

# Kalman Filter

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# 1 Recursive Processing

$$\hat{x}_k = \hat{x}_{k-1} + K_k * (z_k - \hat{x}_{k-1})$$

$$K_k : \text{kalmanGain}$$

- $e_{EST}$ : Estimate error
- $e_{MEA}$ : Measurement error

$$K_k = \frac{e_{EST_{k-1}}}{e_{EST_{k-1}} + e_{MEA_k}}$$

$$\text{Step1} : K_k = \frac{e_{EST_{k-1}}}{e_{EST_{k-1}} + e_{MEA_k}}$$

$$\text{Step2} : \hat{x}_k = \hat{x}_{k-1} + K_k * (z_k - \hat{x}_{k-1})$$

$$\text{Step3} : e_{EST_k} = (1 - K_k) * e_{EST_{k-1}}$$

# 2 Data Fusion

$$z_1 = 30g, \sigma_1 = 2g$$

$$z_2 = 32g, \sigma_2 = 4g$$

$$\hat{z} = z_1 + k(z_2 - z_1)$$

$$k : \text{kalmanGain}, k \in [0, 1]$$

$$k = 0, \hat{z} = z_1; k = 1, \hat{z} = z_2$$

$$k? \Rightarrow \hat{z}_{min} \Rightarrow \sigma_{\hat{z}_{min}}$$

$$\begin{aligned} \sigma_{\hat{z}}^2 &= \text{var}(z_1 + k(z_2 - z_1)) = \text{var}((1 - k)z_1 + kz_2) \\ &= \text{var}((1 - k)z_1) + \text{var}(kz_2) \\ &= (1 - k)^2 \text{var}(z_1) + k^2 \text{var}(z_2) \\ &= (1 - k)^2 \sigma_1^2 + k^2 \sigma_2^2 \end{aligned}$$

$$\frac{d\sigma_{\hat{z}}^2}{dk} = 0$$

$$-2(1 - k)\sigma_1^2 + 2k\sigma_2^2 = 0$$

$$k = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

# 3 Covariance Matrix

$$P = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 \end{bmatrix}$$

- transition matrix

$$a = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$

$$P = \frac{1}{3} \mathbf{a}^\top a$$

## 4 State Space Representation

- Mass-Spring-Damper
  - $K$ : Elastic coefficient
  - $B$ : Damping coefficient
  - $x$ : Mass displacement

$$m\ddot{x} + B\dot{x} + Kx = F$$
$$F \Rightarrow u : \text{Input}$$

- state

$$x_1 = x$$
$$x_2 = \dot{x}$$
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \ddot{x} = \frac{1}{m}u - \frac{B}{m}x_2 - \frac{K}{m}x_1$$

- measure

$$z_1 = x = x_1 : \text{position}$$
$$z_2 = \dot{x} = x_2 : \text{velocity}$$

- state space

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{B}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$z(t) = Hx(t)$$

- Discrete
  - $x_k$ : State variables
  - $A$ : State matrix
  - $B$ : Control matrix
  - $u_k$ : Control variables
  - $\omega_{k-1}$ : Process noise
  - $z_k$ : Measurement variables
  - $H$ : Measurement matrix
  - $v_k$ : Measurement noise

$$x_k = Ax_{k-1} + Bu_{k-1} + \omega_{k-1}$$
$$z_k = Hx_k + v_k$$

## 5 Derivation of Kalman Gain

- $\hat{X}_k^-$ : Priori estimate

$$\begin{aligned}\hat{x}_k^- &= A\hat{x}_{k-1} + Bu_{k-1} \\ z_k = Hx_k &\Rightarrow x_{k_{MEA}} = H^{-1}z_k\end{aligned}$$

- $\hat{x}_k$ : Posterior estimation

$$\begin{aligned}\hat{x}_k &= \hat{x}_k^- + G(H^{-1}z_k - \hat{x}_k^-) \\ G &\in [0, 1] \\ G = 0, \hat{x}_k &= \hat{x}_k^-; G = 1, \hat{x}_k = H^{-1}z_k \\ G &= K_k H \\ \hat{x}_k &= \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-) \\ K_k &\in [0, H^{-1}] \\ K_k = 0, \hat{x}_k &= \hat{x}_k^-; K_k = H^{-1}, \hat{x}_k = H^{-1}z_k \\ K_k? &\Rightarrow \hat{x}_k - x_k\end{aligned}$$

