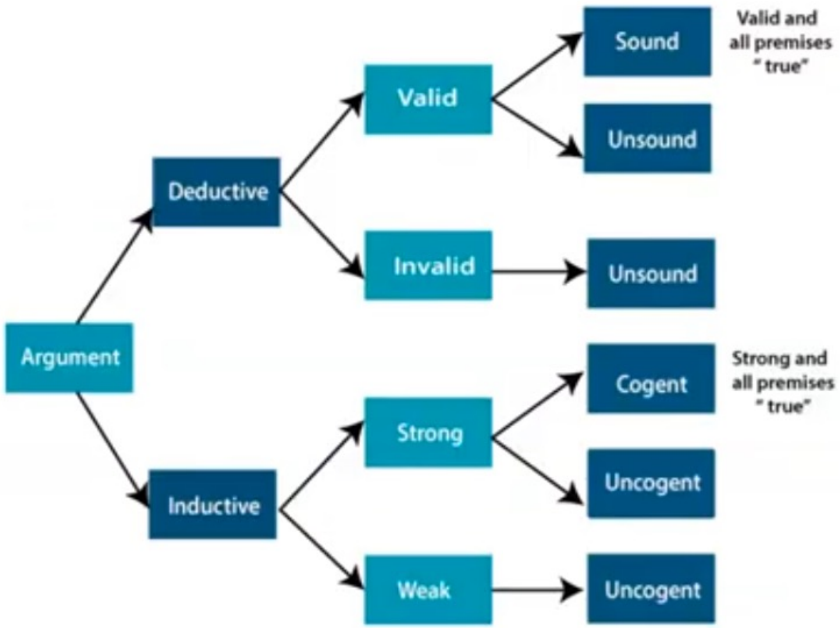
Lecture - 4

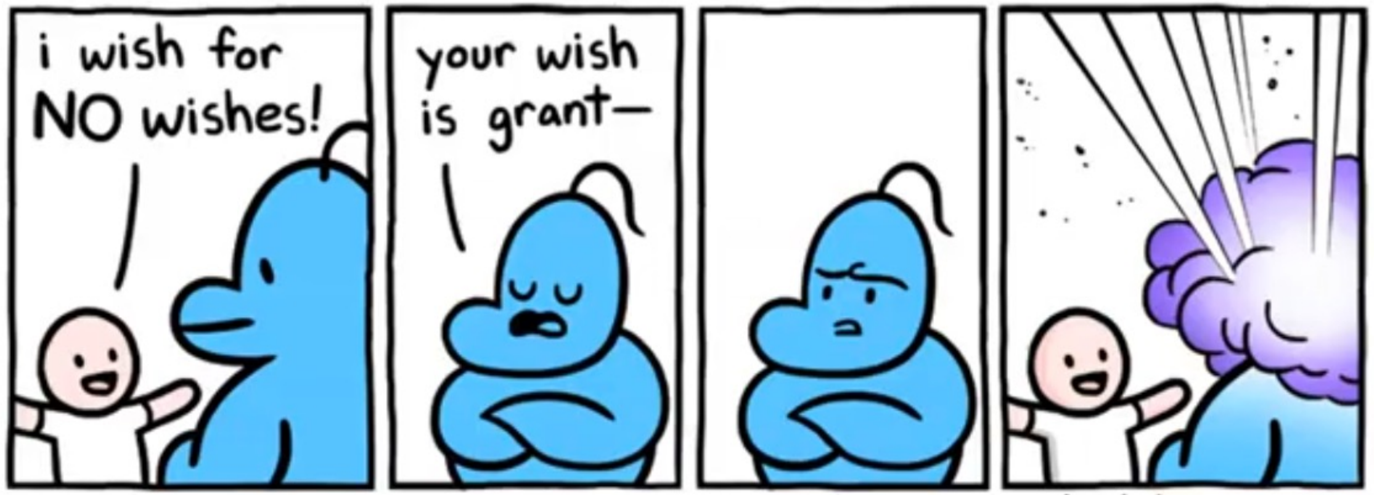
Propositional Logic: Representation

Previously

- AI Concepts:
  - Knowledge Representation
    - Formal representation vs natural language
    - Syntax and mapping with semantics
    - What makes a knowledge representation effective?
  - Basics of Logic
    - Arguments
      - Valid
      - Sound
    - Syllogistic Logic
      - Syllogism
      - Star Test of Validity
  - Overview of Reasoning
    - Deductive, Inductive, Abductive



Lecture Outline



- What is Propositional Logic?
- Propositional Language
- Propositional Truth
- Equivalence
- Logical Calculation

Why Propositional Logic?

- These basics provides the skills needed to more easily working with, manipulate, and reason with more complex knowledge and tasks

- The '*calculus*' skills needed for reasoning

Syllogistic Logic Revisited

- Syllogistic Logic – system of reasoning used by the ancient Greeks

- Examination of the fundamental principles of logical transference: E.g. If A=B and B=C then A=C

- Propositional Logic builds upon Syllogistic Logic

What is Propositional Logic?

Proposition

- A declarative statement either true or false, but not both

- Aka: Formula

- *Atomic proposition* (i.e. 'facts' where truth or falsity doesn't depend on any other propositions)

- Examples:
  - Propositions

• $1 + 1 = 2$	True
• $1 + 1 > 3$	False
• Alison likes patient care	True
• Ryan has an M.D.	False

  - Not Propositions

• $1 + 1 > x$	Unknown
• "What a great book!"	Unknown
• "Is your car red?"	Unknown

What is Propositional Logic?

