Congestion Control in Compartmental Network Systems

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Abstract

In many practical applications of control engineering, the dynamical system under consideration is described by a compartmental network system. This means that the system is governed by a law of mass conservation and that the state variables are constrained to remain non-negative along the system trajectories. In such systems, network congestion arises when the inflow demand exceeds the throughput capacity of the network. When congestion occurs, some links of the network are saturated with the undesirable consequence that there is an overflow of some compartments. Our contribution in this paper is to show that congestion can be automatically prevented by using a nonlinear output feedback controller having an appropriate compartmental structure.

Keywords : compartmental system, congestion control, nonlinear system.

1. INTRODUCTION

In many practical applications of control engineering, the dynamical system under consideration is described by a so-called *compartmental network system* which is *conservative* and *positive*. This means that the system is governed by a law of mass conservation and that the state variables are constrained to remain non-negative along the system trajectories.

The dynamics of compartmental systems with constant inputs have been extensively treated in the literature for more than thirty years (see the tutorial paper [19] and also, for instance, [1], [4], [6], [7], [9], [14], [20], [21], [22], [24], [26]).

In contrast, the control of compartmental systems has received much less attention. Recently, feedback control for *set stabilisation* of positive systems (including compartmental systems) is a topic that has been treated in [2], [3], [5], [17], [18].

In this paper, we are concerned with another issue : the congestion control problem. Network congestion arises in compartmental network systems when the inflow demand exceeds the throughput capacity of the network. When congestion occurs, some links of the network are saturated with the highly undesirable consequence that there is an overflow of some compartments. Our purpose in this paper is to show that congestion can be automatically prevented by using a nonlinear output feedback controller having an appropriate compartmental structure. More precisely we propose an output feedback control scheme able to achieve the objective of congestion avoidance and to satisfy an inflow demand that does not exceeds the transmission capacity.

In order to emphasize the relevance of the congestion control problem addressed in the present paper, we would like to refer to two concrete, although rather different, applications that we have previously treated :

1. The plugging phenomenon in grinding circuits is a well-known critical industrial congestion problem. The dynamics of grinding circuits may typically and efficiently be described by compartmental network systems. The control design presented in this paper is an interesting output feedback alternative to the state feedbak control strategies that have been discussed in [2] and [11].

2. Congestion control in packet switched networks is an issue that has received a lot of attention in the computer science literature. We have shown in [12], [13] that compartmental network systems can constitute a valuable fluid flow modelling approach for such networks and can be used for analysing *hop-by-hop* congestion control strategies. The congestion control approach followed in this paper is different : it illustrates that compartmental fluid flow modelling can also be used to address the so-called *end-to-end* congestion control problem.

Compartmental network systems are defined in Section 2. They have numerous interesting structural properties which are well documented in the literature (see the references). Some of these properties which are useful for our purpose are briefly reviewed in Section 2. In particular, the equilibrium stability properties of cooperative compartmental systems are emphasized. Our contribution is in Section 3 where the proposed controller is presented. The main properties of the closed loop system and the controller are studied. It is shown that the two main objectives of the congestion control are achieved : (i) a demand which is not in excess is automatically satisfied but (ii) in case of an excess demand, an operation without overflow is automatically guaranteed. Furthermore, if the controlled network is cooperative, then the closed loop system has a unique globally asymptotically stable equilibrium. A brief simulation experiment is used in Section 4 to illustrate the validity of the control scheme and some design issues. Some final comments are given in Section 5.

2. COMPARTMENTAL NETWORK SYSTEMS

A compartmental network system is a network of conceptual storage tanks called compartments as illustrated in Fig. 1. Each node of the network represents a compartment which contains a variable quantity $x_i(t)$ of some material or immaterial "species" involved in the

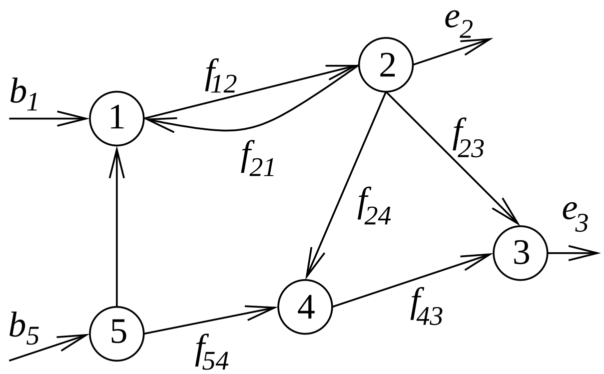


Figure 1: Example of compartmental network

system. The vector $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is the state vector of the system. Each directed arc $i \rightarrow j$ represents a mass transfer which may hold for various transport, transformation or interaction phenomena between the species inside the system. The transfer rate, called *flow* or *flux*, from a compartment i to another compartment j is a function of the state variables denoted $f_{ij}(x(t))$. Additional input and output arcs represent the interactions with the surroundings : either inflows $b_i(t)$ injected from the outside into some compartments or outflows $e_i(x(t))$ from some compartments to the outside.