- $e_k$ : Error
- 0: Target
- $P$ : Covariance matrix

$$\begin{aligned}VAR(x) &= E(x^2) - E^2(x) \\ target = 0 &\Rightarrow E^2(x) = 0 \Rightarrow VAR(x) = E(x^2)\end{aligned}$$

$$\begin{aligned}e_k &= x_k - \hat{x}_k \\ P(e_k) &\sim (0, P)\end{aligned}$$

$$\begin{aligned}P &= E[ee^\top] \\ &= E\left[\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \begin{bmatrix} e_1 & e_2 \end{bmatrix}\right] \\ &= E\left[\begin{bmatrix} e_1^2 & e_1 e_2 \\ e_2 e_1 & e_2^2 \end{bmatrix}\right] \\ &= \begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1} \sigma_{e_2} \\ \sigma_{e_2} \sigma_{e_1} & \sigma_{e_2}^2 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}tr(P) &= \sigma_{e_1}^2 + \sigma_{e_2}^2 \\ tr(P)_{min} &\Rightarrow \sigma_{min} \Rightarrow \hat{x}_k - x_k \\ K_k? &\Rightarrow tr(P)_{min}\end{aligned}$$

$$\begin{aligned}x_k - \hat{x}_k &= x_k - (\hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)) \\ &= x_k - \hat{x}_k^- - K_k z_k + K_k H \hat{x}_k^- \\ &= x_k - \hat{x}_k^- - K_k H x_k - K_k v_k + K_k H \hat{x}_k^- \\ &= (x_k - \hat{x}_k^-) - K_k H (x_k - \hat{x}_k^-) - K_k v_k \\ &= (I - K_k H)(x_k - \hat{x}_k^-) - K_k v_k \\ e_k^- &= x_k - \hat{x}_k^-\end{aligned}$$

- $(AB)^\top = B^\top A^\top$
- $(A + B)^\top = A^\top + B^\top$

$$\begin{aligned}
P_k &= E[ee^\top] \\
&= E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^\top] \\
&= E[(I - K_k H)e_k^- - k_k v_k][(I - K_k H)e_k^- - k_k v_k]^\top] \\
&= E[(I - K_k H)e_k^- - k_k v_k][e_k^{-\top}(I - K_k H)^\top - v_k^\top K_k^\top] \\
&= E[(I - K_k H)e_k^- e_k^{-\top}(I - K_k H)^\top - (I - K_k H)e_k^- v_k^\top K_k^\top \\
&\quad - k_k v_k e_k^{-\top}(I - K_k H)^\top + k_k v_k v_k^\top K_k^\top]
\end{aligned}$$

- $E(AB) = E(A)E(B)$ : A,B Independent

$$\begin{aligned}
E[(I - K_k H)e_k^- v_k^\top K_k^\top] &= (I - K_k H)E[e_k^- v_k^\top]K_k^\top \\
E[e_k^- v_k^\top] &= E[e_k^-] + E[v_k^\top] \\
E[e_k^-] &= 0, E[v_k^\top] = 0 \Rightarrow E[(I - K_k H)e_k^- v_k^\top K_k^\top] = 0 \\
\text{also : } E[k_k v_k e_k^{-\top}(I - K_k H)^\top] &= 0
\end{aligned}$$

$$P_k = (I - K_k H)E(e_k^- e_k^{-\top})(I - K_k H)^\top + K_k E(v_k v_k^\top)K_k^\top$$

- $E(e_k^- e_k^{-\top}) = P_k^-$
- $E(v_k v_k^\top) = R$ : Measurement noise Covariance matrix

$$\begin{aligned}
P_k &= (P_k^- - K_k H P_k^-)(I - H^\top K_k^\top) + K_k R K_k^\top \\
&= P_k^- - K_k H P_k^- - P_k^- H^\top K_k^\top + K_k H P_k^- H^\top K_k^\top + K_k R K_k^\top \\
(P_k^- H^\top K_k^\top)^\top &= K_k (P_k^- H)^\top \\
&= K_k H P_k^- \\
\Rightarrow \text{tr}(K_k H P_k^-) &= \text{tr}(P_k^- H^\top K_k^\top) \\
\text{tr}(P_k) &= \text{tr}(P_k^-) - 2\text{tr}(K_k H P_k^-) \\
&\quad + \text{tr}(K_k H P_k^- H^\top K_k^\top) + \text{tr}(K_k R K_k^\top)
\end{aligned}$$