- A formal modern logic
- Aka: Boolean Logic, Symbolic Logic, Propositional Calculus, Statement Logic, Zeroth-Order Logic
- Relies on the principles established by syllogistic logic
- Concerned with truth and falsehood
  - Of how truth values extend through a series of propositions

TRUE FALSE

  - "Ryan is the course director" AND "Ryan is a magician" → FALSE

- Builds up more complex statements (i.e. sentences) by combining propositions with logical connectives
  - This is the syntax of propositional logic

Propositional Language

Propositional Language: Variables

- To formalize atomic propositions we use simple (i.e. primitive) symbols
  - *Aka: propositional constants*

  - These symbols inherit the same truth values as the statements
  - F as a symbol is a proposition that is always False
  - T as a symbol is a proposition that is always True

- A propositional signature is a set of propositional constants

		Symbols	
"1 + 1 = 2"	=	'P'	True
"1 + 1 > 3"	=	'Q'	False
"Alison likes patient care"	=	'R'	True
"Ryan has an M.D."	=	'S'	False

Propositional Language: Connectives

- Aka: Logical Operators

- The Connectives:
  - Conjunction ($\wedge$, & , •): **AND**
  - Disjunction ($\vee$, | , +): **OR**
  - Negation ($\neg$, ~): **NOT**
  - Implication ($\rightarrow$, $\Rightarrow$, $\supset$): **IF….THEN**
  - Biconditional ($\equiv$, $\Leftrightarrow$): **IF AND ONLY IF**

Propositional Language: Connectives

- Aka: Logical Operators

- Propositional sentence: A statement obtained by using connectives to combine:
  - Members of the propositional signature (i.e. a propositional constant)

$P \wedge Q$

- The Connectives:
  - Conjunction ($\wedge$, & , •): **AND**
  - Disjunction ($\vee$, | , +): **OR**
  - Negation ($\neg$, ~): **NOT**
  - Implication ($\rightarrow$, $\Rightarrow$, $\supset$): **IF....THEN**
  - Biconditional ($\equiv$, $\Leftrightarrow$): **IF AND ONLY IF**

Propositional Language: Connectives

- Aka: Logical Operators

- Propositional sentence: A statement obtained by using connectives to combine:
  - Members of the propositional signature (i.e. a propositional constant)
  - A compound expression formed from members of the propositional signature

$\boxed{P \land Q}$

$\boxed{(S \lor P) \land Q}$

- The Connectives:
  - Conjunction ($\land$, & , •): **AND**
  - Disjunction ($\lor$, | , +): **OR**
  - Negation ($\neg$, ~): **NOT**
  - Implication ($\rightarrow$, $\Rightarrow$, $\supset$): **IF….THEN**
  - Biconditional ($\equiv$, $\Leftrightarrow$): **IF AND ONLY IF**

Truth Tables

- Describes the behavior of a proposition under all possible interpretations of the atomic propositions included

- Table length:
  - Given *n* different atomic propositions in some proposition:
    - 2^n different lines in the truth table for that formula
    - Because each can take one 1 of 2 values— true or false

p	q	~p	~p ∨ q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Truth Tables

- Describes the behavior of a proposition under all possible interpretations of the atomic propositions included

- Table length:
  - Given *n* different atomic propositions in some proposition:
    - 2^n different lines in the truth table for that formula
    - Because each can take one 1 of 2 values— true or false

- Atomic propositions (for following examples):
  - "Finn likes cakes" = 'p'
  - "Finn eats cakes" = 'q'

Atomic Prop.		Propositions	
p	q	~p	~p ∨ q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

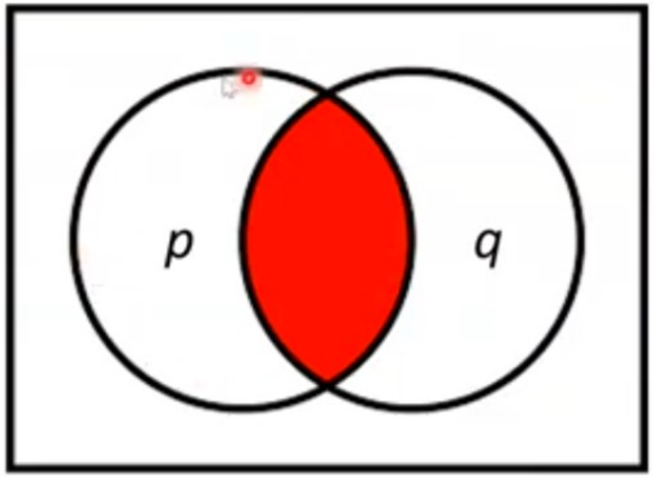
Connectives: Conjunction

- **AND** (∧ , & , •):
  - True only when ‘both’ are true

Truth Table

p	q	$p \land q$
F	F	F
F	T	F
T	F	F
T	T	T

= “Finn likes cakes and Finn eats cakes”



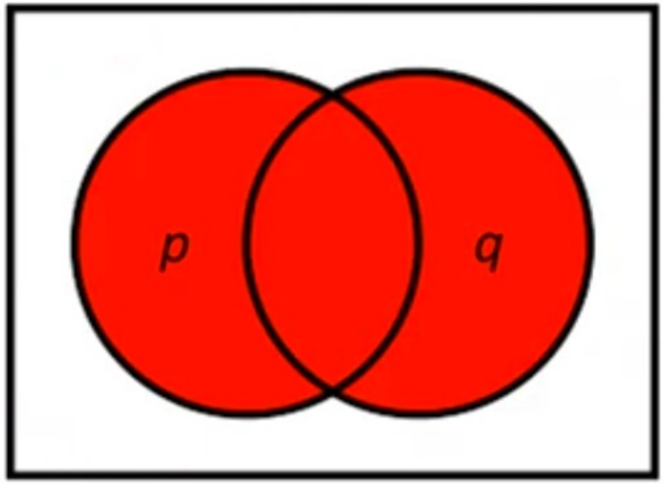
Connectives: Disjunction

- **OR** ($\vee$, | , +):
  - False only when 'both' are false

Truth Table

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

= "Finn likes cakes or Finn eats cakes"



