases} \partial_t u = \mathcal{G}\left(\nu\Delta u - f(u)\right) \\ u(x,0) = u_0 \end{cases}. \tag{1.6}$$

gradienteq

When $\mathcal{G} = id$, the identity map, (1.6) becomes the Allen-Cahn equation; when $\mathcal{G} = (-\Delta)^\alpha$, (1.6) becomes the fractional Cahn-Hilliard equation as discussed above. And the corresponding semi-implicit scheme is

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = \mathcal{G}\left(\nu\Delta u^{n+1} - f(u^n)\right) - A\mathcal{G}(u^{n+1} - u^n)\ ,\ n \geq 0 \\ u^0 = u_0 \end{cases}. \tag{1.7}$$

gradientscheme

The main result of this paper states that for any fixed time step τ, we can always define a large constant A independent of τ in (1.7), such that the numerical solution will be stable in the sense of satisfying the energy-decay condition for "gradient" cases of AC and fractional CH in 2D. In fact our method holds for more general cases including AC on 3D and higher order schemes; we postpone the discussion to a subsequent work.

1.3. **Organization of the presenting paper**

The presenting paper is organized as follows. In Section 2 we list the notation and preliminaries including several useful lemmas. The energy stability of the semi-implicit scheme of the 2D Allen-Cahn will be shown in Section 3 while the error estimate is given in Section 4. The fractional Allen-Cahn case will be discussed in Section 5.

2. Notation and preliminaries

Throughout this paper, for any two (non-negative in particular) quantities X and Y, we denote $X \lesssim Y$ if $X \leq CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \geq CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant C on other parameters or constants are usually clear from the context and we will often suppress this dependence. We shall denote $X \lesssim_{Z_1,Z_2,\cdots,Z_k} Y$ if $X \leq CY$ and the constant C depends on the quantities $Z_1, \cdots, Z_k$.

For any two quantities X and Y, we shall denote $X \ll Y$ if $X \leq cY$ for some sufficiently small constant c. The smallness of the constant c is usually clear from the context. The notation $X \gg Y$

is similarly defined. Note that our use of $\ll$ and $\gg$ here is *different* from the usual Vinogradov notation in number theory or asymptotic analysis.

For a real-valued function $u : \Omega \to \mathbb{R}$ we denote its usual Lebesgue L^p-norm by

$$\|u\|_p = \|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u|^p \ dx\right)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \operatorname{esssup}_{x\in\Omega} |u(x)|, & p = \infty. \end{cases} \tag{2.1}$$ {?}

Similarly, we use the weak derivative in the following sense: For u, $v \in L^1_{loc}(\Omega)$, (i.e they are locally integrable); $\forall \phi \in C_0^\infty(\Omega)$, i.e ϕ is infinitely differentiable (smooth) and compactly supported; and

$$\int_\Omega u(x)\ \partial^\alpha \phi(x)\ dx = (-1)^{\alpha_1+\cdots+\alpha_n} \int_\Omega v(x)\ \phi(x)\ dx,$$

then v is defined to be the weak partial derivative of u, denoted by $\partial^\alpha u$. Suppose $u \in L^p(\Omega)$ and all weak derivatives $\partial^\alpha u$ exist for $|\alpha| = \alpha_1 + \cdots + \alpha_n \le k$, such that $\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \le k$, then we denote $u \in W^{k,p}(\Omega)$ to be the standard Sobolev space. The corresponding norm of $W^{k,p}(\Omega)$ is :

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\le k} \int_\Omega |\partial^\alpha u|^p \ dx\right)^{\frac{1}{p}} .$$

For $p = 2$ case, we use the convention $H^k(\Omega)$ to denote the space $W^{k,2}(\Omega)$. We often use $D^m u$ to denote any differential operator $D^\alpha u$ for any $|\alpha| = m$: D^2 denotes $\partial^2_{x_i x_j} u$ for $1 \le i, j \le d$, as an example.

In this paper we use the following convention for Fourier expansion on $\mathbb{T}^d$:

$$f(x) = \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{Z}^d} \hat{f}(k) e^{ik\cdot x} \ , \ \hat{f}(k) = \int_\Omega f(x) e^{-ik\cdot x} \ dx \ .$$

Taking advantage of the Fourier expansion, we use the well-known equivalent H^s-norm and $\dot{H}^s$-semi-norm of function f by

$$\|f\|_{H^s} = \frac{1}{(2\pi)^{d/2}} \left(\sum_{k\in\mathbb{Z}^d} (1 + |k|^{2s}) |\hat{f}(k)|^2\right)^{\frac{1}{2}} \ , \ \|f\|_{\dot{H}^s} = \frac{1}{(2\pi)^{d/2}} \left(\sum_{k\in\mathbb{Z}^d} |k|^{2s} |\hat{f}(k)|^2\right)^{\frac{1}{2}} .$$

⟨Sobolevineq⟩ **Lemma 2.1** (Sobolev inequality on $\mathbb{T}^d$)**.** Let $0 < s < d$ and $f \in L^q(\mathbb{T}^d)$ for any $\frac{d}{d-s} < p < \infty$, then

$$\| \langle \nabla \rangle^{-s} f \|_{L^p(\mathbb{T}^d)} \lesssim_{s,p,d} \|f\|_{L^q(\mathbb{T}^d)} \ , \ \text{where } \frac{1}{q} = \frac{1}{p} + \frac{s}{d} \ ,$$

where $\langle \nabla \rangle^{-s}$ denotes $(1-\Delta)^{-\frac{s}{2}}$ and $A \lesssim_{s,p,d} B$ is defined as $A \le C_{s,p,d}\ B$ where $C_{s,p,d}$ is a constant dependent on s, p and d.

Remark 2.2. Note that the this Sobolev inequality is a variety of the standard version. Note that on the Fourier side the symbol of $\langle \nabla \rangle^{-s}$ is given by $(1 + |k|^2)^{-\frac{s}{2}}$. In particular, $\|f\|_{\infty(\mathbb{T}^d)} \lesssim \|f\|_{H^2(\mathbb{T}^d)}$, known as Morrey's inequality.

⟨DiscreteGronwall⟩ **Lemma 2.3** (Discrete Grönwall's inequality)**.** Let $\tau > 0$ and $y_n \ge 0$, $\alpha_n \ge 0$, $\beta_n \ge 0$ for $n = 1, 2, 3 \cdots$. Suppose

$$\frac{y_{n+1} - y_n}{\tau} \le \alpha_n y_n + \beta_n \ , \ \forall\ n \ge 0 \ .$$

Then for any $m \ge 1$, we have

$$y_m \le \exp\left(\tau \sum_{n=0}^{m-1} \alpha_n\right) \left(y_0 + \sum_{k=0}^{m-1} \beta_k\right) \ .$$

Remark 2.4. We sketch the proof here. By the assumption, it follows that for $n \ge 0$,

$$y_{n+1} \le (1 + \alpha_n \tau) y_n + \tau \beta_n \le e^{\tau \alpha_n} y_n + \tau \beta_n;$$

therefore one can derive that

$$\exp\left(-\tau\sum_{j=0}^{n}\alpha_j\right)y_{n+1}\le\exp\left(-\tau\sum_{j=0}^{n-1}\alpha_j\right)y_n+\exp\left(-\tau\sum_{j=0}^{n}\alpha_j\right)\beta_n.$$

We thus obtain the desired result by performing a telescoping summation.

3. Stability of a first order semi-implicit scheme on the 2D Allen-Cahn equation

ection:stability2DAC⟩ Recall that the Allen-Cahn equation (AC) is formulated as follows:

$$\begin{cases}\partial_t u=\nu\Delta u-f(u)\\ u(x,0)=u_0\end{cases}.$$

Here $f(u)=u^3-u$, and the spatial domain Ω is taken to be the two dimensional $2\pi-$periodic torus $\mathbb{T}^2$. The corresponding energy is defined by $E(u)=\int_\Omega(\frac{\nu}{2}|\nabla u|^2+F(u))\,dx$, where $F(u)=\frac14(u^2-1)^2$, the anti-derivative of $f(u)$. As is well known, the energy satisfies $E(u(t))\le E(u(s))$, $\forall\, t\ge s$, which gives an *a priori* bound. Recall that we consider the stabilized semi-implicit scheme (1.3):

$$\begin{cases}\dfrac{u^{n+1}-u^n}{\tau}=\nu\Delta u^{n+1}-A(u^{n+1}-u^n)-\Pi_N f(u^n)\\ u^0=\Pi_N u_0\end{cases}. \tag{3.1}$$?1stScheme1?

We aim to show Theorem 1.1. To start with we first introduce a log-type interpolation inequality:

⟨Loginterpolation⟩ **Lemma 3.1** (Log-type interpolation)**.** For all $f\in H^s(\mathbb{T}^2)$, $s>1$, then

$$\|f\|_\infty\le C_s\cdot\left(\|f\|_{\dot H^1}\sqrt{\log(\|f\|_{\dot H^s}+3)}+|\hat f(0)|+1\right).$$

Here C_s is a constant which only depends on s.

Proof. To prove the lemma, we write $f(x)=\frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2}\hat f(k)\;e^{ik\cdot x}$, i.e. the Fourier series of f, which converge pointwisely to f. It then follows that

$$\begin{aligned}\|f\|_\infty&\le\frac{1}{(2\pi)^2}\sum_{k\in\mathbb{Z}^2}|\hat f(k)|\\&\le\frac{1}{(2\pi)^2}\left(|\hat f(0)|+\sum_{0<|k|\le N}|\hat f(k)|+\sum_{|k|>N}|\hat f(k)|\right)\\&\lesssim|\hat f(0)|+\sum_{0<|k|\le N}(|\hat f(k)||k|\cdot|k|^{-1})+\sum_{|k|>N}(|\hat f(k)||k|^s\cdot|k|^{-s})\\&\lesssim|\hat f(0)|+\left(\sum_{0<|k|\le N}|\hat f(k)|^2|k|^2\right)^{\frac12}\cdot\left(\sum_{0<|k|\le N}|k|^{-2}\right)^{\frac12}+\left(\sum_{|k|>N}|\hat f(k)|^2|k|^{2s}\right)^{\frac12}\cdot(\sum_{|k|>N}|k|^{-2s})^{\frac12}\\&\lesssim|\hat f(0)|+\frac{1}{N^{s-1}}\left(\sum_{|k|>N}|\hat f(k)|^2|k|^{2s}\right)^{\frac12}+\left(\sum_{0<|k|\le N}|\hat f(k)|^2|k|^2\right)^{\frac12}\cdot\sqrt{\log(N+3)}\\&\lesssim|\hat f(0)|+\frac{1}{N^{s-1}}\|f\|_{\dot H^s}+\sqrt{\log(N+3)}\|f\|_{\dot H^1}\;.\end{aligned}$$

If $\|f\|_{\dot H^s}\le3$, we can simply take $N=1$; otherwise take N^{s-1} close to $\|f\|_{\dot H^s}$. As a remark, this lemma can be viewed as a variation of the well-known log-type Bernstein's inequality. ☐

We will prove Theorem 1.1 by induction. To start with, let us recall the numerical scheme (1.3):

$$\frac{u^{n+1}-u^n}{\tau}=\nu\Delta u^{n+1}-A(u^{n+1}-u^n)-\Pi_N f(u^n).$$

Here Π_N is truncation of Fourier modes of L^2 functions to $|k|_\infty \leq N$. Multiply the equation by $(u^{n+1}-u^n)$ and integrate over Ω, one has

$$\frac{1}{\tau}\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 = \nu\int_{\mathbb{T}^2}\Delta u^{n+1}(u^{n+1}-u^n) - A\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 - \left(\Pi_N f(u^n), u^{n+1}-u^n\right).$$

Because u^n is periodic, (as $u^n \in X_N$), hence by integration by parts, we have

$$\left(\frac{1}{\tau}+A\right)\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2 + \nu\int_{\mathbb{T}^2}\nabla u^{n+1}\nabla(u^{n+1}-u^n) = -\left(\Pi_N f(u^n), u^{n+1}-u^n\right) \ .$$

Note $\nabla u^{n+1}\nabla(u^{n+1}-u^n) = \frac{1}{2}\left(|\nabla u^{n+1}|^2 - |\nabla u^n|^2 + |\nabla(u^{n+1}-u^n)|^2\right)$, we have

$$\left(\frac{1}{\tau}+A\right)\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2+\frac{\nu}{2}\int_{\mathbb{T}^2}|\nabla u^{n+1}|^2-|\nabla u^n|^2+|\nabla(u^{n+1}-u^n)|^2 = -\left(\Pi_N f(u^n), u^{n+1}-u^n\right) \ .$$