- $\frac{d\text{tr}(AB)}{dA} = B^\top$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\text{tr}(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}$$

$$\frac{d\text{tr}(AB)}{dA} = \begin{bmatrix} \frac{\partial \text{tr}(AB)}{\partial a_{11}} & \frac{\partial \text{tr}(AB)}{\partial a_{12}} \\ \frac{\partial \text{tr}(AB)}{\partial a_{21}} & \frac{\partial \text{tr}(AB)}{\partial a_{22}} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = B$$

- $\frac{d\text{tr}(ABA^\top)}{dA} = 2AB$

$$\frac{d\text{tr}(P_k)}{dK_k} = 0$$

$$\frac{d\text{tr}(P_k)}{dK_k} = 0 - 2(H P_k^-)^\top + 2K_k H P_k^- H^\top + 2K_k R$$

- $P_k^{-\top} = P_k^-$ : Covariance matrix

$$\begin{aligned}
-P_k^- H^\top + K_k (H P_k^- H^\top + R) &= 0 \\
K_k (H P_k^- H^\top + R) &= P_k^- H^\top \\
K_k &= \frac{P_k^- H^\top}{H P_k^- H^\top + R} \\
R \uparrow, K_k &> 0, \hat{x}_k = \hat{x}_k^- \\
R \downarrow, K_k &> H^{-1}, \hat{x}_k = H^{-1} z_k
\end{aligned}$$

## 6 Priori/Posteriori Error Covariance Matrix

- $P(\omega) \sim N(0, Q)$
- $P(v) \sim N(0, R)$

$$\begin{aligned}x_k &= Ax_{k-1} + Bu_{k-1} + \omega_{k-1} \\z_k &= Hx_k + v_k\end{aligned}$$

- Priori estimate

$$\hat{x}_k^- = Ax_{k-1} + Bu_{k-1}$$

- Posteriori estimate

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$$

- Kalman Gain

$$K_k = \frac{P_k^- H^\top}{HP_k^- H^\top + R}$$

- $P_k^-$ ?

$$P_k^- = E[e_k^- e_k^{-\top}]$$

$$e_k^- = x_k - \hat{x}_k^-$$

$$\begin{aligned}e_k^- &= Ax_{k-1} + Bu_{k-1} + \omega_{k-1} - Ax_{k-1} - Bu_{k-1} \\&= A(x_{k-1} - \hat{x}_{k-1}^-) + \omega_{k-1} \\&= Ae_{k-1} + \omega_{k-1}\end{aligned}$$

$$\begin{aligned}P_k^- &= E[(Ae_{k-1} + \omega_{k-1})(Ae_{k-1} + \omega_{k-1})^\top] \\&= E[(Ae_{k-1} + \omega_{k-1})(e_{k-1}^\top A^\top + \omega_{k-1}^\top)] \\&= E[Ae_{k-1}e_{k-1}^\top A^\top + Ae_{k-1}\omega_{k-1}^\top + \omega_{k-1}e_{k-1}^\top A^\top + \omega_{k-1}\omega_{k-1}^\top] \\&= E[Ae_{k-1}e_{k-1}^\top A^\top] + E[Ae_{k-1}\omega_{k-1}^\top] + E[\omega_{k-1}e_{k-1}^\top A^\top] + E[\omega_{k-1}\omega_{k-1}^\top]\end{aligned}$$

- $e_{k-1}, \omega_{k-1}$  Independent
- $E[e_{k-1}] = E[\omega_{k-1}] = 0$
- $E[Ae_{k-1}\omega_{k-1}^\top] = AE[e_{k-1}]E[\omega_{k-1}^\top] = 0$
- *also* :  $E[\omega_{k-1}e_{k-1}^\top A^\top] = 0$

$$\begin{aligned}P_k^- &= AE[e_{k-1}e_{k-1}^\top]A^\top + E[\omega_{k-1}\omega_{k-1}^\top] \\&= AP_{k-1}A^\top + Q\end{aligned}$$