Connectives: Negation

- **NOT** (¬ , ~):
  - Only takes one argument
  - Inversion of truth

Truth Table

p	$\neg p$
F	T
T	F

= "Finn doesn't likes cakes"

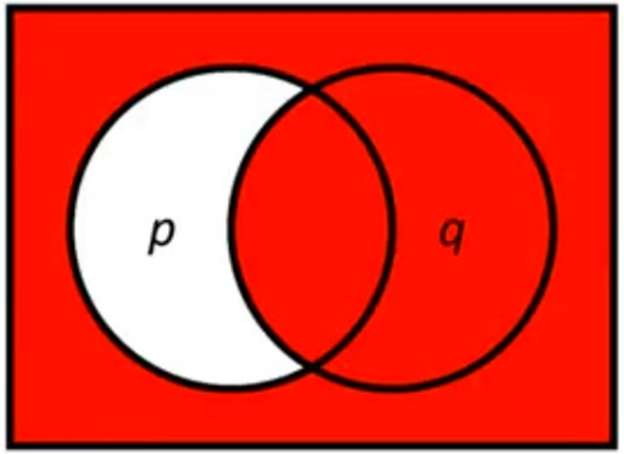
Connectives: Implication

- **IF….THEN** ($\rightarrow$, $\Rightarrow$, $\supset$):
  - True whenever *p* is False or *q* is True
  - Aka. *p* implies *q*

Truth Table

p	q	$p \Rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

= "If Finn likes cakes then Finn eats cakes"



Sufficient and Necessary Conditions

- With respect to implications…

When $p \rightarrow q$, p is called a *sufficient* condition for q , q is a *necessary* condition for p.

- Being Japanese is a sufficient condition for being Asian.

  ≡ if someone is Japanese then s/he will be an Asian

- Being Asian is a necessary condition for being Japanese.

  ≡ 'if someone is not Asian, he can not be Japanese'

Converse, Contrapositive, and Inverse

- From the implication $p \rightarrow q$ we can form new conditional statements
  - Converse = $q \rightarrow p$
  - Contrapositive = $\neg p \rightarrow \neg q$
  - Inverse = $\neg q \rightarrow \neg p$

- Implication: "If it is snowing, then I will not go to school."

  - Converse: "If I do not go to school, then it is snowing."
  - Contrapositive: "If it is not snowing, then I will go to school."
  - Inverse: "If I go to school, then it is not snowing."

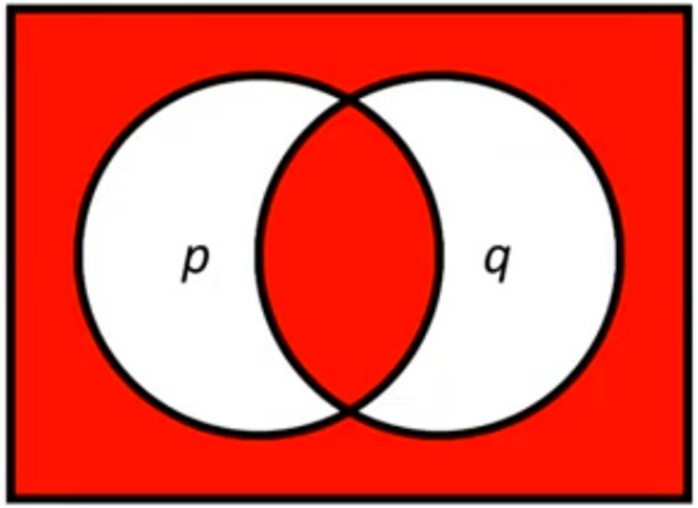
Connectives: Biconditional

- **IF AND ONLY IF** (≡ , ⇔, iff):
  - True if p and q are both either True or False

Truth Table

p	q	$p \Leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

= "If Finn likes cakes then Finn eats cakes, and vice versa"



Propositional Language: Grammar

- Abstract propositions:
  - More general than atomic propositions
  - Can be any syntactically correct proposition
    - i.e. well-formed formula (**wwf**)
  - Can nest complex propositions as deeply as needed
  - Parentheses ( ) are used for grouping

Examples:

a

b

$(\neg a)$

$((\neg a) \wedge b)$

$(((\neg a) \wedge b) \vee b)$

Non-examples:

$a \wedge$

$\Rightarrow\Rightarrow a$

$a \neg b$

Propositional Language: Grammar

- Abstract propositions:
  - More general than atomic propositions
  - Can be any syntactically correct proposition
    - i.e. well-formed formula (**wwf**)
  - Can nest complex propositions as deeply as needed
  - Parentheses ( ) are used for grouping
  - Omit as many parentheses as possible, but without causing ambiguity
  - Negation has highest priority

Examples:

a

b

$(\neg a)$

$((\neg a) \wedge b)$

$(((\neg a) \wedge b) \vee b)$

Non-examples:

$a \wedge$

$\Rightarrow \Rightarrow a$

$a \neg b$

Examples:

$$((\neg a) \vee (\neg b)) \overset{1}{\rightsquigarrow} (\neg a) \vee (\neg b) \overset{2}{\rightsquigarrow} \neg a \vee \neg b$$

$$((\neg a) \wedge b) \overset{1}{\rightsquigarrow} (\neg a) \wedge b \overset{2}{\rightsquigarrow} \neg a \wedge b$$

$$(\neg (a \wedge b)) \overset{1}{\rightsquigarrow} \neg (a \wedge b) \overset{?}{\rightsquigarrow} \cancel{\neg a \wedge b} \quad \text{NO!}$$

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$

then parentheses must be used.