Moreover, every $u^n \in X_N$, we have

$$\left(\frac{1}{\tau}+A\right)\int_{\mathbb{T}^2}|u^{n+1}-u^n|^2+\frac{\nu}{2}\int_{\mathbb{T}^2}|\nabla u^{n+1}|^2-|\nabla u^n|^2+|\nabla(u^{n+1}-u^n)|^2 = -\left(f(u^n), u^{n+1}-u^n\right).$$

To proceed, by the fundamental theorem of calculus and integration by parts,

$$\begin{aligned}
F(u^{n+1}) - F(u^n) &= f(u^n)(u^{n+1}-u^n) + \int_{u^n}^{u^{n+1}} f'(s)(u^{n+1}-s)\ ds \\
&= f(u^n)(u^{n+1}-u^n) + \int_{u^n}^{u^{n+1}} (3s^2-1)(u^{n+1}-s)\ ds \\
&= f(u^n)(u^{n+1}-u^n) + \frac{1}{4}(u^{n+1}-u^n)^2\left(3(u^n)^2+(u^{n+1})^2+2u^nu^{n+1}-2\right) \ .
\end{aligned}$$

Combine previous two equations, and denote $E(u^n)$ by E^n we have

$$\begin{aligned}
&\left(\frac{1}{\tau}+A\right)\|u^{n+1}-u^n\|_{L^2}^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \frac{\nu}{2}\|\nabla u^{n+1}\|_{L^2}^2 - \frac{\nu}{2}\|\nabla u^n\|_{L^2}^2 \\
&+ \int_{\mathbb{T}^2} F(u^{n+1}) - F(u^n) = \frac{1}{4}\left((u^{n+1}-u^n)^2, 3(u^n)^2+(u^{n+1})^2+2u^nu^{n+1}-2\right) \\
&\text{Note } \frac{\nu}{2}\|\nabla u^{n+1}\|_{L^2}^2 + \int_{\mathbb{T}^2}F(u^{n+1}) = E(u^{n+1}) = E^{n+1} \\
&\Longrightarrow \left(\frac{1}{\tau}+A+\frac{1}{2}\right)\|u^{n+1}-u^n\|_{L^2}^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + E^{n+1} - E^n \\
&= \frac{1}{4}\left((u^{n+1}-u^n)^2, 3(u^n)^2+(u^{n+1})^2+2u^nu^{n+1}\right) \\
&\leq \|u^{n+1}-u^n\|_{L^2}^2\left(\|u^n\|_\infty^2 + \frac{1}{2}\|u^{n+1}\|_\infty^2\right) \ .
\end{aligned}$$

To show $E^{n+1} \leq E^n$, clearly it suffices to show

$$\frac{1}{\tau}+A+\frac{1}{2} \geq \frac{3}{2}\max\left\{\|u^n\|_\infty^2\ ,\ \|u^{n+1}\|_\infty^2\right\} \ . \qquad (3.2) \ \text{1stcondtion}$$

Note that $E^0 = E(\Pi_N u_0)$ while $E_0 = E(u_0)$ and in general $E_0 \neq E^0$. We claim the following statement:

Proposition 3.2. Suppose $E^0 = E(\Pi_N u_0)$ and $E_0 = E(u_0)$ as defined above, the following inequality holds:

$$\sup_N\ E(\Pi_N u_0) \lesssim 1 + E_0, \text{where } u_0 \in H^1(\mathbb{T}^2) \ .$$

Proof. We rewrite $\Pi_N u_0$ as $\frac{1}{(2\pi)^2}\sum_{|k|\leq N}\widehat{u_0}(k)e^{ik\cdot x}$, namely the Dirichlet partial sum of u_0.

$$\|\nabla(\Pi_N u_0)\|_{L^2(\mathbb{T}^2)}^2 = \frac{1}{(2\pi)^2}\sum_{|k|\leq N}|k|^2|\widehat{u_0}(k)|^2 \leq \frac{1}{(2\pi)^2}\sum_{|k|\in\mathbb{Z}^2}|k|^2|\widehat{u_0}(k)|^2 = \|\nabla(u_0)\|_{L^2(\mathbb{T}^2)}^2 \ .$$

On the potential energy part, by the Sobolev inequality Lemma 2.1, $\|u_0\|_{L^4(\mathbb{T}^2)} \lesssim \|u_0\|_{H^1(\mathbb{T}^2)}$, this shows $u_0 \in L^4(\mathbb{T}^2)$ and hence the Dirichlet partial sum $\Pi_N u_0$ converges to u_0 in $L^4(\mathbb{T}^2)$. Then $\|\Pi_N u_0\|_{L^4(\mathbb{T}^2)} \to \|u_0\|_{L^4(\mathbb{T}^2)}$, which leads to $\sup_N \|\Pi_N u_0\|_{L^4(\mathbb{T}^2)} < \infty$. By the Uniform Boundedness Principle, we derive $\sup_N \|\Pi_N\| < \infty$, i.e. $\sup_N \|\Pi_N u_0\|_{L^4(\mathbb{T}^2)} \leq c\|u_0\|_{L^4(\mathbb{T}^2)}$ for an absolute constant c. Combining the two estimates above we conclude the proof for the claim. It is worth mentioning that the same claim holds for the 3D case with a similar proof. □

We rewrite the numerical scheme (1.3) as follows:

$$u^{n+1} = \frac{1+A\tau}{1+A\tau-\nu\tau\Delta}u^n - \frac{\tau}{1+A\tau-\nu\tau\Delta}\Pi_N[f(u^n)] \ . \tag{3.3} \{?\}$$

By the interpolation lemma (Lemma 3.1), to control $\|u^{n+1}\|_\infty$ and $\|u^n\|_\infty$, we may consider $\dot{H}^1$-norm and $\dot{H}^{\frac{3}{2}}$-norm together with 0th-mode $|\widehat{u^{n+1}}(0)|$. We start by estimating $|\widehat{u^{n+1}}(0)|$,

$$\begin{aligned} |\widehat{u^{n+1}}(0)| &\leq |\widehat{u^n}(0)| + \frac{\tau}{1+A\tau}|\widehat{f(u^n)}(0)| \\ &\leq |\widehat{u^n}(0)| + \frac{1}{A}|\widehat{f(u^n)}(0)| \\ &\leq \left|\int_{\mathbb{T}^2} u^n \ dx\right| + \left|\int_{\mathbb{T}^2} u^n - (u^n)^3 \ dx\right| \\ &\lesssim 1 + \left|\int_{\mathbb{T}^2} (u^n)^2 \ dx\right|^{\frac{1}{2}} + \left|\int_{\mathbb{T}^2} (1-(u^n)^2)^2 \ dx\right|^{\frac{1}{2}} \\ &\lesssim 1 + \sqrt{E^n} \ . \end{aligned}$$

⟨1stLem⟩ **Lemma 3.3.** There is an absolute constant $c_1 > 0$ such that for any $n \geq 0$

$$\begin{cases} \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \leq c_1 \cdot \left(\dfrac{A+1}{\nu} + \dfrac{1}{\nu\tau}\right) \cdot (E^n + 1) \\ \|u^{n+1}\|_{\dot{H}^1(\mathbb{T}^2)} \leq \left(1 + \dfrac{1}{A} + \dfrac{3}{A}\|u^n\|_\infty^2\right) \cdot \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} \ . \end{cases}$$

Proof. As $0th$-mode will not contribute to $\dot{H}^1$ norm and $\dot{H}^{\frac{3}{2}}$ norm, we can just consider Fourier modes $|k| \geq 1$ from the Fourier side.

Use the symbol $f \lesssim g$ to denote $f \leq c \cdot g$ with c being a constant. We then obtain that

$$\begin{cases} \dfrac{(1+A\tau)|k|^{\frac{3}{2}}}{1+A\tau+\nu\tau|k|^2} \lesssim \dfrac{1+A\tau}{\nu\tau} \\ \dfrac{\tau|k|^{\frac{3}{2}}}{1+A\tau+\nu\tau|k|^2} \lesssim \dfrac{\tau}{\tau\nu}|k|^{-\frac{1}{2}} = \dfrac{1}{\nu}|k|^{-\frac{1}{2}} \ . \end{cases}$$

Hence

$$\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \lesssim \left(\frac{1+A\tau}{\nu\tau}\right)\|u^n\|_{L^2(\mathbb{T}^2)} + \frac{1}{\nu}\|\langle\nabla\rangle^{-\frac{1}{2}}f(u^n)\|_{L^2(\mathbb{T}^2)}. \tag{3.4} \boxed{\texttt{H32norm}}$$

Here the notaion $\langle\nabla\rangle^s = (1-\Delta)^{\frac{s}{2}}$, corresponds to the Fourier side $(1+|k|^2)^{s/2}$. Note that

$$\|u^n\|_{L^2(\mathbb{T}^2)} \lesssim \int_{\mathbb{T}^2} \frac{1}{4}(u^4 - 2u^2 + 1) \ dx + 1 \lesssim E^n + 1$$

by Cauchy-Schwarz inequality. On the other hand, by the Sobolev inequality

$$\begin{aligned} \|\langle\nabla\rangle^{-\frac{1}{2}}f(u^n)\|_{L^2(\mathbb{T}^2)} &\lesssim \|f(u^n)\|_{L^{\frac{4}{3}}(\mathbb{T}^2)} = \|(u^n)^3 - u^n\|_{L^{\frac{4}{3}}(\mathbb{T}^2)} \\ &= \left(\int_{\mathbb{T}^2} ((u^n)^3 - u^n)^{\frac{4}{3}} \ dx\right)^{\frac{3}{4}} \\ &\lesssim E^n + 1 \ . \end{aligned}$$

Therefore (3.4) becomes

$$\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \lesssim \left(\frac{1+A\tau}{\nu\tau} + \frac{1}{\nu}\right)(E^n+1).$$

Similarly, we get

$$\begin{cases} \dfrac{(1+A\tau)|k|}{1+A\tau+\nu\tau|k|^2} \lesssim |k| \\ \dfrac{\tau|k|}{1+A\tau+\nu\tau|k|^2} \lesssim \dfrac{\tau}{\tau A}|k| = \dfrac{1}{A}|k| \ . \end{cases}$$

This implies

$$\begin{aligned} \|u^{n+1}\|_{\dot{H}^1(\mathbb{T}^2)} &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \frac{1}{A}\|f(u^n)\|_{\dot{H}^1(\mathbb{T}^2)} \\ &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \frac{1}{A}\|\nabla(f(u^n))\|_{L^2(\mathbb{T}^2)} \\ &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + \frac{1}{A}\|(3(u^n)^2-1)\cdot(\nabla u^n)\|_{L^2(\mathbb{T}^2)} \\ &\lesssim \|u^n\|_{\dot{H}^1(\mathbb{T}^2)} + (\frac{1}{A} + \frac{3\|u\|_\infty^2}{A})\|u^n\|_{\dot{H}^1(\mathbb{T}^2)} \\ &\lesssim \left(1+\frac{1}{A}+\frac{3\|u\|_\infty^2}{A}\right)\|u^n\|_{\dot{H}^1(\mathbb{T}^2)} \ . \end{aligned}$$

□

Proof of Theorem 1.1. Now we will complete the proof for **Theorem** 1.1 by induction:
Step 1: The induction $n \to n+1$ step. Assuming $E^n \le E^{n-1} \le \cdots \le E^0$ and $E^n \le \sup_N \ E(\Pi_N u_0)$, we will show $E^{n+1} \le E^n$. This implies $\|u^n\|_{\dot{H}^1}^2 = \|\nabla u^n\|_{L^2}^2 \le \frac{2E^n}{\nu} \le \frac{2E^0}{\nu}$.

