 KAVYA SHAMSUNDAR

① Exercise 21.3-3-pg 572

Give a sequence of m MAKE-SET, UNION & FIND-SET operations, n of which are MAKE-SET operations, that takes $\Omega(m \lg n)$ time when we use union by rank only.

Solution: * We need to find a sequence of 'm' operations on 'n' elements that takes $\Omega(m \lg n)$ time.

* Start with 'n' MAKE-SETS to create singleton sets $\{x_1\}, \{x_2\} \ldots \{x_n\}$.

* Next perform the $(n-1)$ UNION operations shown below to create a single set whose tree has depth $\lg n$.

UNION (x_1, x_2)
UNION (x_3, x_4)
UNION (x_5, x_6)
⋮
UNION (x_{n-1}, x_n)

$n/2$ of these

UNION (x_2, x_4)
UNION (x_6, x_8)
UNION (x_{10}, x_{12})
⋮
UNION (x_{n-2}, x_n)

$n/4$ of these

UNION (x_4, x_8)
UNION (x_{12}, x_{16})
UNION (x_{20}, x_{24})
⋮
UNION (x_{n-4}, x_n)

$n/8$ of these

⋮

UNION $(x_{n/2}, x_n) \rightarrow$ 1 of these

Scanned by CamScanner

KAVYA SHAMSUNDARA

* Finally perform $(m-2n+1)$ FIND-SET operations on the deepest element in the tree.
* Each of these FIND-SET operations takes $\Omega(\lg n)$ time. Letting $m \geq 3n$, we have more than $m/3$ FIND-SET operations, so that the total cost is $\Omega(m \lg n)$.

Scanned by CamScanner

KAVYA SHAMSUNDARA

(2) Prove that $A_3(j) > tower(j)$, where

$$tower(n) = \begin{cases} 2^{tower(n-1)} & \text{if } n>0 \\ 1 & \text{if } n=0 \end{cases}$$

Solution

* We know that

$$\boxed{A_k(j) = \begin{cases} j+1 & k=0 \\ A_{k-1}^{(j+1)}(j) & k \geq 1 \end{cases}}$$

I
* Consider $A_3(j)$ for $j=1 \Rightarrow A_k(j) = A_{k-1}^{(j+1)}(j)$ since $k \geq 1$

$A_3(1) = A_2^{(1)}(1)$

$= A_2(1) \Rightarrow$

$$\boxed{\begin{aligned} A_2(1) &= A_1^{(2)}(1) \\ &= A_1(A_1(1)) \\ &= A_1(3) \\ A_2(1) &= 7 \end{aligned}}$$

$\boxed{A_3(1) = 7} \rightarrow$ equation (1)

* Consider $tower(j)$ for $j=1$

From given relation on tower,

$tower(0) = 1.$

$tower(1) = 2^{tower(1-1)}$

$= 2^{tower(0)}$

$= 2^1$

$\boxed{tower(1) = 2} \Rightarrow$ equation (2)

Scanned by CamScanner

KAVYA SHAMSUNDARA

* Comparing equation 1 & equation 2

$$\boxed{A_3(1) > \text{tower}(1)} \rightarrow \text{Conclusion } 1.$$

II * Consider $A_3(j)$ for $j=2 \Rightarrow A_k(j) = A_{k-1}^{(j+1)}(j)$ since $k \geq 1$

$$A_3(2) = A_2^{(3)}(2)$$

$$= A_2(A_2(\underbrace{A_2(2)}_{\downarrow}))$$

$$= A_2(A_2(23))$$

$A_2(j) = 2^{j+1} \cdot (j+1) - 1$

$$= A_2(2^{24} \cdot 24 - 1)$$

$$= 2^{(2^{24} \cdot 24)} \cdot (2^{24} \cdot 24) - 1$$

$$\boxed{A_3(2) \approx 2^{(402653184)} \cdot (402653184) - 1} \rightarrow \text{equation } 3$$

* Consider tower (j) for $j=2$.

From given relation,

$$\text{tower}(0) = 1$$
$$\text{tower}(1) = 2$$

$$\text{tower}(2) = 2^{\text{tower}(2-1)}$$

$$= 2^{\text{tower}(1)}$$

$$= 2^2$$

$$\boxed{\text{tower}(2) = 4} \rightarrow \text{equation } 4$$

* Comparing equation 3 & equation 4

$$\boxed{A_3(2) > \text{tower}(2)} \rightarrow \text{Conclusion } 2$$

Scanned by CamScanner

KAVYA SHAMSUNDARA

III

* If we consider $A_3(j)$ for $j=3$,
the value of $A_3(3)$ will be very huge.

$$A_3(3) = A_2^{(4)}(3)$$
$$= A_2(A_2(A_2(A_2(3))))$$
$$= A_2(A_2(A_2(63)))$$
$$= A_2(A_2(2^{63})) \text{ (approximately)}$$
$$= A_2(2^{2^{63}})$$

$$\boxed{A_3(3) = 2^{2^{2^{63}}}}$$ → approximate value.
→ equation 5

* Consider tower (j) for j=3.
From given relation of tower(n), we have:
tower(0) = 1.
tower(1) = 2
tower(2) = 4

$$\text{tower}(3) = 2^{\text{tower}(3-1)}$$
$$= 2^{\text{tower}(2)}$$

$$\boxed{\text{tower}(3) = 2^4}$$ → equation 6

* Comparing equation 5 & equation 6

$$\boxed{A_3(3) > \text{tower}(3)}$$ → Conclusion 3

Scanned by CamScanner

KAVYA SHAMSUNDARA

* ~~Form~~ FROM Conclusion 1, 2 and 3, we can say that

$$A_3(j) > tower(j)$$

* Also, the values of $A_3(j)$ goes on increasing to huge non-computable values (higher exponential values) with increase in 'j' values.

These values of $A_3(j)$ will always be greater than tower(j) as the value of tower(j) in the given relation of tower(n) will be very small compared to $A_3(j)$.

Hence, we proved that

$$\boxed{A_3(j) > tower(j)}$$

Scanned by CamScanner