Domino Tilings of the Chessboard

An Introduction to Sampling and Counting

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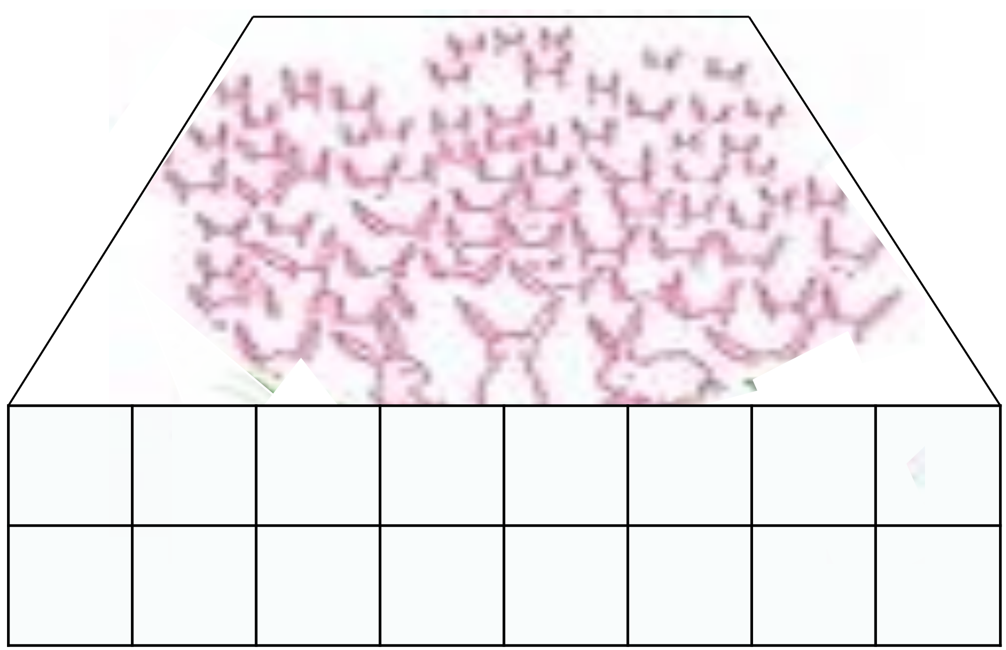
Building short walls

How many ways are there to build a 2 x n wall with 1 x 2 bricks?

1

2

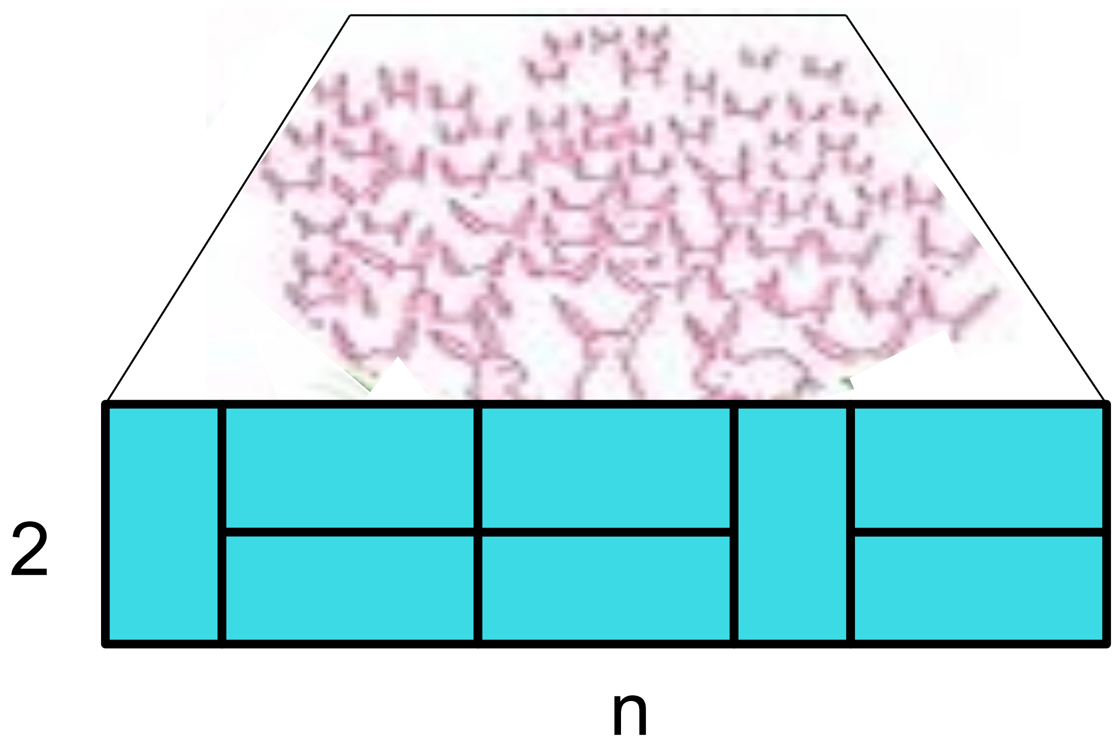


2

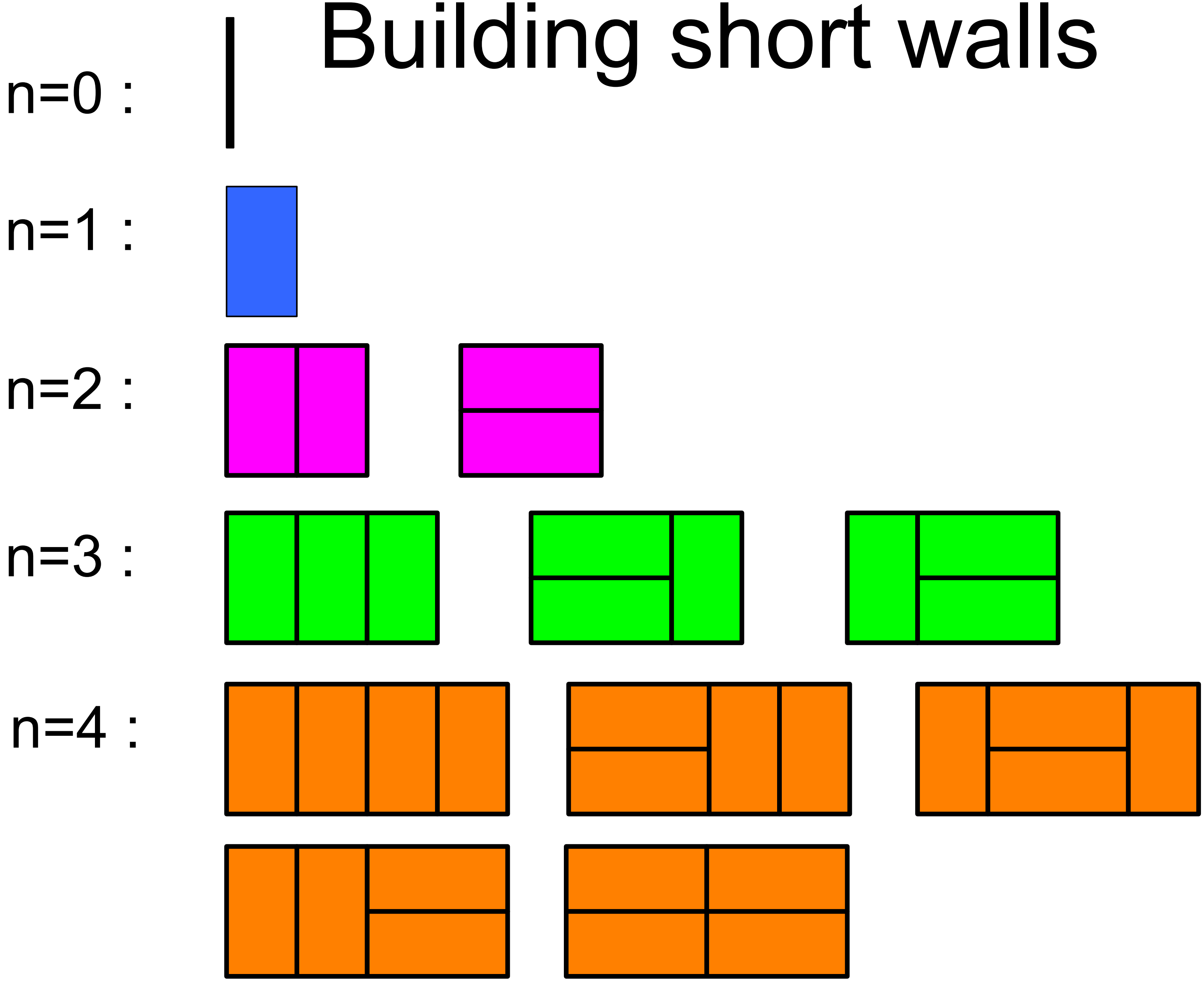
n

Building short walls

How many ways are there to build a 2 x n wall with 1 x 2 bricks?



Building short walls



Building short walls

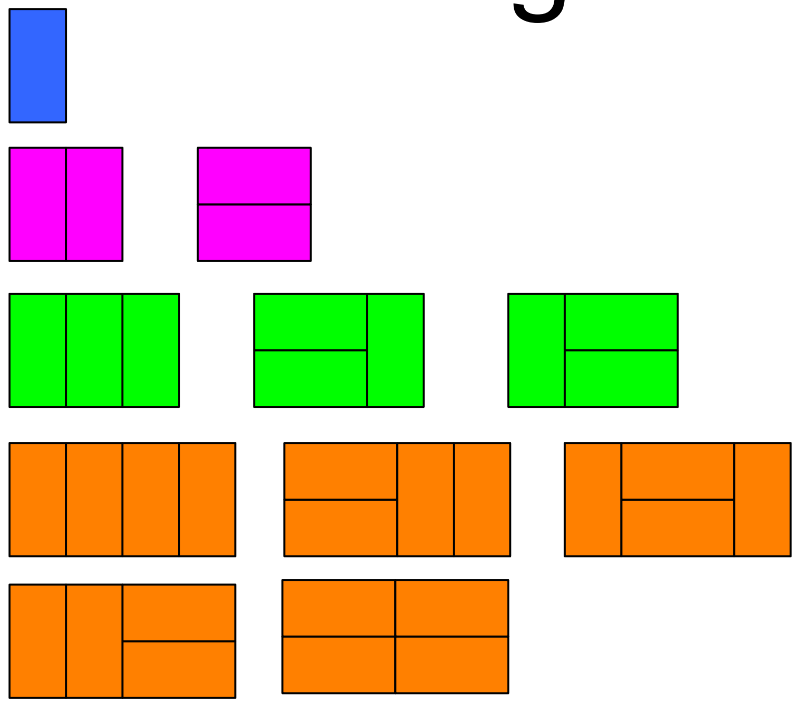
n=0 :

n=1 :

n=2 :

n=3 :

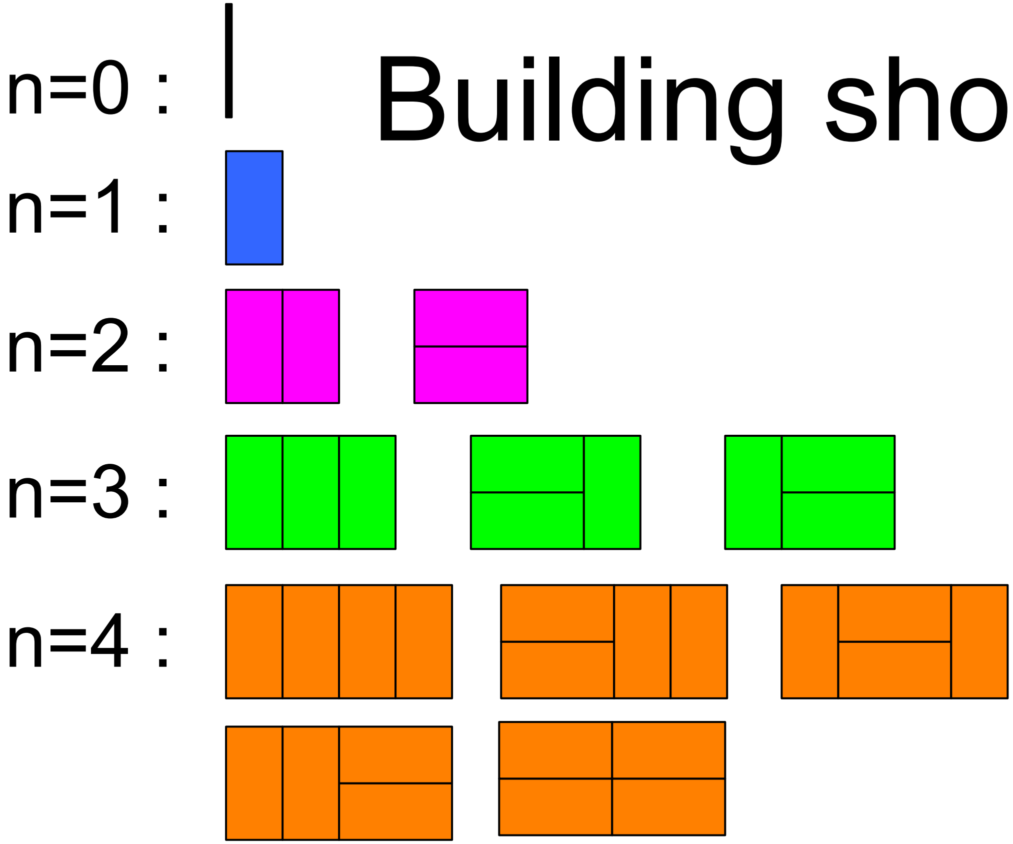
n=4 :

