Problem 1

(LP Formulations) In lecture, Professor Brito mentioned how computational techniques, like interior-point methods, can solve Linear Programming (LP) problems very efficiently and with a high degree of accuracy. Thus, one could argue the more difficult challenge is formulating a problem as an LP, which is exactly the goal of this problem.

Consider a scenario involving an autonomous drone. The goal is to navigate the drone from its initial position to a desired final position while minimizing the total energy consumption. The vehicle operates in 3D space, so its state $\boldsymbol{x}(t) \in \mathbb{R}^6$ is given by $[x, y, z, \dot{x}, \dot{y}, \dot{z}]$ for $t = 0, \ldots, N$. Furthermore, the drone is controlled by four brushless motors, whose actuator signals are given by $\boldsymbol{u}(t) \in \mathbb{R}^4$, for $t = 0, \ldots, N-1$. The dynamics of the system is given by the linear relation

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad t = 0, \ldots, N-1, \text{“} \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{6\times 6}$ and $\boldsymbol{B} \in \mathbb{R}^{6\times 4}$ are given. We assume that the initial state is zero, i.e., $\boldsymbol{x}(0) = \boldsymbol{0}$. Our goal is to choose the inputs $\boldsymbol{u}(0), \ldots, \boldsymbol{u}(N-1)$ to minimize the energy use, which is given by

$$E = \sum_{t=0}^{N-1} \sum_{i=1}^{4} f(u_i(t)), \tag{2}$$

subject to the constraint that $\boldsymbol{x}(N) = \boldsymbol{x}_{\text{stop}}$, where $\boldsymbol{x}_{\text{stop}} \in \mathbb{R}^6$ is a (given) desired target state. The function $f\colon \mathbb{R} \to \mathbb{R}$ is the energy consumption map for an actuator. It represents the amount of energy consumed as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & \text{if } |a| \le 1, \\ 3|a| - 2 & \text{if } |a| > 1. \end{cases} \tag{3}$$

In other words, we set energy use to be proportional to the absolute value of the actuator signal (for small actuator signals; for larger actuator signals the efficiency is reduced by a factor of three.) Your task is as follows: **Formulate this optimal control problem as an LP in standard form, as described in class.** We expect: (1) An explanation regarding the formulation of each of the constraints, (2) A clear description of the formulated LP, and (3) A conversion of the LP to standard form.

Note: Try to keep your answer reasonably concise, with appropriate vector representations of the data.

Problem 2

(Market Factors) Convex Programming is used quite often in financial modeling. We can get a taste of it through the following problem, which assumes no prior experience with the field.

a) Imagine a market operating for a single period in which n different assets are traded. Depending on the events during that single period, there are m possible states of nature at the end of the period. If we invest one dollar in some asset i and the state of nature turns out to be s, we receive a payoff of r_{si}. Thus, each asset i is described by a payoff vector $(r_{1i}, \ldots, r_{mi})$. We can now define an $m \times n$ *payoff matrix*[1] giving the payoffs for each of the n assets:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{bmatrix}.$$

Let x_i be the amount held of asset i. A portfolio is then a vector $\boldsymbol{x} = [x_1, \ldots, x_n]$, where x_i can be positive (bought) or negative (shorted). The income generated from portfolio $\boldsymbol{x}$ in state s is given by:

$$\boldsymbol{w}_s = \sum_{i=1}^{n} r_{si} x_i.$$

Let p_i be the price of asset i in the beginning of the period, and let $\boldsymbol{p} = [p_1, \ldots, p_n]$ be the vector of asset prices. Then, the cost of acquiring portfolio $\boldsymbol{x}$ is given by $\boldsymbol{p}^T \boldsymbol{x}$.

The central problem in asset pricing is to determine what the prices p_i should be. This leads to the *absence of arbitrage* condition[2]: asset prices should always be such that no investor can get a guaranteed nonnegative payoff out of a negative investment. Mathematically, it states that:

$$\text{If } \boldsymbol{R}\boldsymbol{x} \geq \boldsymbol{0}, \text{ then } \boldsymbol{p}^T \boldsymbol{x} \geq 0.$$

Given a particular set of assets, only certain prices $\boldsymbol{p}$ are consistent with the absence of arbitrage. **Prove the following: The absence of arbitrage condition holds if and only if there exists a nonnegative vector $\boldsymbol{q} = [q_1, \ldots, q_m]$ such that the price of each asset i is given by**

$$p_i = \sum_{s=1}^{m} q_s r_{si}.$$

A proof leveraging duality principles or theorems discussed in class is expected.

b) The no-arbitrage condition is very simple, and yet very powerful [3]. Imagine a trio of financial instruments: equities, debt securities, and derivatives working within a single-period financial setting. When the period begins, assume that the equity is priced at S. This price is subject to change and is probabilistically expected to reach either uS (with probability p) or dS (with

[1] You may have seen this matrix before in different contexts, such as Game Theory.

[2] This condition underlies much of finance theory. It is sometimes taken as an axiom.

[3] It is difficult to overstate just how important this fundamental theorem is. It allows us to study asset prices without having to worry about cross-sectional distributions of investors.

probability $1 - p$) by the period's close, where u and d are scaling factors satisfying $d < 1 < u$.

For debt securities, there's no risk involved. Investing a single unit of currency yields a payoff of r units at the period's end, with $r > 1$.

Finally, the derivative (in this case an option) grants you the privilege to acquire a single equity unit at an agreed-upon price K when the period ends. If the market price $\bar{S}$ of the equity exceeds K, you'd exercise the option and quickly liquidate the equity, gaining $\bar{S} - K$ in the process. Otherwise, if $\bar{S} < K$, there's no gain in exercising, leading to a zero payoff. Therefore, the value of the option at the end of the period is equal to $\max\{0, \bar{S} - K\}$. Since the option is itself an asset, it should have a value in the beginning of the time period.

Assuming there are no opportunities for arbitrage, prove the value of the option must be equal to

$$\gamma \max\{uS - K, 0\} + \delta \max\{dS - K, 0\}$$

Here, γ and δ are constants satisfying the linear equations:

$$\begin{aligned} u\gamma + d\delta &= 1, \\ \gamma + \delta &= \frac{1}{r}. \end{aligned}$$

Hint: Write the payoff matrix $\boldsymbol{R}$ and apply part (a).