# Directional Derivative

# Functions of Several Variables

In this section, a certain amount of linear Algebra will be used.

**Derivative in $\mathbb{R}^n$:-** A map $f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$, written as

$$f(x) = f(x_1, x_2, \ldots, x_m) = (f_1(x), f_2(x), \ldots, f_n(x)),$$

is said to be differentiable at $x_0 \in E$ if there exist a linear-mapping $L: R^n \longrightarrow R^m$ such that for every $\varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ such that

if $|x - x_0| < \delta$ then $\boxed{\left|\frac{f(x) - f(x_0)}{x - x_0} - L\right| < \varepsilon}$

The derivative of $f$ at $x_0$ is denoted by $Df(x_0)$ or $f'(x_0)$. instead of $L$.

**Matrix Representation:-** Let $f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a given function then we write $f(x) = f(x_1, x_2, \ldots, x_m) = (f_1(x), f_2(x), \ldots, f_n(x))$ and compute the partial derivatives $\frac{\partial f_j}{\partial x_i}$ $j = 1, 2, \ldots n$, $i = 1, 2, \ldots m$.

Let $B_n = \{e_1, e_2, \ldots, e_n\}$

$B_m = \{e_1, e_2, \ldots, e_m\}$ are the standard basis of $\mathbb{R}^m$ and $\mathbb{R}^n$.

Suppose $f: \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, so the partial derivatives exists and the matrix of the linear map $Df(x)$ with respect to $B_m, B_n$ is given by

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}_{n \times m}$$

**Ex. 1** Let $f: R^2 \to R^3$, $f(x,y) = (\underbrace{x^2}_{f_1}, \underbrace{x^3 y}_{f_2}, \underbrace{x^4 y^2}_{f_3})$. Compute $Df$.

$$Df(x,y) = \begin{bmatrix} 2x & 0 \\ 3x^2 y & x^3 \\ 4x^3 y^2 & 2x^4 y \end{bmatrix}$$ is the matrix of linear map $Df(x)$.

# Existence of Derivative:-

Does existence of the usual partial derivatives imply the existence of the derivative Df ???

**NO:** Not true in general.

Ex. $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ , $f(x,y) = \begin{cases} x & \text{when } y=0 \\ y & \text{when } x=0 \\ 1. & \text{otherwise} \end{cases}$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$ and are equal to 1.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h-0}{h} = 1$$

Similarly $\frac{\partial f}{\partial y} = 1.$

Is $f(x,y)$ is continuous ???

No, So the derivative Df can't exist. at (0,0).

** The partial derivatives depend only on what happens in the direction of the axis. whereas the definition of Df involves the combined behaviour of f in whole nbd of a given point.**

*** The existence of the partial derivatives is not sufficient conditions for the existence of the derivative,
The continuity of these partial derivatives is a sufficient condition.***

**Que.** $f:\mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x^2, y^2 + \sin x)$. [NET DEC-2016].

Then the derivative of f at (x,y) is linear transformation given by

a) $\begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix}$ b) $\begin{pmatrix} 2x & 0 \\ 2y & \cos x \end{pmatrix}$ c) $\begin{pmatrix} 2y & \cos x \\ 2x & 0 \end{pmatrix}$ d) $\begin{pmatrix} 2x & 2y \\ 0 & \cos x \end{pmatrix}$.

**Ans.** $Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \Rightarrow Df(x,y) = \begin{bmatrix} 2x & 0 \\ \cos x & 2y \end{bmatrix}$ option (a) is correct.

**Directional Derivative:-** Let $x_0 \in \mathbb{R}^n$, $e \in \mathbb{R}^n$ (unit vector). Let $f$ be real valued function defined in a nbd of $x_0$. Then

$$\frac{d}{dt}\left(f(x_0+te)\right)\Big|_{t=0} = \lim_{t\to 0} \frac{f(x_0+te)-f(x_0)}{t}$$

is called the directional derivative of $f$ at '$x_0$' in the direction of $e$.

** If $f$ is differentiable at $x_0$, then the directional derivatives also exist but converse need to be true.**

**Ex.** $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y} & : x^2 \neq -y \\ 0 & : x^2 = y \end{cases}$ at $(0,0)$.

Derivative at $(0,0)$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to 0} \frac{xy}{(x^2+y)\sqrt{x^2+y^2}}$

$$= \lim_{r\to 0} \frac{r\cos\theta \, r\sin\theta}{(r^2\cos^2\theta + r\sin\theta)\cdot r}$$

$$= \lim_{r\to 0} \frac{\cos\theta \sin\theta}{r\cos^2\theta + \sin\theta} = \cos\theta$$

depends on $\theta \Rightarrow$ limit does not exist uniquely in nbd of $(0,0)$.