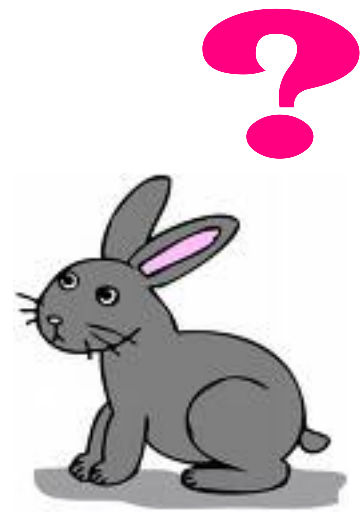


The number of walls equal:

f_n = 1, 1, 2, 3, 5

Building short walls

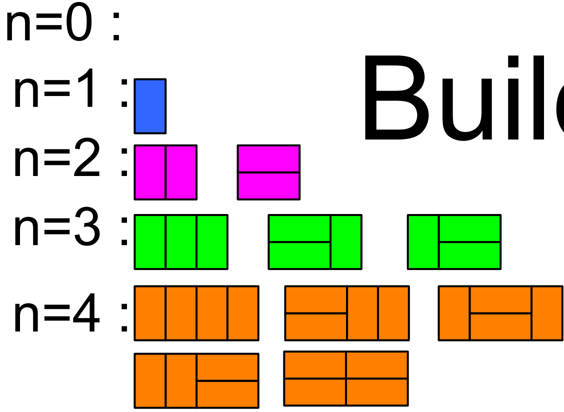


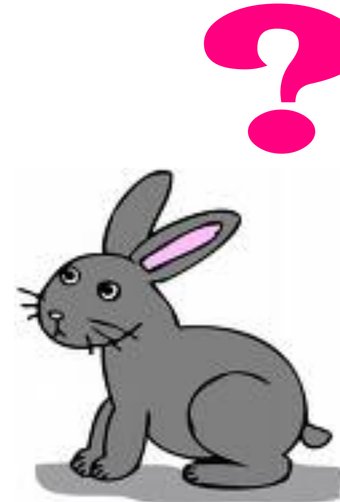


The number of walls equal:

$$f_n = 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

Building short walls

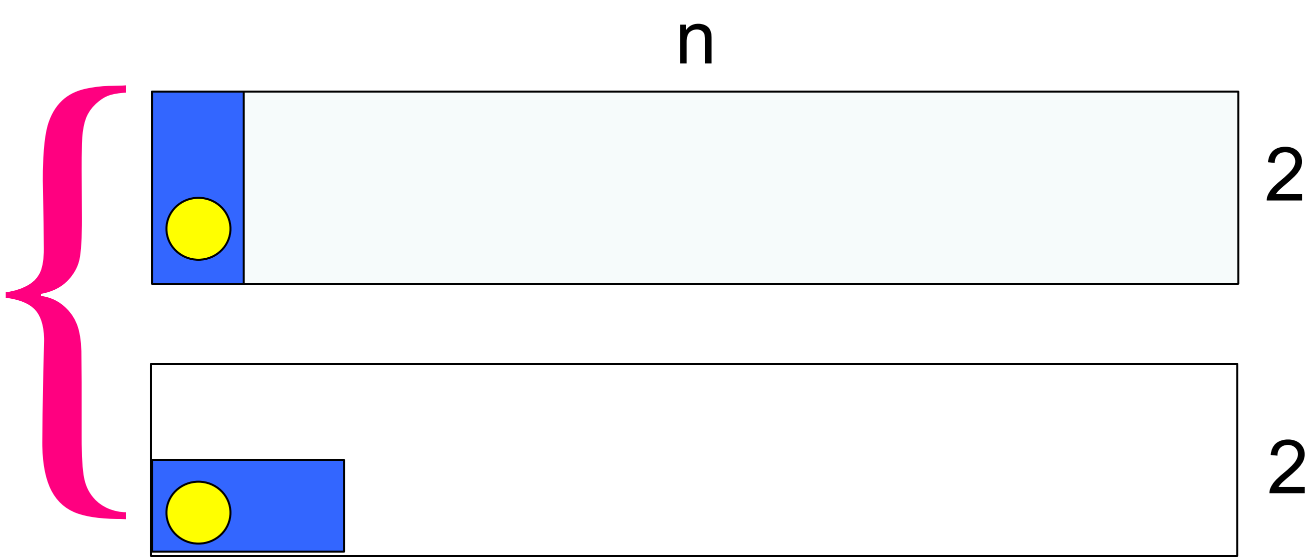


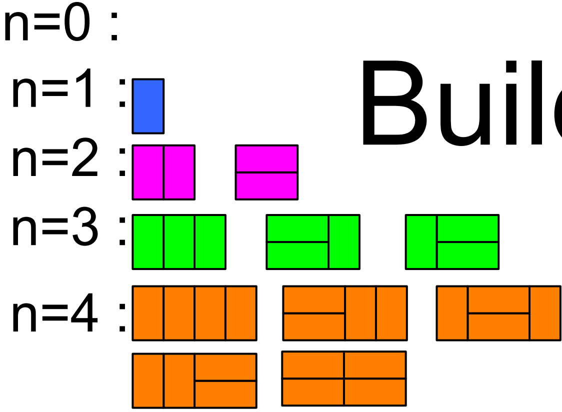


The number of walls equals:

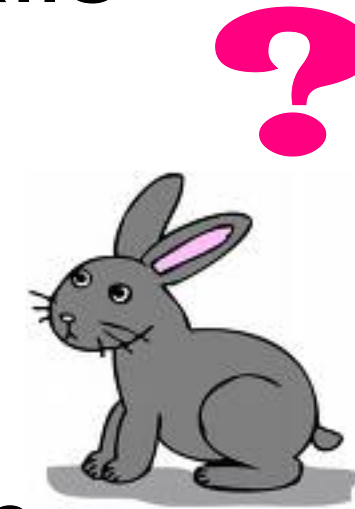
f_n = 1, 1, 2, 3, 5, 8, 13, 21, . . .

f_n =





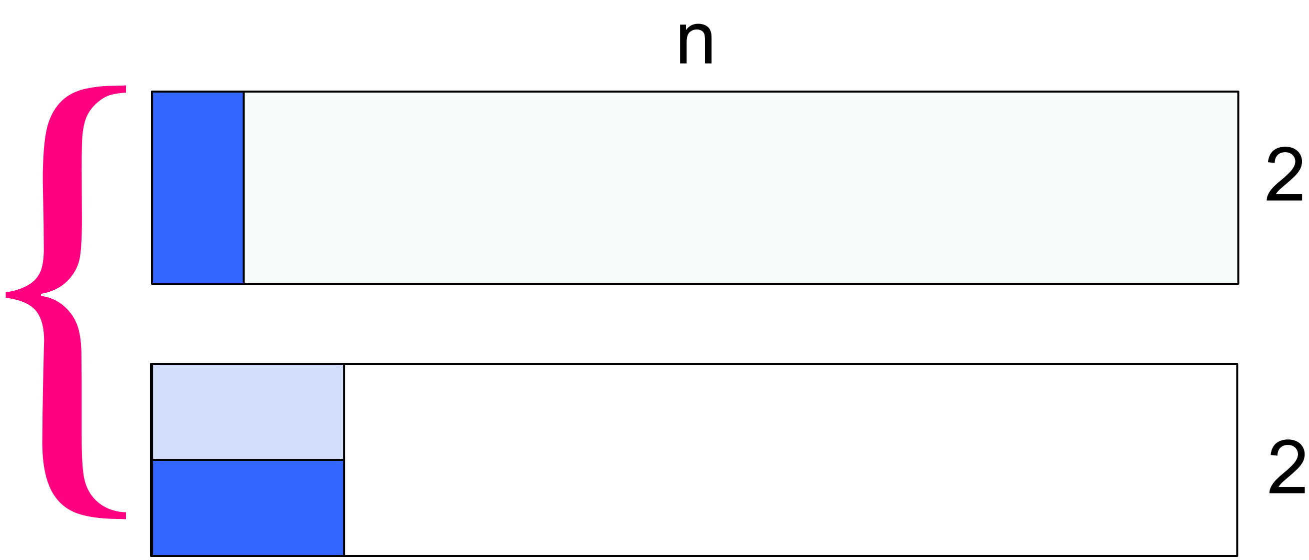
Building short walls



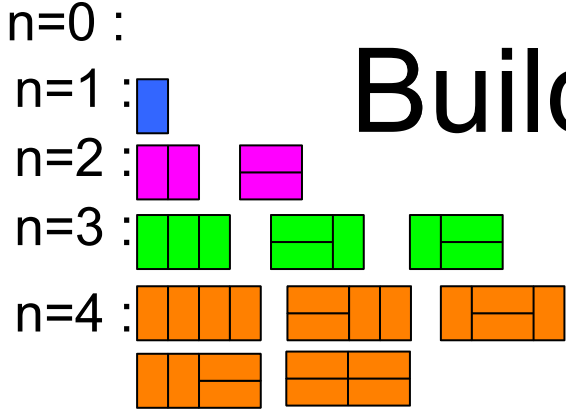
The number of walls equals:

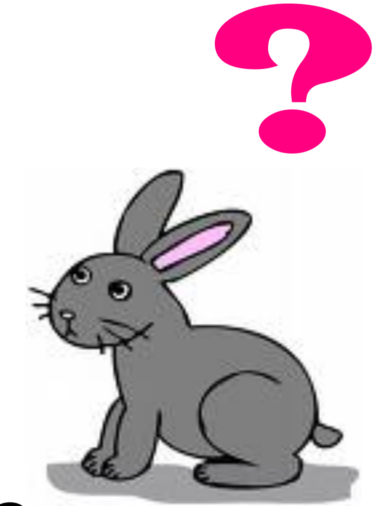
$$f_n = \quad 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

$f_n =$



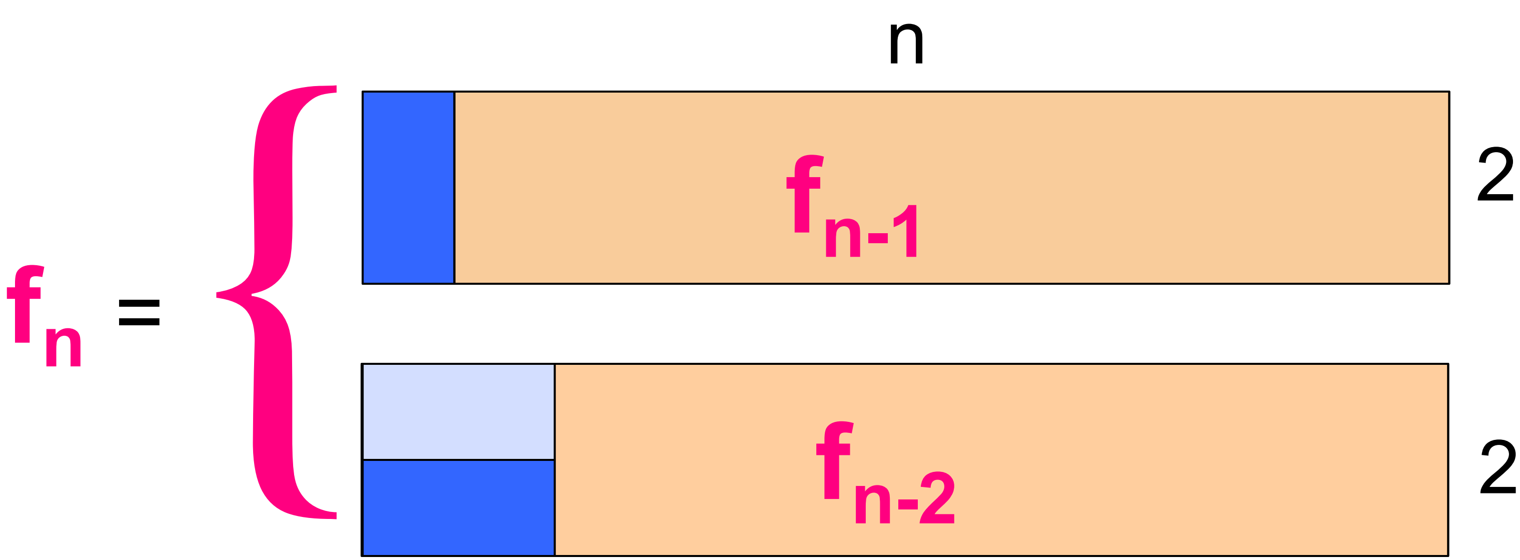
Building short walls



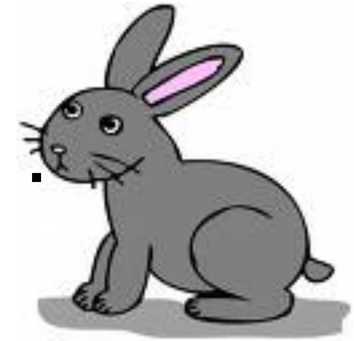


The number of walls equals:

$f_n = \quad 1, \; 1, \; 2, \; 3, \; 5, \; 8, \; 13, \; 21, \; . \, . \, .$



The Fibonacci Numbers



The number of walls equals:

$f_n = \ 1, \ 1, \ 2, \ 3, \ 5, \ 8, \ 13, \ 21, \ . \ . \ .$

$f_n =$

$$f_n = f_{n-1} + f_{n-2}\,, \quad f_0 = f_1 = 1$$

$$f_n = (\varphi^n + (1-\varphi)^n)\,/\sqrt{5}\ ,$$

where: $\varphi = \frac{1+\sqrt{5}}{2}$ ("golden ratio")

Domino Tilings

Given a region R on the infinite chessboard, cover with non-overlapping 2 x 1 dominos.

Where is a tiling? Do any even exist?

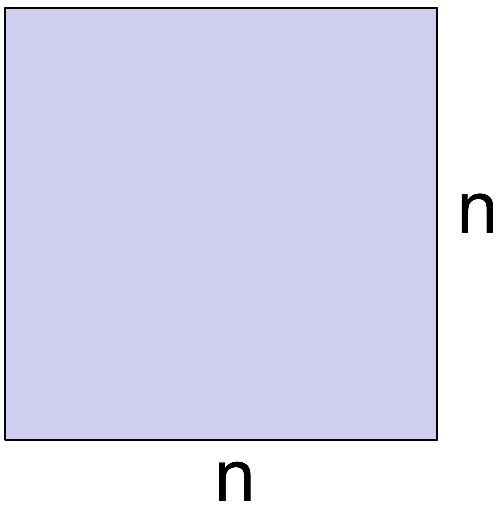
How many tilings are there?

What does a typical tiling look like?

When do we stop our algorithms?

Why do we care?

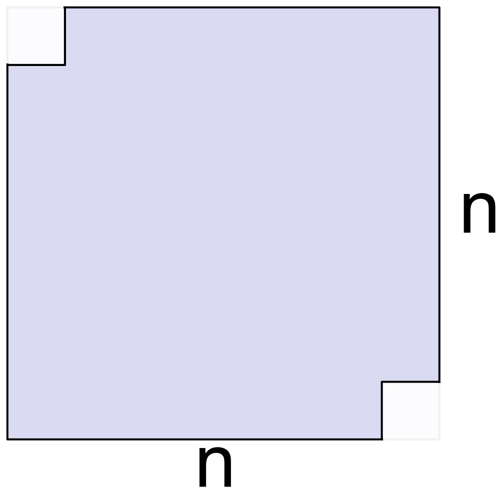
Where is a tiling? Do any exist?



★ Only if n is even!

Where is a tiling? Do any exist?

- The Area of R must be even



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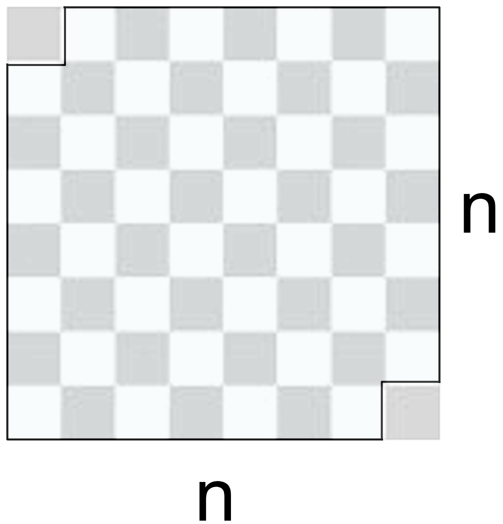


★ There must be an equal number of black and white squares.