Propositional Truth

Computing Truth Tables

- For each (non-atomic) proposition calculate the truth table
- Use respective rules for connectives
- Order of truth table determined by ‘precedence’

$$(p \vee q) \rightarrow \neg r$$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Interpreting Truth Tables

- Every row is a possible interpretation (i.e. model):
  - Interpretations are 'possible worlds'
  - Find the possible truth values of the constituent propositions gives us meaning of the sentence

- Propositional Interpretation:
  - Truth mapping propositional constants

  - *p* = "It is snowing"
  - *q* = "There is a hurricane"
  - *r* = "There is school today"

p	q	r	$p \vee q \rightarrow \neg r$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

Interpreting Truth Tables

- Every row is a possible interpretation (i.e. model):
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  - Find the possible truth values of the constituent propositions gives us meaning of the sentence

- Propositional Interpretation:
  - Truth mapping propositional constants

  - *p* = "It is snowing"
  - *q* = "There is a hurricane"
  - *r* = "There is school today"

- Sentential Interpretation:
  - Truth mapping propositional sentences

p	q	r	$p \vee q \rightarrow \neg r$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

It is not snowing but there is a hurricane therefore there isn't school today

Interpretation Function

- Tree representation of a single row in a truth table

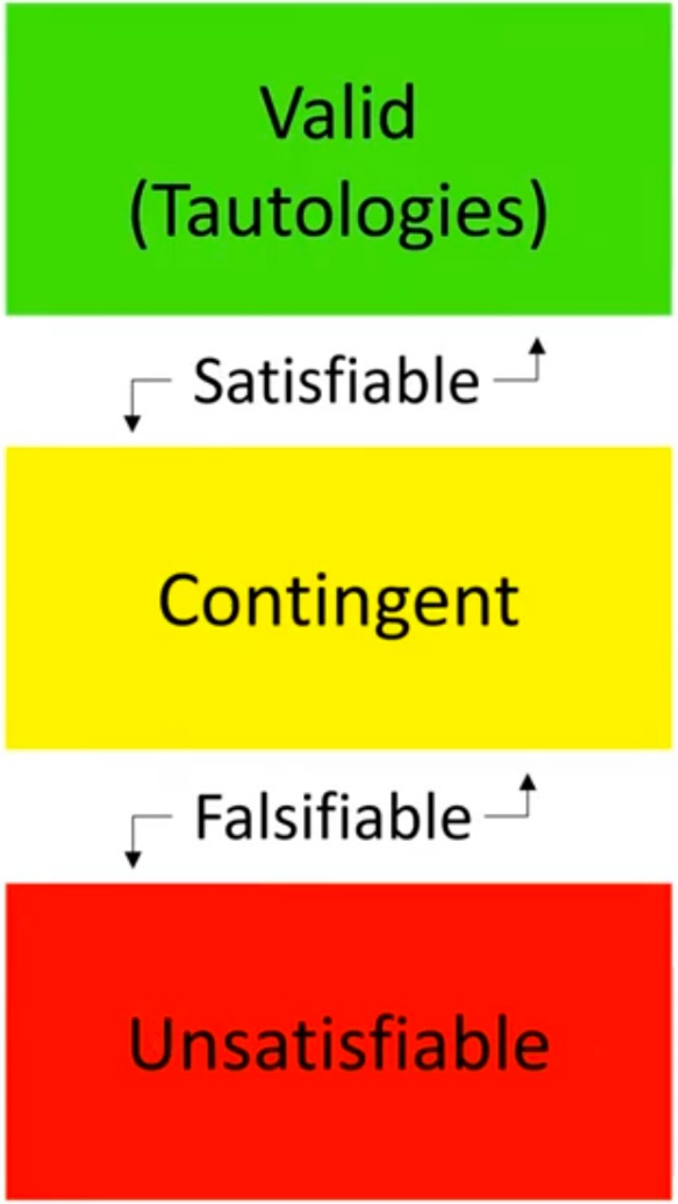
Formula: $f = (\neg A \wedge B) \leftrightarrow C$

Model: $w = \{A : 1, B : 1, C : 0\}$ → Interpretation

Interpretation:

- $\mathcal{I}((\neg A \wedge B) \leftrightarrow C, w) = 1$
  - $\mathcal{I}(\neg A \wedge B, w) = 0$
    - $\mathcal{I}(\neg A, w) = 0$
      - $\mathcal{I}(A, w) = 1$
    - $\mathcal{I}(B, w) = 1$
  - $\mathcal{I}(C, w) = 0$

Properties of Abstract Propositions



- Every interpretation is satisfied (i.e. True)
- Some interpretations satisfy it but others don't
- No interpretations are satisfied

Tautology

"Heads I win, Tails you lose"

- An abstract proposition that is always true

- Example:
  - This course is easy `or' this course is not easy

    $p \vee (\neg p) \equiv T$

- Truth table column is always True
  - (0 = False, 1 = True)

a	b	$b \Rightarrow a$	$a \Rightarrow (b \Rightarrow a)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

Contradiction

- An abstract proposition that is always false

- Example:
  - This course is easy `and' this course is not easy
  - $p \wedge (\neg p) \equiv F$

- Truth table column is always False
  - (0 = False, 1 = True)

a	b	$a \Rightarrow b$	$\neg b$	$a \wedge \neg b$	$(a \Rightarrow b) \wedge (a \wedge \neg b)$
0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	1	0
1	1	1	0	0	0

"What? All I said was I'm proud of my humility."