$\Rightarrow$ Derivative does not exist.

Let $e = (e_1, e_2)$ be an unit vector.

then $$\frac{d}{dt} f(x_0 + te)\Big|_{t=0} = \lim_{t\to 0} \frac{f(te) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(te_1, te_2)}{t} = \lim_{t\to 0} \frac{t^2 e_1 e_2}{t^2 e_1^2 + te_2} \cdot \frac{1}{t}$$

$$= \lim_{t\to 0} \frac{e_1 e_2}{t_1 e_1^2 + e_2} = e_1.$$

thus directional derivative exists at $(0,0)$ but $f$ is not differentiable at $(0,0)$.

# More Approach to Calculate Directional Derivative:-

*) The directional derivative of a function f with respect to unit vector 'v' may be denoted by any of the following-

$\nabla_v f(x)$

$D_v f(x)$

$Df(x)(v)$

$v \cdot \nabla f(x)$

**) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function. If directional derivative exists then we can compute it by the given formulas also.

Let $v = (a, b)$ is a unit vector then

i) $$D_v f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

We can generalize it also

$$D_v f(x,y,z) = f_x a + f_y b + f_z c.$$

(ii) $D_v f(x,y) = Df(x,y)\, v \qquad \{ f: \mathbb{R}^n \to \mathbb{R}^m \}$.

$$= \begin{bmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

**Que:-** $f(x,y) = xy$ $\quad v = (1,2)$ Dir. Vector [NET-DEC-2016]

$a = (a_1, a_2)$ point.

$$D_v f(a) = f_x(a) \cdot 1 + f_y(a) \cdot 2$$

$$= a_2 + 2a_1 \qquad \underline{\underline{\text{Ans.}}}$$

$v = (1,2)$ is not unit vector

$\therefore \dfrac{v}{\|v\|} = \dfrac{1}{\sqrt{5}}(1,2)$

$\therefore D_{\frac{v}{\|v\|}} f(a) = \dfrac{1}{\sqrt{5}}(a_2 + 2a_1)$.

*** In some questions v need not to be unit vector.
So check option carefully if for unit vector, answer is given then select that one.

# JUNE 2020
# (NOV-2020)

**Question Number : 63 Question Id : 802437553 Question Type : MSQ Option Shuffling : No Is Question Mandatory : No**
**Correct Marks : 4.75 Wrong Marks : 0**

. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

Define $g(x,y) = \sum_{n=1}^{\infty} \frac{f\left((x-n),\ (y-n)\right)}{2^n}$.

Which of the following statements are true?

**Options :**

8024372209. The function $h(y) = g(c,y)$ is continuous on $\mathbb{R}$ for all $c$

8024372210. $g$ is continuous from $\mathbb{R}^2$ into $\mathbb{R}$

8024372211. $g$ is not a well-defined function

8024372212. $g$ is continuous on $\mathbb{R}^2 \backslash \{(k,k)\}_{k\in\mathbb{N}}$

**Question Number : 68 Question Id : 802437558 Question Type : MSQ Option Shuffling : No Is Question Mandatory : No**
**Correct Marks : 4.75 Wrong Marks : 0**

Define

$$f(x,y)=\begin{cases}\frac{x^3}{x^2+y^2} & \text{for} \quad (x,y)\neq(0,0)\\ 0 & \text{for} \quad (x,y)=(0,0)\end{cases}$$

Which of the following statements are true?

**Options :**

8024372229. $f$ is discontinuous at $(0,0)$

8024372230. $f$ is continuous at $(0,0)$

8024372231. all directional derivatives of $f$ at $(0,0)$ exist

8024372232. $f$ is not differentiable at $(0,0)$

Question Number : 69 Question Id : 802437559 Question Type : MSQ Option Shuffling : No Is Question Mandatory : No
Correct Marks : 4.75 Wrong Marks : 0

Define

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} \text{ for } (x,y) \neq (0,0) \\ 0 \quad \text{for } (x,y) = (0,0) \end{cases}.$$

Which of the following statements are true?