The instantaneous flow balances around the compartments are expressed by the following set of equations :

$$\dot{x}_i = \sum_{j \neq i} f_{ji}(x) - \sum_{k \neq i} f_{ik}(x) - e_i(x) + b_i \qquad i = 1, \ldots, n \tag{1}$$

In these equations, only the terms corresponding to actual links of the network are made explicit. Otherwise stated, all the b_i, e_i and f_{ij} for non existing links do not appear in the equations.

The model (1) makes sense only if the state variables $x_i(t)$ remain *non-negative*[1] for all t : $x_i(t) \in I\!R_+$. The flow functions f_{ij} and e_i are defined to be non-negative on the non-negative orthant : $f_{ij} : I\!R_+^n \rightarrow I\!R_+$, $e_i : I\!R_+^n \rightarrow I\!R_+$. Similarly the inflows b_i are defined to be non-negative $b_i(t) \in I\!R_+ \forall t$. Moreover, it is obvious that there cannot be a positive flow from an empty compartment :

$$x_i = 0 \quad \Longrightarrow \quad f_{ij}(x) = 0 \quad \text{and} \quad e_i(x) = 0 \tag{2}$$

Under condition (2), if $f_{ij}(x)$ and $e_i(x)$ are differentiable, they can be written as :

$$f_{ij}(x) = r_{ij}(x)x_i \qquad e_i(x) = q_i(x)x_i$$

[1]**Notation.** The set of non-negative real numbers is denoted $I\!R_+ = \{a \in I\!R, a \geq 0\}$ as usual. For any integer n, the set $I\!R_+^n$ is called the "positive orthant".

for appropriate functions $r_{ij}(x)$ and $q_i(x)$ which are defined on $I\!R^n_+$, non-negative and at least continuous. These functions are called *specific flows* (or also *fractional rates*). In this paper, we shall assume that the specific flows $r_{ij}(x)$ and $q_i(x)$ are continuously differentiable and strictly positive functions of their arguments in the positive orthant :

$$r_{ij}(x) > 0 \text{ and } q_i(x) > 0 \ \forall x \in I\!R^n_+$$

In other words, we assume that the flows f_{ij} and e_i vanish only if $x_i = 0$. It is a natural assumptions which is satisfied in many physical and engineering models described by compartmental models.

With these definitions and notations, the compartmental system (1) is written :

$$\dot{x}_i = \sum_{j \neq i} r_{ji}(x)x_j - \sum_{k \neq i} r_{ik}(x)x_i - q_i(x)x_i + b_i \qquad i = 1, \dots, n \tag{3}$$

State-space models of this form are used to represent, for instance, industrial processes (like distillation columns [25], chemical reactors [17], heat exchangers, grinding circuits [11]), queuing systems [8] and communication networks [12], ecological and biological processes [19], [26], etc.

Compartmental network systems have numerous interesting structural properties which are widely documented in the literature (see the references). Some of these properties are listed hereafter.

First of all, as expected, a compartmental system is positive.

Definition 1. ***Positive System*** **(e.g.[23]).** A dynamical system $\dot{x} = f(x,t) \ \ x \in I\!R^n$ is positive if

$$x(0) \in I\!R^n_+ \Longrightarrow x(t) \in I\!R^n_+ \ \ \forall t \geq 0.$$

■

Property 1. ***A compartmental network system is a positive system.*** The system (3) is a *positive* system. Indeed, if $x \in I\!R^n_+$ and $x_i = 0$, then $\dot{x}_i = \sum_{j \neq i} r_{ji}(x)x_j + b_i \geq 0$. This is sufficient to guarantee the forward invariance of the non negative orthant if the functions $r_{ij}(x)$ and $q_i(x)$ are differentiable. ■

The total mass contained in the system is

$$M(x) = \sum_{i=1}^{n} x_i$$

A compartmental system is *mass conservative* in the sense that the mass balance is preserved inside the system. This is easily seen if we consider the special case of a closed system without inflows and outflows.