Contingency

- An abstract proposition that is not a tautology nor a contradiction

Examples:

1. a
2. $a \Rightarrow \neg a$
3. $a \wedge b$
4. $a \vee b$
5. $\neg a \Rightarrow (b \wedge c)$

Equivalence

Equivalent Propositions

- Abstract propositions with identical columns in a truth table are said to be equivalent
  - i.e. the matching truth values in every model

- All tautologies and contradictions are equivalent

a	b	$a \Rightarrow b$	$\neg(a \Rightarrow b)$	$\neg a$	$\neg a \vee b$	$\neg(\neg a \vee b)$
0	0	1	0	1	1	0
0	1	1	0	1	1	0
1	0	0	1	0	0	1
1	1	1	0	0	1	0

Equivalent Propositions: Example

- “Alice is not married but Bob is not single” $(\neg h \wedge \neg b)$
- “Bob is not single and Alice is not married” $(\neg b \wedge \neg h)$
- “Neither Bob is single nor Alice is married” $(\neg(b \vee h))$

Equivalent Propositions: Example

- "Alice is not married but Bob is not single" ($\neg h \land \neg b$)
- "Bob is not single and Alice is not married" ($\neg b \land \neg h$)
- "Neither Bob is single nor Alice is married" ($\neg(b \lor h)$)

- These three statements are equivalent

$$\neg h \land \neg b \equiv \neg b \land \neg h \equiv \neg(b \lor h)$$

b h	$\neg b$ $\neg h$ $b \lor h$	$(\neg h \land \neg b)$	$(\neg b \land \neg h)$	$\neg(b \lor h)$
T T	F F T	F	F	F
T F	F T T	F	F	F
F T	T F T	F	F	F
F F	T T F	T	T	T

Extended Notation

If P is equivalent to Q, then we write $P \stackrel{val}{==} Q$.

Note: $\stackrel{val}{==}$ is not part of the vocabulary of the language of abstract propositions; it is a *meta-symbol*.

So:

$$\underbrace{\underbrace{a \Rightarrow b}_{\text{abstr. prop.}} \quad \stackrel{val}{==} \quad \underbrace{\neg a \vee b}_{\text{abstr. prop.}}}_{\text{meta-formula}}$$

Commutativity and Associativity

- Standard equivalences – i.e. transformation rules

Commutativity:

$$P \wedge Q \stackrel{val}{==} Q \wedge P$$

$$P \vee Q \stackrel{val}{==} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{==} Q \Leftrightarrow P$$

NB: $P \Rightarrow Q \stackrel{val}{\neq} Q \Rightarrow P$

P Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0 1	1	0

Commutativity and Associativity

- Standard equivalences – i.e. transformation rules

Commutativity:

$$P \wedge Q \stackrel{val}{=\!=} Q \wedge P$$
$$P \vee Q \stackrel{val}{=\!=} Q \vee P$$
$$P \Leftrightarrow Q \stackrel{val}{=\!=} Q \Leftrightarrow P$$

Associativity:

$$(P \wedge Q) \wedge R \stackrel{val}{=\!=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \stackrel{val}{=\!=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=\!=} P \Leftrightarrow (Q \Leftrightarrow R)$$

NB: $P \Rightarrow Q \stackrel{val}{\neq} Q \Rightarrow P$

NB: $P \Rightarrow (Q \Rightarrow R) \stackrel{val}{\neq} (P \Rightarrow Q) \Rightarrow R$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	1	1	0

P	Q	R	$P \Rightarrow (Q \Rightarrow R)$	$(P \Rightarrow Q) \Rightarrow R$
0	1	0	1	0

Commutativity and Associativity

- Standard equivalences – i.e. transformation rules

Commutativity:

$$P \wedge Q \stackrel{val}{=\!=} Q \wedge P$$
$$P \vee Q \stackrel{val}{=\!=} Q \vee P$$
$$P \Leftrightarrow Q \stackrel{val}{=\!=} Q \Leftrightarrow P$$

Associativity:

$$(P \wedge Q) \wedge R \stackrel{val}{=\!=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \stackrel{val}{=\!=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=\!=} P \Leftrightarrow (Q \Leftrightarrow R)$$

NB: $P \Rightarrow Q \stackrel{val}{\neq} Q \Rightarrow P$

P Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0 1	1	0

NB: $P \Rightarrow (Q \Rightarrow R) \stackrel{val}{\neq} (P \Rightarrow Q) \Rightarrow R$

P Q R	$P \Rightarrow (Q \Rightarrow R)$	$(P \Rightarrow Q) \Rightarrow R$
0 1 0	1	0

NB: In view of Associativity, we shall write $P \wedge Q \wedge R$ to denote both $(P \wedge Q) \wedge R$ and $P \wedge (Q \wedge R)$.

