13.31 CURL OF A VECTOR FIELD

Consider a vector field $\vec{A}$ the magnitude and direction of which is a function of position co-
ordinates at a point, then the vector (cross) product of the differential operator $\vec{\nabla}$ and the vector
$\vec{A}$ is a *vector* function of position co-ordinates x, y, z and is known as the *curl* of the vector $\vec{A}$.

$$\therefore \qquad Curl\,\vec{A} = \vec{\nabla} \times \vec{A}$$

Now
$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

nd
$$\vec{A} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$

$$\therefore \qquad \vec{\nabla} \times \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$

$$= \frac{\partial A_y}{\partial x}\hat{i} \times \hat{j} + \frac{\partial A_z}{\partial x}\hat{i} \times \hat{k} + \frac{\partial A_x}{\partial y}\hat{j} \times \hat{i} + \frac{\partial A_z}{\partial y}\hat{j} \times \hat{k} + \frac{\partial A_x}{\partial z}\hat{k} \times \hat{i} + \frac{\partial A_y}{\partial z}\hat{k} \times \hat{j}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

The above ... form as under

$$Curl\ \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Physical significance. Consider a vanishingly small area $\Delta \vec{S}$ enclosed by a closed path which forms its boundary. Let $\vec{A}$ be a vector field defined everywhere in the region and $d\vec{l}$ a vector representing a small element of the path, its direction being that of the tangent to the path at the element $d\vec{l}$. If θ is the angle between the vector $\vec{A}$ at any point on the curve and the direction of the element, then

$$\vec{A} \cdot d\vec{l} = A\, dl \cos\theta$$

As $A\cos\theta$ is the component of the vector $\vec{A}$ along the element $d\vec{l}$, the line integral of the vector field $\vec{A}$ for the closed path

$$\oint A\, dl \cos\theta = \oint \vec{A} \cdot d\vec{l}$$

where the symbol $\oint$ represents the integration over the entire closed path.

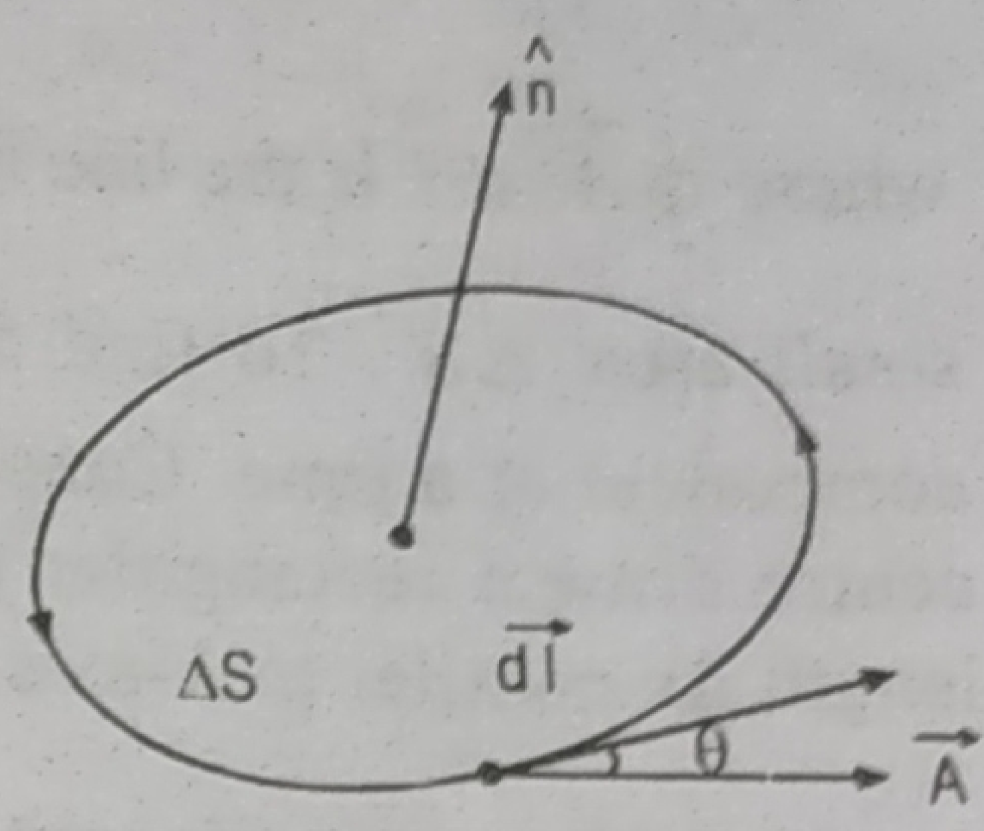

Fig. 13.26

It is a scalar quantity. The path encloses an area $\Delta \vec{S}$.