Where is a tiling? Do any exist?

- The Area of R must be even
- With an equal number of white and black squares

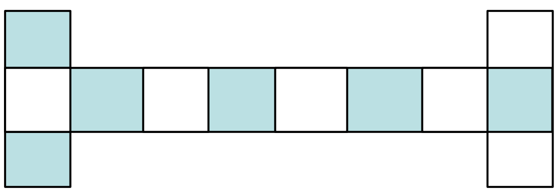
Is this enough?

Where is a tiling? Do any exist?

- The Area of R must be even
- With an equal number of white and black squares

⋮

Is this enough?



?

There is an efficient algorithm to decide if R is tileable and to find one if it is. [Thurston]

Domino Tilings

Where is a tiling? Do any even exist?

How many tilings are there?

What does a typical tiling look like?

When do we stop our algorithms?

Why do we care?

How many tilings are there?

Short 2 x n walls

$\approx \varphi^n$

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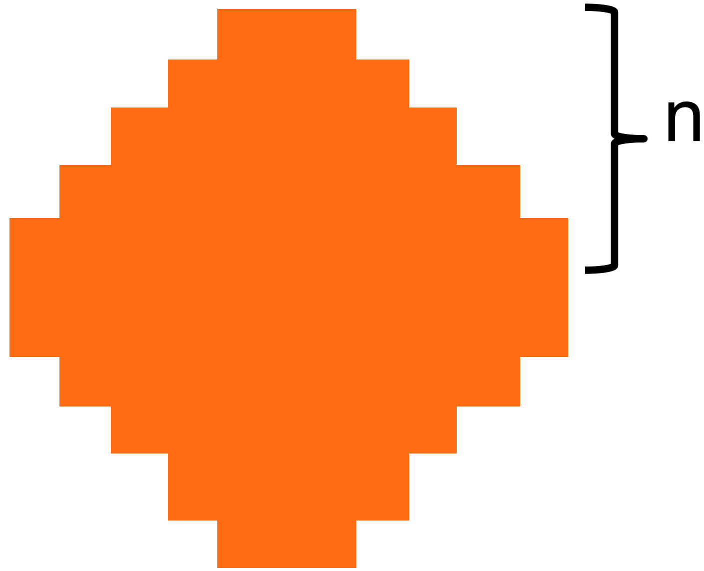
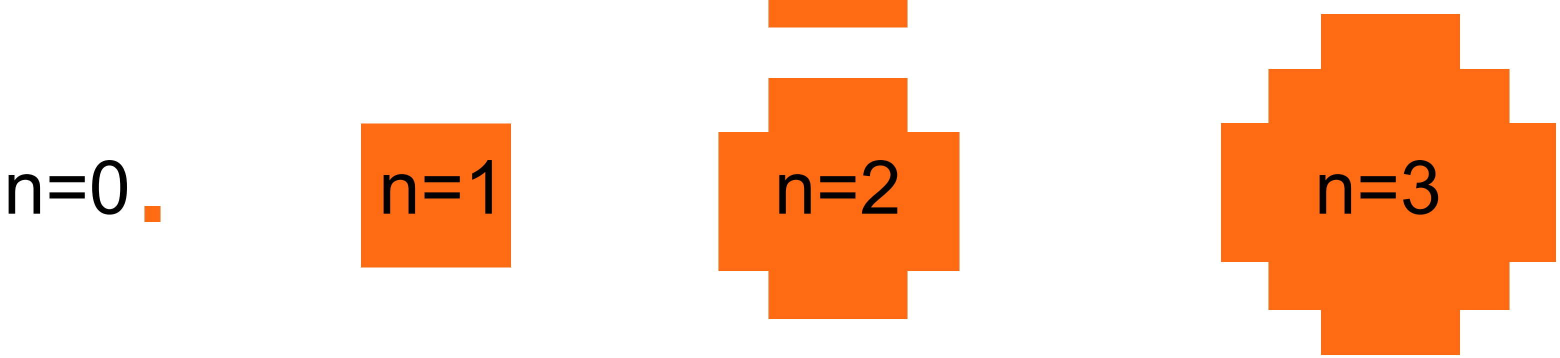
Aztec Diamonds



$= 2^{n(n+1)/2}$

[Elkies, Kuperberg, Larson, Propp]



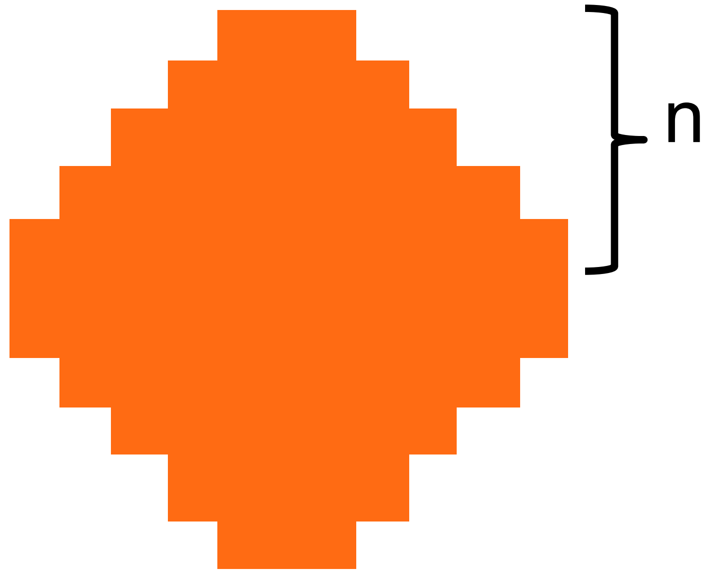
# = 1	2	8	64

How many tilings are there?

Short 2 x n walls

$\approx \varphi^n$

Aztec Diamonds



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How many tilings are there?

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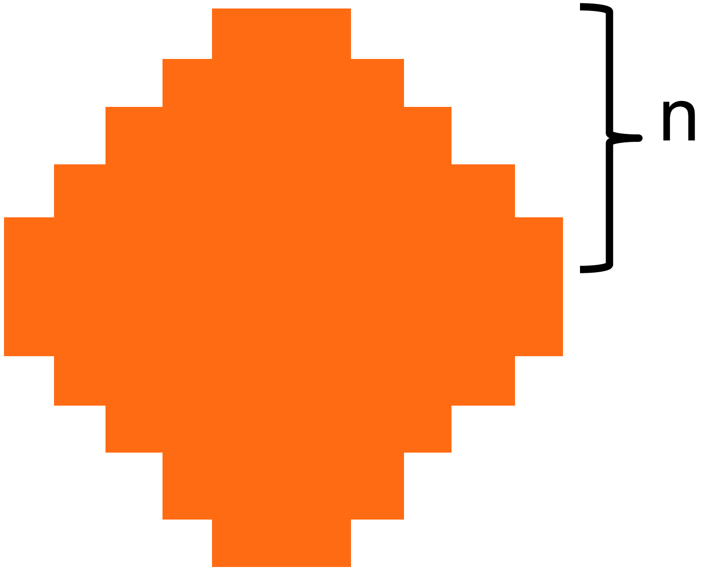
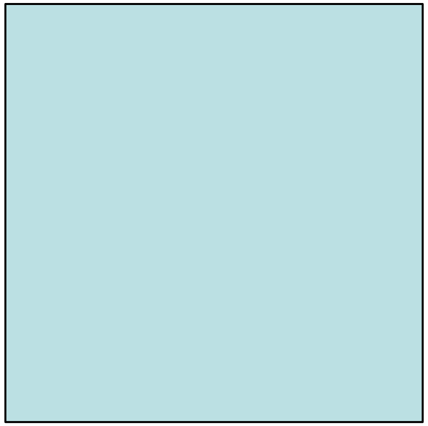


$= 2^{n(n+1)/2}$

Square n x n walls



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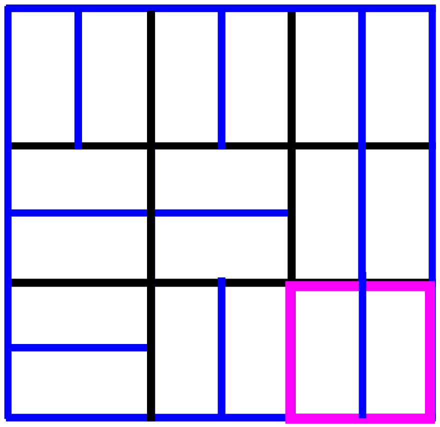
How many tilings are there?

Square n x n walls



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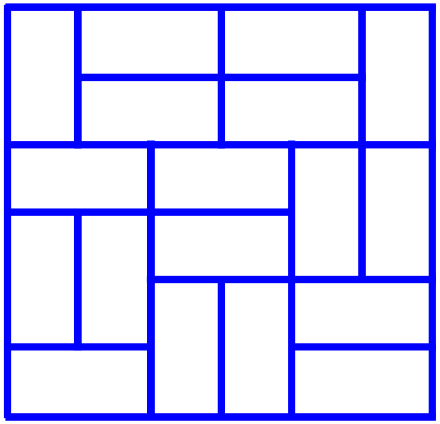
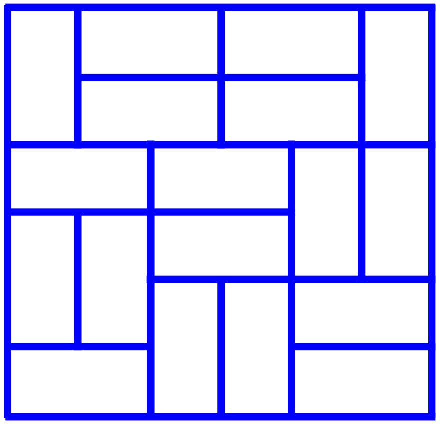


$2^{(\text{Area}/4)} < \#$



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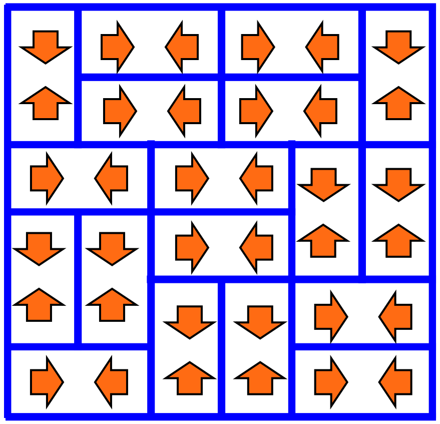


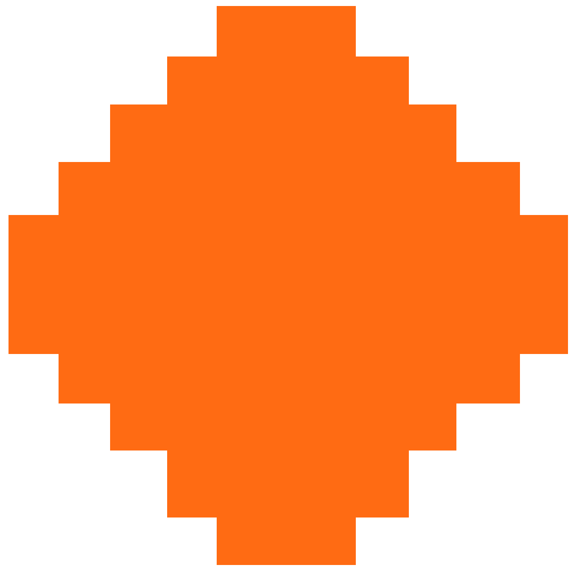
$\# < 4^{(\text{Area})}$

How many tilings are there?

Short 2 x n walls

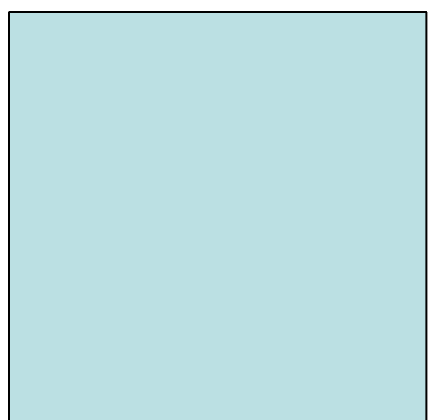
$\# \approx \varphi^n$

Aztec Diamonds



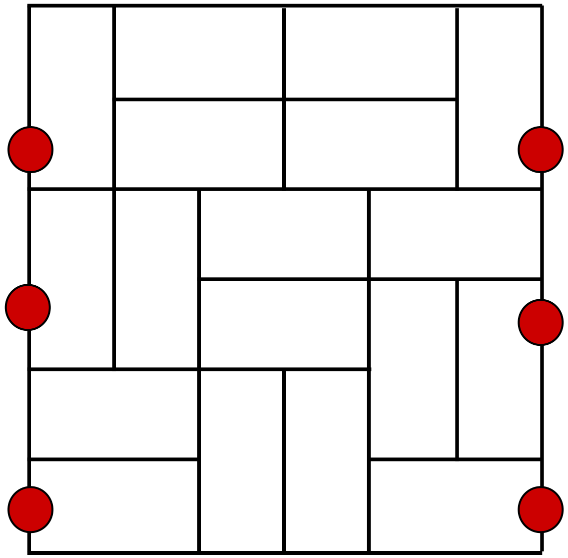
$\# = 2^{n(n+1)/2}$

Square n x n walls



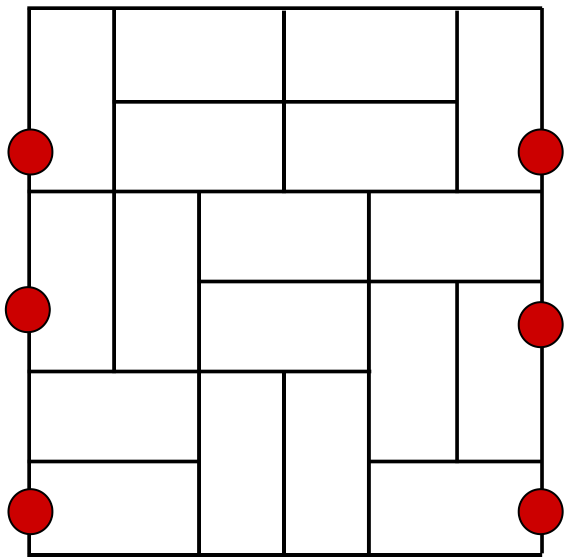
$2^{\mathrm{Area}/4} < \# < 4^{\mathrm{Area}}$

How many: An Algorithm

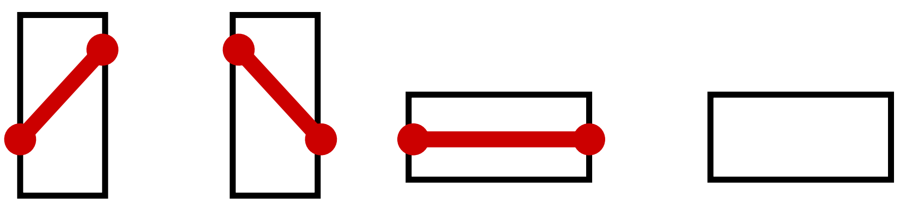


- Mark alternating vertical edges;

