Value-Positivity for Matrix Games: Game-theoretical stability analysis



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Example

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The optimal strategy is given by,

$$p^* = \left(\frac{1}{2}, \frac{1}{2} \right)^\top$$

Therefore,

$$\mathrm{val} M = 0$$

Example, perturbed

Consider $\varepsilon > 0$.

$$M(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} \varepsilon$$

The optimal strategy is given by, for $\varepsilon < 1/2$,

$$p_\varepsilon^* = \left(\frac{1+\varepsilon}{2+3\varepsilon}, \frac{1+2\varepsilon}{2+3\varepsilon} \right)^\top$$

Therefore,

$$\mathrm{val} M(\varepsilon) = \frac{\varepsilon^2}{2+3\varepsilon}$$

Example, perturbed 2

Consider $\varepsilon > 0$.

$$M(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ 0 & -2 \end{pmatrix} \varepsilon$$

The optimal strategy is given by, for $\varepsilon < 2/3$,

$$p_\varepsilon^* = \left(\frac{1-\varepsilon}{2-3\varepsilon}, \frac{1-2\varepsilon}{2-3\varepsilon} \right)^\top$$

Therefore,

$$\operatorname{val} M(\varepsilon) = \frac{\varepsilon^2}{2-3\varepsilon}$$

Questions

Definition (Value-positivity problem)

Is the perturbation benefitial for the row player?
Is value function increasing?

Definition (Functional form problem)

How to play the perturbed game and what is its value?
Value function and some optimal strategy function

Definition (Uniform value-positivity problem)

How to play unaware of the size of ε?
Guaranteeing the unperturbed value in the perturbed game with a fixed strategy

Preliminaries

Matrix Games

Matrix games.

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} & & \\ & m_{i,j} & \\ & & \end{pmatrix} \end{array}$$

Strategies.

$$p \in \Delta[n] \qquad q \in \Delta[n]\,.$$

Value.

$$\operatorname{val} M := \max_{p \in \Delta[n]} \min_{q \in \Delta[n]} p^\top M\, q\,.$$

Perturbed Matrix Games

Polynomial matrix games. Matrix games where payoff entries are given by polynomials.

$$M(\varepsilon) = M_0 + M_1\varepsilon + \ldots + M_K\varepsilon^K \,.$$

Value function.

$$\varepsilon \mapsto \mathrm{val} M(\varepsilon) \,.$$

Questions

Definition (Value-positivity problem)

Is the value function increasing?
$\exists \varepsilon_0 > 0$ such that $\forall \varepsilon \in [0, \varepsilon_0] \quad \mathrm{val} M(\varepsilon) \geq \mathrm{val} M(0)$.

Definition (Functional form problem)

What are the value and some optimal strategy functions?
Return the maps $\mathrm{val} M(\cdot)$ and $p^*(\cdot)$, for $\varepsilon \in [0, \varepsilon_0]$.

Definition (Uniform value-positivity problem)

Can the max-player guarantee $\mathrm{val} M(0)$ with a fixed strategy?
$\exists p_0 \in \Delta[n] \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in [0, \varepsilon_0] \quad \mathrm{val}(M(\varepsilon); p_0) \geq \mathrm{val} M(0)$.

Results

Mills 1956

Theorem

Consider a polynomial matrix game $M(\varepsilon) = M_0 + M_1 \varepsilon$. Then,

$$D \, \mathrm{val}(M(\cdot))|_{\varepsilon = 0} = \max_{p \in P(M_0)} \min_{q \in Q(M_0)} p^\top M_1 \, q \,,$$

and can be computed by solving an LP.

Example, perturbed

Consider $\varepsilon > 0$.

$$M(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} \varepsilon$$

The optimal strategy is given by, for $\varepsilon < 1/2$,

$$p^*_\varepsilon = \left(\frac{1+\varepsilon}{2+3\varepsilon}, \frac{1+2\varepsilon}{2+3\varepsilon} \right)^\top$$

Therefore,

$$\mathrm{val} M(\varepsilon) = \frac{\varepsilon^2}{2+3\varepsilon}$$

Algorithms

Theorem (Poly-time algorithms)

When data is rational, there are polynomial-time algorithms for all three value-positivity problems.

Main ideas

Value-positivity and functional form.
$\varepsilon \mapsto \mathrm{val} M(\varepsilon)$ is rational and have coefficients that are at most exponential.