By Lemma 3.1, use the notation $f \lesssim_{E^0} g$ to denote that $f \le C(E^0)\cdot g$ for some constant $C(E^0)$ depending only on E^0, we have

$$\begin{aligned} \|u^n\|_\infty^2 &\lesssim \|u^n\|_{\dot{H}^1}^2 \left(\sqrt{\log(3+c_1\left(\frac{1}{\nu\tau}+\frac{A+1}{\nu}\right)(E^n+1))}\right)^2 + E^n + 1 \\ &\lesssim \frac{2E^0}{\nu}\left(1+\log(A)+\log(\frac{1}{\nu}) + (\log(1+\frac{1}{\tau}))\right) + E^0 + 1 \\ &\lesssim_{E^0} \nu^{-1}\left(1+\log(A)+\log(\frac{1}{\nu})\right) + \nu^{-1}|\log(\tau)| + 1 \ . \end{aligned} \tag{3.5}$$

1stInfinity_bounc

Define $m_0 := \nu^{-1}(1+\log(A)+|\log(\nu)|)$, and note that $E^0 \le \sup_N \ E(\Pi_N u_0) \lesssim E_0 + 1$, the inequality above (3.5) is then estimated as follows:

$$\|u^n\|_\infty^2 \lesssim_{E_0} m_0 + \nu^{-1}|\log(\tau)| + 1.$$

On the other hand by Lemma 3.3,

$$\begin{aligned} \|u^{n+1}\|_\infty &\lesssim 1 + \|u^{n+1}\|_{\dot{H}^1}\sqrt{\log(3+\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}})} \\ &\lesssim 1 + (\frac{1+\|u^n\|_\infty^2}{A})\|u^n\|_{\dot{H}^1}\sqrt{\log(3+\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}})} \\ &\lesssim_{E_0} 1 + (1+\frac{m_0+\nu^{-1}|\log(\tau)|}{A})(\sqrt{\frac{1}{\nu}}\sqrt{\log(3+\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}})} \\ &\lesssim_{E_0} 1 + (1+\frac{m_0+\nu^{-1}|\log(\tau)|}{A})(\sqrt{m_0+\nu^{-1}|\log(\tau)|}) \\ &\lesssim_{E_0} 1 + \sqrt{m_0+\nu^{-1}|\log(\tau)|} + \frac{(\sqrt{m_0+\nu^{-1}|\log(\tau)|})^3}{A} \\ &\lesssim_{E_0} \sqrt{1+\frac{m_0^3}{A^2}+m_0+\nu^{-3}|\log(\tau)|^3} \ . \end{aligned} \tag{3.6}$$

1stInifinity_bour

The sufficient condition (3.2) thus becomes

$$\begin{cases} A+\dfrac{1}{2}+\dfrac{1}{\tau}\geq C(E_0)\left(m_0+1+\dfrac{m_0^3}{A^2}+\nu^{-3}|\log(\tau)|^3\right) \\ m_0=\nu^{-1}\left(1+\log(A)+|\log(\nu)|\right)\ . \end{cases}$$

We now discuss two cases.

`Case 1:` $\frac{1}{\tau}\geq C(E_0)\nu^{-3}|\log(\tau)|^3$. In this case, we need to choose A such that

$$A\gg_{E_0} m_0=\nu^{-1}\left(1+\log(A)+|\log(\nu)|\right)\ ,$$

where $B\gg_{E_0} D$ means there exists a large constant depending only on E_0. In fact, for $\nu\gtrsim 1$, we can take $A\gg_{E_0} 1$; if $0<\nu\ll 1$, we will choose $A=C_{E_0}\cdot\nu^{-1}|\log\nu|$, where C_{E_0} is a large constant depending only on E_0. Therefore it suffices to choose

$$A=C_{E_0}\cdot\max\left\{\nu^{-1}|\log(\nu)|\ ,\ 1\right\}. \tag{3.7}$$ {?}

`Case 2:` $\frac{1}{\tau}\leq C(E_0)\nu^{-3}|\log(\tau)|^3$. This implies $|\log(\tau)|\lesssim_{E_0} 1+|\log(\nu)|$. Going back to equations (3.5), we have

$$\|u^n\|_\infty^2\lesssim_{E_0} m_0,$$

as $\nu^{-1}|\log(\tau)|$ will be absorbed by m_0, where $m_0=\nu^{-1}\left(1+\log(A)+|\log(\nu)|\right)$. Hence substituting this new bound into (3.6), we get

$$\begin{aligned}\|u^{n+1}\|_\infty &\lesssim 1+\left(\frac{1+\|u^n\|_\infty^2}{A}\right)\|u^n\|_{\dot H^1}\sqrt{\log(3+\|u^{n+1}\|_{\dot H^{\frac32}})}\\ &\lesssim_{E_0} 1+\left(1+\frac{m_0}{A}\right)\sqrt{\frac{1}{\nu}}\sqrt{\log(3+\|u^{n+1}\|_{\dot H^{\frac32}})}\\ &\lesssim_{E_0} 1+\left(1+\frac{m_0}{A}\right)\sqrt{m_0}\\ &\lesssim_{E_0} \sqrt{1+\frac{m_0^3}{A^2}+m_0}\ .\end{aligned}$$

This shows it suffices to take

$$A\geq C_{E_0}m_0,$$

for a large enough constant C_{E_0} depending only on E_0. The same choice of A in `Case 1` (with a larger C_{E_0} if necessary) will still work.

Step 2: We check the induction base step $n=1$. Clearly we only need to check

$$A+\frac{1}{2}+\frac{1}{\tau}\geq\|\Pi_N u_0\|_\infty^2+\frac{1}{2}\|u^1\|_\infty^2.$$

By Lemma 3.3,

$$\begin{aligned}\|u^1\|_{\dot H^1}&\leq\left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\cdot\|u_0\|_{\dot H^1}\\ &\leq\left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\cdot\sqrt{\frac{2E^0}{\nu}}\ .\end{aligned}$$

As a result,

$$\begin{aligned}\|u^1\|_\infty&\lesssim 1+|\widehat{u^1}(0)|+\|u^1\|_{\dot H^1}\sqrt{\log(3+\|u^1\|_{\dot H^{\frac32}})}\\ &\lesssim 1+\sqrt{E^0}+\left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\sqrt{\frac{2E^0}{\nu}}\sqrt{\log\left(3+c_1\left(\frac{A+1}{\nu}+\frac{1}{\nu\tau}\right)(E_0+1)\right)}\\ &\lesssim_{E^0} 1+\left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\cdot\nu^{-\frac12}\cdot\sqrt{1+\log(A)+|\log(\nu)|+|\log(\tau)|}\\ &\lesssim_{E_0}\left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)\cdot\nu^{-\frac12}\cdot\sqrt{1+\log(A)+|\log(\nu)|+|\log(\tau)|}\ .\end{aligned}$$

Thus we need to choose A such that

$$A+\frac{1}{2}+\frac{1}{\tau}\geq\|\Pi_N u_0\|_\infty^2+C_{E_0}\cdot\left(1+\frac{1}{A}+\frac{3}{A}\|\Pi_N u_0\|_\infty^2\right)^2\cdot\nu^{-1}$$
$$\cdot\left(1+\log(A)+|\log(\nu)|+|\log(\tau)|\right)\ ,$$

where C_{E_0} is a large constant depending only on E_0. Note that by Morrey's inequality,

$$\|\Pi_N u_0\|_{L^\infty(\mathbb{T}^2)}\lesssim\|\Pi_N u_0\|_{H^2(\mathbb{T}^2)}\lesssim\|u_0\|_{H^2(\mathbb{T}^2)}.$$

Then it suffices to take A such that

$$A\gg_{E_0}\|u_0\|_{H^2}^2+\nu^{-1}|\log(\nu)|+1\ . \tag{3.8}$$ {?}

This completes the induction and hence proves the theorem. A numerical result is given in Figure 1 where we choose $\nu=0.1$, A=1, $u_0=\sin(x)\sin(y)$, $\tau=0.01$, $N_x=N_y=256$.

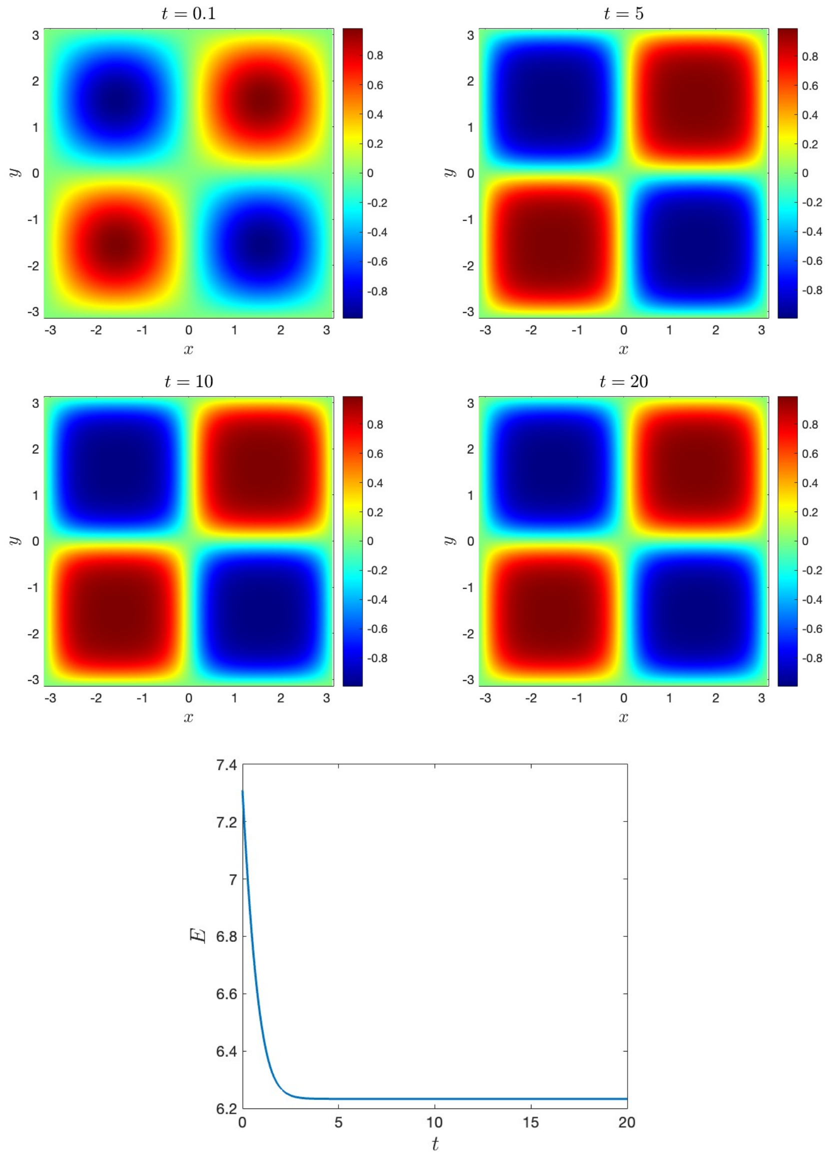

FIGURE 1. Dynamics of 2D Allen-Cahn equation using semi-implicit scheme where $\nu=0.1$, A=1, $u_0=\sin(x)\sin(y)$, $\tau=0.01$, $N_x=N_y=256$.

⟨fig:2dac⟩

□

4. L^2 ERROR ESTIMATE OF THE FIRST ORDER SEMI-IMPLICIT SCHEME FOR THE 2D ALLEN-CAHN EQUATION

?⟨section:error⟩? In this section, we will like to study the L^2 error between the semi-implicit numerical solution and the exact PDE solution in the domain $\mathbb{T}^2$ and eventually prove Theorem 1.3. To start with, we consider the auxiliary L^2 error estimate for near solutions.

4.1. Auxiliary L^2 error estimate for near solutions

Consider the following auxiliary system u^n and v^n for the first order scheme:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = \nu\Delta u^{n+1} - \Pi_N f(u^n) - A(u^{n+1}-u^n) + G_n^1 \\ \dfrac{v^{n+1}-v^n}{\tau} = \nu\Delta v^{n+1} - \Pi_N f(v^n) - A(v^{n+1}-v^n) + G_n^2 \\ u^0 = u_0 \ , \ v^0 = v_0 \ . \end{cases} \tag{4.1}$$ 1st_auxscheme

We define that $G_n = G_n^1 - G_n^2$.

⟨prop_aux⟩ **Proposition 4.1.** For solutions of (4.1), assume for some $N_1 > 0$,

$$\sup_{n\geq 0} \|u^n\|_\infty + \sup_{n\geq 0}\|v^n\|_\infty \leq N_1 \ .$$

Then for any $m \geq 1$,

$$\begin{aligned} &\|u^m - v^m\|_{L^2}^2 = \|e^m\|_{L^2}^2 \\ &\leq \exp\left(m\tau\cdot\left\{C\left(\frac{(1+N_1^2)N_1}{\nu} + N_1^2 + \nu(1+N_1^2)N_1\right) + \frac{B}{\nu}\right\}\right) \\ &\cdot\left((1+A\tau)\|u_0 - v_0\|_{L^2}^2 + B\tau\nu\sum_{n=0}^{m-1}\|G_n\|_{L^2}^2\right) \end{aligned}$$

where $B \ , \ C > 0$ are absolute constants.

