Imagine a particle conducting a walk on the traditional square lattice, starting at the origin $(0,0)$. That is, at any time during the walk, the particle goes one unit distance to either the east, or the west, or the north, or the south. An n-walk is a walk that has taken n steps. The walk is called self-avoiding if the particle does not visit any given state twice.

Let $f(n)$ denote the number of n-walks that are self-avoiding.

Compute $f(n)$ for $n = 2, 3$, and justify how you got these values.

For $n = 2$, the only non self-avoiding paths are those that return to the origin, which are only 4. Then, $f(2) = 4^2 - 4 = 12$.

For $n = 3$, the first two steps of the walk must be a self-avoiding path itself. Then, the third step has three possibilities. Therefore, $f(3) = 3f(2) = 36$.

Compute $f(4)$ if you can, or place it within good lower and upper bounds.

Out of all self-avoiding 3-walks, there are only 8 of them that can complete a square by adding a fourth step, leaving only 2 possibilities to complete a 4-walk. All other self-avoiding 3-walks have 3 possibilities for a fourth step.

Therefore, $f(4) = 3(f(3) - 8) + 2(8) = 84 + 16 = 100$.

Try to give non-trivial lower and upper bounds on $f(n)$ of the form ck^n for $c > 0$ and $k \in \mathbb{N}$.

$f(n) \leq 4 \cdot 3^{n-1} = \frac{4}{3}3^n$, because there are 4 initial directions and each next step has at most 3 possibilities.

$f(n) \geq 4 \cdot (2 \cdot 2^{n-1} - 1) = 4 \cdot 2^n - 4$, because there are 4 initial directions and then using either (i) the same initial direction, or (ii) a perpendicular direction, will result in a self-avoiding walk. This counting procedure repeats the four paths that uses one direction only, therefore we must correct the counting by substracting four. This bounds means that for all N natural number, there exists a constant $c_N > 0$ such that $f(n) \geq c_N \cdot 2^n$ and $\lim_{N\to\infty} c_N = 4$.