Problem 9: Integration by substitution (✓)

Show, by means of a change of variable or otherwise, that

$$\int_0^\infty \mathrm{f}\big((x^2+1)^{\frac12}+x\big)\,\mathrm{d}x = \frac12\int_1^\infty (1+t^{-2})\mathrm{f}(t)\,\mathrm{d}t\,,$$

for any given function f .

Hence, or otherwise, show that

$$\int_0^\infty \big((x^2+1)^{\frac12}+x\big)^{-3}\,\mathrm{d}x = \frac38\,.$$

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Comments

Note that 'by change of variable' means the same as 'by substitution'.

There are two things to worry about when you are trying to find a change of variable to convert one integral to another: you need to make the integrands match up and you need to make the limits match up. Sometimes, the limits give the clue to the change of variable. (For example, if the limits on the original integral were 0 and 1 and the limits on the transformed integral were 0 and $\frac14\pi$, then an obvious possibility would be to make the substitution $t=\tan x$). Here, the change of variable is determined by the integrand, since it must work for all choices of f.

Perhaps you are worried about the infinite upper limit of the integrals. If you are trying to prove some rigorous result about infinite integrals, you might use the definition

$$\int_0^\infty \mathrm{f}\,(x)\,\mathrm{d}x = \lim_{a\to\infty}\int_0^a \mathrm{f}\,(x)\,\mathrm{d}x\,,$$

but for present purposes you just do the integral and put in the limits. The infinite limit will not normally present problems. For example,

$$\int_1^\infty (x^{-2}+\mathrm{e}^{-x})\mathrm{d}x = \big(-x^{-1}-\mathrm{e}^{-x}\big)\Big|_1^\infty = -\frac{1}{\infty}-\mathrm{e}^{-\infty}+\frac11+\mathrm{e}^{-1} = 1+\mathrm{e}^{-1}\,.$$

Don't be afraid of writing $1/\infty=0$. It is perfectly OK to use this as shorthand for $\lim\limits_{x\to\infty} 1/x=0$; but ∞/∞, $\infty-\infty$ and $0/0$ are definitely not OK, because of their ambiguity.