Property 2. ***Mass conservation.*** A compartmental network system (3) is dissipative with respect to the supply rate $w(t) = \sum_i b_i(t)$ with the total mass $M(x)$ as storage function. In the special case of a closed system without inflows ($b_i = 0, \forall i$) and without outflows ($e_i(x) = 0, \forall i$), it is easy to check that $dM(x)/dt = 0$ which shows that the total mass is indeed conserved. ■

The system (3) is written in matrix form as:

$$\dot{x} = A(x)x + b \tag{4}$$

where $A(x)$ is a so-called *compartmental matrix* with the following properties:

1. $A(x)$ is a *Metzler* matrix, i.e. a matrix with non-negative off-diagonal entries:

$$a_{ij}(x) = r_{ji}(x) \geq 0$$

   (note the inversion of indices !)

2. The diagonal entries of $A(x)$ are non-positive:

$$a_{ii}(x) = -q_i(x) - \sum_{j \neq i} r_{ij}(x) \leq 0$$

3. The matrix $A(x)$ is diagonally dominant:

$$|a_{ii}|(x) \geq \sum_{j \neq i} a_{ji}(x)$$

The invertibility and the stability of a compartmental matrix is closely related to the notion of *outflow connectivity* as stated in the following definition.

Definition 2. ***Outflow and inflow connected network.*** A compartment i is said to be *outflow connected* if there is a path $i \rightarrow j \rightarrow k \rightarrow \ldots \rightarrow \ell$ from that compartment to a compartment ℓ from which there is an outflow $q_\ell(x)$. The network is said to be *fully outflow connected* (FOC) if all compartments are outflow connected.

A compartment ℓ is said to be *inflow connected* if there is a path $i \rightarrow j \rightarrow k \rightarrow \ldots \rightarrow \ell$ to that compartment from a compartment i into which there is an inflow b_i. The network is said to be *fully inflow connected* (FIC) if all compartments are inflow connected. ■

Property 3. ***Invertibility and stability of the compartmental matrix*** **([9],[19]).** The compartmental matrix $A(x)$ is non singular and stable $\forall x \in I\!\!R^n_+$ if and only if the compartmental network is fully outflow connected. This shows that the non-singularity and the stability of a compartmental matrix can be directly checked by inspection of the associated compartmental network. ■

The Jacobian matrix of the system (4) is defined as :

$$J(x) = \frac{\partial[A(x)x]}{\partial x}$$

When the Jacobian matrix has a compartmental structure, the off-diagonal entries are non-negative and the system is therefore *cooperative* ([15], [16]). We then have the following interesting stability property.

Property 4. ***Equilibrium stability with a compartmental Jacobian matrix.*** Let us consider the system (4) with *constant* inflows : b_i = constant $\forall i$.

a) If $J(x)$ is a compartmental matrix $\forall x \in I\!R^n_+$, then all bounded trajectories tend to an equilibrium in $I\!R^n_+$.

b) If there is a compact convex set $D \subset I\!R^n_+$ which is *forward invariant* and if $J(x)$ is a *non-singular* compartmental matrix $\forall x \in D$, then there is a *unique* equilibrium $\bar{x} \in D$ which is globally asymptotically stable (GAS) in D. ■

A proof of part a) can be found in [19], Appendix 4 (see also [10],[15]). Part b) is a concise reformulation of a theorem by Rosenbrock [25] (see also [26]).

Property 4 requires that the compartmental Jacobian matrix be invertible in order to have a unique GAS equilibrium. This condition is clearly not satisfied for a *closed* system (without inflows and outflows) that necessarily has a *singular* Jacobian matrix. However the uniqueness of the equilibrium is preserved for closed systems that are strongly connected.

Property 5 ***Equilibrium unicity for a fully connected closed system.*** If a closed system with a compartmental Jacobian matrix is strongly connected (i.e. there is a directed path $i \rightarrow j \rightarrow k \rightarrow \ldots \rightarrow \ell$ connecting any compartment i to any compartment ℓ), then, for any constant $M_0 > 0$, the hyperplane $H = \{x \in I\!R^n_+ : M(x) = M_0 > 0\}$ is forward invariant and there is a unique GAS equilibrium in H. ■

This property is a straightforward extension of Theorem 6 in [24].