Main ideas: Uniform value-positivity

LP solution of Matrix Games.

$$
(P_M) \begin{cases} \max_{p,z} & z & & \\ s.t. & (p^\top M)_j & \geq z & \forall j \in [n] \\ & p & \in \Delta([n]) & \end{cases}
$$

Leading coefficients of a strategy. For a fixed strategy p, we can think about the leading coefficients against every column action

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{array}{c} M_0 \\ M_1 \\ M_2 \\ M_3 \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & \boxed{2} & 0 \\ 0 & -1 & \boxed{-1} \\ \boxed{0} & 0 & 0 \end{pmatrix}
\end{array}
$$

Consequences

Linear Programming

An LP is the following optimization problem.

$$(P)\begin{cases} \min_x & c^\top x & \\ s.t. & Ax & \leq b \\ & x & \geq 0\,, \end{cases}$$

Perturbed LPs

A perturbed LP is the following family of optimization problems.

$$(P_\varepsilon) \begin{cases} \min_x & c(\varepsilon)^\top x \\ s.t. & A(\varepsilon)x \leq b(\varepsilon) \\ & x \geq 0\,, \end{cases}$$

Examples

$$(P_\varepsilon)\begin{cases} \min_x & x \\ s.t. & x \leq -\varepsilon \\ & -x \leq -\varepsilon\,. \end{cases}$$

Examples 2

$$(P_\varepsilon) \begin{cases} \max_{x,y} & x + y \\ s.t. & x \leq 0 \\ & y + \varepsilon x \leq 0 \,. \end{cases}$$

For $\varepsilon < 1$,

$$\mathrm{val}(P_\varepsilon) \equiv 0$$
$$(x, y)^*(\varepsilon) \equiv (0, 0) \,.$$

Sub-class of LPs

Definition (A priori bounded)

The Lp with errors (P_ε) is a priori bounded if both the primal and dual are uniformely bounded for ε small enough.

Questions

Definition (Weakly robust)

Is there a solution?
$\exists \varepsilon_0 > 0$ such that, $\forall \varepsilon \in [0, \varepsilon_0] \quad (P_\varepsilon)$ is feasible.

Definition (Functional form)

What is the solution?
The maps $\mathrm{val}(P_\cdot)$ and $x^*(\cdot)$, for $\varepsilon \in [0, \varepsilon_0]$.

Definition (Strongly robust)

Is there a constant solution?
$\exists x^* \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in [0, \varepsilon_0], \quad x^*$ is also a solution of (P_ε).

Reminder: Equivalence between Matrix Games and LPs

Theorem (Adler03)

Matrix games and LPs a poly-time equivalent.

- [Dantzig51] gives an incomplete proof.
- The reduction depends on the computational model: rational, algebraic or real data.

Results

Theorem (LP with error to polynomial matrix games)

There is a polynomial-time reduction from robustness problems to the respective value-positivity problem, which preserves the degree of the error perturbation, for algebraic data.

Stochastic Games

Matrix games.

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} & & \\ & (m_{i,j}, \rightarrow) & \\ & & \end{pmatrix} \end{array} \qquad \begin{array}{cc} & j \\ i & \begin{pmatrix} & & \\ & (m_{i,j}, \leftarrow) & \\ & & \end{pmatrix} \end{array}$$

Strategies.

$$p \in (\Delta[n])^{|S|} \qquad q \in (\Delta[n])^{|S|} \,.$$

Discounted and limit value. For $\lambda \in (0,1)$,

$$\mathrm{val}_\lambda M := \max_p \min_q \lambda \sum_{i \geq 1} (1-\lambda)^i \left(p_i^\top M^{(i)} q_i \right) .$$

$$\mathrm{val} M := \lim_{\lambda \to 0^+} \mathrm{val}_\lambda M \,.$$

Stochastic Games and Matrix Games

Theorem (Attia and Oliu-Barton 2019)

Consider a Stochastic Game Γ. There exists a parametrized polynomial matrix game

$$M_z = N(\lambda) - z\tilde{N}(\lambda)\,,$$

where $N, \tilde{N}$ are Matrix Games, such that, for all $z \in \mathbb{R}$ and $\lambda \in (0, 1)$,

$$\mathrm{val} M_z(\lambda) \geq 0 \qquad \Leftrightarrow \qquad \mathrm{val}_\lambda \Gamma \geq z\,.$$

Value-positivity for Stochastic Games

Consider a Stochastic game Γ and its parametrized polynomial matrix game $(M_z)_z$.

Lemma (Value-positivity)

For all $z \in \mathbb{R}$, if M_z is value-positive, then, for all λ sufficiently small,

$$\mathrm{val}_\lambda \Gamma \geq z\,.$$

Lemma (Uniform value-positivity)

For all $z \in \mathbb{R}$, if M_z is uniform value-positive, then there exists a fixed strategy $p \in (\Delta[n])^{|S|}$ such that, for all λ sufficiently small,

$$\mathrm{val}_\lambda(\Gamma; p) \geq z\,.$$

Thank you!