Options :

8024372233. $f$ is continuous at $(0,0)$

8024372234. $f$ is bounded in a neighbourhood of $(0,0)$

8024372235. $f$ is not bounded in any neighbourhood of $(0,0)$

8024372236. $f$ has all directional derivatives at $(0,0)$

**Question Number : 70 Question Id : 802437560 Question Type : MSQ Option Shuffling : No Is Question Mandatory : No**
**Correct Marks : 4.75 Wrong Marks : 0**

Let $p: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$p(x,y) = \begin{cases} |x| & if \quad x \neq 0 \\ |y| & if \quad x = 0. \end{cases}$$

Which of the following statements are true?

**Options :**

8024372237. $p(x,y) = 0$ if and only if $x = y = 0$

8024372238. $p(x,y) \geq 0$ for all $x, y$

8024372239. $p(\alpha x, \alpha y) = |\alpha|\, p(x,y)$ for all $\alpha \in \mathbb{R}$ and for all $x, y$

8024372240. $p(x_1 + x_2, y_1 + y_2) \leq p(x_1, y_1) + p(x_2, y_2)$ for all $(x_1, y_1), (x_2, y_2)$

# Dec 2019

**76.** Let $f : [0, 1]^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \frac{xy}{x^2+y^2} \text{ if either } x \neq 0 \text{ or } y \neq 0$$

$$= 0 \text{ if } x = y = 0.$$

Then which of the following statements are true?

(1) f is continuous at (0, 0)

(2) f is a bounded function

(3) $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ exists

(4) f is continuous at (1, 0)

# Dec 2018

**78.** Let A be an invertible real n × n matrix. Define a function $F : \mathbb{R}^n \to \mathbb{R}$ by $F(x, y) = (Ax, y)$ where $(x, y)$ denotes the inner product of x and y. Let $DF(x, y)$ denote the derivative of F at $(x, y)$ which is a linear transformation from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Then

(1) If $x \neq 0$, then $DF(x, 0) \neq 0$

(2) If $y \neq 0$, then $DF(0, y) \neq 0$

(3) If $(x, y) \neq (0, 0)$ then $DF(x, y) \neq 0$

(4) If $x = 0$ or $y = 0$, then $DF(x, y) = 0$

# June 2018

**26.** Let $f(x, y) = \log\left(\cos^2\left(e^{x^2}\right)\right) + \sin(x + y)$.

Then $\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x, y)$ is

(1) $\frac{\cos(e^{x^2}) - 1}{1 + \sin^2(e^{x^2})} - \cos(x + y)$ (2) $0$

(3) $-\sin(x + y)$ (4) $\cos(x + y)$

69. For any $y \in \mathbb{R}$, let $[y]$ denote the greatest integer less than or equal to y. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^{[y]}$. Then

(1) f is continuous on $R^2$

(2) for every $y \in \mathbb{R}$, $x \to f(x,y)$ is continuous on $\mathbb{R}\setminus\{0\}$

(3) for every $x \in \mathbb{R}$, $y \to f(x,y)$ is continuous on $\mathbb{R}$

(4) f is continuous at no point of $\mathbb{R}^2$

# Dec 2017

**67.** Let $f(x, y) = \dfrac{1-\cos(x+y)}{x^2+y^2}$ if $(x, y) \neq (0, 0)$ $f(0, 0) = \dfrac{1}{2}$

and $\quad g(x, y) = \dfrac{1-\cos(x+y)}{(x+y)^2} \qquad$ if $x + y \neq 0$

$\qquad g(x, y) = \dfrac{1}{2} \qquad$ if $x + y = 0$

Then

(1) f is continuous at (0, 0)

(2) f is continuous everywhere except at (0, 0)

(3) g is continuous at (0, 0)

(4) g is continuous everywhere

**68.** Let $f : \mathbb{R}^4 \to \mathbb{R}$ be defined by $f(x) = x^t Ax$, where $A$ is a $4 \times 4$ matrix with real entries and $x^t$ denotes the transpose of $x$. The gradient of $f$ at a point $x_0$ necessarily is

(1) $2Ax_0$

(2) $Ax_0 + A^t x_0$

(3) $2A^t x_0$

(4) $Ax_0$

# June 2017

**68.** Suppose that $f:\mathbb{R}^n \to \mathbb{R}$ is given by $f(\underline{x}) = a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$, where $\underline{x} = (x_1, x_2, \cdots, x_n)$ and at least one $a_j$ is not zero. Then we can conclude that

1. $f$ is not everywhere differentiable
2. the gradient $(\nabla f)(\underline{x}) \neq 0$ for every $\underline{x} \in \mathbb{R}^n$
3. if $\underline{x} \in \mathbb{R}^n$ is such that $(\nabla f)(\underline{x}) = 0$ then $f(\underline{x}) = 0$
4. if $\underline{x} \in \mathbb{R}^n$ is such that $f(\underline{x}) = 0$ then $(\nabla f)(\underline{x}) = 0$

