Closure Properties of Context-Free Languages (CFLs)

Context-Free Languages (CFLs) have several interesting closure properties. Here is a summary of these properties:

Operation	**Closure**
Union	Yes
Concatenation	Yes
Kleene Star	Yes
Reversal	Yes
Intersection	No
Complement	No
Substitution	Yes
Homomorphism	Yes
Inverse Homomorphism	Yes

Table 1: Closure properties of Context-Free Languages (CFLs)

Proof Outlines

Closure Under Union

CFLs are closed under union.

Proof Outline: If L_1 and L_2 are two context-free languages, then there exists a context-free grammar G that generates the language $L_1 \cup L_2$.

Closure Under Concatenation

CFLs are closed under concatenation.

Proof Outline: If L_1 and L_2 are two context-free languages, then there exists a context-free grammar G that generates the language $L_1 L_2$.

Closure Under Kleene Star

CFLs are closed under Kleene star.

Proof Outline: If L is a context-free language, then there exists a context-free grammar G that generates the language L^*.

Closure Under Reversal

CFLs are closed under reversal.

Proof Outline: If L is a context-free language, then the reverse of L, denoted as L^R, is also a context-free language.

Non-Closure Under Intersection

CFLs are not closed under intersection.

Proof Outline: The intersection of two context-free languages is not necessarily a context-free language. For example, $L_1 = \{a^n b^n c^m \mid n, m \geq 0\}$ and $L_2 = \{a^m b^n c^n \mid m, n \geq 0\}$ are both context-free, but their intersection $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free.

Non-Closure Under Complement

CFLs are not closed under complement.

Proof Outline: The complement of a context-free language is not necessarily a context-free language. This follows from the non-closure under intersection and De Morgan's laws.

Closure Under Substitution

CFLs are closed under substitution.

Proof Outline: If L is a context-free language and ϕ is a substitution such that $\phi(a)$ is a context-free language for each $a \in \Sigma$, then $\phi(L)$ is also a context-free language.

Closure Under Homomorphism

CFLs are closed under homomorphism.

Proof Outline: If L is a context-free language and h is a homomorphism, then $h(L)$ is also a context-free language.

Closure Under Inverse Homomorphism

CFLs are closed under inverse homomorphism.

Proof Outline: If L is a context-free language and h is a homomorphism, then $h^{-1}(L)$ is also a context-free language.