Task 1 To derive propagation and correction equations for

(A) EKF, (B) UKF, (C) PF, (d) RI-EKF

Given, "Velocity Motion Model"

$$x_{k+1} = x' = x_k - \frac{\hat{v}}{\hat{\omega}} \sin(\theta_k) + \frac{\hat{v}}{\hat{\omega}} \sin(\theta_k + \hat{\omega}\Delta t)$$

$$y_{k+1} = y' = y + \frac{\hat{v}}{\hat{\omega}} \cos(\theta_k) - \frac{\hat{v}}{\hat{\omega}} \cos(\theta_k + \hat{\omega}\Delta t)$$

$$\theta_{k+1} = \theta' = \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t$$

where γ is always zero.

"Motion Model using SE(2)"

$$X_{k+1} = X_k \exp(\xi_k \Delta t)$$

$$\Sigma_{k+1} = \Sigma_k + Ad_{X_k} Q_k Ad_{X_k}^T,$$

where, $X_k = \begin{bmatrix} R_k & P_k \\ 0 & 1 \end{bmatrix} \in SE(2)$, twist $\xi_k = \begin{bmatrix} \omega_k & v_k \\ 0 & 0 \end{bmatrix} \in se(2)$

$$\omega_k \sim N(0, Q_k)$$

(A) Extended Kalman Filter

Generic model is given by,

→ Prediction : $\bar{\mu}_t = g(u_t, \mu_{t-1})$ {process model}

$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$$

→ Correction : $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$

$$\mu_t = \bar{\mu}_t + K_t . v$$

where $v = z_t - h(\bar{\mu}_t)$ measurement model

↳ innovation

$$\Sigma_t = (I - K_t H_t)\bar{\Sigma}_t$$

From the velocity motion model;

$$\underbrace{\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix}}_{x_t} = \underbrace{\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} -\frac{\hat{v}}{\hat{\omega}} \sin\theta + \frac{\hat{v}}{\hat{\omega}} \sin(\theta + \hat{\omega}\Delta t) \\ \frac{\hat{v}}{\hat{\omega}} \cos\theta - \frac{\hat{v}}{\hat{\omega}} \cos(\theta + \hat{\omega}\Delta t) \\ \hat{\omega}\Delta t \end{bmatrix}}_{g(u_t, x_{t-1})} + \mathcal{N}(0, R_t)$$

This represents the random noise component.

Similarly,

$$\begin{bmatrix} \hat{v} \\ \hat{\omega} \end{bmatrix} = \begin{bmatrix} v \\ \omega \end{bmatrix} + \mathcal{N}(0, M_t)$$

represents the random noise component.

where, $M_t = \begin{bmatrix} \alpha_1 v_t^2 + \alpha_2 \omega_t^2 & 0 \\ 0 & \alpha_3 v_t^2 + \alpha_4 \omega_t^2 \end{bmatrix}$

By Taylor expansion,

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t (x_{t-1} - \mu_{t-1})$$

Jacobian.

where $G_t = \dfrac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} = \begin{bmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{bmatrix}$

$$\Rightarrow G_t = \begin{bmatrix} 1 & 0 & \frac{v_t}{\omega_t}\left(-\cos\mu_{t-1,\theta} + \cos(\mu_{t-1,\theta} + \omega_t \Delta t)\right) \\ 0 & 1 & \frac{v_t}{\omega_t}\left(-\sin\mu_{t-1,\theta} + \sin(\mu_{t-1,\theta} + \omega_t \Delta t)\right) \\ 0 & 0 & 1 \end{bmatrix}$$

Covariance matrix M_t is defined in control space, but the motion model requires it to be mapped to state space. Therefore it is linearised by V_t, defined as,

$$V_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} = \begin{bmatrix} \frac{-\sin\theta + \sin(\theta + \omega_t \Delta t)}{\omega_t} & \frac{V_t}{\omega_t}\left[\frac{(\sin\theta - \sin(\theta + \omega \Delta t))}{\omega_t} + \cos(\theta + \omega_t \Delta t)\Delta t\right] \\ \frac{\cos\theta - \cos(\theta + \omega_t \Delta t)}{\omega_t} & \frac{V_t}{\omega_t}\left[\frac{-\cos\theta + \cos(\theta + \omega \Delta t)}{\omega_t} + \sin(\theta + \omega_t \Delta t)\Delta t\right] \\ 0 & \Delta t \end{bmatrix}$$

The multiplication $V_t M_t V_t^T$ transforms M_t to the state space.

