Sub. Code: 216

NEB - GRADE XII
Mathematics
Model Question (For 2077)

Candidates are required to give their answer in their own words as far as practicable. The figures in the margin indicate full marks

Time - 1.30 hrs. Full Marks – 40

Note: Group A is compulsory and select another one Group either B or C

Group 'A'

Attempt **all** the questions.

1. a) Show that $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots=1$ (2)

Solution

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots=\frac{2-1}{2!}+\frac{3-1}{3!}+\frac{4-1}{4!}+\ldots$$
$$=\frac{2}{2!}-\frac{1}{2!}+\frac{3}{3!}-\frac{1}{3!}+\frac{4}{4!}-\frac{1}{4!}+\ldots$$
$$=\frac{2}{2\times1!}-\frac{1}{2!}+\frac{3}{3\times2!}-\frac{1}{3!}+\frac{4}{4\times3!}-\frac{1}{4!}+\ldots$$
$$=1-\frac{1}{2!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{4!}-\ldots\quad\left[\because \frac{1}{4!}\text{ will be next term in the series}\right]$$
$$=1$$

b. Find the ratio in which the line joining the points $P(-2, 4, 7)$ and $Q(3, -5, -1)$ is divided by ZX plane. (2)

Solution

Let ZX plane divides the line segment in $k:1$ ratio.

In ZX plane, $y = 0$

So, $\frac{ky_2+y_1}{k+1}=0$ $\left[\because y=\frac{ky_2+y_1}{k+1}\right]$

or, $\frac{k(-5)+4}{k+1}=0$

or, $-5k+4=0$

or, $-5k=-4$

$\therefore$ $k=\frac{4}{5}$

Hence, the required ratio is 4:5.

c) If $\vec{a} = \hat{i} + 2j - k$ and $\vec{b} = \hat{i} - j + k$ find the projection of $\vec{b}$ on $\vec{a}$. (2)

Solution

Here, $\vec{a} = \hat{i} + 2j - k$ and $\vec{b} = \hat{i} - j + k$

Now,

$$\text{Projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$$

$$= \frac{(\hat{i} + 2j - k).(\hat{i} - j + k)}{\sqrt{1^2 + 2^2 + (-1)^2}}$$

$$= \frac{1-2-1}{\sqrt{1+4+1}}$$

$$= \frac{-2}{\sqrt{6}}$$

The negative sign indicates that the projection has an opposite direction with respect to *a*.

2. a) Solve: $\frac{dy}{dx} + \frac{1 + \cos 2y}{1 - \cos 2x} = 0$ (2)

Solution

Here,

$$\frac{dy}{dx} + \frac{1 + \cos 2y}{1 - \cos 2x} = 0$$

$$\text{or, } \frac{dy}{dx} + \frac{2\cos^2 y}{2\sin^2 x} = 0$$

$$\text{or, } \frac{dy}{dx} + \frac{\cos^2 y}{\sin^2 x} = 0$$

$$\text{or, } \frac{dy}{dx} = -\frac{\cos^2 y}{\sin^2 x}$$

$$\text{or, } -\cos^2 y dx = \sin^2 x dy$$

$$\text{or, } -\frac{dx}{\sin^2 x} = \frac{dy}{\cos^2 y}$$

$$\text{or, } -\text{cosec}^2 x dx = \sec^2 y dy$$

Integrating on both sides, we get

$$\int -\text{cosec}^2 x dx = \int \sec^2 y dy$$

$$\text{or, } \cot x = \tan y + c$$

$$\therefore \cot x - \tan y = c$$

b. Calculate the mean deviation from mean of the data: **(2)**
3, 5, 9, 11, 7, 6

Solution

Given data: 3, 5, 9, 11, 7, 6

Computation of MD:

x	$\lvert x-\bar{x}\rvert = \lvert x-6.83\rvert$
3	3.83
5	1.83
9	2.17
11	4.17
7	0.17
6	0.83
$\sum x = 41$	$\sum\lvert x-\bar{x}\rvert = 13$