Idempotence and Double Negation

- Standard equivalences – i.e. transformation rules

Idempotence:

$$P \wedge P \stackrel{val}{==} P$$

$$P \vee P \stackrel{val}{==} P$$

NB: $P \Rightarrow P \stackrel{val}{\neq} P$

$P \Leftrightarrow P \stackrel{val}{\neq} P$ (It turns out == to True)

Idempotence and Double Negation

- Standard equivalences – i.e. transformation rules

Idempotence:

$$P \wedge P \stackrel{val}{==} P$$

$$P \vee P \stackrel{val}{==} P$$

NB: $P \Rightarrow P \stackrel{val}{\neq} P$

$P \Leftrightarrow P \stackrel{val}{\neq} P$ (It turns out == to True)

Double Negation:

$$\neg\neg P \stackrel{val}{==} P$$

'It is **not** that I **don't** like spinach'

(NB: in propositional logic the intended nuance cannot be captured.)

Adsorption

- Standard equivalences – i.e. transformation rules
- Simplification/Reduction

$$p \vee (p \wedge q) \equiv p$$
$$p \wedge (p \vee q) \equiv p$$

True and False (T & P)

- Standard equivalences – i.e. transformation rules
- Simplifications of abstract propositions to truth constants

Inversion:

$$\neg\texttt{True} \stackrel{val}{==} \texttt{False}$$

$$\neg\texttt{False} \stackrel{val}{==} \texttt{True}$$

True and False (T & P)

- Standard equivalences – i.e. transformation rules
- Simplifications of abstract propositions to truth constants

Inversion:

$$\neg\mathtt{True} \stackrel{val}{=\!=} \mathtt{False}$$

$$\neg\mathtt{False} \stackrel{val}{=\!=} \mathtt{True}$$

Contradiction:

$$P \wedge \neg P \stackrel{val}{=\!=} \mathtt{False}$$

True and False (T & P)

- Standard equivalences – i.e. transformation rules
- Simplifications of abstract propositions to truth constants

Inversion:

$$\neg\texttt{True} \stackrel{val}{=\!=} \texttt{False}$$

$$\neg\texttt{False} \stackrel{val}{=\!=} \texttt{True}$$

Contradiction:

$$P \wedge \neg P \stackrel{val}{=\!=} \texttt{False}$$

Excluded Middle:

$$P \vee \neg P \stackrel{val}{=\!=} \texttt{True}$$

True and False (T & P)

- Standard equivalences – i.e. transformation rules
- Simplifications of abstract propositions to truth constants

Inversion:

$$\neg\texttt{True} \stackrel{val}{==} \texttt{False}$$

$$\neg\texttt{False} \stackrel{val}{==} \texttt{True}$$

Negation:

$$\neg P \stackrel{val}{==} P \Rightarrow \texttt{False}$$

Contradiction:

$$P \wedge \neg P \stackrel{val}{==} \texttt{False}$$

Excluded Middle:

$$P \vee \neg P \stackrel{val}{==} \texttt{True}$$

True and False (T & P)

- Standard equivalences – i.e. transformation rules
- Simplifications of abstract propositions to truth constants

Inversion:

$$\neg\texttt{True} \stackrel{val}{=\!=} \texttt{False}$$

$$\neg\texttt{False} \stackrel{val}{=\!=} \texttt{True}$$

Negation:

$$\neg P \stackrel{val}{=\!=} P \Rightarrow \texttt{False}$$

Contradiction:

$$P \wedge \neg P \stackrel{val}{=\!=} \texttt{False}$$

True/False-elimination:

$$P \wedge \texttt{True} \stackrel{val}{=\!=} P$$

$$P \wedge \texttt{False} \stackrel{val}{=\!=} \texttt{False}$$

$$P \vee \texttt{True} \stackrel{val}{=\!=} \texttt{True}$$

$$P \vee \texttt{False} \stackrel{val}{=\!=} P$$

Excluded Middle:

$$P \vee \neg P \stackrel{val}{=\!=} \texttt{True}$$

Distributivity

Distributivity:

$$P \wedge (Q \vee R) \overset{val}{=\!=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \overset{val}{=\!=} (P \vee Q) \wedge (P \vee R)$$

- Standard equivalences – i.e. transformation rules

- May recognize names and patterns from algebra

$$A \cdot (B + C) = (A \cdot B) + (A \cdot C)$$

$$(A \cdot B) + (A \cdot C) = A \cdot (B + C)$$

De Morgan's Laws

De Morgan:

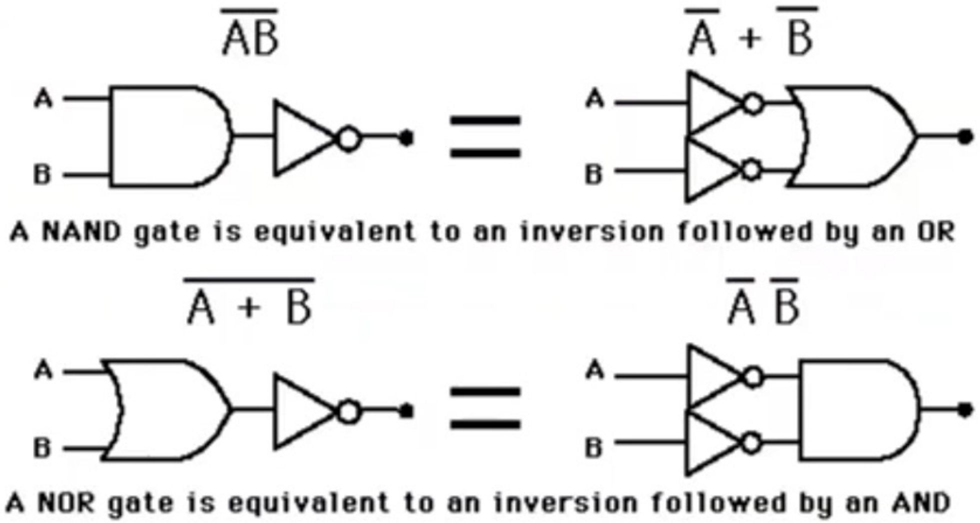
$$\neg(P \wedge Q) \overset{val}{=\!=} \neg P \vee \neg Q$$

$$\neg(P \vee Q) \overset{val}{=\!=} \neg P \wedge \neg Q$$

- Standard equivalences – i.e. transformation rules

p q	¬p ¬q p ∧ q	¬(p ∧ q)	¬p ∨ ¬q
T T	F F T	F	F
T F	F T F	T	T
F T	T F F	T	T
F F	T T F	T	T