Proof. Write $e^n = u^n - v^n$. Then

$$\frac{e^{n+1}-e^n}{\tau} = \nu\Delta e^{n+1} - A(e^{n+1}-e^n) - \Pi_N\left(f(u^n) - f(v^n)\right) + G_n \ .$$

Taking L^2-inner product with e^{n+1} on both sides and recalling similar computations in previous section, we have

$$\begin{aligned} &\frac{1}{2\tau}\left(\|e^{n+1}\|_{L^2}^2 - \|e^n\|_{L^2}^2 + \|e^{n+1}-e^n\|_{L^2}^2\right) + \nu\|\nabla e^{n+1}\|_{L^2}^2 + \frac{A}{2}(\|e^{n+1}\|_{L^2}^2 - \|e^n\|_{L^2}^2 + \\ &\|e^{n+1}-e^n\|_{L^2}^2) = \left(G_n, \ e^{n+1}\right) + \left(f(u^n) - f(v^n), \ \Pi_N e^{n+1}\right) \end{aligned}$$

where $(. \ , \ .)$ denotes the L^2 inner product and the last term is because Π_N is a self-adjoint operator $(\Pi_N f, \ g) = (f, \ \Pi_N g)$, since it is just an N-th Fourier mode truncation. By Hölder's inequality, we obtain that

$$|\left(G_n, \ e^{n+1}\right)| \leq \|e^{n+1}\|_{L^2}\|G_n\|_{L^2} \leq 2B\left(\nu\|G_n\|_{L^2}^2 + \frac{\|e^{n+1}\|_{L^2}^2}{\nu}\right).$$

Next, by the fundamental theorem of calculus, we have

$$\begin{aligned} f(u^n) - f(v^n) &= \int_0^1 f'(v^n + se^n)ds \ e^n \\ &= (a_1 + a_2(v^n)^2)e^n + a_3 v^n(e^n)^2 + a_4(e^n)^3 \ , \end{aligned}$$

where a_i are constants can be computed. Note that we will denote C to be an absolute constant whose value may vary in different lines:

$$\begin{aligned}|\left((a_1+a_2(v^n)^2)e^n\ ,\ e^{n+1}\right)| &\le C(1+\|v^n\|_\infty^2)\|e^{n+1}\|_{L^2}\|e^n\|_{L^2}\\ &\le C(1+N_1^2)N_1(\frac{\|e^{n+1}\|_{L^2}^2}{\nu}+\nu\|e^n\|_{L^2}^2)\\ &\le \frac{C(1+N_1^2)N_1}{\nu}\|e^{n+1}\|_{L^2}^2+\nu\cdot C(1+N_1^2)N_1\|e^n\|_{L^2}^2\ ,\end{aligned}$$

moreover, the other two terms can be estimated similarly:

$$\begin{aligned}|\left(a_3v^n(e^n)^2\ ,\ e^{n+1}\right)| &\le C\|v^n\|_\infty\|e^{n+1}\|_\infty\|e^n\|_{L^2}^2\\ &\le CN_1^2\|e^n\|_{L^2}^2\ ,\\ |\left(a_4(e^n)^3\ ,\ e^{n+1}\right)| &\le C\|e^{n+1}\|_\infty\|e^n\|_\infty\|e^n\|_{L^2}^2\\ &\le CN_1^2\|e^n\|_{L^2}^2\ .\end{aligned}$$

To simplify the formula, we use the notation $\|u\|_2$ to denote the L^2 norm. Collecting all estimates, we get

$$\frac{\|e^{n+1}\|_2^2-\|e^n\|_2^2}{\tau}+A(\|e^{n+1}\|_2^2-\|e^n\|_2^2)\le B\nu\|G_n\|_2^2+\frac{B}{\nu}\|e^{n+1}\|_2^2$$
$$+C\left(\nu(1+N_1^2)N_1+N_1^2\right)\|e^n\|_2^2+\frac{C(1+N_1^2)N_1}{\nu}\|e^{n+1}\|_2^2$$

where B and C are two absolute constants that can be computed exactly. Recalling that A is chosen larger than $O(\nu^{-1}|\log\nu|)$ for ν small, we derive that

$$\frac{\|e^{n+1}\|_2^2-\|e^n\|_2^2}{\tau}+\left(A-\frac{C(1+N_1^2)N_1}{\nu}-\frac{B}{\nu}\right)(\|e^{n+1}\|_2^2-\|e^n\|_2^2)\le B\nu\|G_n\|_2^2+$$
$$\left\{C\left(\frac{(1+N_1^2)N_1}{\nu}+N_1^2+\nu(1+N_1^2)N_1\right)+\frac{B}{\nu}\right\}\|e^n\|_2^2.$$

Define

$$\begin{aligned}y_n &= \left(1+\left(A-\frac{C(1+N_1^2)N_1}{\nu}-\frac{B}{\nu}\right)\tau\right)\|e^n\|_2^2\ ,\\ \alpha &= C\left(\frac{(1+N_1^2)N_1}{\nu}+N_1^2+\nu(1+N_1^2)N_1\right)+\frac{B}{\nu}\ ,\\ \beta_n &= B\nu\|G_n\|_2^2\ .\end{aligned}$$

Then for ν small, we have

$$\frac{y_{n+1}-y_n}{\tau}\le\alpha y_n+\beta_n\ .$$

Applying discrete Grönwall's inequality in Lemma 2.3, we have

$$\begin{aligned}\|u^m-v^m\|_2^2 =&\|e^m\|_2^2\le y_m\\ \le&\exp\left(m\tau\cdot\left\{C\left(\frac{(1+N_1^2)N_1}{\nu}+N_1^2+\nu(1+N_1^2)N_1\right)+\frac{B}{\nu}\right\}\right)\\ &\cdot\left(\left(1+\left(A-\frac{C(1+N_1^2)N_1}{\nu}-\frac{B}{\nu}\right)\tau\right)\|u_0-v_0\|_2^2+B\tau\nu\sum_{n=0}^{m-1}\|G_n\|_2^2\right)\\ \le&\exp\left(m\tau\cdot\left\{C\left(\frac{(1+N_1^2)N_1}{\nu}+N_1^2+\nu(1+N_1^2)N_1\right)+\frac{B}{\nu}\right\}\right)\\ &\cdot\left((1+A\tau)\|u_0-v_0\|_2^2+B\tau\nu\sum_{n=0}^{m-1}\|G_n\|_2^2\right)\ .\end{aligned} \tag{4.2) \{?\}$$

□

4.2. **L^2 error estimate of the 2D Allen-Cahn equation**

In this section, to simplify the notation, we will write $x \lesssim y$ if $x \leq C(\nu, u_0)\ y$ for a constant C depending on ν and u_0. We consider the system

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = \nu\Delta u^{n+1} - \Pi_N f(u^n) - A(u^{n+1}-u^n) \\ \partial_t u = \nu\Delta u - f(u) \\ u^0 = \Pi_N u_0\ ,\ u(0) = u_0\ . \end{cases} \tag{4.3}$$?1st_contscheme?

In order to prove Theorem 1.3, it is clear that we shall estimate G_n introduced in (4.1) from the previous proposition. Note that for a one-variable function $h(t)$, one has the formula:

$$\begin{cases} \dfrac{1}{\tau}\displaystyle\int_{t_n}^{t_{n+1}} h(t) = h(t_n) + \dfrac{1}{\tau}\int_{t_n}^{t_{n+1}} h'(t)\cdot(t_{n+1}-t)\ dt \\ \dfrac{1}{\tau}\displaystyle\int_{t_n}^{t_{n+1}} h(t) = h(t_{n+1}) + \dfrac{1}{\tau}\int_{t_n}^{t_{n+1}} h'(t)\cdot(t_n-t)\ dt\ . \end{cases} \tag{4.4}$$ hform

Using the formula (4.4) above and integrating the Allen-Cahn equation (AC) on the time interval $[t_n, t_{n+1}]$, we get

$$\begin{aligned} \frac{u(t_{n+1})-u(t_n)}{\tau} =& \nu\Delta u(t_{n+1}) - A\left(u(t_{n+1})-u(t_n)\right) \\ &- \Pi_N f(u(t_n)) - \Pi_{>N} f(u(t_n)) + G_n\ , \end{aligned} \tag{4.5}$$ {?}

where $\Pi_{>N} = id - \Pi_N$, the large mode truncation operator, and

$$G_n = \frac{\nu}{\tau}\int_{t_n}^{t_{n+1}} \partial_t\Delta u\cdot(t_n-t)\ dt - \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \partial_t(f(u))(t_{n+1}-t)\ dt + A\int_{t_n}^{t_{n+1}} \partial_t u\ dt\ . \tag{4.6}$$ {?}

To bound $\|G_n\|_2$, we introduce some useful lemmas.

4.3. **Bounds on the Allen-Cahn exact solution and numerical solution**

⟨MP_ac⟩ **Lemma 4.2** (Maximum principle for smooth solutions to the Allen-Cahn equation)**.** Let $T > 0$, $d \leq 3$ and assume $u \in C_x^2 C_t^1(\mathbb{T}^d \times [0,T])$ is a classical solution to Allen-Cahn equation with initial data u_0. Then

$$\|u(.\ ,\ t)\|_\infty \leq \max\{\|u_0\|_\infty\ ,\ 1\}\ ,\ \forall 0 \leq t \leq T\ .$$

Remark 4.3. As proved in [16], there exists a global $H_x^4 C_t^1$ solution to Allen-Cahn equation. In fact as pointed out by Li et al. in [24, 29], the regularity will be higher due to the smoothing effect. Therefore we assume a smooth solution here.

Proof. We define $f(x,t) = u(x,t)^2$ and $f^\epsilon(x,t) = f(x,t) - \epsilon t$. Since f^ϵ is a continuous function on the compact domain $\mathbb{T}^d \times [0,T]$, it achieves maximum at some point (x_*, t_*), i.e.

$$\max_{\substack{0\leq t\leq T\ ,\\ x\in\mathbb{T}^d}} f^\epsilon(x,t) = f^\epsilon(x_*, t_*) := M_\epsilon\ .$$

We discuss several cases.

`Case 1:` $0 < t_* \leq T$ and $M_\epsilon > 1$. This shows $\nabla f^\epsilon(x_*, t_*) = 0\ ,\ \Delta f^\epsilon(x_*, t_*) \leq 0$. Note that

$$\nabla f^\epsilon = 2u\nabla u\ ,\ \Delta f^\epsilon = 2|\nabla u|^2 + 2u\Delta u\ ,$$

this shows $\nabla u(x_*, t_*) = 0$, $u\Delta u(x_*, t_*) < 0$. However, we also have

$$\begin{aligned} \partial_t f^\epsilon(x_*, t_*) &= 2u(x_*, t_*)\partial_t u(x_*, t_*) - \epsilon \\ &= 2u(x_*, t_*)(\nu\Delta u(x_*, t_*) - u^3(x_*, t_*) + u(x_*, t_*)) - \epsilon \\ &< -2u^4(x_*, t_*) + 2u^2(x_*, t_*) - \epsilon \\ &< -2(u^2(x_*, t_*) - \frac{1}{2})^2 + \frac{1}{2} - \epsilon \\ &< -\epsilon < 0 \end{aligned}$$

as $u^2(x_*, t_*) > 1$ by assumption. This contradicts the hypothesis that f^ϵ achieves its maximum at (x_*, t_*).
`Case 2:` $0 < t_* \le T$ and $M_\epsilon \le 1$. In this case we obtain

$$\max_{0\le t\le T\ ,\ x\in\mathbb{T}^d} f(x,t) \le 1 + \epsilon T\ ,$$

letting $\epsilon \to 0$, we obtain $f(x,t) \le 1$.
`Case 3:` $t_* = 0$, then

$$\max_{0\le t\le T\ ,\ x\in\mathbb{T}^d} f(x,t) \le \max_{x\in\mathbb{T}^d} f(x,0) + \epsilon T\ ,$$

sending ϵ to 0, we obtain $f(x,t) \le f(x,0)$.

This concludes $\|u\|_\infty \le \max\{\|u_0\|_\infty\ ,\ 1\}$. □

⟨H^kregularity_AC⟩ **Lemma 4.4** (H^k boundedness of the exact solution)**.** Assume $u(x,t)$ is a smooth solution to the Allen-Cahn equation in $\mathbb{T}^d$ with $d = 1,2,3$ and the initial data $u_0 \in H^k(\mathbb{T}^d)$ for $k \ge 2$. Then,

$$\sup_{t\ge 0} \|u(t)\|_{H^k(\mathbb{T}^d)} \lesssim_k 1 \tag{4.7}$$ {?}

where we omit the dependence on ν and u_0.