Using the above derived variables, prediction step is formulated as -

$$\bar{\mu}_t = \mu_{t-1} + \begin{bmatrix} \frac{v_t}{\omega_t}\left(-\sin\theta + \sin(\theta + \omega \Delta t)\right) \\ \frac{v}{\omega_t}\left(\cos\theta - \cos(\theta + \omega \Delta t)\right) \\ \omega \Delta t \end{bmatrix}$$

$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + V_t M_t V_t^T$$

Measurement model $h(x_t, j, m)$ is given as

$$\hat{z}_t^i = \begin{bmatrix} r_t^i \\ \phi_t^i \end{bmatrix} = \begin{bmatrix} \sqrt{q} \\ \operatorname{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \end{bmatrix}$$

where $q = (m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2$

Using Taylor expansion,

$$h(x_t, j, m) \approx h(\bar{\mu}_t, j, m) + H_t^i (x_t - \mu_t)$$

The jacobian H_t^i is given as, $H_t^i = \frac{\partial h(\bar{\mu}_t, j, m)}{\partial x_t}$

$$\Rightarrow H_t^i = \begin{bmatrix} -\frac{m_{j,x} - \bar{\mu}_{t,x}}{\sqrt{q}} & -\frac{m_{j,y} - \bar{\mu}_{t,y}}{\sqrt{q}} & 0 \\ \frac{m_{j,y} - \bar{\mu}_{t,y}}{q} & -\frac{m_{j,x} - \bar{\mu}_{t,x}}{q} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

And the measurement covariance Q_t is given as,

$$Q_t = \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_\phi^2 & 0 \\ 0 & 0 & \sigma_s^2 \end{bmatrix}$$

Hence, the correction steps can be written as,

$$\left.\begin{aligned} S_t^i &= H_t^i \bar{\Sigma}_t (H_t^i)^T + Q_t \\ K_t^i &= \bar{\Sigma}_t (H_t^i)^T (S_t^i)^{-1} \\ \bar{\mu}_t &= \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^i) \\ \bar{\Sigma}_t &= (I - K_t^i H_t^i) \bar{\Sigma}_t \end{aligned}\right\} \text{ iterated over all observed features } z_t^i = \begin{bmatrix} r_t^i \\ \phi_t^i \\ s_t^i \end{bmatrix}$$

Then at the end of iteration

$$\mu_t = \bar{\mu}_t \text{ and } \Sigma_t = \bar{\Sigma}_t \quad \boxed{I}$$

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(B) Unscented Kalman Filter

In UKF we consider an augmented state, given by sum of state, control and measurement dimensions. $(3+2+2)$

$$\therefore \quad \chi^{a}_{t-1} = \begin{bmatrix} \chi^{x\,T}_{t-1} \\ \chi^{u\,T}_{t} \\ \chi^{z\,T}_{t} \end{bmatrix}$$

Hence, $\mu^{a}_{t-1} = \left(\mu^{T}_{t-1} \;\; (0\,0)^{T} \;\; (0\,0)^{T}\right)^{T}$

$$\Sigma^{a}_{t-1} = \begin{bmatrix} \Sigma_{t-1} & 0 & 0 \\ 0 & M_t & 0 \\ 0 & 0 & Q_t \end{bmatrix}$$

where M_t and Q_t are same as defined previously for EKF.

Next step is to generate sigma points and weights,

$\chi^{[0]} = \mu$, $\quad x^{i} = \mu + l_i'$ for $i = 1, \ldots n$

$x^{i} = \mu - l_{i-n}$ for $i = n+1, \ldots 2n$

l_i' is the i-th column of L'

where $L' = \sqrt{(n+K)}\, L$ and

$\Sigma = LL^{T}$ by Cholesky decomp.

$$\omega^{[0]} = \frac{K}{n+K}$$

$$\omega^{[i]} = \frac{1}{2(n+K)} \qquad i = 1, \ldots, 2n$$

Then the sigma points are passed through the motion model.

$$\bar{\mathcal{X}}_t^x = g(\underbrace{u_t + \mathcal{X}_t^u}_{\hat{u}_t}, \underbrace{\mathcal{X}_{t-1}^x}_{\hat{\mu}_t})$$

Predicted mean and covariances:

$$\bar{\mu}_t = \sum_{i=0}^{2L} w_i \bar{\mathcal{X}}_{i,t}^x \quad , \quad \bar{\Sigma}_t = \sum_{i=0}^{2L} w_i (\bar{\mathcal{X}}_{i,t}^x - \bar{\mu}_t)(\bar{\mathcal{X}}_{i,t}^x - \bar{\mu}_t)^T$$

Predicted measurements,

$$\bar{Z}_t = h(\bar{\mathcal{X}}_t^x) + \mathcal{X}_t^z$$

$$\hat{z}_t = \sum_{i=0}^{2L} w_i \bar{Z}_{i,t}$$

$$S_t = \sum_{i=0}^{2L} w_i (\bar{Z}_{i,t} - \hat{z}_t)(\bar{Z}_{i,t} - \hat{z}_t)^T$$

$$\Sigma_t^{x,z} = \sum_{i=0}^{2L} w_i (\bar{\mathcal{X}}_{i,t}^x - \bar{\mu}_t)(\bar{Z}_{i,t} - \hat{z}_t)^T$$

Correction $\longrightarrow$ $K_t = \Sigma_t^{x,z} S_t^{-1}$

$$\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$$

$$\Sigma_t = \bar{\Sigma}_t - K_t S_t K_t^T \quad \square$$

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C. Particle Filter (PF)

We start of by drawing samples from the posterior

$x_t := x_t^{[1]}, x_t^{[2]}, \ldots, x_t^{[m]}$; where $x_t^{[m]}$ is a particle at time

$1 \leq m < M$

where 'M' is the no. of particles in a set.