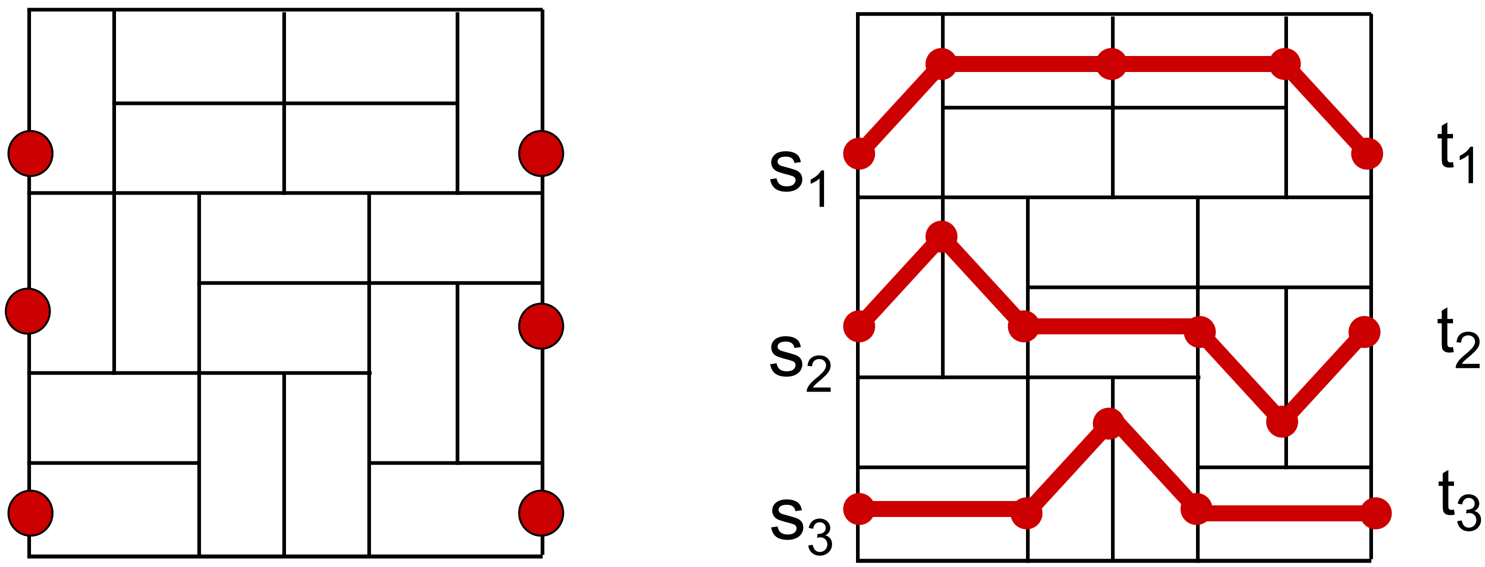
How many: An Algorithm

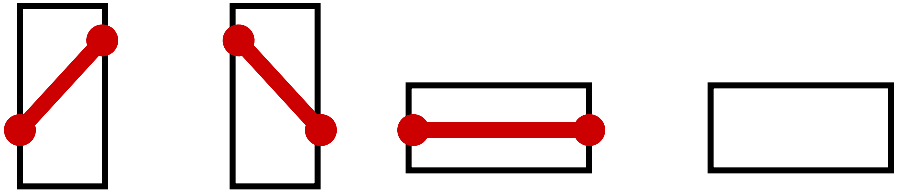


- Mark alternating vertical edges;
- Use marked tiles;
- Markings must line up!

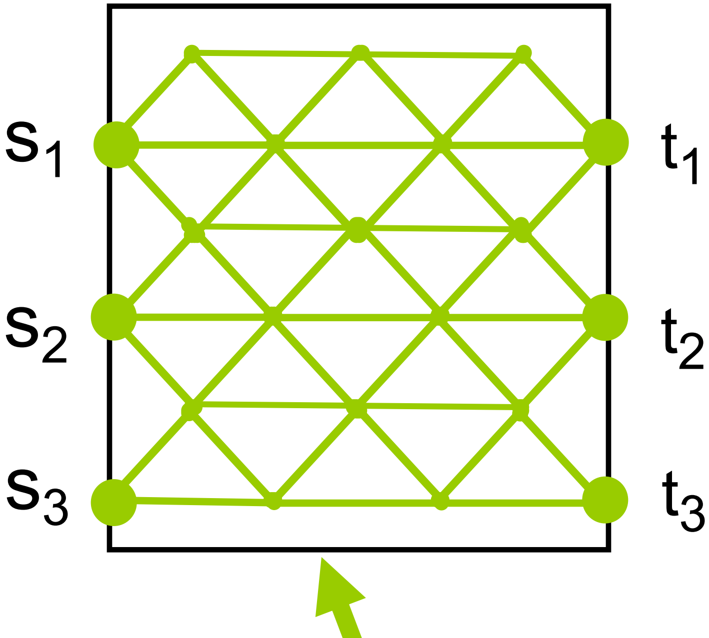


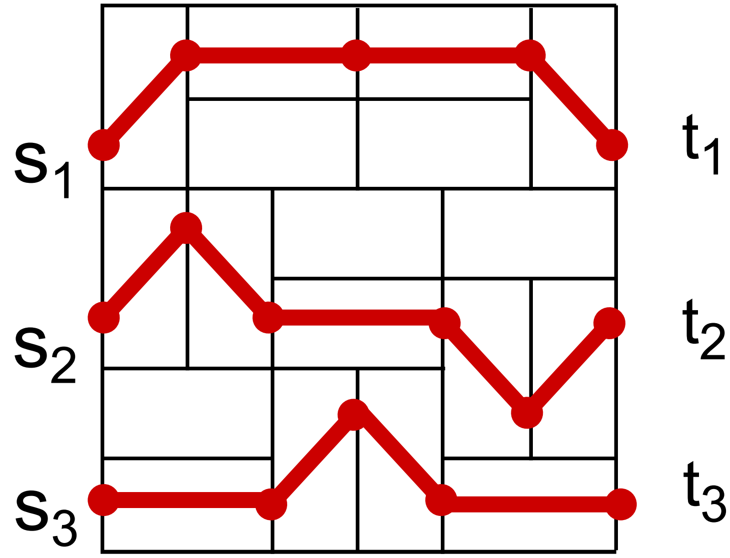
How many: An Algorithm





How many: An Algorithm

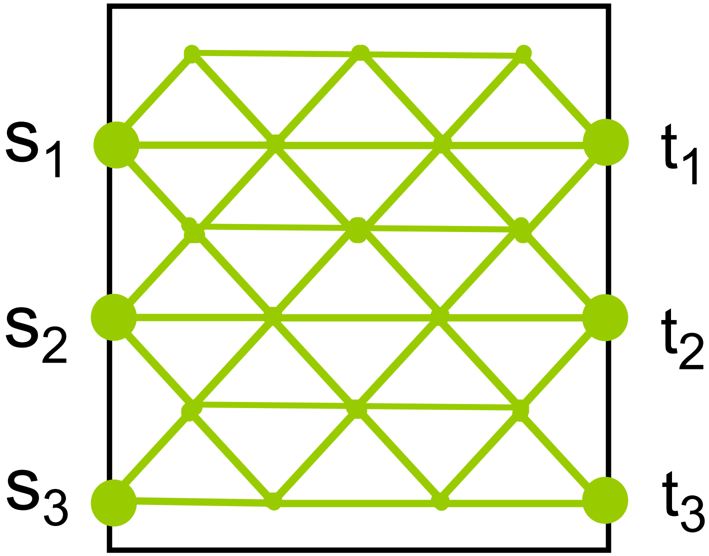




We want to count non-intersecting sets of paths from s_i to t_i.

[R.]

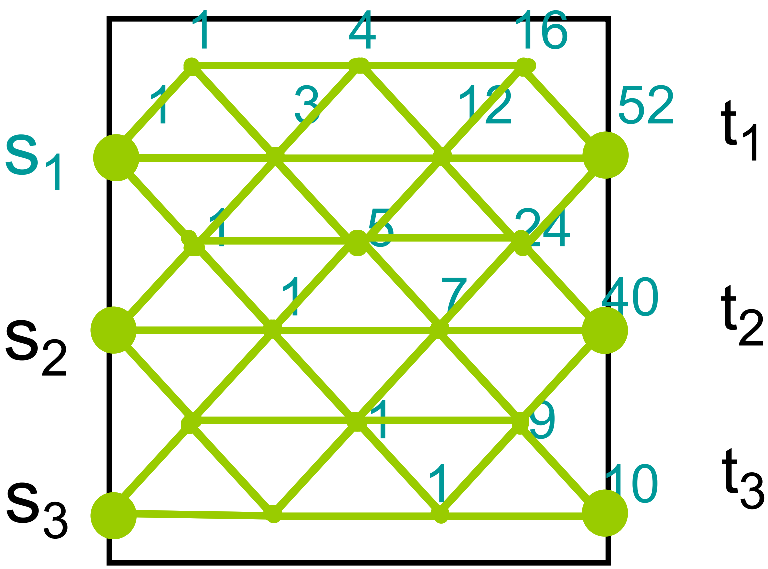
How many: An Algorithm



We want to count non-intersecting sets of paths from s_i to t_i.

Let a_{ij} be the number of paths from s_i to t_j.

How many: An Algorithm



We want to count non-intersecting sets of paths from s_i to t_i.

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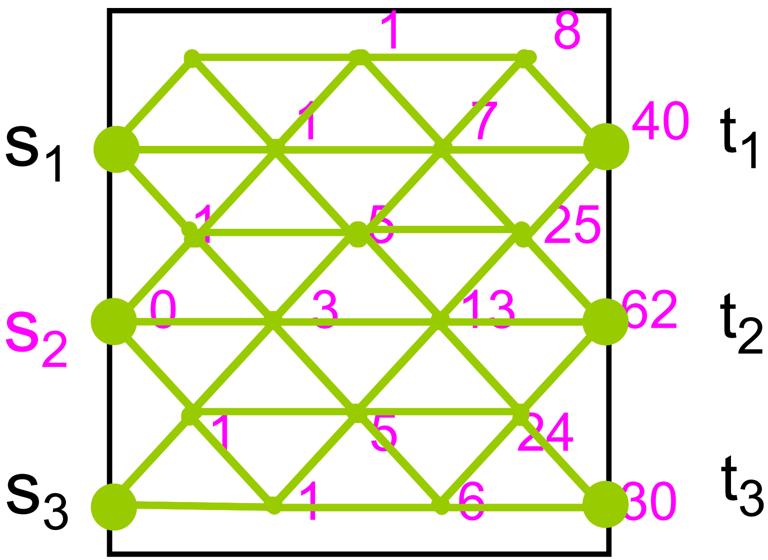
52 40 10

How many: An Algorithm



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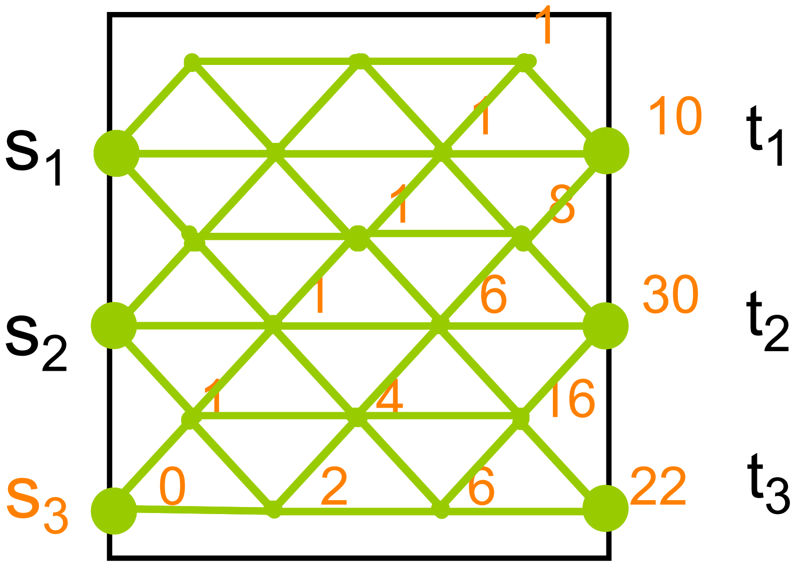
52 40 10

40 62 30

How many: An Algorithm



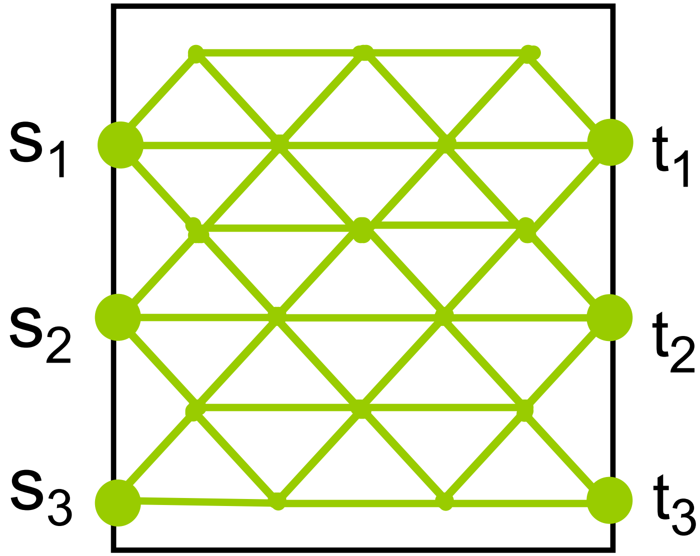
We want to count non-intersecting sets of paths from s_i to t_i.

Let a_{ij} be the number of paths from s_i to t_j.

$$\begin{matrix} 52 & 40 & 10 \\ 40 & 62 & 30 \\ 10 & 30 & 22 \end{matrix}$$

How many: An Algorithm

[Gessel, Viennot]



We want to count non-intersecting sets of paths from s_i to t_i.

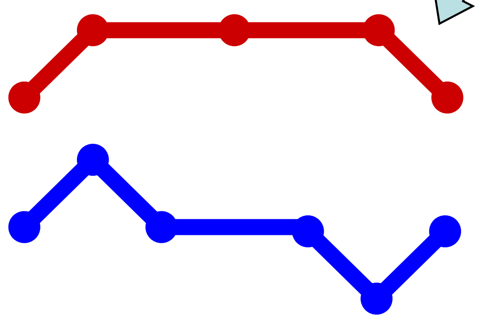
Let a_{ij} be the number of paths from s_i to t_j.