$\therefore$ Line integral per unit area $= \dfrac{1}{\Delta S} \oint \vec{A} \cdot d\vec{l}$,

Now, consider a unit vector $\hat{n}$ in a direction perpendicular to the area S, the positive direction of $\hat{n}$ being related to the direction of the line integral by the right handed screw rule. *The curl of the vector field $\vec{A}$ (curl $\vec{A}$) is defined as the vector, the magnitude of whose component in the direction of the unit vector $\hat{n}$ is given by the line integral of the vector $\vec{A}$ for the closed path per unit area enclosed by it when the area becomes vanishingly small.* As the magnitude of the component of a vector in the direction of the unit vector is given by the scalar product of the vector and the unit vector $\hat{n}$.

$$\therefore \quad Curl\ \vec{A} \cdot \hat{n} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint \vec{A} \cdot d\vec{l}$$

As $\Delta S \to 0$ let it be denoted by dS, then

$$Curl\ \vec{A} \cdot \hat{n}\, dS = \oint \vec{A} \cdot d\vec{l}$$

$$\therefore \quad Curl\ \vec{A} \cdot d\vec{S} = \oint \vec{A} \cdot d\vec{l}$$

$$Curl\ \vec{A} = \frac{1}{d\vec{S}} \oint \vec{A} \cdot d\,l$$

or

***Curl* of a vector field is a vector.** *Curl* of a vector field $\vec{A}$ is given by the cross product of the operator $\vec{\nabla}$ which behaves as a vector and $\vec{A}$ *i.e.*,

$$Curl\ \vec{A} = \vec{\nabla} \times \vec{A}$$

$\therefore$ *Curl* $\vec{A}$ is a *vector*.

Relation (*i*) also shows that *Curl* $\vec{A}$ is a vector.

13.32 CURL OF A VECTOR IN CARTESIAN CO-ORDINATES

To calculate the value of the curl of a vector we use the definition

$$Curl\ \vec{A} = \lim_{\Delta S \to 0} \frac{1}{\Delta \vec{S}} \oint \vec{A} \cdot d\vec{l}$$

where $\oint \vec{A} \cdot d\vec{l}$ is the line integral of the vector field $\vec{A}$ for the closed path enclosing a vanish small area $\Delta \vec{S}$. To find the value of *Curl* $\vec{A}$ in Cartesian co-ordinates we shall take up component at a time. Consider a point P having co-ordinates x, y, z. For simplicity, with P as centre draw a rectangular path $ABCD$ of surface parallel to XY plane having its surface AB length Δx parallel to X-axis and surface BC of length Δy parallel to Y-axis.

In such a case the unit vector $\hat{n}$ = unit vector $\hat{k}$ parallel to Z axis and we shall get the component of the *Curl* $\vec{A}$ denoted as $curl_z\ \vec{A}$.

According to right hand screw rule, the direction of line integral around the closed rectangular path must be clockwise as seen by someone looking up in the direction $\hat{n}$. Let A_x, A_y, A_z be the components of the vector $\vec{A}$ in the directionn of X, Y, Z, axes respectively at the point x, y, z.

