Advanced Algorithms: Homework 1

Due on Jan. 24, 2024 at 2:29pm EST

Professor Dana Randall Spring 2024

Submit Q1 as your submission for hw1part1. Submit the rest of the homework as your submission for hw1part2

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Exercise 1

(KT Chapter 1, Exercise 1) True or false? (If true, give a brief explanation, if false, give a counterexample): In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

Exercise 2

(KT Chapter 1, Exercise 2) True or false? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.

Exercise 3

(KT Chapter 1, Exercise 3) There are many other settings in which we can ask questions related to some type of "stability" principle. Here's one, involving competition between two enterprises.

Suppose we have two television networks whom we'll call A and B. There are n prime-time programming slots, and each network has n TV shows. Each network wants to devise a schedule – an assignment of each show to a distinct slot – so as to attract as much market share as possible.

Each show has a fixed rating, which is based on the number of people who watched it last year; we'll assume that no two shows have exactly the same rating. A network *wins* a given time slot if the show that it schedules for that time slot has a larger rating than the other network schedules for that time slot. The goal is to win as many time slots as possible.

We'll say a pair of schedules (S, T) is *stable* if neither network can unilaterally change its own schedule and win more time slots.

For every set of TV shows and ratings, is there always a stable pair of schedules? Resolve this question by doing one of two things:

1. Give an algorithm that, for any TV shows and associated ratings, produces a pair of stable schedules; or
2. Give an example of TV shows and ratings for which there is no stable pair of schedules.

Exercise 4

Give an input to the Stable Marriage Problem with n men and n women that has a unique stable pairing. Now give an input with n men and n women that has an exponential number (in n) of stable pairings.

Exercise 5

Let $[m]$ denote the set $\{0, 1, \ldots, m-1\}$. For each of the following families of hash functions, say whether it is 2-universal or not, and specify how many random bits are needed to sample a hash function from the family, and then justify your answer.

(a) $H_a = \{h(x_1, x_2) = a_1x_1 + a_2x_2 \pmod m) \mid a_1, a_2 \in [m]\}$, where m is some fixed prime. (Note that for each $h \in H_a$ we have $h : [m]^2 \to [m]$, i.e., it maps a pair of integers in $[m]$ to a single integer in $[m]$.)

(b) $H_b = \{h(x_1, x_2) = a_1x_1 + a_2x_2 \pmod m)\} : a_1, a_2 \in [m]\}$, where $m = 2^k$ is some fixed power of two.

(c) The set of all functions $f : [m] \to [m-1]$.