Proof. We write the solution u in the mild form

$$u(t) = e^{\nu t\Delta}u_0 + \int_0^t e^{\nu(t-s)\Delta}(u-u^3)\ ds\ .$$

We will prove this argument inductively. By previous lemma 4.2, we have $\|u\|_2 \lesssim 1$ as $\|u\|_\infty \lesssim 1$ and we will show $\|u\|_{H^1} \lesssim 1$ for any $t \ge 1$. Then by taking the spatial derivative and L^2 norm in the formula above, we derive

$$\|Du\|_2 \le \|De^{\nu t\Delta}u_0\|_2 + \int_0^t \|De^{\nu(t-s)\Delta}(u-u^3)\|_2\ ds$$

where $D^m u$ denotes any differential operator $D^\alpha u$ for any $|\alpha| = m$.

First, we consider the nonlinear part.

$$\|De^{\nu(t-s)\Delta}(u-u^3)\|_2 \lesssim \|De^{\nu(t-s)\Delta}(u-u^3)\|_\infty \lesssim |K_1 * (u-u^3)|\ ,$$

where K_1 is the kernel corresponding to $De^{\nu(t-s)\Delta}$. Therefore we estimate that

$$\begin{aligned}|K_1 * (u-u^3)| &\le \|K_1\|_2 \cdot \|u-u^3\|_2 \\ &\lesssim \|K_1\|_2 \cdot \|u\|_2\end{aligned}$$

by the boundedness of $\|u\|_\infty$. Note that

$$\begin{aligned}\|K_1\|_2 &\lesssim \left(\sum_{k\in\mathbb{Z}^d} |k|^2 e^{-2\nu(t-s)|k|^2}\right)^{\frac12} \\ &= \left(\sum_{|k|\ge 1} |k|^2 e^{-2\nu(t-s)|k|^2}\right)^{\frac12} \\ &\lesssim \left(\int_1^\infty e^{-2\nu(t-s)r^2} r^{d+1}\ dr\right)^{\frac12}\ .\end{aligned}$$

The estimates for different dimensions are different. Now we will assume $t \ge 1$ because the other case $t < 1$ is much easier.
`Case 1:` $d = 1$. $\int_1^\infty e^{-2\nu(t-s)r^2} r^2\ dr \lesssim \frac{e^{-2\nu(t-s)}}{t-s} + \frac{\mathrm{erf}(\sqrt{2\nu(t-s)})}{(t-s)^{3/2}}$, where $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\ dt$, the complementary error function. Letting $\gamma = t - s$,

$$\int_0^t \|De^{\nu(t-s)\Delta}u\|_2\ ds \lesssim \left(\int_0^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}} + \frac{(\mathrm{erf}(\sqrt{\nu\gamma}))^{1/2}}{\gamma^{3/4}}\ d\gamma\right) \cdot \|u\|_2\ .$$

For γ small enough, $\frac{(\mathrm{erf}(\sqrt{\nu\gamma}))^{1/2}}{\gamma^{3/4}}$ will dominate the estimate and for γ away from 0, $\frac{e^{-\nu\gamma}}{\gamma^{1/2}}$ shall dominate the estimate. Then we split the integral as below (recall that $t \geq 1$):

$$\begin{aligned}\int_0^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}} + \frac{(\mathrm{erf}(\sqrt{\nu\gamma}))^{1/2}}{\gamma^{3/4}}\ d\gamma &\lesssim \int_0^1 \frac{1}{\gamma^{3/4}}\ d\gamma + \int_1^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\ d\gamma \\ &\lesssim 1 + \int_0^\infty \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\ d\gamma \\ &\lesssim 1\ .\end{aligned}$$

`Case 2:` $d = 2$. $\int_1^\infty e^{-2\nu(t-s)r^2} r^3\ dr \lesssim \frac{e^{-2\nu(t-s)}}{(t-s)^2} + \frac{e^{-2\nu(t-s)}}{t-s}$. Similar to Case 1, we will split the integral as well. Letting $\gamma = t - s$, we have

$$\begin{aligned}\int_1^t \frac{e^{-\nu\gamma}}{\gamma} + \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\ d\gamma &\lesssim \int_1^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\ d\gamma \\ &\lesssim \int_0^\infty \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\ d\gamma \\ &\lesssim 1\ .\end{aligned}$$

However, the estimate in Case 1 does not work for $\gamma \leq 1$. Now we estimate $\|K_1 * (u - u^3)\|_{L^2(\mathbb{T}^d)}$ differently. We compute from the Fourier side:

$$\begin{aligned}\|K_1 * (u - u^3)\|_{L^2(\mathbb{T}^d)}^2 &= \sum_{|k|\geq 1} |k|^2 e^{-2\nu(t-s)|k|^2} |\widehat{u - u^3}(k)|^2 \\ &\leq \max_{|k|\geq 1} \left\{ |k|^2 e^{-2\nu(t-s)|k|^2} \right\} \cdot \sum_{|k|\geq 1} |\widehat{u - u^3}(k)|^2 \\ &\lesssim \max_{|k|\geq 1} \left\{ |k|^2 e^{-2\nu(t-s)|k|^2} \right\} \cdot \|u\|_{L^2(\mathbb{T}^d)}^2\ .\end{aligned}$$

Define $g(x) = x^2 e^{-2\nu\gamma x^2}$, where $x \geq 0$. Then,

$$g'(x) = 2xe^{-2\nu\gamma x^2}\left(1 - 2\nu\gamma x^2\right)\ ,$$

which shows the maximum is achieved at $x = \frac{1}{\sqrt{2\nu\gamma}}$ and hence

$$g(x) \leq g(\frac{1}{\sqrt{2\nu\gamma}}) \lesssim \frac{1}{\gamma}\ .$$

Therefore

$$\|De^{\nu(t-s)\Delta}(u - u^3)\|_{L^2(\mathbb{T}^d)} \lesssim \frac{1}{\sqrt{t-s}}\|u\|_{L^2(\mathbb{T}^d)}\ .$$

Note that this proof works for any dimension. As a result,

$$\int_0^1 \|De^{\nu\gamma\Delta}u\|_2\ d\gamma \lesssim \int_0^1 \frac{1}{\sqrt{\gamma}}\ d\gamma \cdot \|u\|_2 \lesssim 1\ .$$

This shows $\int_0^t \|De^{\nu(t-s)\Delta}u\|_2\ ds \lesssim 1$.

`Case 3:` $d = 3$. As proved in previous case, we will only need to check the case $\gamma \geq 1$. Note that $\int_1^\infty e^{-2\nu\gamma r^2} r^4\ dr \lesssim \frac{e^{-2\nu\gamma}}{\gamma}$ for $\gamma \geq 1$. This shows that

$$\begin{aligned}\int_1^t \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\ d\gamma &\lesssim \int_0^\infty \frac{e^{-\nu\gamma}}{\gamma^{1/2}}\ d\gamma \\ &\lesssim 1\ .\end{aligned}$$

For the case where $t \leq 1$, it is easier because we do not need to split the integral and all integrals from 0 to t can be bounded by the integral from 0 to 1.

Now for the linear part, by Duhamel's Principle, $e^{\nu t\Delta}u_0$ denotes the solution to the heat equation. As is well known, every spatial derivative of the solution $e^{\nu t\Delta}u_0$ solves the heat equation, hence by the energy decay property, we have $\|e^{\nu t\Delta}u_0\|_{H^m} \lesssim \|u_0\|_{H^m}$ for any $1 \leq m \leq k$. Combining the nonlinear and linear parts, we obtain that $\|u\|_{H^1} \lesssim 1$ independent of $t \geq 0$ and hence $\sup_{t\geq 0}\|u\|_{H^1} \lesssim 1$.

Assume that we have $\sup_{t\geq 0}\|u\|_{H^{m-1}}\lesssim 1$, then the estimate follows by repeating the process above:

$$
\begin{aligned}
\|D(D^{m-1}u)\|_2 &\leq \|De^{\nu t\Delta}D^{m-1}u_0\|_2+\int_0^t \|De^{\nu(t-s)\Delta}D^{m-1}u\|_2\ ds\\
&\lesssim \|u_0\|_{H^m}+\int_0^1 \|De^{\nu\gamma\Delta}D^{m-1}u\|_2\ d\gamma+\int_1^t \|De^{\nu\gamma\Delta}D^{m-1}u\|_2\\
&\lesssim 1+\int_0^1 \frac{1}{\sqrt{\gamma}}\ d\gamma\cdot\|D^{m-1}u\|_2+\int_0^\infty \frac{e^{-\nu\gamma}}{\sqrt{\gamma}}\ d\gamma\cdot\|D^{m-1}u\|_2\\
&\lesssim 1\ ,
\end{aligned}
$$

We finally obtain that

$$
\sup_{t\geq 0}\|u\|_{H^k(\mathbb{T}^d)}\lesssim_k 1\ . \tag{4.8}
$$
{?}

□

te_H^k regularity_AC⟩ **Lemma 4.5** (Discrete version H^k boundedness)**.** Suppose $u_0\in H^k(\mathbb{T}^d)$ with $d\leq 3$ and $k\geq 2$. Then, suppose u^n is the numerical solution that satisfies

$$
\begin{cases}
\dfrac{u^{n+1}-u^n}{\tau}=\nu\Delta u^{n+1}-A(u^{n+1}-u^n)-\Pi_N f(u^n)\\
u^0=\Pi_N u_0\ ,
\end{cases}
$$

then

$$
\sup_{n\geq 0}\|u^n\|_{H^k(\mathbb{T}^d)}\lesssim_{A,k} 1\ .
$$

Remark 4.6. The bound on u^n is independent of time step τ and truncation number N.

Remark 4.7. The proof uses the energy decay property of the numerical scheme, so we will assume this property for now. In fact the case for the energy decay in 3D case will be discussed in a subsequent work [11].

Proof. To simplify the notation, we will use "$\lesssim$" instead of "$\lesssim_{\nu,u_0,A,k}$" only in this lemma. We will use a similar method to the one provided in [29].

We can write the scheme as follows:

$$
\begin{aligned}
u^{n+1}&=\frac{1+A\tau}{1+A\tau-\nu\tau\Delta}u^n+\frac{-\tau\Pi_N}{1+A\tau-\nu\tau\Delta}f(u^n)\\
&:=L_1(u^n)+L_2(f(u^n))\\
&=L_1\left(L_1u^{n-1}+L_2f(u^{n-1})\right)+L_2f(u^n)\\
&=L_1^{m_0+1}u^{n-m_0}+\sum_{l=0}^{m_0}L_1^lL_2f(u^{n-l})\ ,
\end{aligned}
\tag{4.9}
$$
{?}

where m_0 will be chosen later.

Similar to the continuous version, we prove inductively. To demonstrate the idea, we show

$$
\sup_{n\geq 0}\|u^n\|_{H^3(\mathbb{T}^d)}\lesssim 1\ .
$$

Recall $\sup_{n\geq 0}\|u^n\|_{H^1}\lesssim 1$ and $\sup_{n\geq 0}\|f(u^n)\|_2\lesssim 1$ by the energy decay property, then we only need to consider the $\dot{H}^3$ semi norm.

We discuss 2 cases:

`Case 1:` $A\tau\geq\frac{1}{10}$. Then for $0\neq k\in\mathbb{Z}^d$:

$$
\begin{aligned}
\left|\widehat{L_1}(k)\right|&=\frac{1+A\tau}{1+A\tau+\nu\tau|k|^2}\\
&\leq\frac{11A\tau}{A\tau+\nu\tau|k|^2}\\
&\lesssim\frac{1}{1+|k|^2}\ ,
\end{aligned}
$$

and

$$\left|\widehat{L_2}(k)\right| = \frac{\tau}{1 + A\tau + \nu\tau|k|^2} \lesssim \frac{1}{1+|k|^2} \ .$$

To conclude the result for this case, we prove the following bootstrap argument:

Claim. Let L_1 and L_2 be defined as above. Suppose given some $s > 0$,

$$\sup_{0\le j\le n} \|\widehat{f^j}(k)|k|^s\|_{l^\infty} \le \alpha < \infty,$$

where $f^j := f(u^j)$. Define

$$v = \sum_{j=1}^{n} L_1^j L_2 f^j.$$

We have

$$\|\hat{v}(k)|k|^{s+2}\|_{l^\infty} \le \frac{\alpha}{\nu},$$

moreover for $d \le 3$,

$$\|v\|_{H^{s+0.4}(\mathbb{T}^d)} \le \beta < \infty,$$

where $\beta > 0$ depends only on α and ν.