# Dec 2016

**23.** Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = (x^2, y^2 + \sin x).$$

Then the derivative of $f$ at $(x, y)$ is the linear transformation given by

1. $\begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix}$
2. $\begin{pmatrix} 2x & 0 \\ 2y & \cos x \end{pmatrix}$
3. $\begin{pmatrix} 2y & \cos x \\ 2x & 0 \end{pmatrix}$
4. $\begin{pmatrix} 2x & 2y \\ 0 & \cos x \end{pmatrix}$

**24.** A function $f: \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x, y) = xy$. Let $v = (1, 2)$ and $a = (a_1, a_2)$ be two elements of $\mathbb{R}^2$. The directional derivative of $f$ in the direction of $v$ at $a$ is:

1. $a_1 + 2a_2$
2. $a_2 + 2a_1$
3. $\frac{a_2}{2} + a_1$
4. $\frac{a_1}{2} + a_2$

**68.** A function $f(x, y)$ on $\mathbb{R}^2$ has the following partial derivatives

$$\frac{\partial f}{\partial x}(x, y) = x^2, \quad \frac{\partial f}{\partial y}(x, y) = y^2.$$

Then

1. $f$ has directional derivatives in all directions everywhere.
2. $f$ has a derivative at all points.
3. $f$ has directional derivative only along the direction (1,1) everywhere.
4. $f$ does not have directional derivatives in any direction everywhere.

# June 2016

**24.** Consider the function

$$f(x, y) = \frac{x^2}{y^2}, (x, y) \in [1/2, 3/2] \times [1/2, 3/2]$$

The derivative of the function at $(1, 1)$ along the direction $(1, 1)$ is:

1. $0$
2. $1$
3. $2$
4. $-2$

# Dec 2015

**69.** Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by the formula
$f(x,y) = (3x + 2y + y^2 + |xy|,\ 2x + 3y + x^2 + |xy|)$.
Then,

1. $f$ is discontinuous at $(0,0)$.
2. $f$ is continuous at $(0,0)$ but not differentiable at $(0,0)$.
3. $f$ is differentiable at $(0,0)$.
4. $f$ is differentiable at $(0,0)$ and the derivative $Df(0,0)$ is invertible.

# June 2015

68. Let $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function $F(x, y) = \langle Ax, y \rangle$, where $\langle , \rangle$ is the standard inner product of $\mathbb{R}^n$ and $A$ is a $n \times n$ real matrix. Here $D$ denotes the total derivative. Which of the following statements are correct?

1. $\big(DF(x, y)\big)(u, v) = \langle Au, y \rangle + \langle Ax, v \rangle$.
2. $\big(DF(x, y)\big)(0,0) = 0$.
3. $DF(x, y)$ may not exist for some $(x, y) \epsilon \mathbb{R}^n \times \mathbb{R}^n$.
4. $DF(x, y)$ does not exist at $(x, y) = (0,0)$.

# Dec 2014

25. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f:\Omega \to \mathbb{R}$ be a differentiable function such that $(Df)(x) = 0$ for all $x \in \Omega$. Then which of the following is true?

1. $f$ must be a constant function
2. $f$ must be constant on connected components of $\Omega$
3. $f(x) = 0$ or $1$ for $x \in \Omega$
4. The range of the function $f$ is a subset of $\mathbb{Z}$

# June 2014

77. Define $f: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x,y) = (x + 2y + y^2 + |xy|, 2x + y + x^2 + |xy|)$$

for $(x,y) \in \mathbb{R}^2$. Then

1. $f$ is discontinuous at $(0,0)$.
2. $f$ is continuous at $(0,0)$ but not differentiable at $(0,0)$.
3. $f$ is differentiable at $(0,0)$.
4. $f$ is differentiable at $(0,0)$ and the derivative $Df(0,0)$ is invertible.

78. Let $A = \{(x,y) \in \mathbb{R}^2 : x + y \neq -1\}$.
Define $f : A \to \mathbb{R}^2$ by
$f(x,y) = \left(\frac{x}{1+x+y}, \frac{y}{1+x+y}\right)$. Then,

1. the Jacobian matrix of $f$ does not vanish on $A$.
2. $f$ is infinitely differentiable on $A$.
3. $f$ is injective on $A$.
4. $f(A) = \mathbb{R}^2$.