3. CONGESTION CONTROL

Network congestion arises in compartmental network systems when the inflow demand exceeds the throughput capacity of the network. The most undesirable symptom of this kind of instability is an unbounded accumulation of material in the system inducing an overflow of the compartments. Our purpose in this paper is to show that congestion can be automatically prevented by using a nonlinear output feedback controller having an appropriate compartmental structure.

The congestion control problem is formulated as follows. We consider a compartmental network system with n compartments, m inflows and p outflows, and we assume that :

1. The network is FIC and FOC;
2. The links of the network have a maximal tranfer capacity : $0 \leq f_{ij}(x) \leq f_{ij}^{max}$ and $0 \leq e_i(x) \leq e_i^{max}$, $\forall x \in I\!R^n_+$;
3. The compartments of the network have a maximal capacity : x_i^{max}, $i = 1, \ldots, n$;
4. There is an *inflow demand* denoted d_i on each input of the network : it is the inflow rate that the user would like to inject into the system or, otherwise stated, that the user would like to assign to the inflow rate b_i.

Then, congestion may occur in the system if the total demand exceeds the maximal achievable throughput capacity of the network which is limited by the maximal transfer capacity of the links. When congestion occurs, some links of the network are saturated with the highly undesirable consequence of an overflow of the compartments that supply the congested links.

In order to allow for congestion control, we assume that, when necessary, the inflow rates $b_i(t)$ injected into the network may be mitigated and made lower than the demand $d_i(t)$. This is expressed as $b_i(t) = u_i(t)d_i(t)$, $0 \leq u_i(t) \leq 1$ where $u_i(t)$ represents the fraction of the inflow demand $d_i(t)$ which is actually injected in the network. We assume furthermore that the outflow rates $e_i(x(t)) \triangleq y_i(t)$ are the measurable outputs of the system. With these definitions and notations, the model is written in state space form :

$$\begin{aligned} \dot{x} &= A(x)x + B(d)u && (5) \\ y &= C(x)x && (6) \end{aligned}$$

with obvious definitions of the matrices $B(d), C(x)$ and the vectors d, u, y.

The control objective is then to define an output feedback controller that is able to achieve the demand as best as possible while avoiding overflows. In order to solve this problem we propose a dynamical nonlinear controller of the following form :

$$\begin{aligned} \dot{z}_i &= y_i - \phi(z_i) \sum_{k \in Q_i} \alpha_{ki} d_k \quad (i \in \mathcal{I}_{out}) \\ u_j(z) &= \sum_{k \in P_j} \alpha_{jk} \phi(z_k) \qquad (j \in \mathcal{I}_{in}) \end{aligned}$$

with the following notations and definitions :

(a) $\mathcal{I}_{in}$ is the index set of the input nodes ($|\mathcal{I}_{in}| = m$);

(b) $\mathcal{I}_{out}$ is the index set of the output nodes ($|\mathcal{I}_{out}| = p$);

(c) $\mathcal{R}$ is the set of node pairs (j, k) (with $j \in \mathcal{I}_{in}$ and $k \in \mathcal{I}_{out}$) such that there is a directed path in the network from the input node j to the output node k;

(d) $P_j = \{k : (j,k) \in \mathcal{R}\} \subset \mathcal{I}_{out}$ is the index set of the output nodes that are reachable from the input node j;

(e) $Q_i = \{k : (k,i) \in \mathcal{R}\} \subset \mathcal{I}_{in}$ is the index set of the input nodes from which the output node i is reachable;

(f) α_{jk} (with $(j,k) \in \mathcal{R}$) are design parameters such that $0 \leq \alpha_{jk} \leq 1$ and $\sum_{k \in P_j} \alpha_{jk} = 1$;

(g) $\phi : I\!R_+ \rightarrow I\!R_+$ is a monotonically increasing and continuously differentiable function with $\phi(0) = 0$ and $\phi(+\infty) = 1$.