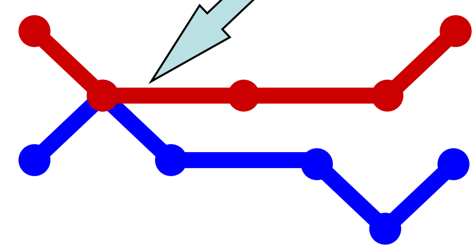
$$\text{Det}\begin{vmatrix} 52 & 40 & 10 \\ 40 & 62 & 30 \\ 10 & 30 & 22 \end{vmatrix} = \mathbf{1728.}$$

This is the number domino tilings!!

Proof sketch for two paths:

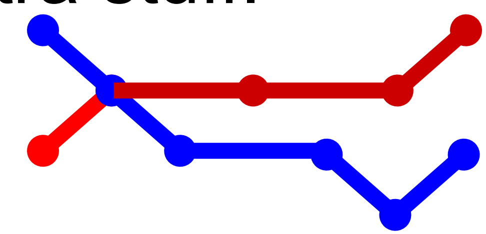
a_{11} x a_{22} counts what we want + extra stuff.





= #

a_{12} x a_{21} also counts the extra stuff.



Therefore $(a_{11}$ x $a_{22})$ - $(a_{12}$ x $a_{21})$ counts real tilings.

(This is the 2 x 2 determinant!)

Domino Tilings

Where is a tiling? Do any even exist?

How many tilings are there?

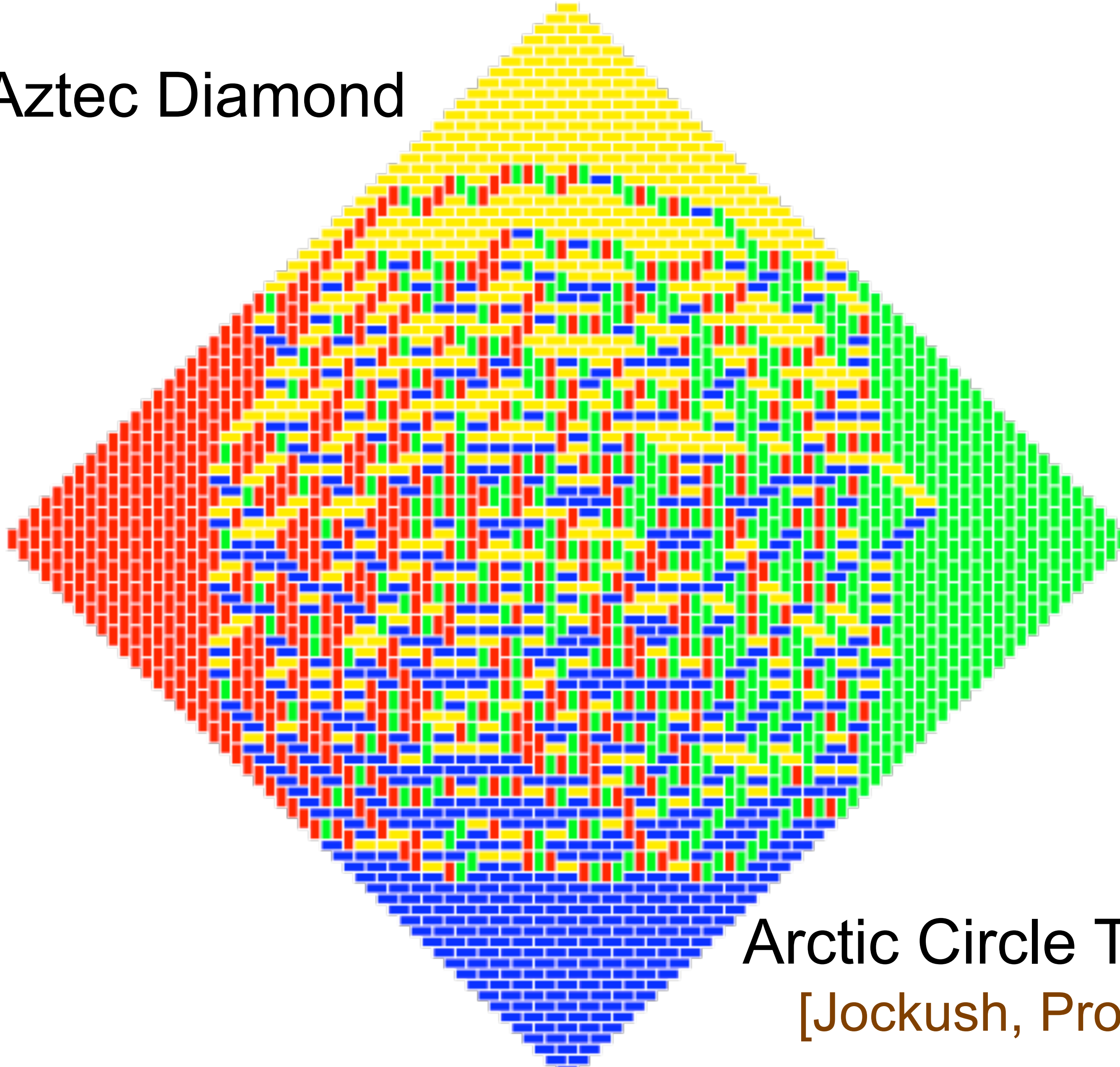
What does a typical tiling look like?

When do we stop our algorithms?

Why do we care?

Why Mathematicians Care

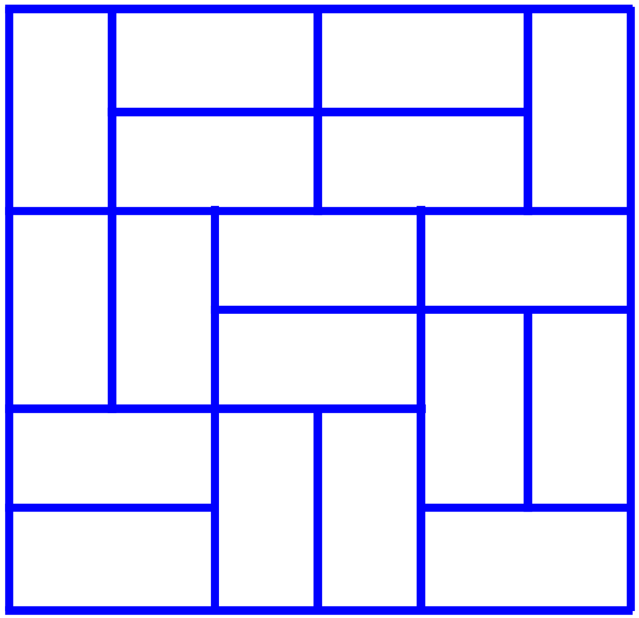
The Aztec Diamond



Arctic Circle Theorem

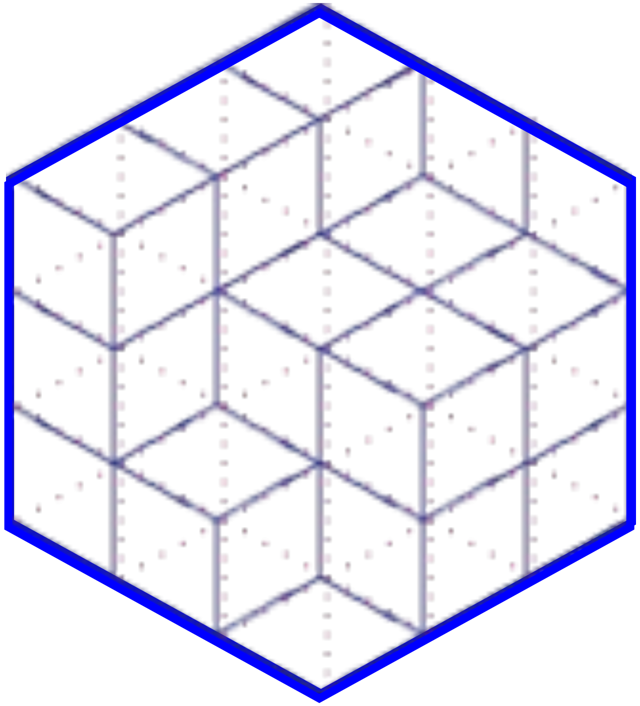
[Jockush, Propp, Shor]

What about tilings on lattices?



On the chessboard

“Domino tilings”

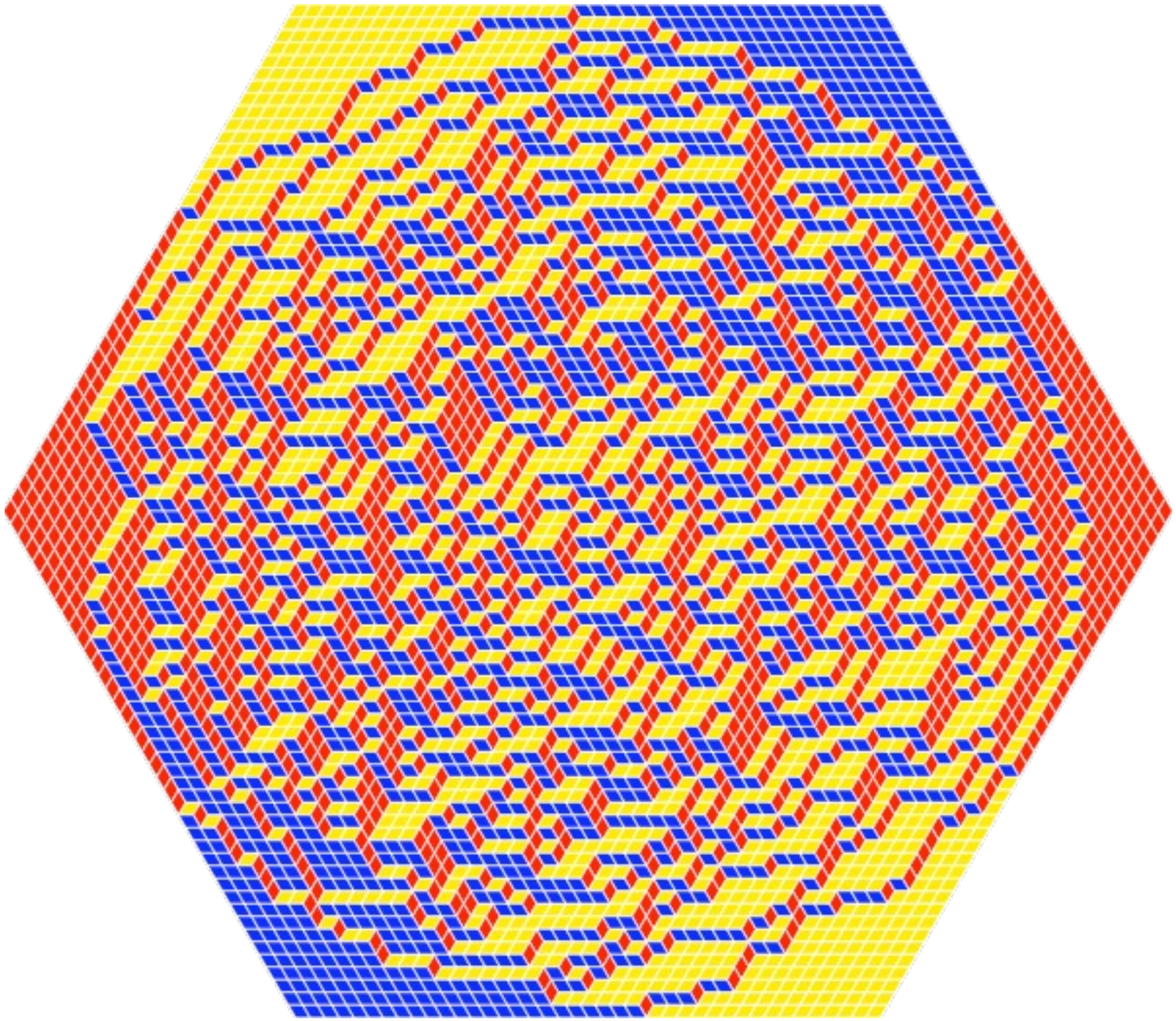


On the hexagonal lat.

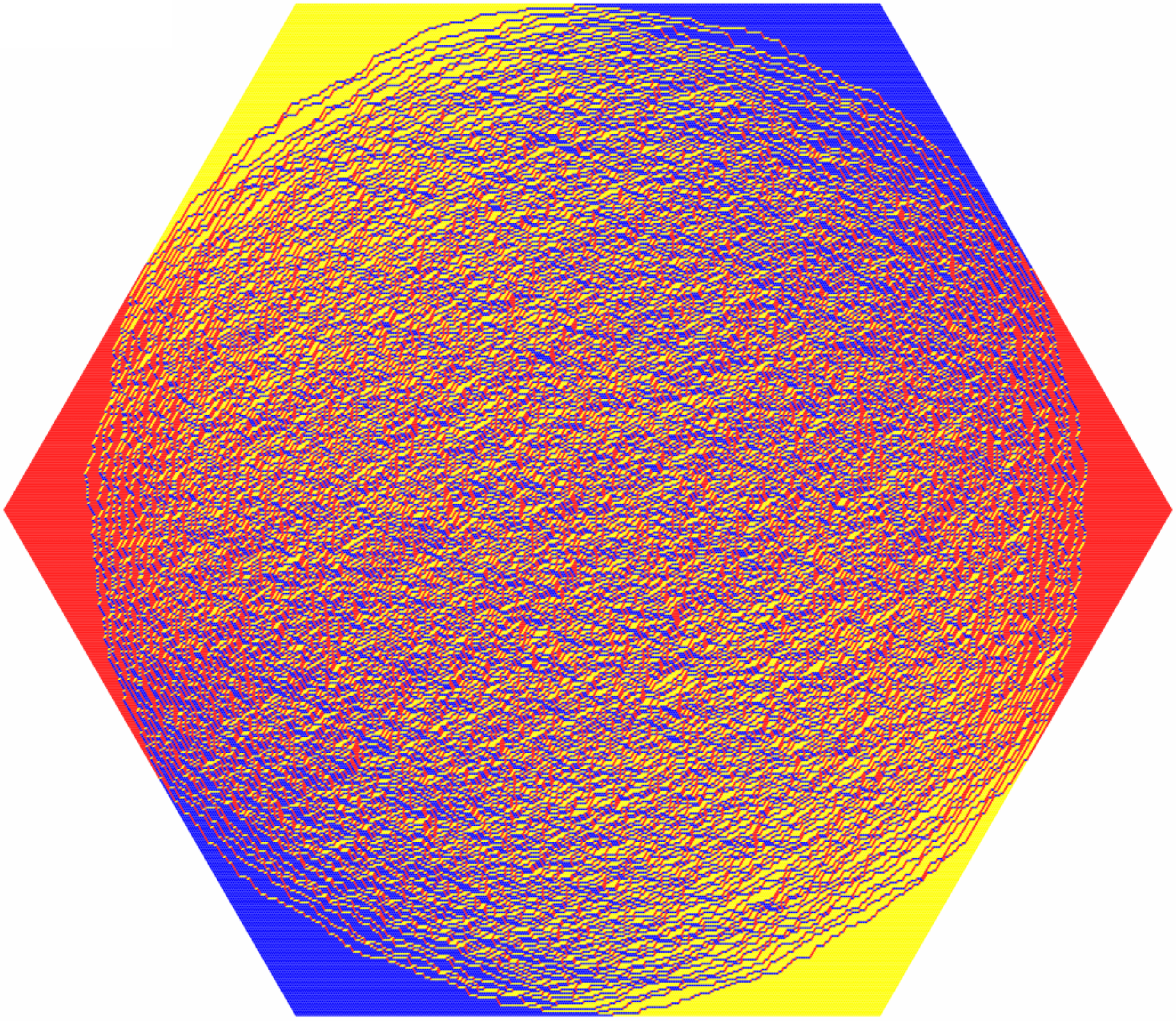
“Lozenge tilings”

