Kalman Filter

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1 Recursive Processing

$$\hat{x_k} = \hat{x_{k-1}} + K_k * (z_k - \hat{x_{k-1}})$$
$$K_k : kalmanGain$$

- e_{EST}: Estimate error
- e_{MEA}: Measurement error

$$K_k = \frac{e_{EST_{k-1}}}{e_{EST_{k-1}} + e_{MEA_k}}$$

$$Step1 : K_k = \frac{e_{EST_{k-1}}}{e_{EST_{k-1}} + e_{MEA_k}}$$
$$Step2 : \hat{x_k} = \hat{x_{k-1}} + K_k * (z_k - \hat{x_{k-1}})$$
$$Step3 : e_{EST_k} = (1 - K_k) * e_{EST_{k-1}}$$

2 Data Fusion

$$z_1 = 30g, \sigma_1 = 2g$$
$$z_2 = 32g, \sigma_2 = 4g$$
$$\hat{z} = z_1 + k(z_2 - z_1)$$
$$k : kalmanGain, k \in [0, 1]$$
$$k = 0, \hat{z} = z_1; k = 1, \hat{z} = z_2$$
$$k? \Rightarrow \hat{z}_{min} \Rightarrow \sigma_{\hat{z}_{min}}$$

$$\begin{aligned}
\sigma_{\hat{z}}^2 &= var(z_1 + k(z_2 - z_1)) = var((1-k)z_1 + kz_2) \\
&= var((1-k)z_1) + var(kz_2) \\
&= (1-k)^2 var(z_1) + k^2 var(z_2) \\
&= (1-k)^2 \sigma_1^2 + k^2 \sigma_2^2
\end{aligned}$$

$$\frac{\mathrm{d}\sigma_{\hat{z}}^2}{\mathrm{d}k} = 0$$
$$-2(1-k)\sigma_1^2 + 2k\sigma_2^2 = 0$$
$$k = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

3 Covariance Matrix

$$P = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 \end{bmatrix}$$

- transition matrix

$$a = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$
$$P = \frac{1}{3}\mathbf{a}^\top a$$

4 State Space Representation

- Mass-Spring-Damper
  - K: Elastic coefficient
  - B: Damping coefficient
  - x: Mass displacement

$$m\ddot{x} + B\dot{x} + Kx = F$$
$$F \Rightarrow u : Input$$

- state

$$\begin{aligned} &x_1 = x \\ &x_2 = \dot{x} \\ &\dot{x_1} = x_2 \\ &\dot{x_2} = \ddot{x} = \frac{1}{m}u - \frac{B}{m}x_2 - \frac{K}{m}x_1 \end{aligned}$$

- measure

$$z_1 = x = x_1 : positin$$
$$z_2 = \dot{x} = x_2 : velocity$$

- state space

$$\begin{aligned} &\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{B}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ &\dot{x}(t) = Ax(t) + Bu(t) \\ &\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &z(t) = Hx(t) \end{aligned}$$

- Discrete
  - x_k: State variables
  - A: State matrix
  - B: Control matrix
  - u_k: Control variables
  - ω_{k-1}: Process noise
  - z_k: Measurement variables
  - H: Measurement matrix
  - v_k: Measurement noise

$$\begin{aligned} x_k &= Ax_{k-1} + Bu_{k-1} + \omega_{k-1} \\ z_k &= Hx_k + v_k \end{aligned}$$

5 Derivation of Kalman Gain

- $\hat{X_k^-}$: Priori estimate

$$\hat{x_k^-} = A\hat{x_{k-1}} + Bu_{k-1}$$
$$z_k = Hx_k \Rightarrow \hat{x_{k_{MEA}}} = H^{-1}z_k$$

- $\hat{x_k}$: Posterior estimation

$$\hat{x_k} = \hat{x_k^-} + G(H^{-1}z_k - \hat{x_k^-})$$
$$G \in [0,1]$$
$$G = 0, \hat{x_k} = \hat{x_k^-}; G = 1, \hat{x_k} = H^{-1}z_k$$

$$G = K_k H$$
$$\hat{x_k} = \hat{x_k^-} + K_k(z_k - H\hat{x_k^-})$$
$$K_k \in [0, H^{-1}]$$
$$K_k = 0, \hat{x_k} = \hat{x_k^-}; K_k = H^{-1}, \hat{x_k} = H^{-1}z_k$$
$$K_k? \Rightarrow \hat{x_k} -> x_k$$