Proof of claim. Notice that for each $k \ne 0$,

$$|\widehat{L_1^j}(k)\widehat{L_2}(k)| = \left(\frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2}\right)^j \cdot \frac{\tau}{1 + A\tau + \nu\tau|k|^2};$$

therefore for each $k \ne 0$,

$$\begin{aligned}|\hat{v}(k)| \le& \alpha|k|^{-s} \sum_{j=1}^{n} \left(\frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2}\right)^j \cdot \frac{\tau}{1 + A\tau + \nu\tau|k|^2} \\ \le& \alpha|k|^{-s} \cdot \frac{1}{\nu|k|^2}.\end{aligned}$$

This concludes that $\|\hat{v}(k)|k|^{s+2}\|_{l^\infty} \le \frac{\alpha}{\nu}$ and therefore

$$\|v\|_{H^{s+0.4}(\mathbb{T}^d)} \le \beta$$

for some β depending on α and ν only.

Recall that $\sup_{n\ge 0} \|u^n\|_{H^1} \lesssim 1$, we have

$$\||k|\widehat{f^j}(k)\|_{l^\infty} \lesssim 1,$$

therefore applying this claim, we get $\||k|^3\widehat{u^{n+1}}\|_{l^\infty} \lesssim 1$. It then follows that $\|u^{n+1}\|_{H^3} \lesssim 1$.
`Case 2:` $A\tau < \frac{1}{10}$. Take m_0 to be one integer such that $\frac{1}{2} \le m_0\tau < 1$ and thus $m_0 \ge 5$.

$$\begin{aligned}\left|\widehat{L_1^{m_0+1}}(k)\right| &\le \left(\frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2}\right)^{m_0+1} \\ &\le \left(\frac{1 + A\tau}{1 + A\tau + \nu\tau|k|^2}\right)^{m_0} \\ &= \left(1 + \frac{\nu\tau|k|^2}{1 + A\tau}\right)^{-m_0} \ .\end{aligned}$$

Recall $A\tau < \frac{1}{10} < 1$, then

$$\left(1 + \frac{\nu\tau|k|^2}{1 + A\tau}\right)^{-m_0} \le \left(1 + \frac{\nu\tau|k|^2}{2}\right)^{-m_0} \ ,$$

define $t_0 := m_0\tau$ and we derive

$$\left|\widehat{L_1^{m_0+1}}(k)\right| \le \left(1 + \frac{1}{2}\nu|k|^2\frac{t_0}{m_0}\right)^{-m_0} \ .$$

For any $a > 0$, we consider the function $h(x) = -x \log\left(1 + \frac{a}{x}\right)$, $x > 0$. Then

$$h'(x) = -\log\left(1 + \frac{a}{x}\right) + \frac{a}{a+x}$$
$$h''(x) = \frac{a}{x+a}\left(\frac{1}{x} - \frac{1}{x+a}\right) > 0 \ .$$

By direct computation, $h(x)$ decreases on $(0, \infty)$. Therefore, recalling $m_0 \geq 5$,

$$\left|\widehat{L_1^{m_0+1}}(k)\right| \leq \left(1 + \frac{1}{2}\nu|k|^2 \frac{t_0}{m_0}\right)^{-m_0} \leq \left(1 + \frac{1}{2}\nu|k|^2 \cdot \frac{t_0}{5}\right)^{-5} \ .$$

As a direct result, we have

$$\begin{aligned}
\left|\widehat{L_2}(k)\right| \cdot \sum_{l=0}^{m_0} \left|\widehat{L_1}(k)\right|^l &\leq \left|\widehat{L_2}(k)\right| \cdot \frac{1}{1 - \left|\widehat{L_1}(k)\right|} \\
&= \frac{\tau}{1 + A\tau + \nu\tau|k|^2} \cdot \frac{1}{1 - \frac{1+A\tau}{1+A\tau+\nu\tau|k|^2}} \\
&= \frac{1}{\nu|k|^2} \\
&\lesssim \frac{1}{|k|^2} \ .
\end{aligned}$$

Therefore for $n \geq m_0$,

$$\|u^{n+1}\|_{\dot{H}^2} \lesssim \|u^{n-m_0}\|_2 + \sup_{0 \leq l \leq m_0} \|f(u^{n-l})\|_2 \lesssim 1 \ .$$

For $1 \leq n \leq m_0 + 1$, we apply

$$u^n = L_1^n u^0 + \sum_{l=0}^{n-1} L_1^l L_2 f(u^{n-1-l}) \ .$$

Hence we get

$$\|u^n\|_{\dot{H}^2} \lesssim \|u^0\|_{\dot{H}^2} + \sup_{0 \leq l \leq n-1} \|f(u^{n-l-1})\|_2 \lesssim 1 \ .$$

Then by the bootstrap lemma, we can conclude

$$\|u^{n+1}\|_{H^3(\mathbb{T}^d)} \lesssim 1 \ .$$

By the energy decay property, the constant depends only on ν, u_0 and A, we can conclude that

$$\sup_{n \geq 0} \|u^n\|_{H^3(\mathbb{T}^d)} \lesssim 1 \ . \tag{4.10}$$ {?}

To obtain a different H^k norm control, we can repeat the bootstrap lemma above and derive the desired result.

□

Remark 4.8. The H^k-boundedness for the exact solution and the numerical solution is similar in the sense that a smoothing effect will take place after a short time period.

4.4. **Proof of L^2 error estimate of 2D Allen-Cahn equation**

Proof of Theorem 1.3. By Lemma 4.5, $\sup_{n \geq 0} \|u^n\|_\infty \lesssim 1$ using Morrey's inequality. Thus the assumptions of Proposition 4.1 (auxiliary L^2 error estimate proposition) are satisfied. Recall that

$$G_n = \frac{\nu}{\tau}\int_{t_n}^{t_{n+1}} \partial_t \Delta u \cdot (t_n - t)\ dt - \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \partial_t(f(u))(t_{n+1} - t)\ dt + A\int_{t_n}^{t_{n+1}} \partial_t u\ dt \ .$$

Then we can estimate that

$$\begin{aligned}\|G_n\|_2 &\lesssim \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \, dt + \int_{t_n}^{t_{n+1}} \|\partial_t (f(u))\|_2 \, dt + A \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2 \, dt \\ &\lesssim \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \, dt + \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2 \, dt \cdot \left(A + \|f'(u)\|_{L_t^\infty L_x^\infty}\right) \\ &:= I_1 + I_2.\end{aligned} \tag{4.11}$$ eq:G_nest

Note that $\partial_t u = \nu \Delta u - u + u^3$ and hence by Lemma 4.4,

$$\|\partial_t u\|_2 \lesssim 1 \ , \quad \|f'(u)\|_\infty \lesssim 1 \ .$$

Recall the energy decay property:

$$\frac{dE}{dt} = -\|\partial_t u\|_2^2 \ .$$

This shows

$$\int_0^\infty \|\partial_t u\|_2^2 \, dt \lesssim 1 \ .$$

Note that by the Gagliardo-Nirenberg interpolation inequality, we have

$$\|\partial_t \Delta u\|_2 \lesssim \|\langle \nabla \rangle^3 \partial_t u\|_2^{\frac{2}{3}} \cdot \|\partial_t u\|_2^{\frac{1}{3}} \lesssim \|\partial_t u\|_2^{\frac{1}{3}} \ .$$

This implies

$$\begin{aligned}&\int_0^\infty \|\partial_t \Delta u\|_2^6 \, dt \lesssim 1 \ , \\ \Longrightarrow &\int_0^T \|\partial_t \Delta u\|_2^2 \, dt \lesssim \left(\int_0^T \|\partial_t \Delta u\|_2^6 \, dt\right)^{\frac{1}{3}} \cdot \left(\int_0^T 1 \, dt\right)^{\frac{2}{3}} \lesssim 1 + T^{\frac{2}{3}} \ .\end{aligned}$$

Therefore we can estimate (4.11) as following

$$I_1 = \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2 \, dt \lesssim \left(\int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2^2 \, dt\right)^{\frac{1}{2}} \cdot \sqrt{\tau} \ .$$

Similarly for I_2, we obtain that

$$I_2 \lesssim (1+A) \cdot \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2 \, dt \lesssim (1+A) \cdot \left(\int_{t_n}^{t_{n+1}} \|\partial_t u\|_2^2 \, dt\right)^{\frac{1}{2}} \cdot \sqrt{\tau} \ .$$

Hence for $t_m \geq 1$,

$$\begin{aligned}\sum_{n=0}^{m-1} \|G_n\|_2^2 &\lesssim \sum_{n=0}^{m-1} \left((I_1)^2 + (I_2)^2\right) \\ &\lesssim \sum_{n=0}^{m-1} \left(\tau \int_{t_n}^{t_{n+1}} \|\partial_t \Delta u\|_2^2 \, dt + (1+A)^2 \tau \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2^2 \, dt\right) \\ &\lesssim \tau \int_0^{t_m} \|\partial_t \Delta u\|_2^2 \, dt + (1+A)^2 \tau \int_0^{t_m} \|\partial_t u\|_2^2 \, dt \\ &\lesssim \tau (1 + t_m) + (1+A)^2 \tau \\ &\lesssim (1+A)^2 \tau \cdot (1 + t_m) \ .\end{aligned} \tag{4.12}$$ {?}

On the other hand, by the high Sobolev bound lemma (Lemma 4.4) $\sup_{t\geq 0} \|u(t)\|_{H^s} \lesssim_s 1$, we have $\sup_{n\geq 0} \|f(u(t_n))\|_{H^s} \lesssim_s 1$. We can then derive that

$$\begin{aligned}
\|\Pi_{>N} f(u(t_n))\|_2^2 &= \sum_{|k|>N} \left| \widehat{f(u(t_n))}(k) \right|^2 \\
&\leq \sum_{|k|>N} |k|^{2s} \left| \widehat{f(u(t_n))}(k) \right|^2 \cdot |k|^{-2s} \\
&\lesssim N^{-2s} \cdot \sum_{|k|>N} |k|^{2s} \left| \widehat{f(u(t_n))}(k) \right|^2 \\
&\lesssim N^{-2s} \cdot \|f(u(t_n))\|_{H^s}^2 \\
&\lesssim N^{-2s} \ ,
\end{aligned}$$

thus

$$\sum_{n=0}^{m-1} \|\Pi_{>N} f(u(t_n))\|_2^2 \lesssim_s m \cdot N^{-2s} \lesssim \frac{t_m N^{-2s}}{\tau} \ .$$

Therefore,

$$\tau \sum_{n=0}^{m-1} \left(\|G_n\|_2^2 + \|\Pi_{>N} f(u(t_n))\|_2^2 \right) \lesssim_s (1+t_m)(\tau^2 + N^{-2s})(1+A)^2 \ .$$

Similarly, we have

$$\|u^0 - u(0)\|_2^2 = \|\Pi_N u_0 - u_0\|_2^2 \lesssim N^{-2s} \ .$$

Applying the auxiliary solutions estimate in Proposition 4.1 and noting that $t_m = m\tau$, we can get

$$\|u^m - u(t_m)\|_2^2 \lesssim_s (1+A)^2 e^{Ct_m} \left(N^{-2s} + \tau \cdot N^{-2s} + (1+t_m)(\tau^2 + N^{-2s}) \right) \ .$$

Note that

$$\begin{cases} \tau \cdot N^{-2s} \lesssim \tau^2 + N^{-4s} \lesssim \tau^2 + N^{-2s} \\ 1 + t_m \lesssim e^{C' t_m} \ , \end{cases}$$

which leads to

$$\|u^m - u(t_m)\|_2^2 \lesssim_s (1+A)^2 e^{Ct_m} \left(N^{-2s} + \tau^2 \right) \ .$$

Thus

$$\|u^m - u(t_m)\|_2 \leq (1+A) \cdot C_2 \cdot e^{C_1 t_m} \left(N^{-s} + \tau \right) \ , \tag{4.13}$$ {?}

where $C_1 > 0$ is a constant depending on ν , u_0; $C_2 > 0$ is a constant depending on s , ν and u_0. This completes the proof of L^2 error estimate. □

5. Stability of a first order semi-implicit scheme for the 2D fractional Cahn-Hilliard equation

In this section, we will show Theorem 1.4. As mentioned earlier, the fractional Cahn-Hilliard equation behaves as an “interpolation” between Allen-Cahn equation and original Cahn-Hilliard equation:

$$\begin{cases} \partial_t u = \nu \Delta \left((-\Delta)^\alpha u + (-\Delta)^{\alpha-1} f(u) \right) \ , \ 0 < \alpha \leq 1 \\ u(x,0) = u_0. \end{cases}$$