(little “cubes”)

Why Mathematicians Care



Why Mathematicians Care

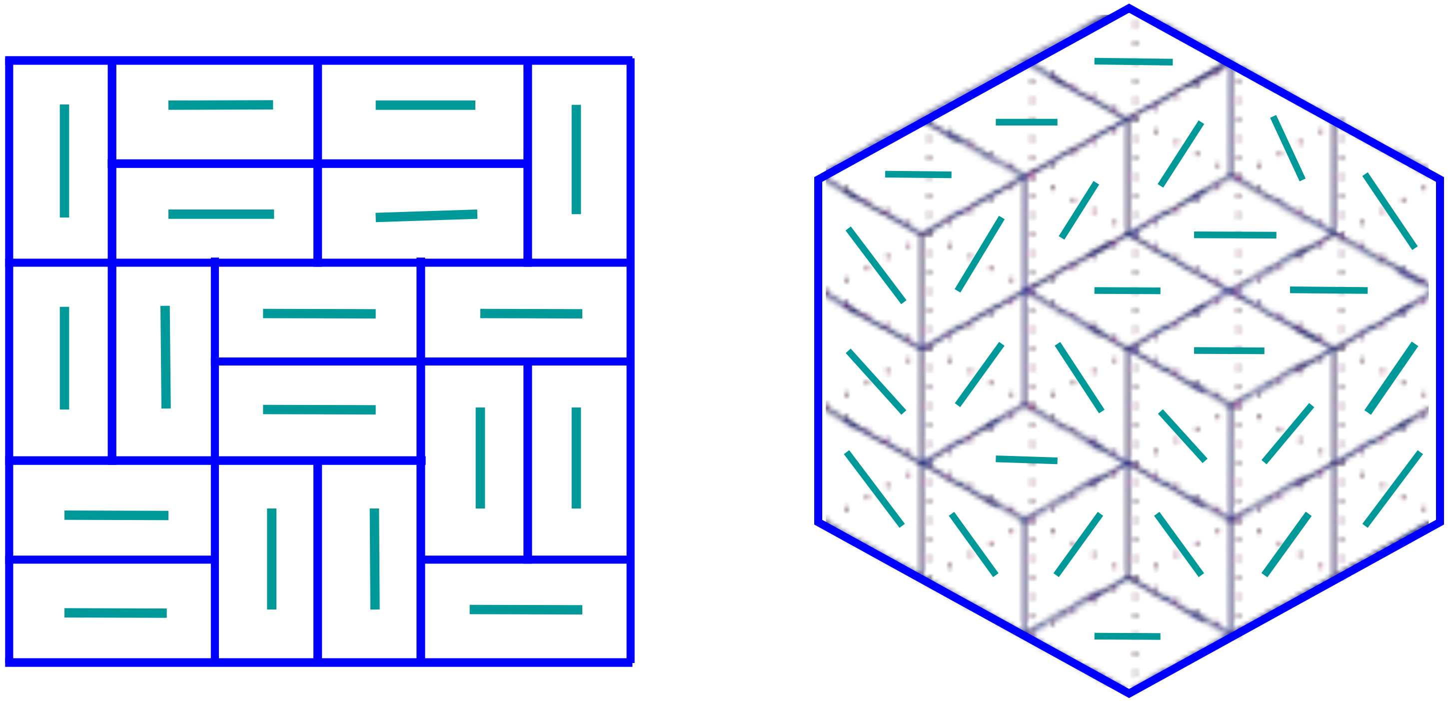


Why do we care?

- Mathematics: Discover patterns
- Chemistry, Biology: Estimate probabilities
- Physics: Count and calculate other functions

  to study a physical system
- Nanotechnology: Model growth processes

Why Physicists Care



"Dimer models": diatomic molecules adhering to the surface of a crystal.

The count ("partition function") determines:

specific heat, entropy, free energy, …

What does "nature" compute?

Domino Tilings

Where is a tiling? Do any even exist?

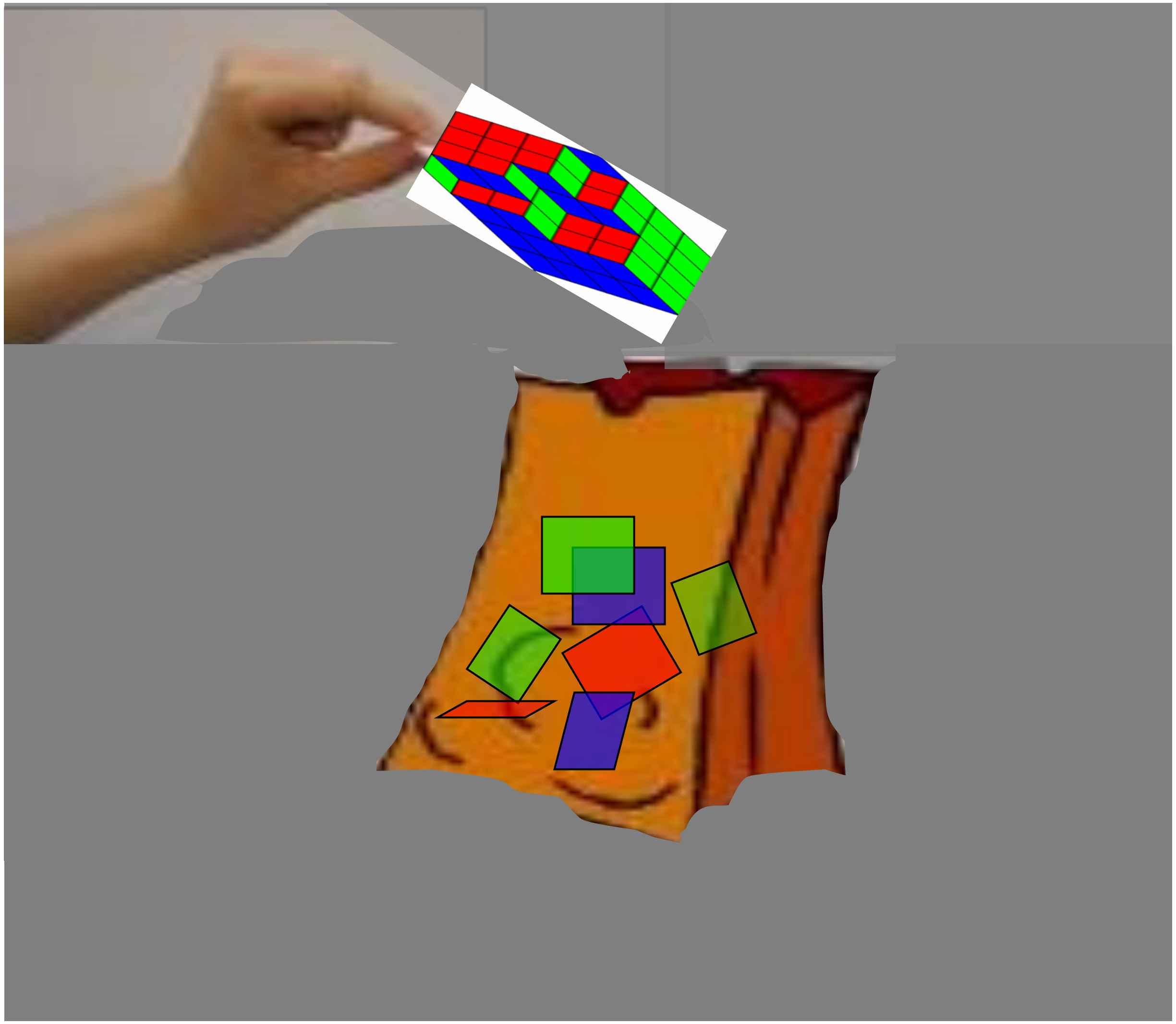
How many tilings are there?

What does a typical tiling look like?

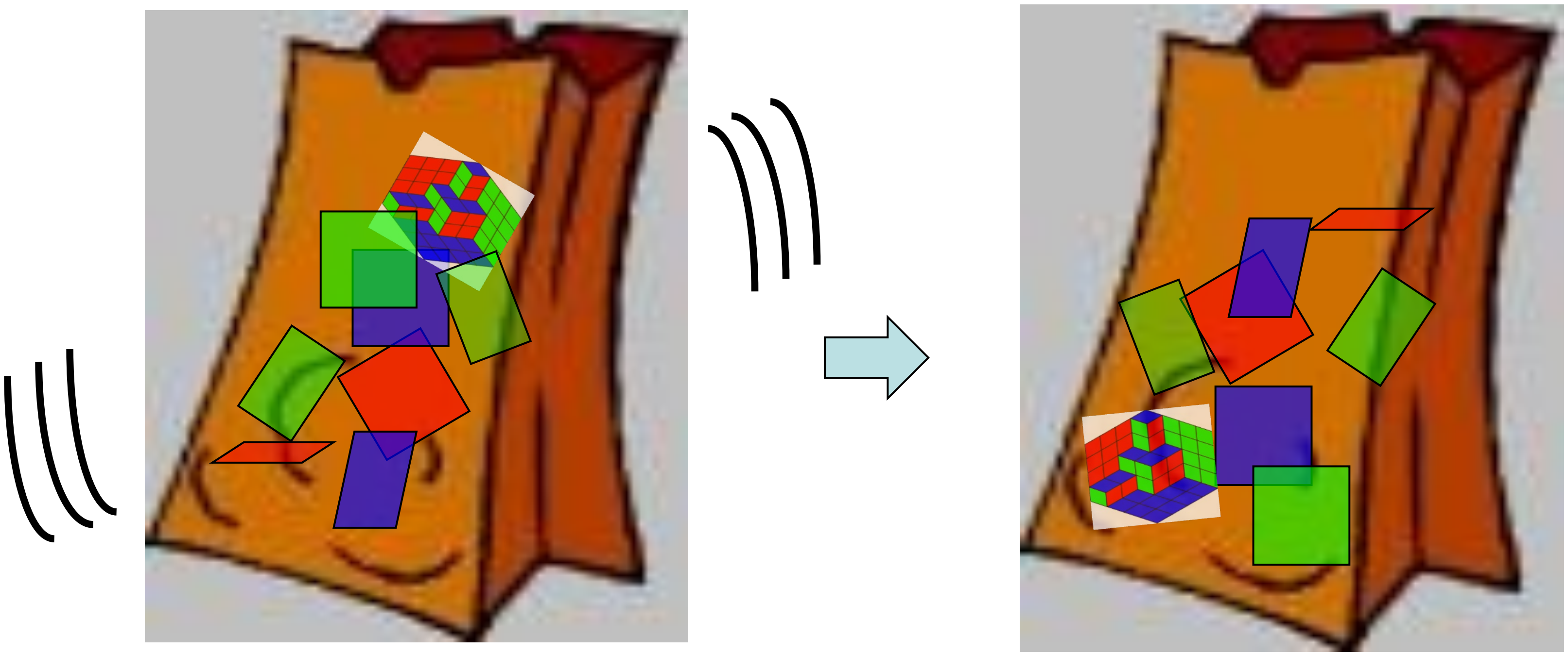
When do we stop our algorithms?

Why do we care?

What does a *typical* tiling look like?

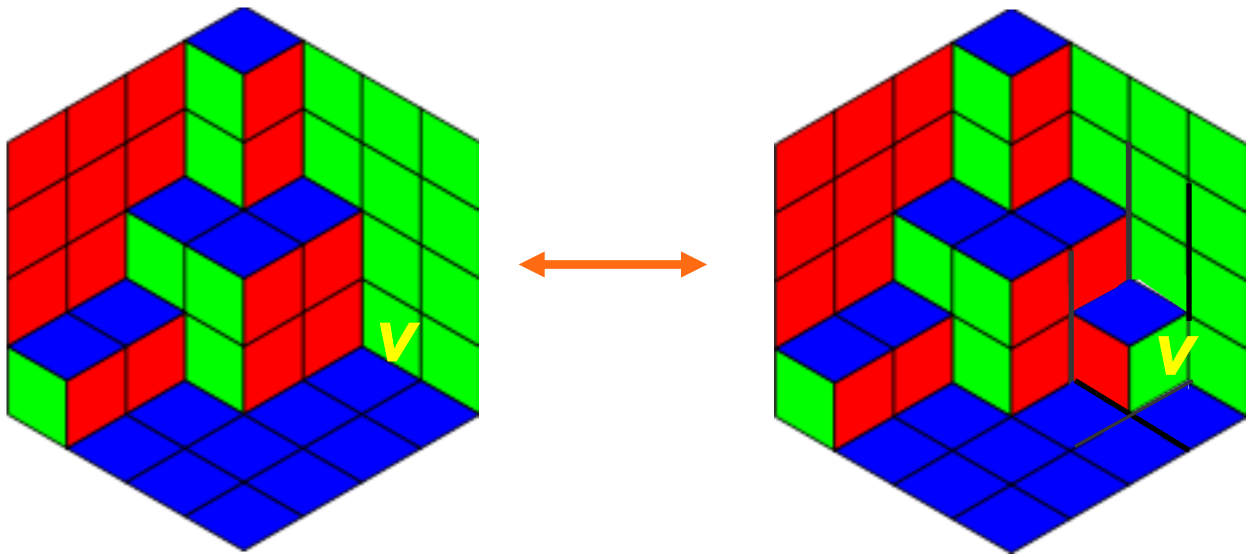


What does a *typical* tiling look like?



"Mix" them up!

