**Constructing finite difference schemes.**

Example: 4th order, CIS finite difference scheme for the first derivative: fi

Start with:

(1) fm+2 = fm +fim{2D/1!} +fiim{4D2/2!} +fiiim{8D3/3!} +fivm{16D4/4!} +fvm{32D5/5!}+…

(2) fm+1 = fm +fim{D/1!} +fiim{D2/2!} +fiiim{D3/3!} +fivm{D4/4!} +fvm{D5/5!} + …

(3) fm-1 = fm -fim{D/1!} +fiim{D2/2!} -fiiim{D3/3!} +fivm{D4/4!} -fvm{D5/5!} + …

(4) fm-2 = fm -fim{2D/1!} +fiim{4D2/2!} -fiiim{8D3/3!} +fivm{16D4/4!} -fvm{32D5/5!} +…

Consider the sum: P\*(1) + Q\*(2) + R\*(3) + S\*(4) as follows:

P\* [ fm+2 = fm+fim{2D/1!}+fiim{4D2/2!}+fiiim{8D3/3!}+fivm{16D4/4!}+fvm{32D5/5!}+…]

Q\*[fm+1 = fm +fim{D/1!} +fiim{D2/2!} +fiiim{D3/3!} +fivm{D4/4!} +fvm{D5/5!}+ … ]

R\*[fm-1 = fm -fim{D/1!} +fiim{D2/2!} -fiiim{D3/3!} +fivm{D4/4!} -fvm{D5/5!} + … ]

S\*[fm-2 = fm -fim{2D/1!} +fiim{4D2/2!}-fiiim{8D3/3!} +fivm{16D4/4!}-fvm{32D5/5!}+… ]

Summing up the columns above gives:

column on the RHS =

(5) P\* fm+2 + Q\*fm+1 + R\*fm-1 + S\*fm-2

1st column on LHS =

(6) (P + Q + R + S)\*fm

2nd column on LHS =

(7) fim\*{D/1!}\*(2\*P + Q – R – 2\*S)

3rd column on LHS =

(8) fiim\*{D2 /2!}\*(4\*P + Q + R + 4\*S)

4th column on LHS =

(9) fiiim\*{D3 /3!}\*(8\*P + Q – R – 8\*S)

5th column on LHS =

(10) fivm\*{D4 /4!}\*(16\*P + Q + R + 16\*S)

4th column on LHS =

(11) fvm\*{D5 /5!}\*(32\*P + Q – R – 32\*S)

To eliminate a term set the sum in the ( ) =0. for that column.

For a 4th order centered in space scheme, set the ( ) terms separately to zero in (8) – (10):

(12) 4\*P + Q + R + 4\*S = 0.

(13) 8\*P + Q – R – 8\*S = 0.

(14) 16\*P + Q + R + 16\*S = 0.

And, set the ( ) terms in (7) = 1.

(15) 2\*P + Q – R – 2\*S = 1.

Solve for P, Q, R, and S. Note that

(14) - (12) = 12\*P + 12S = 0. 🡪 P = - S (16)

Also,

4\*(12)-(14) = 3\*Q + 3\*R =0. 🡪 Q = - R (17)

Substitute (16) and (17) into the eqns not yet used: (13) and (15) to get:

16\*P + 2\*Q = 0. 🡪 8\*P + Q = 0. (18)

4\*P + 2\*Q = 1 . 🡪 2\*P + Q = 1. (19)

Thus:

(18)-(19) = 6\*P = -1 🡪 P = -1./6. (20)

From (16) 🡪 S = 1./6. (21)

From (18) -8/6 + Q = 0. 🡪 Q = 4./3. (22)

From (17) 🡪 R = -4./3. (23)

Reconstructing the finite difference scheme, using (5)-(7):

fim = { (5) - (6) }/ D

(24) fim = { P\* fm+2 + Q\*fm+1 + R\*fm-1 + S\*fm-2 - (P + Q + R + S)\*fm }/ D

Substituting (20)-(23) yields the desired finite difference formula:

(25) fim = { -1/6 fm+2 +4/3 fm+1 -4/3 fm-1 +1/6 fm-2 }/ D

where it turns out that P + Q + R + S = 0.

*Note this simple check: if you sum the coefficients up inside the { } in (25) they should and do equal zero.*