In this section, we stick to the same region, two dimensional 2π-periodic torus $\mathbb{T}^2 = \mathbb{R}^2 / 2\pi\mathbb{Z}^2$. $f(u) = u^3 - u$ and the energy $E(u) = \int_{\mathbb{T}^2} \left(\frac{\nu}{2} |\nabla u|^2 + F(u) \right) \, dx$, with $F(u) = \frac{1}{4}(u^2-1)^2$. Recall that the semi-implicit scheme (1.4) is given by the following:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = -\nu (-\Delta)^{\alpha+1} u^{n+1} - (-\Delta)^\alpha A (u^{n+1} - u^n) - (-\Delta)^\alpha \Pi_N f(u^n) \\ u^0 = \Pi_N u_0. \end{cases}$$

Proof of Theorem 1.4. The proof adopts a similar computation given in previous section. We recall the scheme (1.4):

$$\frac{u^{n+1}-u^n}{\tau} = -\nu(-\Delta)^{\alpha+1}u^{n+1} - (-\Delta)^{\alpha}A(u^{n+1}-u^n) - (-\Delta)^{\alpha}\Pi_N f(u^n) \ .$$

Now we multiply the equation by $(-\Delta)^{-\alpha}(u^{n+1}-u^n)$ and apply the fundamental theorem of calculus as in section 3. We then obtain

$$\begin{aligned}&\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2}^2 + \frac{\nu}{2}\left(\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \|\nabla u^{n+1}\|_{L^2}^2 - \|\nabla u^n\|_{L^2}^2\right)\\ &+A\|u^{n+1}-u^n\|_{L^2}^2 = -\left(f(u^n), u^{n+1}-u^n\right) \ .\end{aligned}$$

This then implies that

$$\begin{aligned}&\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2}^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 + \left(A+\frac{1}{2}\right)\|u^{n+1}-u^n\|_{L^2}^2 + E^{n+1} - E^n\\ &\le \|u^{n+1}-u^n\|_{L^2}^2\left(\|u^n\|_\infty^2 + \frac{1}{2}\|u^{n+1}\|_\infty^2\right) \ .\end{aligned} \tag{5.1}$$

Fracineq_1

It is clear that the first two norms $\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2}^2$ and $\frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2$ will be hard to control as we will expect more help from $\|u^{n+1}-u^n\|_{L^2}^2$.

Lemma 5.1. There exists a constant $C_{\alpha\nu\tau}$ that is determined by α, ν and τ, such that

$$\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2 \ge C_{\alpha\nu\tau}\|u^{n+1}-u^n\|_{L^2(\mathbb{T}^2)}^2 \ .$$

Proof. It is natural to examine the above norms $\frac{1}{\tau}\||\nabla|^{-\alpha}(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2$ and $\frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_{L^2(\mathbb{T}^2)}^2$ on the Fourier side. Then we obtain that

$$\begin{aligned}&\frac{1}{\tau}\sum_{k\neq 0}|k|^{-2\alpha}|\widehat{u^{n+1}}(k)-\widehat{u^n}(k)|^2 + \frac{\nu}{2}\sum_{k\neq 0}|\widehat{u^{n+1}}(k)-\widehat{u^n}(k)|^2\\ &= \sum_{k\neq 0}|\widehat{u^{n+1}}(k)-\widehat{u^n}(k)|^2\cdot\left(\frac{|k|^{-2\alpha}}{\tau}+\frac{\nu|k|^2}{2}\right) \ .\end{aligned}$$

We apply standard Young's inequality for product to estimate: $ab \le \frac{a^\gamma}{\gamma} + \frac{b^\beta}{\beta}$, with $\frac{1}{\gamma}+\frac{1}{\beta}=1$. We then take $a=|k|^p$, $b=|k|^q$, where $p+q=0$. To fulfill the condition, we choose γ, β, p and q as follows:

$$\begin{cases} p = \dfrac{-2\alpha}{\alpha+1}\\ q = \dfrac{2\alpha}{\alpha+1}\\ \gamma = \alpha+1\\ \beta = \dfrac{\alpha+1}{\alpha} \ . \end{cases} \implies \begin{cases} -2\alpha = p\gamma\\ 2 = q\beta \end{cases} .$$

Therefore, we have

$$\begin{cases} a^\gamma = |k|^{p\gamma} = |k|^{-2\alpha}\\ b^\beta = |k|^{q\beta} = |k|^2 \end{cases} .$$

As a result, we obtain that

$$\begin{aligned}&\sum_{k\neq 0}|\widehat{u^{n+1}}(k)-\widehat{u^n}(k)|^2\cdot\left(\frac{|k|^{-2\alpha}}{\tau}+\frac{\nu|k|^2}{2}\right)\\ =&\sum_{k\neq 0}|\widehat{u^{n+1}}(k)-\widehat{u^n}(k)|^2\cdot\left[\frac{\alpha+1}{\tau}\cdot\left(\frac{|k|^{-2\alpha}}{\alpha+1}\right)+\frac{\nu(\alpha+1)}{2\alpha}\cdot\left(\frac{|k|^2}{\frac{\alpha+1}{\alpha}}\right)\right]\\ \ge&\sum_{k\neq 0}|\widehat{u^{n+1}}(k)-\widehat{u^n}(k)|^2\cdot\left(\frac{\alpha+1}{\tau}\right)^{\frac{1}{\alpha+1}}\cdot\left(\frac{\nu(\alpha+1)}{2\alpha}\right)^{\frac{\alpha+1}{\alpha}} \ .\end{aligned}$$

Clearly it suffices to take $C_{\alpha\tau\nu} = \left(\frac{\alpha+1}{\tau}\right)^{\frac{1}{\alpha+1}} \cdot \left(\frac{\nu(\alpha+1)}{2\alpha}\right)^{\frac{\alpha+1}{\alpha}}$. □

Remark 5.2. In the proof above, $C_{\alpha\tau\nu} \to \infty$ as $\alpha \to 0$. As a result, the method above will not work for the (AC) case.

Back to the proof of Theorem 1.4, (5.1) leads to

$$(A + \frac{1}{2} + C_{\alpha\tau\nu})\|u^{n+1} - u^n\|_{L^2}^2 + E^{n+1} - E^n \le \|u^{n+1} - u^n\|_{L^2}^2 \left(\|u^n\|_\infty^2 + \frac{1}{2}\|u^{n+1}\|_\infty^2\right) .$$

To prove $E^{n+1} \le E^n$, it suffices to show $A + \frac{1}{2} + C_{\alpha\tau\nu} \ge \frac{3}{2}\max\left\{\|u^{n+1}\|_\infty^2 \ , \ \|u^n\|_\infty^2\right\}$. We rewrite the scheme (1.4) as

$$u^{n+1} = \frac{1 + A\tau(-\Delta)^\alpha}{1 + \tau\nu(-\Delta)^{\alpha+1} + A\tau(-\Delta)^\alpha}u^n - \frac{\tau(-\Delta)^\alpha}{1 + \tau\nu(-\Delta)^{\alpha+1} + A\tau(-\Delta)^\alpha}\Pi_N[f(u^n)] .$$

Similarly, we can still apply Lemma 3.1 under the assumption u_0 satisfies zero-mean condition. Recall that

$$\|u^{n+1}\|_\infty \lesssim \|u^{n+1}\|_{\dot{H}^1}\sqrt{\log(\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}} + 3)} .$$

We will estimate $\|u^{n+1}\|_{\dot{H}^1}$ and $\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}}$. As we did in section 3,

$$\begin{cases} \dfrac{1 + A\tau|k|^{2\alpha}}{1 + A\tau|k|^{2\alpha} + \nu\tau|k|^{2+2\alpha}} \cdot |k| \lesssim |k| \\ \dfrac{\tau|k|^{2\alpha}}{1 + A\tau|k|^{2\alpha} + \nu\tau|k|^{2+2\alpha}} \cdot |k| \lesssim \dfrac{\tau}{\tau A}|k| = \dfrac{1}{A}|k| \end{cases} .$$

Hence we derive

$$\|u^{n+1}\|_{\dot{H}^1(\mathbb{T}^2)} \lesssim \left(1 + \frac{1}{A} + \frac{3\|u\|_\infty^2}{A}\right)\|u^n\|_{\dot{H}^1(\mathbb{T}^2)} ,$$

which is the same argument as before. Similarly, we can derive

$$\|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^2)} \lesssim \left(\frac{1 + A\tau}{\nu\tau} + \frac{1}{\nu}\right)(E^n + 1) .$$

We then prove by induction again.

Step 1: The induction $n \to n+1$ step. Assume $E^n \le E^{n-1} \le \cdots \le E^0$ and $E^n \le \sup_N \ E(\Pi_N u_0)$, we will show $E^{n+1} \le E^n$. This implies $\|u^n\|_{\dot{H}^1}^2 = \|\nabla u^n\|_{L^2}^2 \le \frac{2E^n}{\nu} \le \frac{2E^0}{\nu}$. By applying the main lemma carefully and $E^0 \lesssim E_0 + 1$,

$$\|u^n\|_\infty^2 \lesssim_{E_0} \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right) + \nu^{-1}|\log(\tau)| + 1 \quad .$$

Define $m_0 := \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right)$, then the inequality above can be written as

$$\|u^n\|_\infty^2 \lesssim_{E_0} m_0 + \nu^{-1}|\log(\tau)| + 1 . \tag{5.2}$$ 1stFrac_Infinity_

Similarly,

$$\|u^{n+1}\|_\infty^2 \lesssim_{E_0} 1 + \frac{m_0^3}{A^2} + m_0 + \nu^{-3}|\log(\tau)|^3. \tag{5.3}$$ 1stFrac_Infinity_

Therefore we require the following condition:

$$\begin{cases} A + \dfrac{1}{2} + (\dfrac{\alpha+1}{\tau})^{\frac{1}{\alpha+1}} \cdot (\dfrac{\nu(\alpha+1)}{2\alpha})^{\frac{\alpha+1}{\alpha}} \ge C(E_0)\left(m_0 + 1 + \dfrac{m_0^3}{A^2} + \nu^{-3}|\log(\tau)|^3\right) \\ m_0 = \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right) \ . \end{cases}$$

Now we discuss 2 cases again:

`Case 1:` $(\frac{\alpha+1}{\tau})^{\frac{1}{\alpha+1}} \cdot (\frac{\nu(\alpha+1)}{2\alpha})^{\frac{\alpha+1}{\alpha}} \ge C(E_0)\nu^{-3}|\log(\tau)|^3$. In this case, it suffices to choose A such that

$$A \gg_{E_0} m_0 = \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right) .$$

In fact, for $\nu \gtrsim 1$, we can take $A \gg_{E_0} 1$; if $0 < \nu \ll 1$, we will choose $A = C_{E_0} \cdot \nu^{-1}|\log\nu|$, where C_{E_0} is a large constant depending only on E_0. Therefore in both cases it suffices to choose

$$A = C_{E_0} \cdot \max\left\{\nu^{-1}|\log(\nu)| \ , \ 1\right\} .$$

`Case 2`:$(\frac{\alpha+1}{\tau})^{\frac{1}{\alpha+1}} \cdot (\frac{\nu(\alpha+1)}{2\alpha})^{\frac{\alpha+1}{\alpha}} \leq C(E_0)\nu^{-3}|\log(\tau)|^3$. This implies $(\frac{1}{\tau})^{\frac{1}{\alpha+1}} \lesssim (\frac{1}{\nu})^{-4-\frac{1}{\alpha}}$, hence $|\log(\tau)| \lesssim_{E_0} 1 + |\log(\nu)|$ for fixed $0 < \alpha \leq 1$. Now going back to equations (5.2), we have

$$\|u^n\|_\infty^2 \lesssim_{E_0} m_0$$

as $\nu^{-1}|\log(\tau)|$ will be dominated by m_0, recall that $m_0 = \nu^{-1}\left(1 + \log(A) + |\log(\nu)|\right)$. Substituting this new bound to (5.3), we derive that

$$\begin{aligned}
\|u^{n+1}\|_\infty^2 &\lesssim \left(1 + (\frac{1 + \|u^n\|_\infty^2}{A})\|u^n\|_{\dot{H}^1}\sqrt{\log(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}})}\right)^2 \\
&\lesssim_{E_0} \left(1 + (1 + \frac{m_0}{A})(\sqrt{\frac{1}{\nu}}\sqrt{\log(3 + \|u^{n+1}\|_{\dot{H}^{\frac{3}{2}}})}\right)^2 \\
&\lesssim_{E_0} \left(1 + (1 + \frac{m_0}{A})\sqrt{m_0}\right)^2 \\
&\lesssim_{E_0} 1 + \frac{m_0^3}{A^2} + m_0 \ .
\end{aligned}$$

Thus it suffices to take

$$A \geq C_{E_0} m_0 \ . \tag{5.4}$$ {?}

For the induction base **Step 2**, the proof is exactly the same as in Section 3 and this shows stability of the semi-implicit scheme in the fractional Cahn-Hilliard case. A numerical result is given in Figure 2 where we choose $\nu = 0.1$, A=1, $u_0 = \sin(x)\sin(y)$, $\tau = 0.01$, $N_x = N_y = 256$ and $\alpha = 0.5$. □

6. Concluding remark

Throughout this paper, we discussed certain first order semi-implicit Fourier spectral methods on the Allen-Cahn equation and the fractional Cahn-Hilliard equation in a two dimensional torus. We proved the stability (energy decay) of the first order numerical scheme by adding a stabilizing term $A(u^{n+1} - u^n)$ and $(-\Delta)^\alpha A(u^{n+1} - u^n)$ with a large constant A at least of size $O(\nu^{-1}|\log(\nu)|)$. Note that this stability is preserved independent of time step τ. We also proved a L^2 error estimate between numerical solutions from the semi-implicit scheme and exact solutions.