Markov chain for Lozenge Tilings

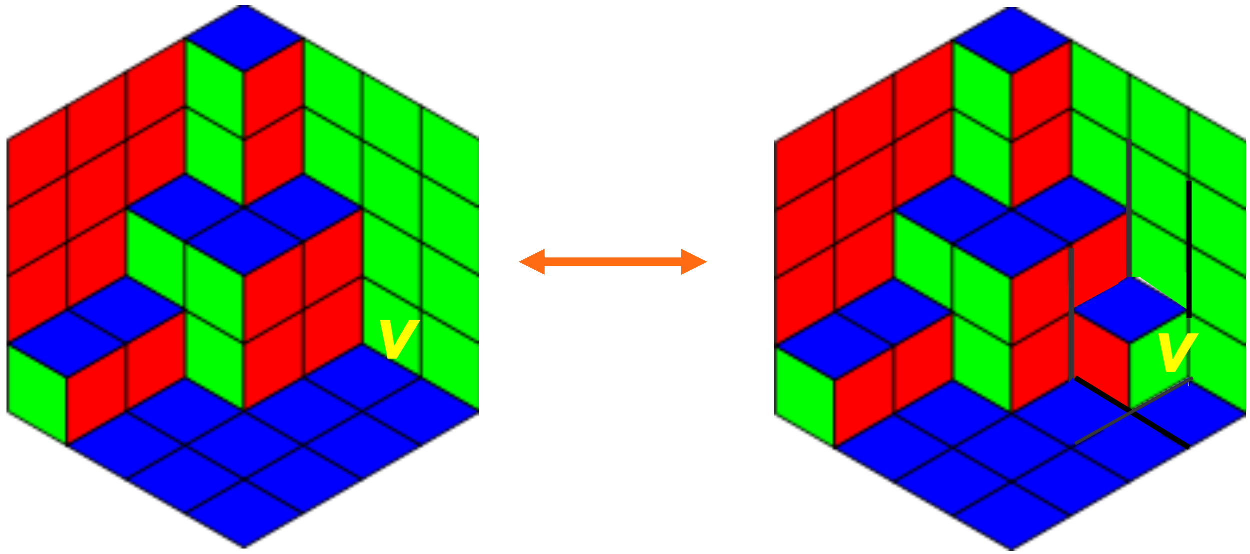


Repeat:

- Pick *v* in the lattice region;
- Add / remove the ``cube'' .

  at *v* w.p. ½, if possible.

Markov chain for Lozenge Tilings



1. The state space is connected.
2. If we do this long enough, each tiling will be equally likely.
3. *How long* is “long enough” ?

Domino Tilings

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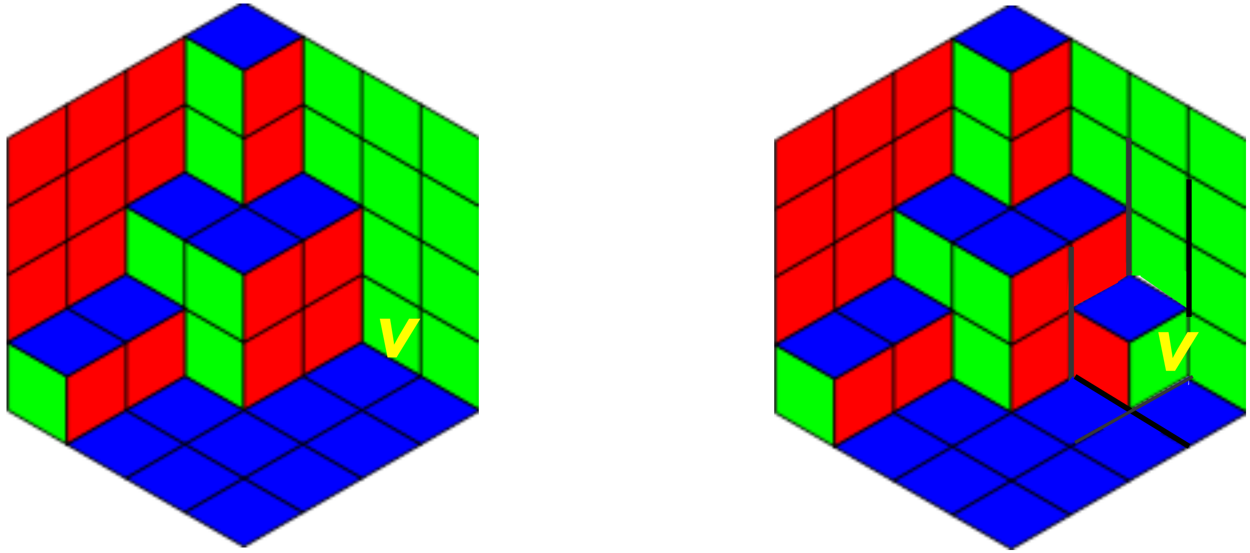
When do we stop our algorithms?



3. *How long* is "long enough" ?

Thm: The lozenge Markov chain is "rapidly mixing."

[Luby, R., Sinclair]

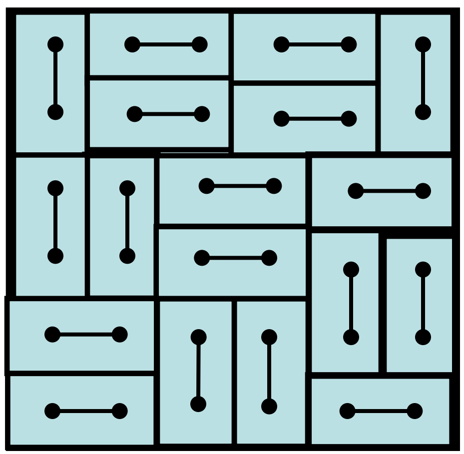
2^n, $3^n n^2$, $10^{\sqrt{n}}$, ..

(exponential)

n^2, $n \log n$, n^{10}, …

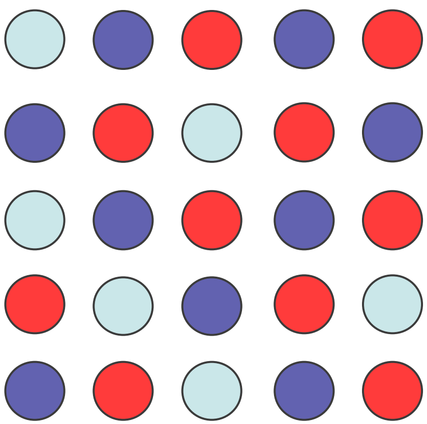
(polynomial)

What about other models?



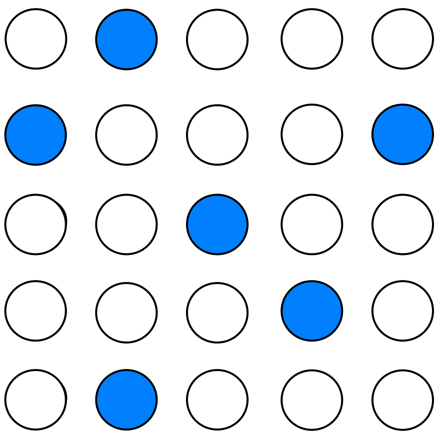
Dimer model

Domino tilings



Potts model

k-colorings

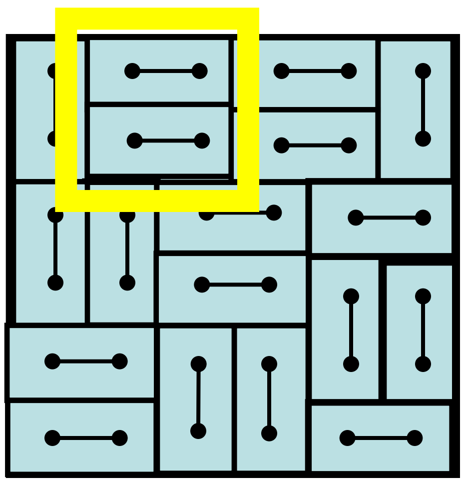


Hardcore model

Independent sets

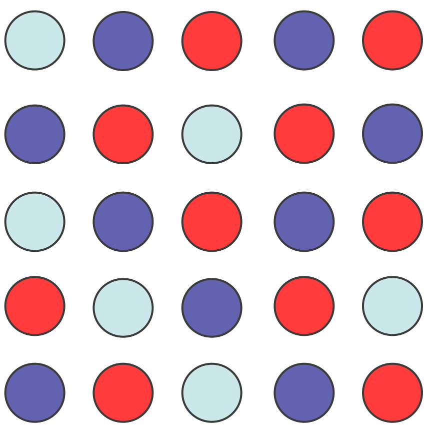
What about other models?

- Pick a 2 x 2 square;



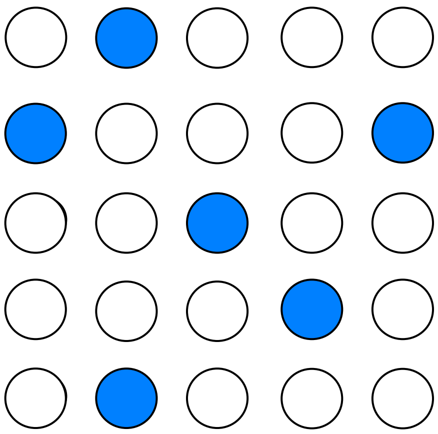
Dimer model

Domino tilings



Potts model

k-colorings

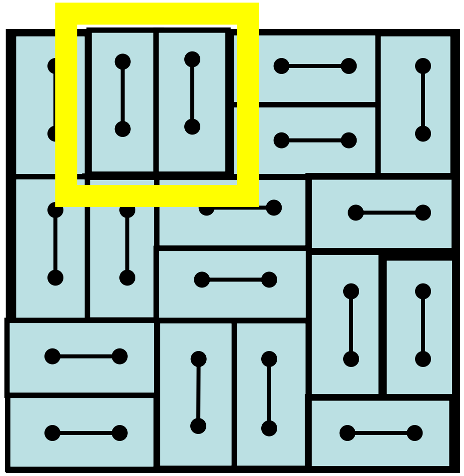


Hardcore model

Independent sets

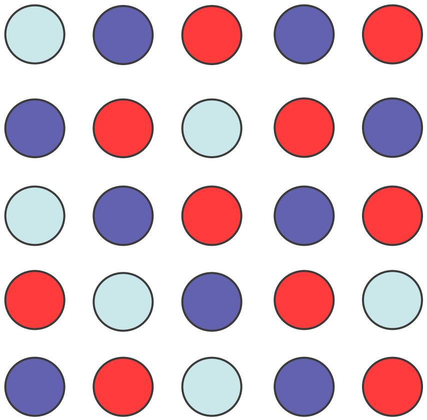
What about other models?

- Pick a 2 x 2 square;
- Rotate, if possible;



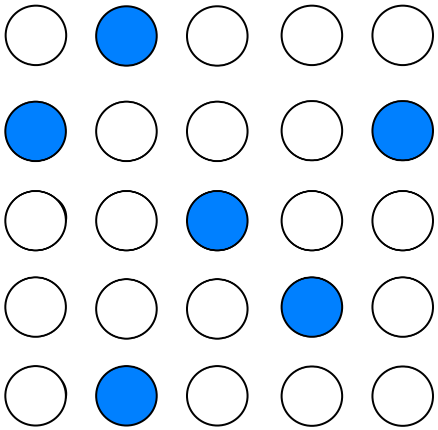
Dimer model

Domino tilings



Potts model

k-colorings

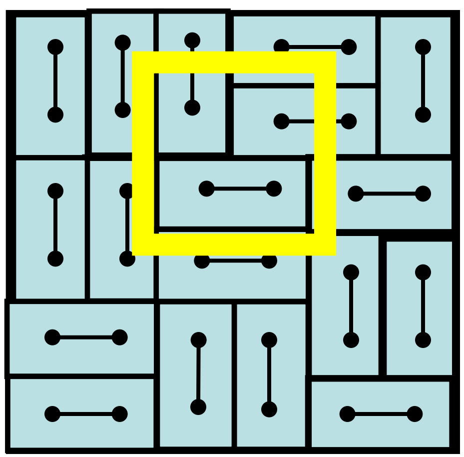
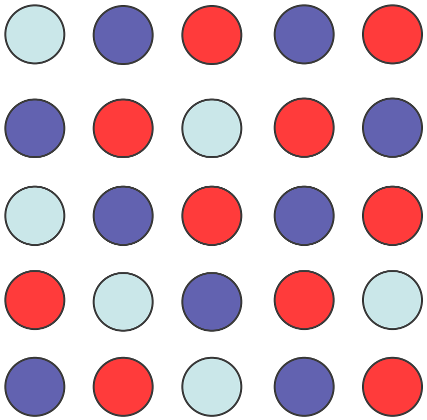
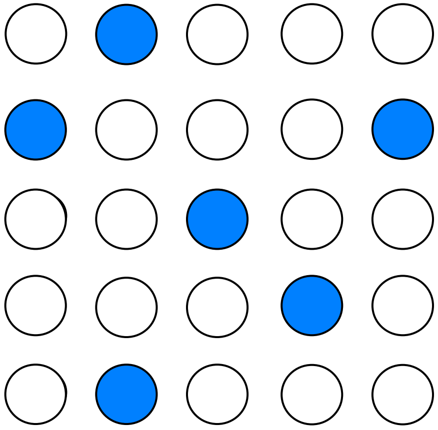


Hardcore model

Independent sets

What about other models?

- Pick a 2 x 2 square;
- Rotate, if possible;
- Otherwise do nothing.

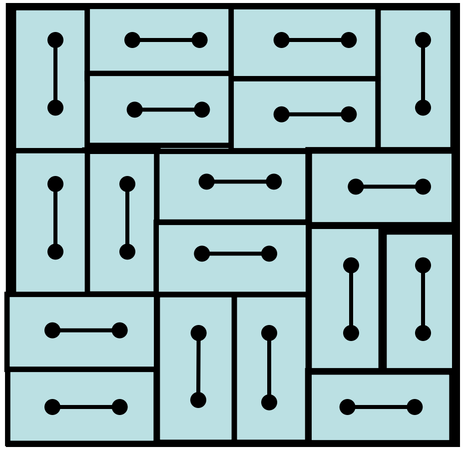




Dimer model | Potts model | Hardcore model

Domino tilings | k-colorings | Independent sets

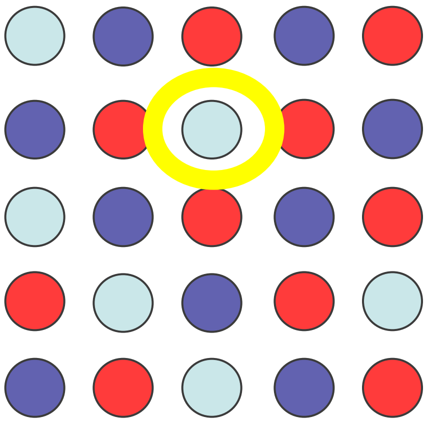
What about other models?

- Pick a vtx and a color;



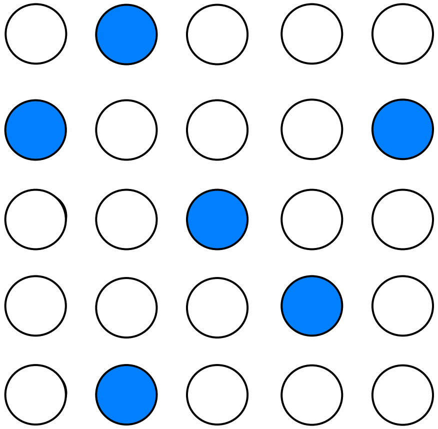
Dimer model

Domino tilings



Potts model

k-colorings

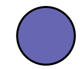


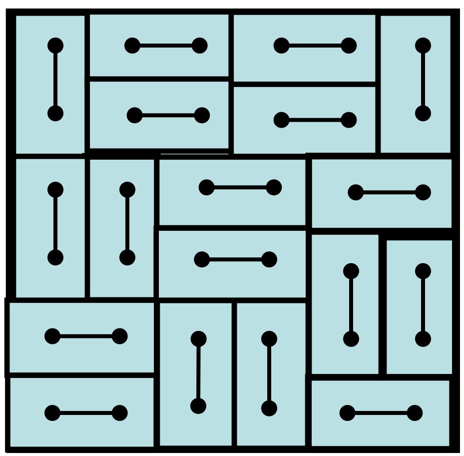
Hardcore model

Independent sets

What about other models?