The rationale behind the construction of this control law is illustrated in Fig.2. The controller

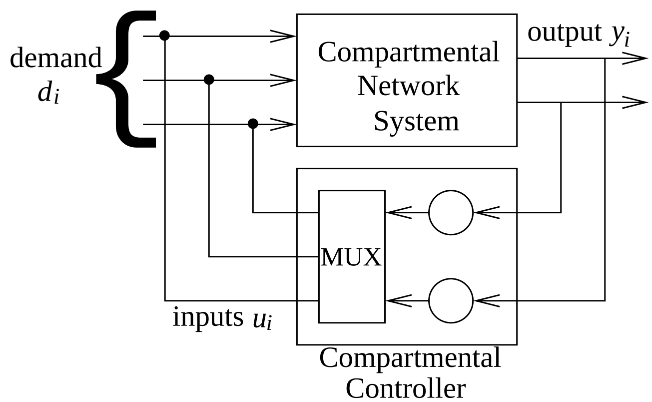


Figure 2: Structure of the closed loop system

has a compartmental structure with as many compartments as outputs y_i in the controlled network. Each compartment of the controller is virtually fed with a copy of one of the outflows of the controlled network. Then, the flows going out of the controller compartments are distributed among the control inputs u_j (this is represented by a multiplexer in Fig. 2) in such a way that there is exactly one connection from a network output k to a network input j through the controller for each $(j,k) \in \mathcal{R}$ (i.e. if there is an inverse connection between a network input j and a network output k through the controlled network).

In matrix form, the control law is written :

$$\dot{z} = G(d)F(z)z + y \tag{7}$$

$$u = K(z)z \tag{8}$$

with $G(d) = \text{diag}\{\sum_{k \in Q_i}(-\alpha_{ki}d_k), i \in \mathcal{I}_{out}\}$ and obvious definitions for the vector z and the matrices $F(z)$ and $K(z)$. It follows that the closed loop system obtained by combining the network (5)-(6) with the controller (7)-(8) is written :

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A(x) & B(d)K(z) \\ C(x) & G(d)F(z) \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \triangleq L(x,z) \begin{pmatrix} x \\ z \end{pmatrix} \tag{9}$$

Let us now analyse the main properties of the control law (7)-(8) and of the closed loop control system (9).

1) We first observe that the matrix $L(x,z)$ in (9) is a compartmental matrix parametrized by d. The closed loop (9) is thus a *closed* compartmental network system. The closed loop system is therefore a positive system (Property 1). Moreover, since the system is closed, the storage function

$$M(x,z) = \sum_{i=1}^{n} x_i + \sum_{j=1}^{p} z_j$$

is invariant (Property 2) and the state trajectories with non-negative initial conditions are confined in the compact invariant set :

$$H = \{(x,z) \in I\!R_+^n \times I\!R_+^p : M(x,z) = M(x(0),z(0)) = \sigma > 0\}$$

2) It follows readily that the state variables are bounded :

$$0 \leq x_i(t) \leq \sigma \;\; (i = 1, n) \quad \text{and} \quad 0 \leq z_j(t) \leq \sigma \;\; (j \in \mathcal{I}_{out})$$

Hence, **the first objective of the congestion control is achieved with the proposed controller** : provided σ is smaller than the maximal capacity of the compartments x_i^{max}, we have the guarantee that no overflow can occur. Furthermore, we observe that the value of σ depends on the initial conditions $(x(0), z(0))$. In many practical applications, it is a natural operation to start the system with empty compartments $x(0) = 0$. The value of σ is then freely assigned by the user which selects the initial conditions of the controller state variables $z_j(0)$ and hence the value of $\sigma = \sum_{j=1}^{p} z_j(0)$.

3) As expected, the controls $u_j(z)$ (i.e. the fractions of the inflow demand achieved by the controller) are confined in the interval $[0,1]$. Indeed, under condition (g) above we have $0 \leq \phi(z_k) \leq 1 \;\; \forall z_k \in I\!R_+$ which, together with condition (f), implies :

$$0 \leq u_j(z) = \sum_{k \in P_j} \alpha_{jk}\phi(z_k) \leq \sum_{k \in P_j} \alpha_{jk} = 1$$

4) Because the controlled network is FIC and FOC, and due to the structure of the controller, it is readily verified that the closed loop system (9) is necessarily a *strongly connected* closed compartmental system (or is a partition of separate strongly connected closed compartmental systems). On the other hand, if the controlled network has a compartmental Jacobian matrix, then the closed loop system also has a compartmental Jacobian matrix. Then, for a constant inflow demand d, the closed loop system has a single GAS equilibrium in the positive orthant (Property 5).