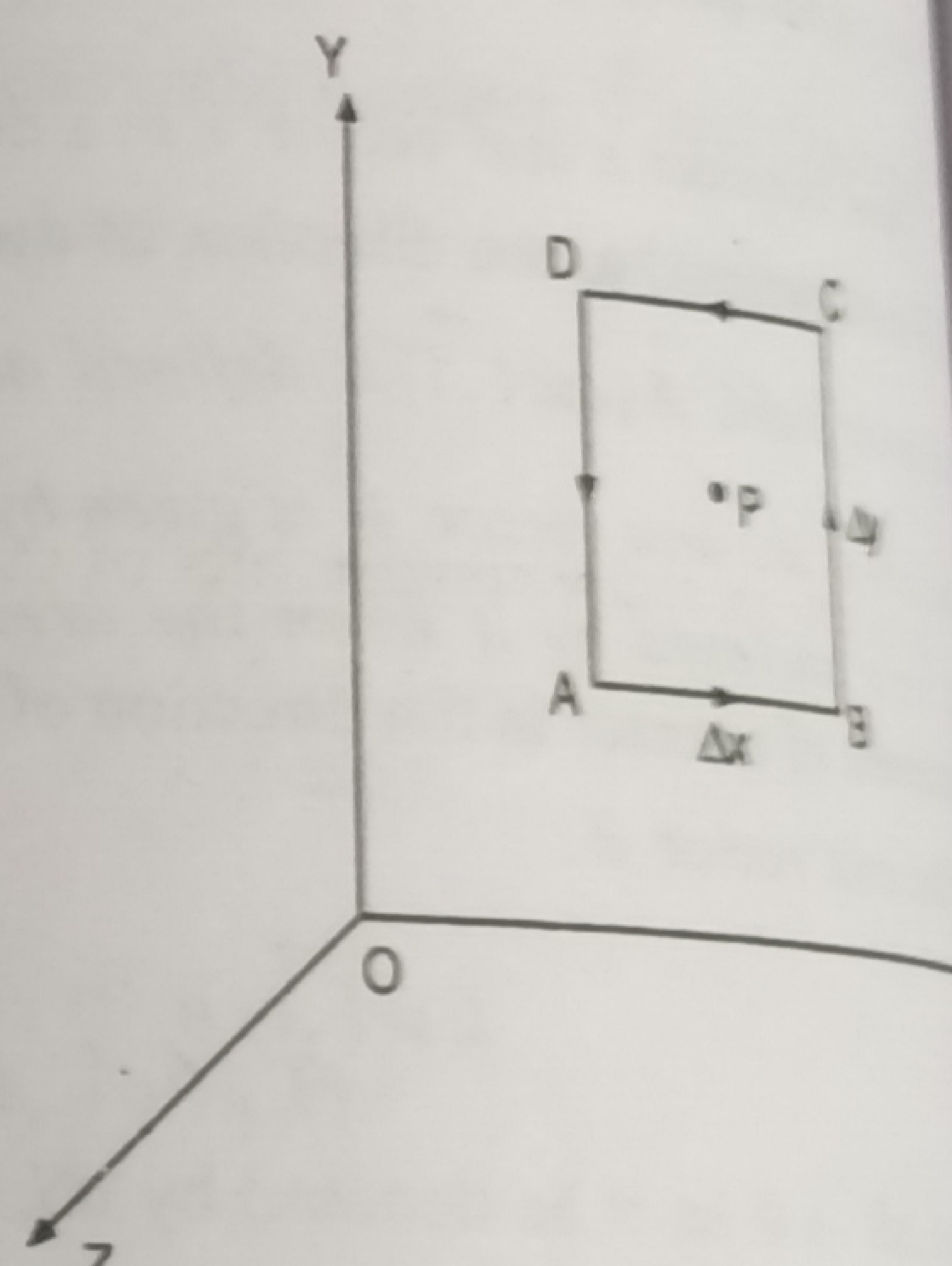

Fig. 13.27

The co-ordinates of a point at the centre of AB are $x, \left(y - \frac{\Delta y}{2}\right), z$. If the rate of change of A_x along the Y-axis is $\frac{\partial A_x}{\partial y}$ then the value of A_x at the centre of AB

$$= A_x - \frac{\partial A_x}{\partial y} \frac{\Delta y}{2}$$

Similarly, the value of A_x at the centre of CD

$$= A_x + \frac{\partial A_x}{\partial y} \frac{\Delta y}{2}$$

The co-ordinates of a point at the centre of BC are $\left(x+\frac{\Delta x}{2}\right), y, z$. If the rate of change of A_y along the Y-axis is $\left(\frac{\partial A_y}{\partial x}\right)$ then the value of A_y at the centre of BC

$$= A_y + \frac{\partial A_y}{\partial x}\frac{\Delta x}{2}$$

Similarly, the value of A_y at the centre of DA

$$= A_y - \frac{\partial A_y}{\partial x}\frac{\Delta x}{2}$$

Hence

$$\oint \vec{A}\cdot d\vec{l} = \left(A_x - \frac{\partial A_x}{\partial y}\frac{\Delta y}{2}\right)\Delta x + \left(A_y + \frac{\partial A_y}{\partial x}\frac{\Delta x}{2}\right)\Delta y - \left(A_x + \frac{\partial A_x}{\partial y}\frac{\Delta y}{2}\right)\Delta x - \left(A_y - \frac{\partial A_y}{\partial x}\frac{\Delta x}{2}\right)\Delta y$$

$$= \frac{\partial A_y}{\partial x}\Delta x\,\Delta y - \frac{\partial A_x}{\partial y}\Delta x\,\Delta y$$

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta x\,\Delta y$$

$$Curl_z\, A = \lim_{\Delta S \to 0}\frac{1}{\Delta \vec{S}}\oint \vec{l}\cdot d\vec{l}$$

$$= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \qquad \ldots [\because \Delta S = \Delta x\,\Delta y]$$

Similarly, x and y components of the curl $\vec{A}$ are

$$Curl_x\, A = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$Curl_y\, A = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$Curl\, \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

$$= \vec{\nabla}\times\vec{A}$$