In the future work, more cases can be discussed on other gradient cases (as mentioned in Remark 1.8) such as general nonlocal Allen-Cahn and Cahn-Hilliard equations, MBE equations, and other equations describing phenomena of interest in material sciences. Higher order schemes and more nonlinear numerical framework will be considered.

Acknowledgement

The author was partially supported by the Shanghai “Super Postdoc” Incentive Plan (No. 2021014) and China Postdoctoral Science Foundation (Grant No. 2022M710796, 2022T150139). This work was partially completed when the author was at the University of British Columbia and the author would like to thank Prof. Brian Wetton for the fruitful discussion.

References

FracCH [1] M. Ainsworth and Z. Mao. Analysis and Approximation of a Fractional Cahn–Hilliard Equation. *SIAM J. Numer. Anal.*, 55(4): 1689–1718, 2017.

ABC [2] N. Alikakos, P. Bates, and X. Chen. Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Arch. Ration. Mech. An.*, 128(2): 165–205, 1994.

OG_AC [3] S. M. Allen and J. W. Cahn. Ground state structures in ordered binary alloys with second neighbor interactions. *Acta Metallurgica.*, 20(3): 423–433, 1972.

BLW22 [4] G. Bai, B. Li, and Y. Wu. A constructive low-regularity integrator for the 1d cubic nonlinear Schrödinger equation under the Neumann boundary condition. *IMA J. Numer. Anal.*, to appear.

Bertozzi [5] A. Bertozzi, N. Ju, and H.-W. Lu. A biharmonic-modified forward time stepping method for fourth order nonlinear diffusion equations. *Discrete Cont. Dyn. Syst.*, 29(4): 1367–1391, 2011.

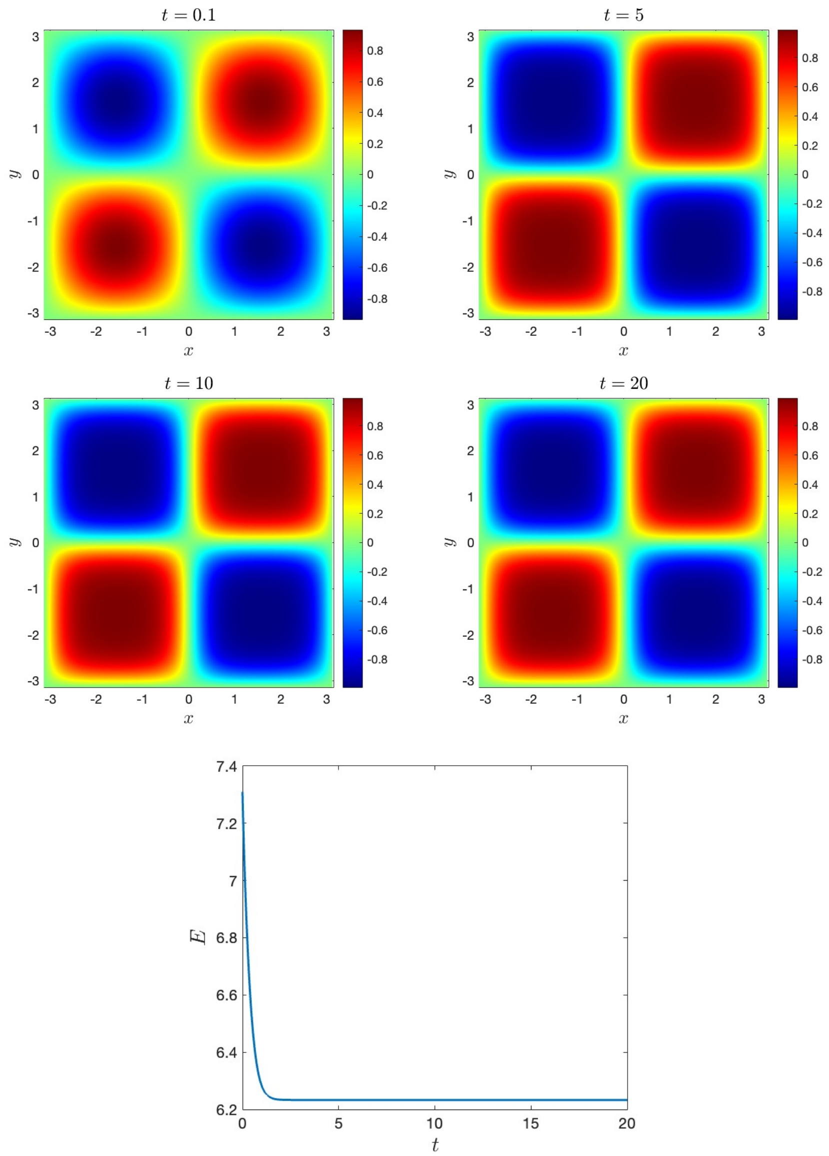

FIGURE 2. Dynamics of 2D fractional Cahn-Hilliard equation using semi-implicit scheme where $\nu = 0.1$, A=1, $u_0 = \sin(x)\sin(y)$, $\tau = 0.01$, $N_x = N_y = 256$ and $\alpha = 0.5$.

⟨fig:2dfch⟩

FracCH_2 [6] J. Bosch and M. Stoll. A fractional inpainting model based on the vector-valued Cahn-Hilliard equation. *SIAM J. Imaging. Sci.*, 8(4): 2352–2382, 2015.

Bourgain1 [7] J. Bourgain and D. Li. Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces. *Invent. Math.*, 201(1): 97–157, 2015.

Bourgain2 [8] J. Bourgain and D. Li. Strong illposedness of the incompressible Euler equation in integer C^m spaces. *Geom. Funct. Anal.*, 25(1): 1–86, 2015.

OG [9] J. W. Cahn and J. E. Hilliard. Free Energy of a Nonuniform System. I. Interfacial Free Energy. *J. Chem. Phys.*, 28(2): 258–267, 1958.

ShenChen [10] L.-Q. Chen and J. Shen. Applications of semi-implicit Fourier-spectral method to phase field equations. *Comput. Phys. Comm.*, 108(2–3): 147–158, 1998.

C23b [11] X. Cheng. Energy stable semi-implicit schemes for the 3D Allen-Cahn equation. Preprint.

?C23c? [12] X. Cheng. Second order energy stable semi-implicit schemes for the 2D Allen-Cahn equation. Preprint.

CLPW21 [13] X. Cheng, D. Li, K. Promislow and B. Wetton. Asymptotic Behaviour of Time Stepping Methods for Phase Field Models, *J. Sci. Comput.*, 86(3): 1–34, 2021.

CLQY21 [14] X. Cheng, D. Li, C. Quan and W. Yang. On a parabolic Sine-Gordon model, *Numerical Mathematics: Theory, Methods and Applications.*, 14(4): 1068–1084, 2021.

Brian [15] A. Christlieb, J. Jones, K. Promislow, B. Wetton, and M. Willoughby. High accuracy solutions to energy gradient flows from material science models. *J. Comput. Phys.*, 257: 193–215, 2014.

H4 [16] C. M. Elliott and S. Zheng. On the Cahn-Hilliard equation. *Arch. Ration. Mech. An.*, 96(4): 339–357, 1986.

E98 [17] D. J. Eyre. Unconditionally gradient stable time marching the Cahn-Hilliard equation. *MRS Proc.* 529(39), 1998.

FLWZ23 [18] M. Fei, F. Lin, W. Wang, and Z. Zhang. Matrix-valued Allen–Cahn equation and the Keller–Rubinstein–Sternberg problem. *Inventiones mathematicae*, 1-80, 2023.

FengProhl [19] X. Feng and A. Prohl. Error analysis of a mixed finite element method for the Cahn-Hilliard equation. *Numerische Mathematik.*, 99(1): 47–84, 2004.

Keith [20] G. Gavish, J. Jones, Z. X. Xu, A. Christlieb, and K. Promislow. Variational Models of Network Formation and Ion Transport: Applications to Perfluorosulfonate Ionomer Membranes. *Polymers.*, 4(1): 630–655, 2012.

HeLiuTang [21] Y. He, Y. Liu, and T. Tang. On large time-stepping methods for the Cahn–Hilliard equation. *Appl. Numer. Math.*, 57(5–7): 616–628, 2007.

AC_MCF [22] T. Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differ. Geom.*, 38: 417–461, 1993.

bL22 [23] B. Li. Maximal regularity of multistep fully discrete finite element methods for parabolic equations. *IMA J. Numer. Anal.*, 42: 1700–1734, 2022.

L22 [24] D. Li. Why large time-stepping methods for the Cahn-Hilliard equation is stable. *Math. Comp.*, 91: 2501–2515, 2022.

LQT22 [25] D. Li, C. Quan, and T. Tang. Stability and convergence analysis for the implicit-explicit method to the Cahn-Hilliard equation. *Math. Comp.*, 91(334): 785–809, 2022.

LQTY22 [26] D. Li, C. Quan, T. Tang, and W. Yang, On symmetry breaking of Allen-Cahn. *CSIAM Trans. Appl. Math.*, 3: 221–243, 2022.

LQX22a [27] D. Li, C. Quan, and J. Xu. Stability and convergence of Strang splitting. Part I: Scalar Allen-Cahn equation. *J. Comput. Phys.*, 458(111087), 2022.

LQX22b [28] D. Li, C. Quan, and J. Xu. Stability and convergence of Strang splitting. Part II: Tensorial Allen-Cahn equations. *J. Comput. Phys.*, 454(110985), 2022.

LiTao2020_2 [29] D. Li and T. Tang. Stability of semi-implicit methods for the Cahn-Hilliard equation with logarithmic potentials. *Ann. Appl. Math..*, 37(1): 31–60, 2021.

?LJYC23? [30] Q. Liu, J. Jing, M. Yuan and W. Chen. A Positivity-preserving, energy stable BDF2 scheme with variable steps for the Cahn–Hilliard equation with logarithmic potential. *J. Sci. Comput.*, 95(37), 2023.

Pego [31] R. Pego. Front migration in the nonlinear Cahn-Hilliard equation. *Proc. Roy. Soc. London.*, A(422): 261–278, 1989.

ShenYang10_2 [32] J. Shen and X. Yang. Energy stable schemes for Cahn-Hilliard phase-field model of two-phase incompressible flows. *Chinese Ann. Math.*, 31(5): 743–758, 2010.

ShenYang10 [33] J. Shen and X. Yang. Numerical approximations of Allen–Cahn and Cahn–Hilliard equations. *Discrete Cont. Dyn. Syst.*, 28(4): 1669–1691, 2010.

TangYang16 [34] T. Tang and J. Yang. Implicit-explicit scheme for the Allen-Cahn equation preserves the maximum Principle. *J. Comput. Math.*, 34(5): 471–481, 2016.

Taotang [35] C. Xu and T. Tang. Stability analysis of large time-stepping methods for epitaxial growth models. *SIAM J. Numer. Anal.*, 44(4): 1759–1779, 2006.

Shen [36] J. Zhu, L.-Q. Chen, J. Shen, and V. Tikare. Coarsening kinetics from a variable-mobility Cahn-Hilliard equation: Application of a semi-implicit Fourier spectral method. *Phys. Rev. E.*, 60(4): 3564–3572, 1999.

Xinyu Cheng
School of mathetmatical sciences, Fudan University, Shanghai, China
xycheng@fudan.edu.cn