Particle filter varies from EKF and UKF specially in the determination of weights, where the weights are calculated as the pdf of a multivariate normal distribution with measurement noise covariance Q_t. The above weights are normalized as well.

The weight probabilities are calculated as follows,

$$w_t^{[i]} = w_{t-1}^{[i]} \cdot p(z_t^{[i]} | x_t^{[i]}) \; ; \; w_t^{[i]} = \frac{w_t^{[i]}}{\sum_{i=1}^{M} w_t^{[i]}}$$

Resampling → Next part is to draw particles with high weights. A low-covariance sampling algorithm is used for resampling.

n- particles are drawn based on probability propotional to the normalised weights. Hence, particles with high normalised weights are drawn frequently and thus only these high weights are considered.

[algorithm] →

draw a random no. r between 0 to $1/m$,

Count = 1,

for $j = 1:m$

$u = r + (i-1)\,1/m$

while $u > w^{Count}$: Count = Count + 1

$x_t^{[j]} = x_t^{[i]}$, $w_t^{[j]} = 1/m$

Correction & Updation

Based on the new particles, Corrected mean

$$\mu_{t,x,y} = \frac{\sum_{i=1}^{N} \bar{x}_t^{[i]}}{N},$$

we get

$$\mu_{t,\theta} = \frac{\sum_{i=1}^{N} \sin x_{t,\theta}^{[i]}}{\sum_{i=1}^{N} \cos x_{t,\theta}^{[i]}}$$

$$\mu_{t,\theta} = \operatorname{atan2}\left(\sum_{i=1}^{N} \sin x_{t,\theta}^{[i]}, \sum_{i=1}^{N} \cos x_{t,\theta}^{[i]}\right)$$

$$\Rightarrow \mu_t = [\mu_{t,x,y} \;\; \mu_{t,\theta}]^T$$

$$\Rightarrow \Sigma_t = \sum_{i=1}^{M} (x_t^{[i]} - \mu_t)(x_t^{[i]} - \mu_t)^T / M \quad \square$$

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(D) Right invariant EKF (RI-EKF)

μ is in SE(2) model and can be written as,

$$\mu = \begin{bmatrix} \cos(\mu_\theta) & -\sin(\mu_\theta) & \mu_x \\ \sin(\mu_\theta) & \cos(\mu_\theta) & \mu_y \\ 0 & 0 & 1 \end{bmatrix}$$

Prediction

→ Generating the twist matrix

$$\mu_c = [x \ y \ \theta]^T , \quad \bar{\mu}_c = g(\mu_c, u)$$

↳ predicted mean is computed using the non-linear action model.

$$\xi^\wedge = \log(\mu \bar{\mu}^{-1}) .$$

Propagation

$\xi_{noise} = \text{chol}(Q).\text{randn}(3,1)$; where Q is the control input cov.

$\xi^\wedge = \xi^\wedge + \xi^\wedge_{noise}$, where $^\wedge$ is the wedge operator

i.e. if $\xi_k = \begin{bmatrix} \omega_k \\ v_k \end{bmatrix}$, $\xi^\wedge_k = \begin{bmatrix} \hat{\omega}_k & v_k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 1 \end{bmatrix}$

where ω is 1×1

v is 2×1

$$\bar{\mu} = \mu \exp(\xi^\wedge)$$

$\bar{\Sigma} = \Sigma + Ad_\mu Q Ad_\mu^T$, where Ad is the adjoint in SE(2)

$$Ad_{\mu} = \begin{bmatrix} \cos(\mu_\theta) & -\sin(\mu_\theta) & \mu_x \\ \sin(\mu_\theta) & \cos(\mu_\theta) & \mu_y \\ 0 & 0 & 1 \end{bmatrix}$$

Correction

$$H = \begin{bmatrix} -1 & 0 & L_{y_1} \\ 0 & -1 & -L_{x_1} \\ -1 & 0 & L_{y_2} \\ 0 & -1 & -L_{x_2} \end{bmatrix} \quad b_1 = \begin{bmatrix} L_{x_1} \\ L_{y_1} \\ 1 \end{bmatrix} \quad b_2 = \begin{bmatrix} L_{x_2} \\ L_{y_2} \\ 1 \end{bmatrix}$$

incorporating sensor noise.

$$S_{noise} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}$$ where N is top 2×2 matrix of $\bar{\mu} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \bar{\mu}^T$

$$S = H \bar{\Sigma} H^T + S_{noise}$$

$$L = \bar{\Sigma} H^T S^{-1}$$

innovation $\rightarrow$ $\nu_c = \begin{bmatrix} \nu_{1c} \\ \nu_{2c} \end{bmatrix}$ where ν_{ic} is top 2×1 vector of $\bar{\mu} Y_i - b_i$

$$\nu = (L \nu_c)^\wedge$$

Update $\rightarrow$ $\mu = \exp(\nu) \bar{\mu}$

$$\Sigma = (I - LH) \bar{\Sigma} (I - LH)^T + L S_{noise} L^T$$

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