Math 6643, Numerical Linear Algebra
HW 1, Due on Wednesday, September 20

Notice: For the computational problems, you must design your own ways, such as using tables and plots etc, to present the results you obtain. And you must show the procedure you take, and explain the results you get, and draw conclusions if you have any. Computer code is not considered as an explanation. Please upload your code in T-square, and return your HW in class on or before the Due date.

- **Problem 1**: Prove if A is a lower triangular matrix with nonzero diagonal entries, then A^{-1} is also lower triangular.

- **Problem 2**: If $\|A\|$ is an induced matrix norm for $m \times m$ matrix A, show that $\rho(A) \le \|A\|$, where $\rho(A)$ is the spectral radius of A.

- **Problem 3**: Suppose A is a 202×202 matrix with $\|A\|_2 = 100$ and $\|A\|_F = 101$, give the sharpest possible lower bound on the 2-norm condition number $\kappa(A)$.

- **Problem 4**: Perform the following numerical experiments for random square matrices. Here, we define a random matrix to be a $m \times m$ matrix whose entries are independent random numbers from real normal distribution with mean zero and standard deviation $\sqrt{m}$. In each of the following items, you must repeat the experiment for at least 100 different random matrices to collect the statistical information.
  (1) Compute the $p-$norm, with $p = 1, 2$, and infinite norm, for the random matrices with sizes $m = 100, 200, 300, \cdots$. For each of the random matrices, compute the ratios between 1-norm and 2-norm if your GT ID# is an even number, or 2-norm and infinite norm if your GT ID# is an odd number. Find out how those quantities behave as $m \to \infty$.
  (2) Compute the 1-norm condition number for the random matrices with size $m = 100, 200, 300, \cdots$, and find out its behavior as $m \to \infty$.

- **Problem 5**: Write your own code for the naive Gaussian Elimination, and Gaussian Elimination with partial pivoting. Apply both algorithms to the following problem:
  Discretize one-dimensional equation

$$\begin{cases} -u'' + \lambda u = 3x - \frac{1}{2} & x \in [0,1] \\ u(0) = 0, \quad u(1) = -2 \end{cases}$$

  by centered difference scheme with n interior mesh points. Here λ is the constant. Perform the following experiments for $\lambda = 0$ and $\lambda = 2$ respectively.

  Form the linear system $A\vec{x} = \vec{b}$. Solve the discrete linear system by your GE code for $n = 200, 400, 800, 1000, 2000, 4000$ and larger if your computer allows. Plot the results and calculate the cpu time. Comments on your results.

  Hint: you must find ways to demonstrate how the computation cost is associated the size of the system, and how to compare the numerical solutions with the exact solution if you can find it. If you can't find the exact solution, what is your remedy to perform the comparison.