- e_k: Error
- 0: Target
- P: Covariance matrix

$$VAR(x) = E(x^2) - E^2(x)$$
$$target = 0 \Rightarrow E^2(x) = 0 \Rightarrow VAR(x) = E(x^2)$$

$$e_k = x_k - \hat{x_k}$$
$$P(e_k) \sim (0, P)$$

$$\begin{aligned} P &= E[ee\top] \\ &= E[\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \begin{bmatrix} e_1 & e_2 \end{bmatrix}] \\ &= E[\begin{bmatrix} e_1^2 & e_1e_2 \\ e_2e_1 & e_2^2 \end{bmatrix}] \\ &= \begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1}\sigma_{e_2} \\ \sigma_{e_2}\sigma_{e_1} & \sigma_{e_2}^2 \end{bmatrix} \end{aligned}$$

$$tr(P) = \sigma_{e_1}^2 + \sigma_{e_2}^2$$
$$tr(P)_{min} \Rightarrow \sigma_{min} \Rightarrow \hat{x_k} -> x_k$$
$$K_k? \Rightarrow tr(P)_{min}$$

$$\begin{aligned} x_k - \hat{x_k} &= x_k - (\hat{x_k^-} + K_k(z_k - H\hat{x_k^-})) \\ &= x_k - \hat{x_k^-} - K_k z_k + K_k H\hat{x_k^-} \\ &= x_k - \hat{x_k^-} - K_k H x_k - K_k v_k + K_k H\hat{x_k^-} \\ &= (x_k - \hat{x_k^-}) - K_k H(x_k - \hat{x_k^-}) - K_k v_k \\ &= (I - K_k H)(x_k - \hat{x_k^-}) - K_k v_k \\ e_k^- &= x_k - \hat{x_k^-} \end{aligned}$$

- $(AB)\top = B\top A\top$
- $(A+B)\top = A\top + B\top$

$$
\begin{aligned}
P_k &= E[ee\top] \\
&= E[(x_k - \hat{x_k})(x_k - \hat{x_k})\top] \\
&= E[[(I - K_k H)e_k^- - k_k v_k][(I - K_k H)e_k^- - k_k v_k]\top] \\
&= E[[(I - K_k H)e_k^- - k_k v_k][e_k^-\top(I - K_k H)\top - v_k\top K_k\top]] \\
&= E[(I - K_k H)e_k^- e_k^-\top(I - K_k H)\top - (I - K_k H)e_k^- v_k\top K_k\top \\
&- \ k_k v_k e_k^-\top(I - K_k H)\top + k_k v_k v_k\top K_k\top]
\end{aligned}
$$

- $E(AB) = E(A)E(B)$: A,B Independent

$$
\begin{gathered}
E[(I - K_k H)e_k^- v_k\top K_k\top] = (I - K_k H)E[e_k^- v_k\top]K_k\top \\
E[e_k^- v_k\top] = E[e_k^-] + E[v_k\top] \\
E[e_k^-] = 0, E[v_k\top] = 0 \Rightarrow E[(I - K_k H)e_k^- v_k\top K_k\top] = 0 \\
also: E[k_k v_k e_k^-\top(I - K_k H)\top] = 0
\end{gathered}
$$

$$
P_k = (I - K_k H)E(e_k^- e_k^-\top)(I - K_k H)\top + K_k E(v_k v_k\top)K_k\top
$$

- $E(e_k^- e_k^-\top) = P_k^-$

- $E(v_k v_k\top) = R$: Measurement nosie Covariance matrix

$$
\begin{aligned}
P_k &= (P_k^- - K_k H P_k^-)(I\top - H\top K_k\top) + K_k R K_k\top \\
&= P_k^- - K_k H P_k^- - P_k^- H\top K_k\top + K_k H P_k^- H\top K_k\top + K_k R K_k\top
\end{aligned}
$$

$$
\begin{aligned}
(P_k^- H\top K_k\top)\top &= K_k(P_k^- H\top)\top \\
&= K_k H P_k^- \\
\Rightarrow tr(K_k H P_k^-) &= tr(P_k^- H\top K_k\top)
\end{aligned}
$$

$$
\begin{aligned}
tr(P_k) = tr(P_k^-) &- 2tr(K_k H P_k^-) \\
&+ \ tr(K_k H P_k^- H\top K_k\top) + tr(K_k R K_k\top)
\end{aligned}
$$