Circuit: Logic Gates



Revisiting Implication and Contraposition

- Standard equivalences – i.e. transformation rules

Implication:

$P \Rightarrow Q \overset{val}{=\!=} \neg P \vee Q$

$$
\begin{aligned}
& \neg P \Rightarrow Q \\
\overset{val}{=\!=} \quad & \{ \text{ Implication } \} \\
& \neg\neg P \vee Q \\
\overset{val}{=\!=} \quad & \{ \text{ Double Negation } \} \\
& P \vee Q
\end{aligned}
$$

Calculation
using the equivalence

Revisiting Implication and Contraposition

- Standard equivalences – i.e. transformation rules

Implication:

$$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q$$

$$\begin{array}{ll} & \neg P \Rightarrow Q \\ \stackrel{val}{==} & \{\text{ Implication }\} \\ & \neg\neg P \vee Q \\ \stackrel{val}{==} & \{\text{ Double Negation }\} \\ & P \vee Q \end{array}$$

Calculation
using the equivalence

Contraposition:

$$P \Rightarrow Q \stackrel{val}{==} \neg Q \Rightarrow \neg P$$

NB: $P \Rightarrow Q \stackrel{val}{\neq} \neg P \Rightarrow \neg Q$

(NB: 'if . . . , then . . . ' in natural language often means 'if and only if'.)

Bi-implication and Self-Equivalence

- Standard equivalences – i.e. transformation rules

Bi-implication:

$$P \Leftrightarrow Q \stackrel{val}{=\!=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Bi-implication and Self-Equivalence

- Standard equivalences – i.e. transformation rules

Bi-implication:

$$P \Leftrightarrow Q \stackrel{val}{==} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence:

$$P \Leftrightarrow P \stackrel{val}{==} \texttt{True}$$

Summary: Equivalences for Connectives

Distributivity:

$P \land (Q \lor R) \stackrel{val}{==} (P \land Q) \lor (P \land R),$

$P \lor (Q \land R) \stackrel{val}{==} (P \lor Q) \land (P \lor R)$

Inversion:

$\neg\texttt{True} \stackrel{val}{==} \texttt{False},$

$\neg\texttt{False} \stackrel{val}{==} \texttt{True}$

Contradiction:

$P \land \neg P \stackrel{val}{==} \texttt{False}$

Excluded Middle:

$P \lor \neg P \stackrel{val}{==} \texttt{True}$

Bi-implication:

$P \Leftrightarrow Q \stackrel{val}{==} (P \Rightarrow Q) \land (Q \Rightarrow P)$

Double Negation:

$\neg\neg P \stackrel{val}{==} P$

Negation:

$\neg P \stackrel{val}{==} P \Rightarrow \texttt{False}$

Associativity:

$(P \land Q) \land R \stackrel{val}{==} P \land (Q \land R),$

$(P \lor Q) \lor R \stackrel{val}{==} P \lor (Q \lor R),$

$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{==} P \Leftrightarrow (Q \Leftrightarrow R)$

De Morgan:

$\neg(P \land Q) \stackrel{val}{==} \neg P \lor \neg Q,$

$\neg(P \lor Q) \stackrel{val}{==} \neg P \land \neg Q$

Implication:

$P \Rightarrow Q \stackrel{val}{==} \neg P \lor Q$

Contraposition:

$P \Rightarrow Q \stackrel{val}{==} \neg Q \Rightarrow \neg P$

Self-equivalence:

$P \Leftrightarrow P \stackrel{val}{==} \texttt{True}$

True/False-elimination:

$P \land \texttt{True} \stackrel{val}{==} P,$

$P \land \texttt{False} \stackrel{val}{==} \texttt{False},$

$P \lor \texttt{True} \stackrel{val}{==} \texttt{True},$

$P \lor \texttt{False} \stackrel{val}{==} P$

Commutativity:

$P \land Q \stackrel{val}{==} Q \land P,$

$P \lor Q \stackrel{val}{==} Q \lor P,$

$P \Leftrightarrow Q \stackrel{val}{==} Q \Leftrightarrow P$

Idempotence:

$P \land P \stackrel{val}{==} P,$

$P \lor P \stackrel{val}{==} P$

Absorption Laws

$p \lor (p \land q) \equiv p$

$p \land (p \lor q) \equiv p$

Logical Calculation

Calculation – 'Logic Calculus'

Recall the following *calculation*:

$$
\begin{aligned}
& \neg P \Rightarrow Q \\
\stackrel{val}{=\!=} \quad & \{ \text{ Implication } \} \\
& \neg\neg P \vee Q \\
\stackrel{val}{=\!=} \quad & \{ \text{ Double Negation } \} \\
& P \vee Q
\end{aligned}
$$

Can we conclude

$$\neg P \Rightarrow Q \stackrel{val}{=\!=} P \vee Q \ ?$$

YES!