(Note: there is no need to arrange the given data in order for calculating mean as shown above)

We know,

$$\bar{x} = \frac{\sum x}{N} = \frac{41}{6} = 6.83$$

Now,

$$\text{Mean deviation from mean} = \frac{\sum\lvert x-\bar{x}\rvert}{N} = \frac{13}{6} = 2.17$$

3. Define abelian group. If (G,*) is an abelian group, prove that $(a*b)^{-1} = a^{-1}*b^{-1}\ \forall\ a, b \in G$.

Solution **(4)**

Abelian group:

A group (G, *) is said to be an abelian group if $a * b = b * a$ for all $a, b \in G$.

Next Part:

$$\begin{aligned}(a*b)*(a^{-1}*b^{-1}) &= ((a*b)*a^{-1})*b^{-1} && \text{[By associative law]}\\ &= ((b*a)*a^{-1})*b^{-1} && [G \text{ is abelian group, so } a*b=b*a]\\ &= (b*(a*a^{-1}))*b^{-1} && [a*a^{-1}=e]\\ &= (b*e)*b^{-1}\\ &= b*b^{-1} && [b*b^{-1}=e]\\ &= e\end{aligned}$$

Similarly, $(a^{-1}*b^{-1})*(a*b) = e$

$\therefore a^{-1}*b^{-1}$ is inverse of $a*b$

i.e. $(a*b)^{-1} = a^{-1}*b^{-1}$

4. Find the condition that a line $ax + by + c = 0$ may be normal to the parabola $y^2 = 4mx$.

Or

Find the vertices and foci of the ellipse $\frac{(x+2)^2}{16} + \frac{(y-5)^2}{9} = 1$ (4)

Solution

The equation of normal to the parabola $y^2 = 4ax$ is

$$y = m_1x - 2am_1 - am_1^3$$

$$\text{or, } -m_1x + y + am_1(2 + m_1^2) = 0 \qquad \text{...... (1)}$$

Comparing $y^2 = 4ax$ with $y^2 = 4mx$, we get,

$$a = m$$

Substituting the value of $a = m$ in eqn. (1), we get

$$-m_1x + y + mm_1(2 + m_1^2) = 0 \qquad \text{...... (2)}$$

The given equation of line is

$$ax + by + c = 0 \qquad \text{...... (3)}$$

Eqn. (2) and (3) represent same line if

$$-\frac{m_1}{a} = \frac{1}{b} = \frac{mm_1(2 + m_1^2)}{c}$$

From first two ratios,

$$-\frac{m_1}{a} = \frac{1}{b}$$

$$\text{or, } m_1 = -\frac{a}{b}$$

From last two ratios,

$$\frac{1}{b} = \frac{mm_1(2 + m_1^2)}{c}$$

$$\text{or, } c = bmm_1(2 + m_1^2)$$

$$\text{or, } c = \cancel{b}m.-\frac{a}{\cancel{b}}\left(2 + \frac{a^2}{b^2}\right) \quad \left[\because m_1 = -\frac{a}{b}\right]$$

$$\text{or, } c = -am\left(\frac{2b^2 + a^2}{b^2}\right)$$

$$\text{or, } b^2c = -am(a^2 + 2b^2)$$

$\therefore$ $am(a^2 + 2b^2) + b^2c = 0$, is the required condition.

OR Question Solution

The equation of ellipse is

$$\frac{(x+2)^2}{16}+\frac{(y-5)^2}{9}=1$$

Comparing given equation with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, we get

$h=-2, k=5, a^2=16$ and $b^2=9$,

so, $a = 4$ and $b = 3$

Now,

$$e=\sqrt{1-\frac{b^2}{a^2}}$$
$$=\sqrt{1-\frac{9}{16}}$$
$$=\sqrt{\frac{16-9}{16}}$$
$$=\sqrt{\frac{7}{16}}$$
$$=\frac{\sqrt{7}}{4}$$