- Kalman formula

– Predict

- Priori Estimate
- Priori Error Covariance Matrix

– Correction

- Kalman Gain

- Posteriori Estimate
- Posteriori Error Covariance Matrix

Variables	Meaning
A	state matrix
B	control matrix
Q	process noise covariance matrix
R	measurement noise covariance matrix
H	measurement matrix
$P_k^-/P_k$	priori/posteriori error covariance matrix

$$\hat{x}_k^- = Ax_{k-1} + Bu_{k-1}$$

$$P_k^- = AP_{k-1}A^\top + Q$$

$$K_k = \frac{P_k^- H^\top}{HP_k^- H^\top + R}$$

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$$

$$\begin{aligned}
P_k &= P_k^- - K_k H P_k^- - P_k^- H^\top K_k^\top + K_k H P_k^- H^\top K_k^\top + K_k R K_k^\top \\
&= P_k^- - K_k H P_k^- - P_k^- H^\top K_k^\top + K_k (H P_k^- H^\top + R) K_k^\top \\
&= P_k^- - K_k H P_k^- - P_k^- H^\top K_k^\top + \frac{P_k^- H^\top}{HP_k^- H^\top + R} (H P_k^- H^\top + R) K_k^\top \\
&= P_k^- - K_k H P_k^- \\
&= (I - K_k H) P_k^-
\end{aligned}$$

## 7 Extended Kalman Filter

- Nonlinear
  - $P(\omega) \sim N(0, Q)$
  - $P(v) \sim N(0, R)$

$$\begin{aligned}
x_k &= f(x_{k-1}, u_{k-1}, \omega_{k-1}) \\
z_k &= h(x_k, v_k)
\end{aligned}$$

A normally distributed random variable is no longer normally distributed after passing through a nonlinear system

- Linearization
  - Taylor Series:  $f(x) = f(x_0) + \frac{\partial f}{\partial x}(x - x_0)$
  - Operating Point:
    - $x_k : \hat{x}_{k-1}$
    - $z_k : \tilde{x}_k$

$$x_k = f(\hat{x}_{k-1}, u_{k-1}, \omega_{k-1}) + A(x_k - \hat{x}_{k-1}) + W\omega_{k-1}$$

$$\text{suppose} : \omega_{k-1} = 0$$

$$f(\hat{x}_{k-1}, u_{k-1}, 0) = \tilde{x}_k$$

$$A = \frac{\partial f}{\partial x} \Big|_{\hat{x}_{k-1}, u_{k-1}}$$

$$W = \frac{\partial f}{\partial \omega} \Big|_{\hat{x}_{k-1}, u_{k-1}}$$

eg :

$$x_1 = x_1 + \sin x_2 = f_1$$

$$x_2 = x_1^2 = f_2$$

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & \cos x_2 \\ 2x_1 & 0 \end{bmatrix} \Big|_{\hat{x}_{k-1}, u_{k-1}}$$

$$A_k = \begin{bmatrix} 1 & \cos \hat{x}_{2k-1} \\ 2\hat{x}_{1k-1} & 0 \end{bmatrix}$$

$$z_k = h(\tilde{x}_k, v_k) + H(x_k - \tilde{x}_k) + Vv_k$$

$$\text{suppose} : v_k = 0$$

$$h(\tilde{x}_k, 0) = \tilde{z}_k$$

$$H = \frac{\partial h}{\partial x} \Big|_{\tilde{x}_k}$$

$$V = \frac{\partial h}{\partial v} \Big|_{\tilde{x}_k}$$

- $P(W\omega) \sim N(0, WQW^\top)$

- $P(Vv) \sim N(0, VRV^\top)$

$$x_k = \tilde{x}_k + A(x_k - \hat{x}_{k-1}) + \omega\omega_{k-1}$$

$$z_k = \tilde{z}_k + H(x_k - \tilde{x}_k) + Vv_k$$

- Extended Kalman formula

$$\hat{x}_k^- = f(\hat{x}_{k-1}, u_{k-1}, 0)$$

$$P_k^- = AP_{k-1}A^\top + WQW^\top$$

$$K_k = \frac{P_k^- H^\top}{HP_k^- H^\top + VRV^\top}$$

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - h(\hat{x}_k^-, 0))$$

$$P_k = (I - K_k H)P_k^-$$