5) The choice of the function ϕ is free provided it satisfies the above condition (g). An appropriate choice is to select an hyperbolic function of the form :

$$\phi(z_j) = \frac{z_j}{z_j + \varepsilon}$$

with ε a small positive constant. This function is of interest because it can be made arbitrarily close to a unit step function by taking ε small enough. In more mathematical terms, for any arbitrarily small $\delta > 0$, there exist a small enough $\varepsilon > 0$ such that $|1 - \phi(z_j)| \leq \delta \quad \forall z_j \geq \delta$. Let us now assume that, for a given constant inflow demand d, the closed loop system (9) has a stable equilibrium $(\bar{x}, \bar{z}) \in H$ with $\bar{z}_i \geq \delta$. Then, for this equilibrium, we have :

$$\begin{aligned}
\sum_{i \in \mathcal{I}_{out}} \bar{y}_i = \sum_{i \in \mathcal{I}_{out}} e_i(\bar{x}) &= \sum_{i \in \mathcal{I}_{out}} \sum_{k \in Q_i} \alpha_{ki} \phi(\bar{z}_i) d_k \\
&\simeq \sum_{i \in \mathcal{I}_{out}} \sum_{k \in Q_i} \alpha_{ki} d_k \quad (\text{because } \phi(\bar{z}_i) \simeq 1) \\
&= \sum_{k \in \mathcal{I}_{in}} (\sum_{i \in P_k} \alpha_{ki}) d_k \\
&= \sum_{k \in \mathcal{I}_{in}} d_k \quad (\text{because } \sum_{i \in P_k} \alpha_{ki} = 1)
\end{aligned}$$

In that case, we see that the total outflow $\sum_{i \in \mathcal{I}_{out}} \bar{y}_i$ is arbitrarily close to the total inflow demand $\sum_{k \in \mathcal{I}_{in}} d_k$. Consequently, **the second objective of the congestion control may be achieved with the proposed controller** : a demand which is not in excess can automatically be satisfied by the feedback controller. It must however be emphasized that this property is **not independent** from the choice of the design parameters α_{ki}. Indeed, for each steady-state output $\bar{y}_i$ at the equilibrium, we have :

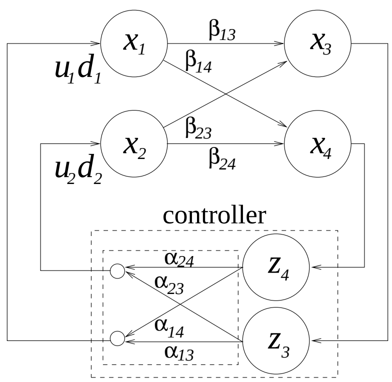
$$\bar{y}_i = e_i(\bar{x}) = \sum_{k \in Q_i} \alpha_{ki} \phi(\bar{z}_i) d_k$$

It follows that the condition $\phi(\bar{z}_i) \simeq 1$ may be satisfied only if each parameter α_{ki} is closed to the steady-state flow fraction that would go from input k to output i in the open-loop system. In less technical terms, the control parameters α_{ki} must be *adapted* to the network in order to achieve the demand as best of possible. If those parameters are not well adapted, there can be a performance degradation which is the price to pay in order to control the congestion and avoid buffer overflows.

6) The proposed congestion controller has an interesting robustness property. In order to build the control law (7)-(8) only the *structure* of the controlled compartmental network must be known. But the control law is independent from the knowledge of the specific flow functions $r_{ij}(x)$ and $e_i(x)$. This is quite important because in many practical applications, an accurate knowledge of these functions is precisely a critical modelling issue.

4. A SIMULATION EXPERIMENT

In this section, a numerical example that illustrates the properties of our controller is presented. The ability of the control law to prevent overflows during congestion periods is first validated and the performance of the controller is then discussed. The topology used for this example is shown in Fig. 3.



$$\begin{aligned}\beta_{13} &= 0.2\\ \beta_{14} &= 0.8\\ \beta_{23} &= 0.4\\ \beta_{24} &= 0.6\end{aligned}$$

Figure 3: Topology used for the numerical example.

