Chapter 6 - Exotic options

Modelling Derivatives in C++

1 Implied volatility

Black-Scholes models assume that volatility is constant, this is not realistic as:

1. Historical volatility is not constant over time
2. Options with strikes and maturities don't have the same implied volatility
3. Risk neutral probability distributions of future asset prices are not lognormal

The straightforward solution is to model volatility with a completely separate diffusion process.

The implied volatility in the BS model is the σ that solves the following equation between model and market price:

$$C_{BS}(\sigma) = C_{market}$$

This can only be calculated through numerical methods (e.g., Newton-Raphson).

2 Volatility skews and smiles

Volatility skews and smiles depict the different implied volatilities of options as a result of differing strike prices. This is a direct violation of the Black-Scholes assumption of constant implied volatility across strike prices.

3 Empirical explanations

There are a few reasons that can explain this violation, e.g., return distributions are not exactly normal.

4 Implied volatility surfaces

Volatility surfaces add another dimension - maturities in the volatility smile chart (i.e., a term-structure). Usually - higher maturies imply higher volatilities. Many numerical and machine learning methods can be used to model volatility surfaces for the purpose of examining arbitrage opportunities.

5 One-factor models

With the inconsistencies between implied volatility and the BS assumptions - the BS model can be relaxed to allow time-varying volatility, or volatility depending on the price of the underlying.

6 Constant elasticity of variance models

For example, a simple extension to the volatility equation such that it depends on the underlying is:

$$\sigma(S) = \sigma_0 \left(\frac{S}{S_0} \right)^{\gamma - 1}$$

Cox and Rubinstein provided a closed-form analytic formula for the European call using the CEV (note, long formula).

7 Recovering implied volatility surfaces

Not too relevant as mentioned before.

8 Local volatility surfaces

There are parametric and non-parametric methods to recover the volatility surfaces mentioned here, to be assessed under robustness checks.

9 Jump-diffusion models

This section considers the implementation of finding the volatility surfaces given price diffusions with jumps. Similar thinking to before.

10 Two-factor models

For a more advanced method of finding the implied volatility surface (e.g., not the BS or BS-CEV model), you can also model the volatility as a diffusion process. An implementation is provided in this section.

11 Hedging with stochastic volatility

With stochastic volatility constructed, we can construct portfolios that are hedged against price movements. This depends on the Greeks - and the Greeks depend on the stochastic volatility of the financial instrumen.