The vertices of ellipse

$$=(h\pm a, k)$$
$$=(-2\pm 4, 5)$$
$$=(-6,5) \text{ and } (2,5)$$

The foci of ellipse

$$=(h\pm ae, k)$$
$$=\left(-2\pm 4.\frac{\sqrt{7}}{4}, 5\right)$$
$$=\left(-2\pm\sqrt{7}, 5\right)$$

5. Evaluate: $\int \frac{dx}{1+\sin x+\cos x}$ **(4)**

Solution

Let $I = \int \frac{dx}{1+\sin x+\cos x}$

$= \int \frac{dx}{(1+\cos x)+\sin x}$

$= \int \frac{dx}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}.\cos\frac{x}{2}}$

$= \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2}+\sin\frac{x}{2}.\cos\frac{x}{2}}$

$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}.dx}{\cos^2\frac{x}{2}.\sec^2\frac{x}{2}+\sin\frac{x}{2}.\cos\frac{x}{2}.\sec^2\frac{x}{2}}$ $\left[\because \text{Multiplying numerator and denomerator by } \sec^2\frac{x}{2}\right]$

$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}.dx}{1+\sin\frac{x}{2}.\sec\frac{x}{2}}$

$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}.dx}{1+\sin\frac{x}{2}/\cos\frac{x}{2}}$

$= \int \frac{\frac{1}{2}\sec^2\frac{x}{2}.dx}{1+\tan\frac{x}{2}}$

Let $y = 1+\tan\frac{x}{2}$ then $dy = \frac{1}{2}\sec^2\frac{x}{2}dx$

Now,

$I = \int \frac{dy}{y}$

$= \log y + c$

$= \log\left(1+\tan\frac{x}{2}\right)+c$

6. From definition, find the derivative of $e^{\tan x}$ **(6)**

Or

State Mean value theorem. Verify it for the function $f(x) = 2x^2 - 10x + 29$ in [2, 7]

Solution

Let $f(x) = e^{\tan x}$ then $f(x+h) = e^{\tan(x+h)}$

From the definition,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{\tan(x+h)}-e^{\tan x}}{h} \text{............(1)}$$

Let $\tan x = u$ and $\tan(x+h) = u+k$ then $k = \tan(x+h) - \tan x$

When $h \to 0, k \to 0$. Then eqn. (1) becomes

$$f'(x) = \lim_{h\to 0}\frac{e^{u+k}-e^{u}}{h}$$

$$= \lim_{h\to 0}\frac{e^{u}(e^{k}-1)}{h}$$

$$= e^{u}.\lim_{h\to 0}\frac{(e^{k}-1)}{k}.\frac{k}{h}$$

$$= e^{u}.\lim_{k\to 0}\frac{(e^{k}-1)}{k}.\lim_{h\to 0}\frac{k}{h}$$

$$= e^{\tan x}.1.\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h} \qquad \left[\because \lim_{k\to 0}\frac{e^{k}-1}{k}=1\right]$$

$$= e^{\tan x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$= e^{\tan x}\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{\cos(x+h)\cos x}\right]$$

$$= e^{\tan x}\lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$= e^{\tan x}\lim_{h\to 0}\frac{\sin h}{h\cos(x+h)\cos x}$$

$$= e^{\tan x}\lim_{h\to 0}\frac{\sin h}{h}.\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}$$

$$= e^{\tan x}.1.\frac{1}{\cos x.\cos x}$$

$$= e^{\tan x}.1.\frac{1}{\cos^2 x}$$

$$\therefore \frac{d}{dx}(e^{\tan x}) = e^{\tan x}\sec^2 x$$

OR Question Solution

Mean Value Theorem:

It states that, "If a function $f(x)$ is

a) continuous in the closed interval [a, b], and
b) differentiable in the open interval (a, b)

then there exists at least one value $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}."$$

Next Part:

Here, $f(x) = 2x^2 - 10x + 29$

Since $f(x)$ is a polynomial function, it is continuous in [2, 7].

And $f'(x) = 4x - 10$, which exists for all $x \in (2, 7)$

$\therefore f(x)$ is differentiable in (2, 7).