The closed-loop compartmental system is defined as follows :

$$\begin{cases}\dot{x}_1 &= d_1u_1(z) - v_1(x_1)\\ \dot{x}_2 &= d_2u_2(z) - v_2(x_2)\\ \dot{x}_3 &= \beta_{13}v_1(x_1) + \beta_{23}v_2(x_2) - v_3(x_3)\\ \dot{x}_4 &= \beta_{14}v_1(x_1) + \beta_{24}v_2(x_2) - v_4(x_4)\\ \dot{z}_3 &= v_3(x_3) - \phi(z_3)(\alpha_{13}d_1 + \alpha_{23}d_2)\\ \dot{z}_4 &= v_4(x_4) - \phi(z_4)(\alpha_{14}d_1 + \alpha_{24}d_2)\end{cases}$$

with

$$\begin{aligned}u_1(z) &= \alpha_{13}\phi(z_3) + \alpha_{14}\phi(z_4)\\ u_2(z) &= \alpha_{23}\phi(z_3) + \alpha_{24}\phi(z_4)\end{aligned}$$

and

$$v_i(x_i) = \frac{\mu_i x_i}{1+x_i}, \mu_i = 120 \quad i = 1, , 4 \quad \phi(z_j) = \frac{z_j}{\epsilon + z_j}, \epsilon = 10^{-3} \quad j = 1, 2$$

4.1 Congestion control

The parameters $\mu_i = 120$ can be interpreted as the maximum output flow of each compartment. The demands $d_1(t), d_2(t)$ are shown in Fig. 4(A) where it can be seen that $d_1(t)$ is set to a constant ($d_1 = 100$) and that $d_2(t)$ is piecewise constant and jumps from $d_2 = 50$ to $d_2 = 100$ at time $t = 5$. The maximum inflow rate at compartment 4 is $\beta_{14}d_1 + \beta_{24}d_2 = 140$

for $t > 5$ which is greater than the maximum output rate of this compartment. Consequently, in open loop, network congestion starts at $t = 5$. This is observed in Fig. 4(B) where we can see that from $t > 5$, the state variable x_4 increases almost linearly and without bound.

In contrast, the closed loop behaviour may be observed in Fig. 4(D) where it appears that all the state variables, including x_4 remain bounded. This figure is obtained with the adapted parameters :

$$\alpha_{13} = \beta_{13} \quad \alpha_{14} = \beta_{14} \quad \alpha_{23} = \beta_{23} \quad \alpha_{24} = \beta_{24} \tag{10}$$

The initial conditions are set to $x(0) = [0, 0, 0, 0]^T$ and $z(0) = [30, 30]^T$. It can be verified that $x_4(t)$ is bounded by a value smaller than $\sigma = 60$. Fig. 4(C) shows how the controlled demand is modulated in order to prevent the overflow. Interestingly, the control variables u_1 and u_2 both converge to the value 0.84 yielding a total inflow rate at the compartment of 117.8 that is to say smaller, but very close to, the maximum possible outflow rate of that compartment.

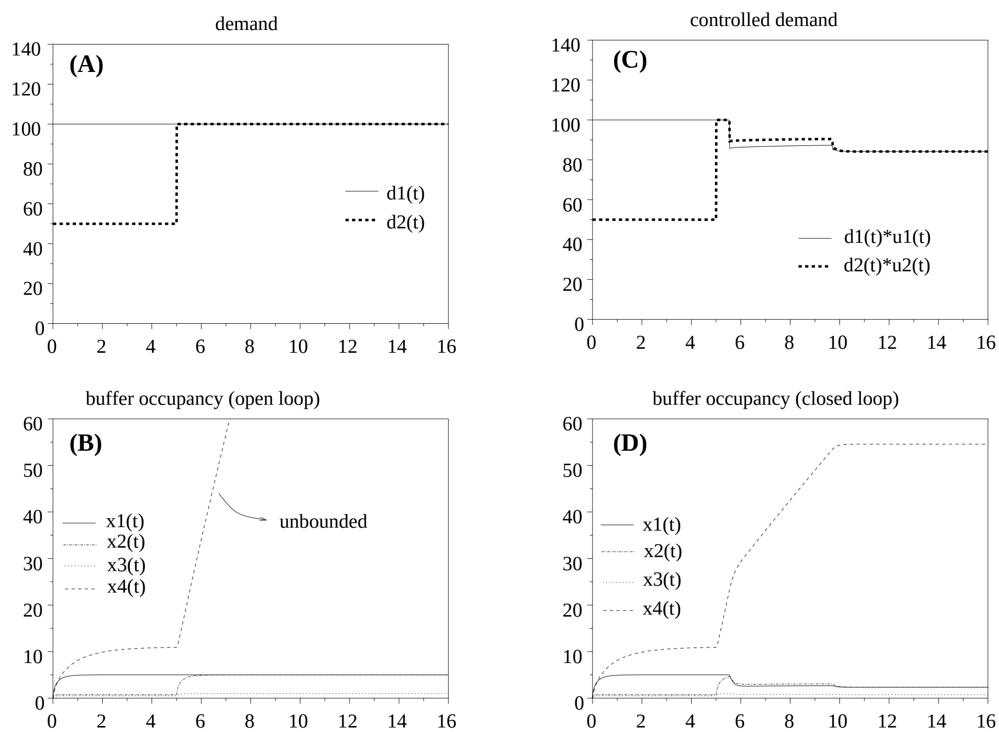


Figure 4: Demand (or controlled demand) and compartment occupancy in open loop (left) compared to the closed loop case (right).