- $\frac{\mathrm{d}tr(AB)}{\mathrm{d}A} = B\top$

$$
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}
$$

$$
tr(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}
$$

$$
\frac{\mathrm{d}tr(AB)}{\mathrm{d}A} = \begin{bmatrix} \frac{\partial tr(AB)}{\partial a_{11}} & \frac{\partial tr(AB)}{\partial a_{12}} \\ \frac{\partial tr(AB)}{\partial a_{21}} & \frac{\partial tr(AB)}{\partial a_{22}} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = B
$$

- $\frac{\mathrm{d}tr(ABA\top)}{\mathrm{d}A} = 2AB$

$$
\begin{gathered}
\frac{\mathrm{d}tr(P_k)}{\mathrm{d}K_k} = 0 \\
\frac{\mathrm{d}tr(P_k)}{\mathrm{d}K_k} = 0 - 2(HP_k^-)\top + 2K_k H P_k^- H\top + 2K_k R
\end{gathered}
$$

- $P_k^-\top = P_k^-$: Covariance matrix

$$
\begin{gathered}
-P_k^- H\top + K_k(HP_k^- H\top + R) = 0 \\
K_k(HP_k^- H\top + R) = P_k^- H\top \\
K_k = \frac{P_k^- H\top}{HP_k^- H\top + R} \\
R\uparrow, K_k -> 0, \hat{x_k} = \hat{x_k^-} \\
R\downarrow, K_k -> H^{-1}, \hat{x_k} = H^{-1}z_k
\end{gathered}
$$

6 Priori/Posteriori Error Covariance Matrix

- $P(\omega) \sim N(0, Q)$
- $P(v) \sim N(0, R)$

$$
\begin{aligned}
x_k &= Ax_{k-1} + Bu_{k-1} + \omega_{k-1} \\
z_k &= Hx_k + v_k
\end{aligned}
$$

- Priori estimate

$$\hat{x_k^-} = A\hat{x_{k-1}} + Bu_{k-1}$$

- Posteriori estimate

$$\hat{x_k} = \hat{x_k^-} + K_k(z_k - H\hat{x_k^-})$$

- Kalman Gain

$$K_k = \frac{P_k^- H\top}{HP_k^- H\top + R}$$

- P_k^-?

$$
\begin{aligned}
P_k^- &= E[e_k^- e_k^- \top] \\
e_k^- &= x_k - \hat{x_k^-}
\end{aligned}
$$

$$
\begin{aligned}
e_k^- &= Ax_{k-1} + Bu_{k-1} + \omega_{k-1} - A\hat{x_{k-1}} - Bu_{k-1} \\
&= A(x_{k-1} - \hat{x_{k-1}}) + \omega_{k-1} \\
&= Ae_{k-1} + \omega_{k-1}
\end{aligned}
$$

$$
\begin{aligned}
P_k^- &= E[(Ae_{k-1} + \omega_{k-1})(Ae_{k-1} + \omega_{k-1})\top] \\
&= E[(Ae_{k-1} + \omega_{k-1})(e_{k-1}\top A\top + \omega_{k-1}\top)] \\
&= E[Ae_{k-1}e_{k-1}\top A\top + Ae_{k-1}\omega_{k-1}\top + \omega_{k-1}e_{k-1}\top A\top + \omega_{k-1}\omega_{k-1}\top] \\
&= E[Ae_{k-1}e_{k-1}\top A\top] + E[Ae_{k-1}\omega_{k-1}\top] + E[\omega_{k-1}e_{k-1}\top A\top] + E[\omega_{k-1}\omega_{k-1}\top]
\end{aligned}
$$

- e_{k-1}, ω_{k-1} Independent
- $E[e_{k-1}] = E[\omega_{k-1}] = 0$
- $E[Ae_{k-1}\omega_{k-1}\top] = AE[e_{k-1}]E[\omega_{k-1}\top] = 0$
- $also: E[\omega_{k-1}e_{k-1}\top A\top] = 0$

$$
\begin{aligned}
P_k^- &= AE[e_{k-1}e_{k-1}\top]A\top + E[\omega_{k-1}\omega_{k-1}\top] \\
&= AP_{k-1}A\top + Q
\end{aligned}
$$

- Kalman formula
  - Predict
    - Priori Estimate
    - Priori Covariance Matrix
  - Correction
    - Kalman Gain