Since all the condition of mean value theorem is satisfied, there exists at least a point $c \in (2, 7)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\text{or, } 4c - 10 = \frac{f(7) - f(2)}{7 - 2}$$

$$\text{or, } 4c - 10 = \frac{(2.7^2 - 10.7 + 29) - (2.2^2 - 10.2 + 29)}{5}$$

$$\text{or, } 4c - 10 = \frac{(98 - 70 + 29) - (8 - 20 + 29)}{5}$$

$$\text{or, } 4c - 10 = \frac{98 - 70 + 29 - 8 + 20 - 29}{5}$$

$$\text{or, } 4c - 10 = \frac{40}{5}$$

$$\text{or, } 4c - 10 = 8$$

$$\text{or, } 4c = 10 + 8$$

$$\text{or, } 4c = 18$$

$$\text{or, } c = \frac{18}{4}$$

$$\text{or, } c = \frac{9}{2}$$

$$\therefore \quad c = 4.5 \in (2,7)$$

Hence, mean value theorem is verified.

Group 'C'

10. Examine whether the system of equations $3x + 12y - z = 28$, $x + 4y + 7z = 2$ and $10x + 4y - 2z = 20$ is diagonally dominant. (2)

Solution

$$\text{Here, } 3x+12y-z=28$$
$$x+4y+7z=2$$
$$10x+4y-2z=20$$

The given system is not diagonally dominant because

$$|3| \ngeq |12|+|-1|$$
$$|4| \ngeq |1|+|7|$$
$$|-2| \ngeq |10|+|4|$$

11. Use Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6$ in (0, 1). (4)

Solution

Watch Best Calculator Tricks for Bisection Method: ***https://youtu.be/Q3rXNNEKpiY***

Here, $f(x) = x^3 - 7x^2 + 14x - 6$

$f(0) = -6$ (−ve)

$f(1) = 1 - 7 + 14 - 6 = 2$ (+ve)

Since, $f(0).f(1) = -6.2 = -12 < 0$, a root lies between 0 and 1.

Let $a(-ve) = 0$ and $b(+ve) = 1$

*(Note: If **f (x)** is − ve, change the value of **a** equal to **x**$_n$ keeping value of **b** same in next iteration and if **f (x**$_n$**)** is **+ ve**, change the value of **b** equal to **x**$_n$ keeping value of **a** same in next iteration)*

Using bisection method,

Iteration	a (− ve)	b (+ ve)	Midpoint, $x_n = \frac{a+b}{2}$	$f(x_n)$
1	0	1	0.500	− 0.625
2	0.500	1	0.750	0.984
3	0.500	0.750	0.625	0.260
4	0.500	0.625	0.563	− 0.162
5	0.563	0.625	0.594	0.054
6	0.563	0.594	0.578	− 0.053
7	0.578	0.594	**0.58**6	0.001
8	0.578	0.586	**0.58**2	− 0.026

In iteration 7 and 8, the value of x is same upto two decimal (10^{-2}), so the approximate root in (0,1) interval is $x = 0.58$.

12. By simplex method maximize
$F = 15x_1 + 10x_2$ subject to $2x_1 + x_2 \le 10$, $x_1 + 3x_2 \le 10$; $x_1, x_2 \ge 0$ **(6)**

Solution

Watch Best Calculator Tricks for Simplex Method: ***https://youtu.be/WYjq4lqd1UQ***

Step 1: Let s_1 and s_2 be non-negative slack variables. Expressing the given LP in equation form:

$$2x_1 + x_2 + s_1 = 10 \quad ...... (1)$$
$$x_1 + 3x_2 + s_2 = 10 \quad ...... (2)$$
$$-15x_1 - 10y + F = 0 \quad ...... (3)$$
$$x_1, x_2, s_1, s_2 \ge 0$$

Step 2: Constructing initial simplex tableau

Table 1: Initial Simplex Tableau

	BV	x_1	x_2	s_1	s_2	F	RHS	Ratio
R_1	s_1	2	1	1	0	0	10	$10/2 = 5 \rightarrow$
R_2	s_2	1	3	0	1	0	10	$10/1 = 10$
R_3	F	−15	−10	0	0	1	0	
		↑						

From initial simplex tableau, the initial basic solution is

$x_1 = 0, x_2 = 0$ (Non-basic variables)

$s_1 = 10, s_2 = 10$ (Basic variables)

$\therefore F = 15.0 + 10.0 = 0$

This is not the optimal solution as last row (objective function row *i.e.* *z* row) contains negative entry.