4.2 Performance

The role of the selection of the control parameters α_{ij} may be appreciated in Fig. 5 which compares the evolution of the control variables $u_1(t)$ and $u_2(t)$ when these parameters are adapted to the topology and when they are not. The adapted case corresponds to eq. (10) and the non-adapted case is given by :

$$\alpha_{13} = 0.3 \quad \alpha_{14} = 0.7 \quad \alpha_{23} = 0.4 \quad \alpha_{24} = 0.6 \tag{11}$$

During the first five seconds when the system is not congested it can be seen that the control variables take a value very close to $u_1 = u_2 = 1$ for the adapted case as expected. The demand is therefore satisfied and the controller is transparent. In contrast, for the non-adapted case, the control variables take a value close to 0.9 even though there is no congestion in the system. It means that the controller slightly limits the achievable performance of the system in the absence of congestion : it is the price to pay in order to avoid the risk of congestion when the controller parameters are not well adapted. But obviously in both cases, the controller is able to maintain the stability of the system and the boundeness of the state during the congested period. It is also worth noting that the parameter adaptation requires the knowledge of the flow fractions β_{jk} only but neither the values of the demands d_1, d_2 nor the knowledge of the rate functions $v_i(x_i)$.

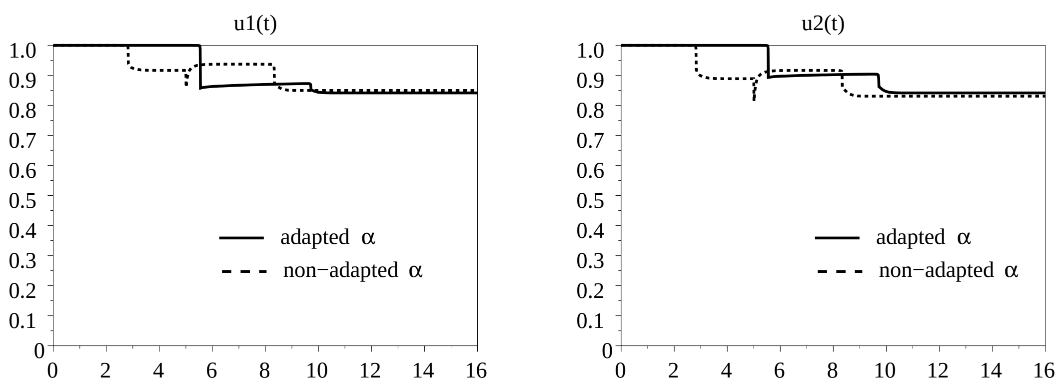


Figure 5: Evolution of the control variables $u_1(t)$ (left) and $u_2(t)$(right) when the controller is adapted to the topology compared to the non-adapted case.

5. CONCLUSIONS

Our contribution in this paper was to show that congestion in compartmental network systems can be automatically prevented by using a nonlinear output feedback controller having an appropriate compartmental structure. We have shown that two main objectives are achieved with the proposed controller : a demand which is not in excess is automatically satisfied but, in case of an excess demand, an operation without overflow is guaranteed provided the design parameter σ is smaller than the maximal capacity of the compartments.

There are obviously many additionnal issues that could be investigated regarding this congestion control strategy. We may mention for instance the influence of the choice of the parameters α_{jk} on the performance of the congestion control. A relevant issue is certainly to design on-line parameter adaptation schemes in order to optimize some performance criterion. In particular, the proposed control law should be able to efficiently accomodate a bursty demand (i.e. a piecewise constant demand with short peaks in excess). Another issue is the extension of the congestion control to compartmental network systems with lags as discussed e.g. in [22] and [20]. Moreover, the congestion control should certainly be improved by combining state and output feedback in the realistic case where the content x_i of some internal compartments is accessible for measurements. Finally an open issue is also to derive decentralised extensions of the control with a view to the application to large-scale networks.

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