Chapter 1 - Black Scholes and Pricing Fundamentals

Modelling Derivatives in C++

1 Forward contracts

Derivatives - a security whose value is contingent on the value of an underlying security/macroeconomic variable.

Simplest derivative - forward contract, an agreement between two parties to buy and sell an asset at a certain time $T > 0$ and a certain delivery price K at t_0.

Payoffs (on the delivery date):

1. Long position: $f_T = S_T - K$
2. Short position: $f_T = K - S_T$
3. Forward price at time t for a short position: $f_{t,T} = S_t - e^{r(T-t_0)}S_0$

2 Black-Scholes PDE

What should our price process satisfy? They need to satisfy 3 requirements:

1. The price process is always greater than zero.
2. Once the price process hits 0, it will never rise again.
3. Expected percentage required by investors are independent of the stock's price. Risk-averse investors require a rate of return $m = r + r_e$ for the stock, where r_e is the excess return above the risk-free rate.

The resultant diffusion processes that needs to be used to be able to attain the above price process:

1. In general: $dS_t = mS_t dt + b(S_t, t)dz_t$
2. In its simplest form (the BS diffusion process): $dS_t = mS_t dt + \sigma S_t dz_t$
3. Its solution: $lnS_t|S_t \sim N(lnS_t + m(T-t), \sigma\sqrt{T-t})$
4. The option price f given the underlying S_t must satisfy the PDE:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = -\frac{\partial f}{\partial t}$$

3 Risk-neutral pricing

BS price for the call option:

1. $C(S_t, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$
2. $d_1 = \frac{ln\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$
3. $d_2 = d_1 - \sigma\sqrt{T-t}$

Intuitive interpretation of the BS price:

1. First term - discounted stock price $\times$ probability of obtaining the stock
2. Second term - discounted strike price $\times$ probability of obtaining the stock

4 Black-Scholes and diffusion process implementation

Strategy (classes to define):

1. General diffusion process
   (a) Black-Scholes process
   (b) Ornstein-Uhlenbeck Process
   (c) Square-root process
2. Instrument
   (a) Options
      i. Vanilla Option
         A. Black-Scholes Option