Step 3: Finding pivot column, pivot row, pivot element and performing row operation

Pivot column = Most Negative in last row = x_1 column = entering variable

Pivot row = Minimum Ratio row = R_1 row = s_1 row = leaving variable

Pivot element = Intersection of Pivot column and Pivot Row = 2

<u>For pivot row:</u>

New Pivot Row → Pivot row / Pivot element

$\therefore R_1(\text{new}) \rightarrow R_1(\text{old}) / 2 \Rightarrow \boxed{R_2 \rightarrow R_1 / 2}$

<u>For other row:</u>

$$\text{New Row} \rightarrow \text{Old Row} - \frac{\text{Pitvot column coefficient}}{\text{Pivot element}} \times \text{Pivot row}$$

$$R_2(\text{new}) \rightarrow R_2(\text{old}) - \frac{1}{2} \times R_1(\text{old}) \Rightarrow \boxed{R_2 \rightarrow R_2 - 0.5R_1}$$

$$R_3(\text{new}) \rightarrow R_3(\text{old}) - \frac{-15}{2} \times R_1(\text{old}) \Rightarrow \boxed{R_3 \rightarrow R_3 + 7.5R_1}$$

Using above row operation, constructing new table

Table 2: x_1 enters, s_1 leaves

	BV	x_1	x_2	s_1	s_2	F	RHS	Ratio
R_1	x_1	1	0.5	0.5	0	0	5	5/0.5 = 10
R_2	s_2	0	2.5	−0.5	1	0	5	5/2.5 = 2 →
R_3	F	0	−2.5	7.5	0	1	75	

This is still not the optimal solution as last row (objective function row) contains negative entry.

Step 4: The above solution is not optimal, so repeating as in **Step 3**.
Pivot column = Most Negative in last row = x_2 column = entering variable
Pivot row = Minimum Ratio row = R_2 row = s_2 row = leaving variable
Pivot element = Intersection of Pivot column and Pivot Row = 2.5

For pivot row:

New Pivot Row → Pivot row / Pivot element

$$R_2(\text{new}) \to R_2(\text{old})/2.5 \Rightarrow \boxed{R_2 \to R_1/2.5}$$

For other row:

$$\text{New Row} \to \text{Old Row} - \frac{\text{Pitvot column coefficient}}{\text{Pivot element}} \times \text{Pivot row}$$

$$R_1(\text{new}) \to R_1(\text{old}) - \frac{0.5}{2.5} \times R_2(\text{old}) \Rightarrow \boxed{R_1 \to R_1 - 0.2R_2}$$

$$R_3(\text{new}) \to R_3(\text{old}) - \frac{-2.5}{2.5} \times R_2(\text{old}) \Rightarrow \boxed{R_3 \to R_3 + R_2}$$

Using above row operation, constructing new table

Table 3: x_2 enters, s_2 leaves

	BV	x_1	x_2	s_1	s_2	F	RHS	Ratio
R_1	x_1	1	0	0.6	− 0.2	0	4	
R_2	x_2	0	1	− 0.2	0.4	0	2	
R_3	F	0	0	7	1	1	80	

Since, all the entries in last row is non-negative, the present solution is optimal.

$\therefore x_1 = 4, x_2 = 2$ and $\max F = 80$

$$\begin{aligned}\text{Also, max } z &= 15x_1 + 10x_2 \\ &= 15.4 + 10.2 \\ &= 80 \text{ which is true}\end{aligned}$$