Other Basic Equivalences

1. (Reflexivity:) $P \stackrel{val}{=\!=} P$
2. (Symmetry:) If $P \stackrel{val}{=\!=} Q$, then $Q \stackrel{val}{=\!=} P$
3. (Transitivity:) If $P \stackrel{val}{=\!=} Q$ and $Q \stackrel{val}{=\!=} R$, then $P \stackrel{val}{=\!=} R$

Substitution

- The replacement of all occurrences of a 'letter' by a formula

Examples:

1. If we substitute $Q \wedge P$ for P in the valid equivalence

$$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q \ ,$$

then we get the valid equivalence

$$(Q \wedge P) \Rightarrow Q \stackrel{val}{==} \neg(Q \wedge P) \vee Q \ .$$

- This would be true for some substitution for Q
- Or both P and Q simultaneously

SUBSTITUTION PRESERVES EQUIVALENCE

Leibniz's rule

- The replacement of a subformula by an equivalent subformula

Example:

From the valid equivalence

$$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q$$

we can *make new valid equivalences* by replacing $P \Rightarrow Q$ in some complex formula by $\neg P \vee Q$, for instance:

$$(\neg P \wedge (P \Rightarrow Q)) \vee R \stackrel{val}{==} (\neg P \wedge (\neg P \vee Q)) \vee R$$

Proving Tautologies (Example 1)

Prove with a calculation that $\neg(P \wedge \neg P)$ is a tautology.

We have the following calculation:

$$
\begin{array}{ll}
 & \neg(P \wedge \neg P) \\
\stackrel{val}{==} & \{\text{ De Morgan }\} \\
 & \neg P \vee \neg\neg P \\
\stackrel{val}{==} & \{\text{ Double Negation }\} \\
 & P \vee \neg P \\
\stackrel{val}{==} & \{\text{ Excluded Middle }\} \\
 & \texttt{True}
\end{array}
$$

So $\neg(P \wedge \neg P)$ is a tautology.

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 & \mathtt{True}
\end{array}
$$

$$\neg(P \wedge Q) \stackrel{val}{==} \neg P \vee \neg Q$$

$$\neg\neg P \stackrel{val}{==} P$$

$$P \vee \neg P \stackrel{val}{==} \mathtt{True}$$

So $\neg(P \wedge \neg P)$ is a tautology.

Proving Tautologies (Example 2)

Prove with a calculation that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

First, we establish, with a calculation, that $\neg(Q \Rightarrow R) \stackrel{val}{=\!=} (\neg R \wedge Q)$:

$$
\begin{array}{ll}
 & \neg(Q \Rightarrow R) \\
\stackrel{val}{=\!=} & \{ \text{ Implication } \} \\
 & \neg(\neg Q \vee R) \\
\stackrel{val}{=\!=} & \{ \text{ De Morgan } \} \\
 & \neg\neg Q \wedge \neg R \\
\stackrel{val}{=\!=} & \{ \text{ Double negation } \} \\
 & \neg R \wedge Q
\end{array}
$$

Proving Tautologies (Example 2)

Prove with a calculation that $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

First, we establish, with a calculation, that $\neg(Q \Rightarrow R) \stackrel{val}{==} (\neg R \wedge Q)$:

$\neg(Q \Rightarrow R)$

$\stackrel{val}{==}$ { Implication } With Leibniz's rule

$\neg(\neg Q \vee R)$

$\stackrel{val}{==}$ { De Morgan } With substitution rule

$\neg\neg Q \wedge \neg R$

$\stackrel{val}{==}$ { Double negation }

$\neg R \wedge Q$

$$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q$$

$$\neg(P \wedge Q) \stackrel{val}{==} \neg P \vee \neg Q$$

$$\neg\neg P \stackrel{val}{==} P$$

Proving Tautologies (Example 2)

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\stackrel{val}{==} & \{ \text{ De Morgan } \} \\
 & \neg\neg Q \wedge \neg R \\
\stackrel{val}{==} & \{ \text{ Double negation } \} \\
 & \neg R \wedge Q
\end{array}
$$

With Leibniz's rule

With substitution rule

$$P \Rightarrow Q \stackrel{val}{==} \neg P \vee Q$$

$$\neg(P \wedge Q) \stackrel{val}{==} \neg P \vee \neg Q$$

$$\neg\neg P \stackrel{val}{==} P$$

From $\neg(Q \Rightarrow R) \stackrel{val}{==} \neg R \wedge Q$ it follows $\neg(Q \Rightarrow R) \Leftrightarrow (\neg R \wedge Q)$ is a tautology.

Summary of Propositional Logic Rep.

1. Language of logic (as a representation)
   - Atomic propositions
     - NB: Symbols with a fixed meaning (i.e. these are not ‘variables’)

2. Connectives, and how they transform truthfulness across a propositional sentence

3. Truth tables, tautologies, and contradictions are important concepts for summarizing truth and falsehood across a propositional sentence
   - Here we deal with single propositional sentences at a time

4. Equivalence – understanding when separate propositions mean the same thing (and could be exchanged for one another)
   - How equivalence can be put to use (e.g. in proving a tautology)

Assignments

- Assignment 2:

- Goals:
  - Propositional Logic & First Order Logic
    - Practice working with the syntax and semantics of logic
    - Practice manipulating logical expressions in an ‘algebraic’ manner
    - Apply logic to making inferences and determining ‘truth’
    - Understand two of the simplest inferences approaches that we will focus on primarily in this course:
      - Forward chaining
      - Backward chaining