- Posteriori Estimate
- Posteriori Covariance Matrix

Variables	Meaning
A	state transition matrix
B	control input matrix
Q	process noise covariance matrix
R	measurement noise covariance matrix
H	measurement matrix
P_k^-/P_k	priori/posteriori error covariance matrix

$$\hat{x_k^-} = A\hat{x_{k-1}} + Bu_{k-1}$$

$$P_k^- = AP_{k-1}A\top + Q$$

$$K_k = \frac{P_k^- H\top}{HP_k^- H\top + R}$$

$$\hat{x_k} = \hat{x_k^-} + K_k(z_k - H\hat{x_k^-})$$

$$\begin{aligned}
P_k &= P_k^- - K_k H P_k^- - P_k^- H\top K_k\top + K_k H P_k^- H\top K_k\top + K_k R K_k\top \\
&= P_k^- - K_k H P_k^- - P_k^- H\top K_k\top + K_k(HP_k^- H\top + R)K_k\top \\
&= P_k^- - K_k H P_k^- - P_k^- H\top K_k\top + \frac{P_k^- H\top}{HP_k^- H\top + R}(HP_k^- H\top + R)K_k\top \\
&= P_k^- - K_k H P_k^- \\
&= (I - K_k H)P_k^-
\end{aligned}$$

7 Extended Kalman Filter

- Nonlinear
  - $P(\omega) \sim N(0, Q)$
  - $P(v) \sim N(0, R)$

$$\begin{aligned}
x_k &= f(x_{k-1}, u_{k-1}, \omega_{k-1}) \\
z_k &= h(x_k, v_k)
\end{aligned}$$

  A normally distributed random variable is no longer normally distributed after passing through a nonlinear system

- Linearzation
  - Taylor Series: $f(x) = f(x_0) + \frac{\partial f}{\partial x}(x - x_0)$
  - Operating Point:
    - $x_k : \hat{x_{k-1}}$
    - $z_k : \tilde{x_k}$

$$
\begin{aligned}
&x_k = f(x_{\hat{k-1}}, u_{k-1}, \omega_{k-1}) + A(x_k - x_{\hat{k-1}}) + W\omega_{k-1} \\
&suppose : \omega_{k-1} = 0 \\
&f(x_{\hat{k-1}}, u_{k-1}, 0) = \tilde{x_k} \\
&A = \frac{\partial f}{\partial x}|_{x_{\hat{k-1}}, u_{k-1}} \\
&W = \frac{\partial f}{\partial \omega}|_{x_{\hat{k-1}}, u_{k-1}}
\end{aligned}
$$

$$
\begin{aligned}
&eg : \\
&x_1 = x_1 + \sin x_2 = f_1 \\
&x_2 = x_1^2 = f_2 \\
&A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & \cos x_2 \\ 2x_1 & 0 \end{bmatrix}|_{x_{\hat{k-1}}, u_{k-1}} \\
&A_k = \begin{bmatrix} 1 & \cos x_{2_{\hat{k-1}}} \\ 2x_{1_{\hat{k-1}}} & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&z_k = h(\tilde{x_k}, v_k) + H(x_k - \tilde{x_k}) + Vv_k \\
&suppose : v_k = 0 \\
&h(\tilde{x_k}, 0) = \tilde{z_k} \\
&H = \frac{\partial h}{\partial x}|_{\tilde{x_k}} \\
&V = \frac{\partial h}{\partial v}|_{\tilde{x_k}}
\end{aligned}
$$

- $P(W\omega) \sim N(0, WQW\top)$

- $P(Vv) \sim N(0, VRV\top)$

$$
\begin{aligned}
&x_k = \tilde{x_k} + A(x_k - x_{\hat{k-1}}) + W\omega_{k-1} \\
&z_k = \tilde{z_k} + H(x_k - \tilde{x_k}) + Vv_k
\end{aligned}
$$

- Extended Kalman Formula

$$
\begin{gathered}
\hat{x_k^-} = f(x_{\hat{k-1}}, u_{k-1}, 0) \\
P_k^- = AP_{k-1}A\top + WQW\top \\
K_k = \frac{P_k^- H\top}{HP_k^- H\top + VRV\top} \\
\hat{x_k} = \hat{x_k^-} + K_k(z_k - h(\hat{x_k^-}, 0)) \\
P_k = (I - K_k H)P_k^-
\end{gathered}
$$