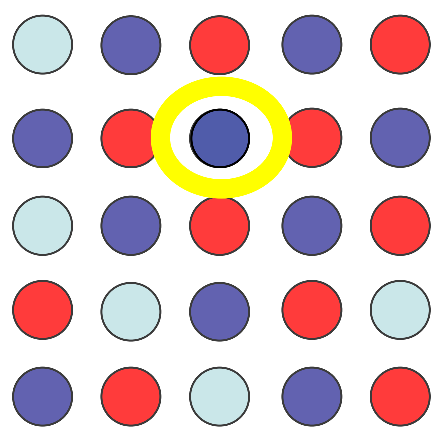
- Pick a vtx and a color;
- Recolor, if possible;





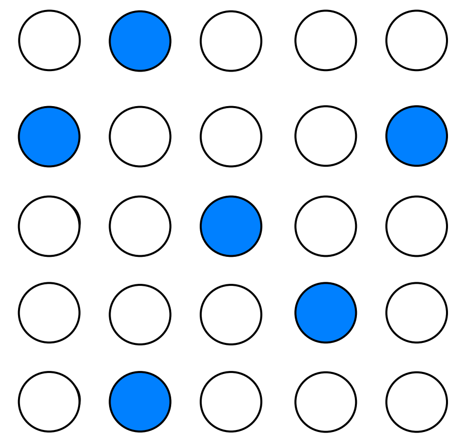
Dimer model

Domino tilings



Potts model

k-colorings

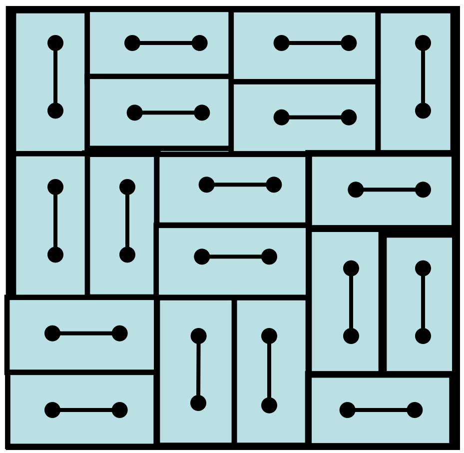


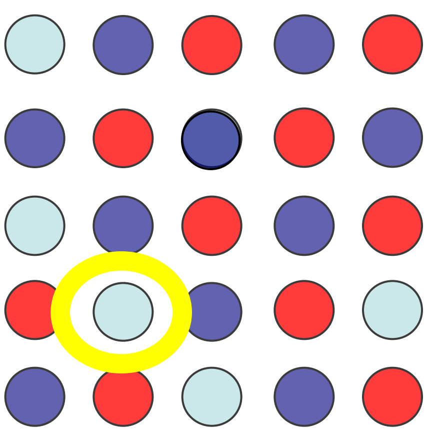
Hardcore model

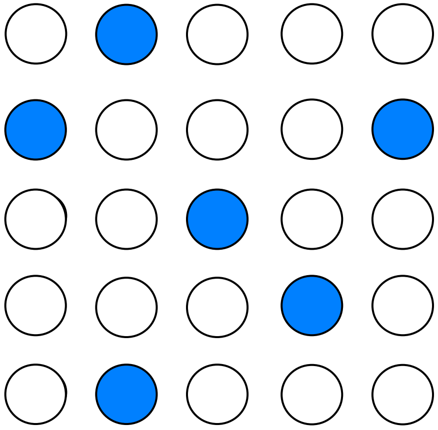
Independent sets

What about other models?

- Pick a vtx and a color;
- Recolor, if possible;
- Otherwise do nothing.







Dimer model

Domino tilings

Potts model

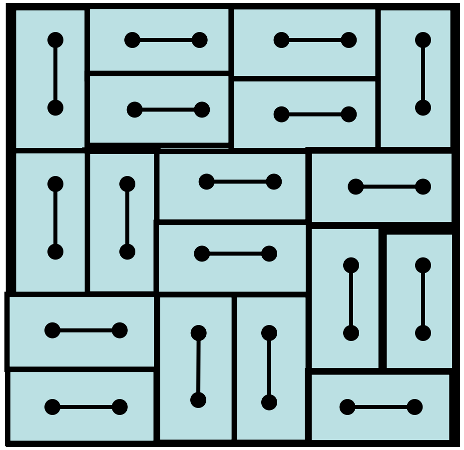
k-colorings

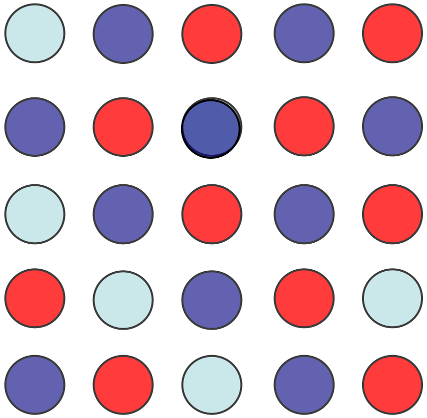
Hardcore model

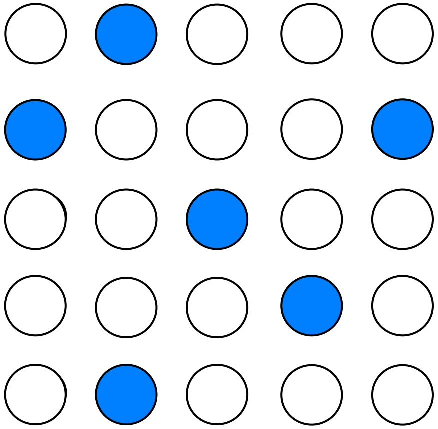
Independent sets

What about other models?

- Pick a vtx v and a bit b;







Dimer model

Domino tilings

Potts model

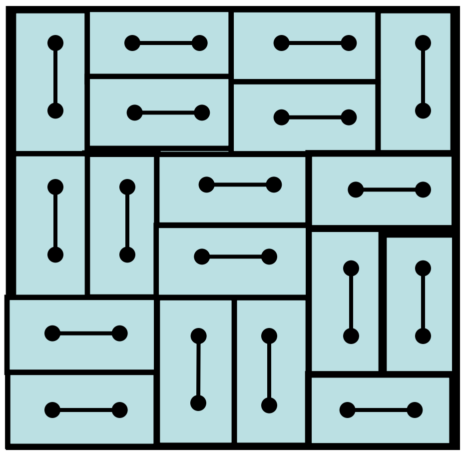
k-colorings

Hardcore model

Independent sets

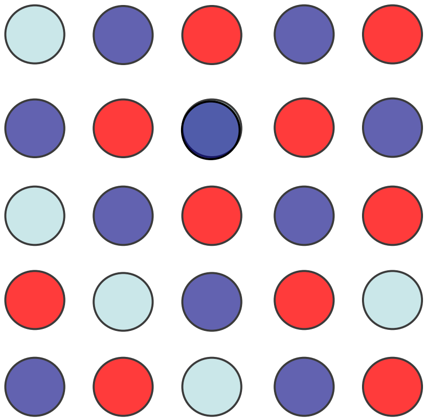
What about other models?

- Pick a vtx v and a bit b;
- If b=1, try to add v



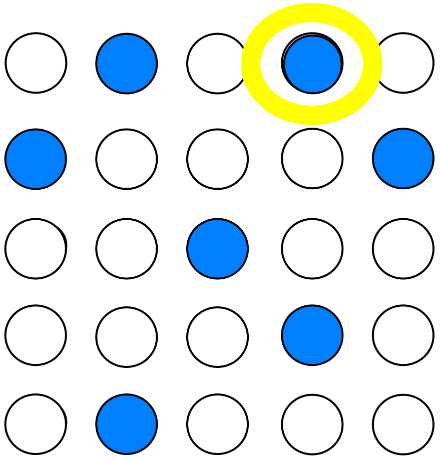
Dimer model

Domino tilings



Potts model

k-colorings

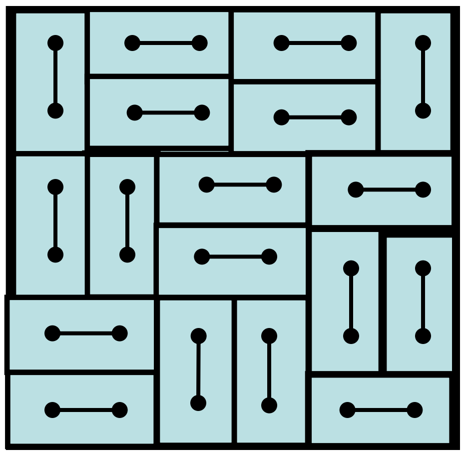


Hardcore model

Independent sets

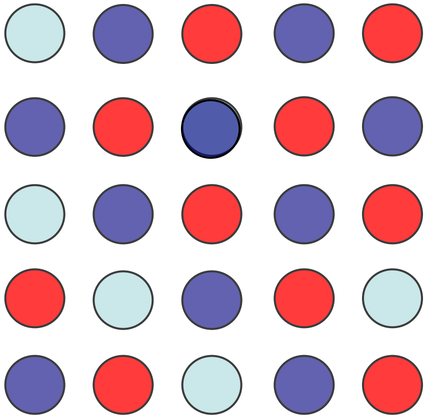
What about other models?

- Pick a vtx v and a bit b;
- If b=1, try to add v;
- If b=0, try to remove v;



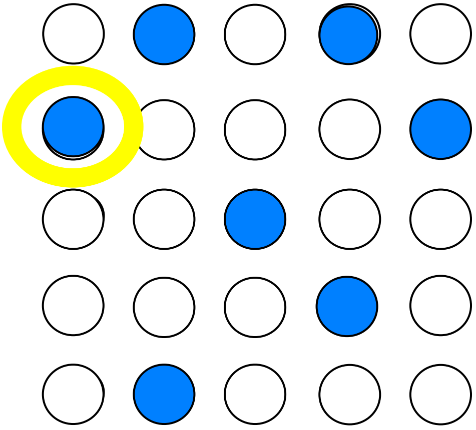
Dimer model

Domino tilings



Potts model

k-colorings

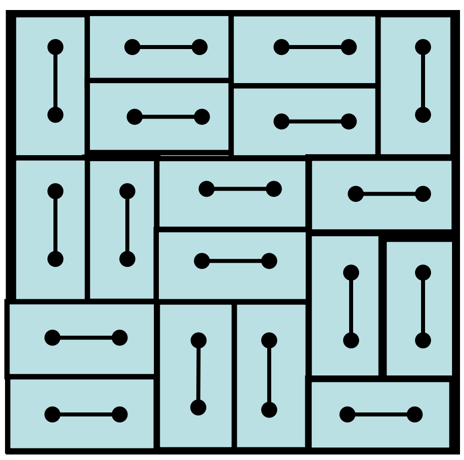


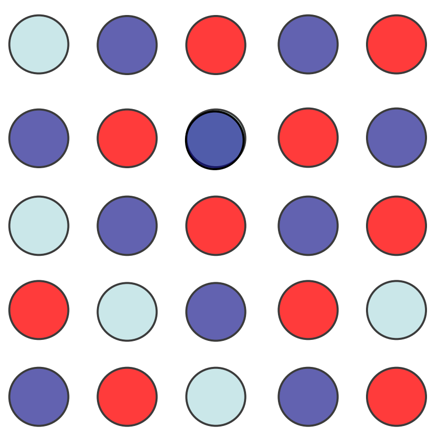
Hardcore model

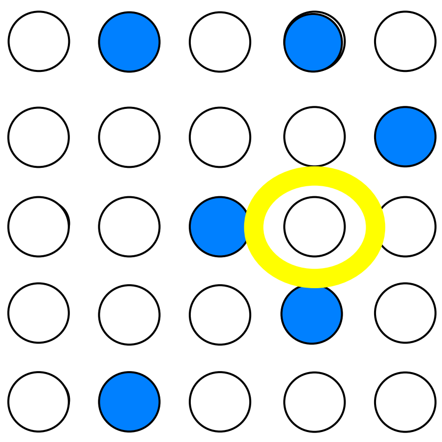
Independent sets

What about other models?

- Pick a vtx v and a bit b;
- If b=1, try to add v;
- If b=0, try to remove v;
- O.w. do nothing.







Dimer model

Domino tilings

Potts model

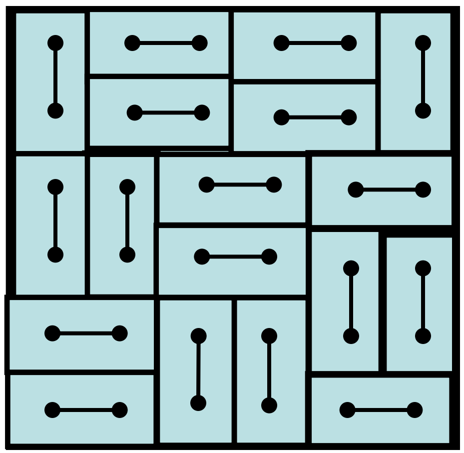
k-colorings

Hardcore model

Independent sets

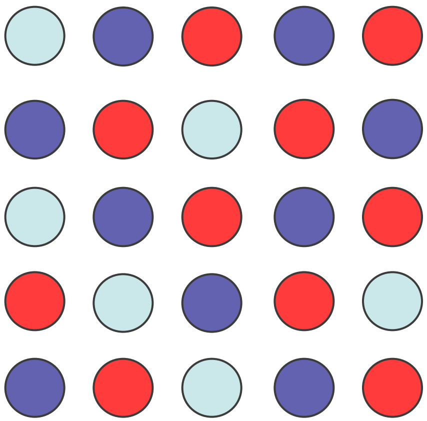
What about other models?

<u>Thm</u>: **All** of these chains are rapidly mixing.



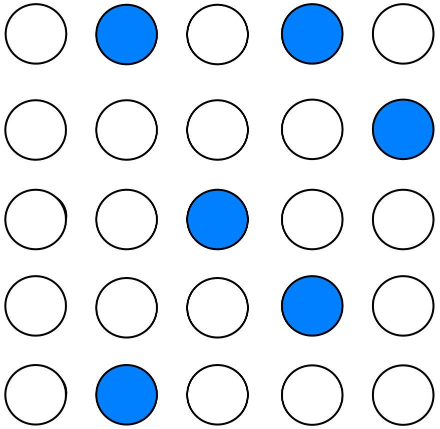
Dimer model

Domino tilings



Potts model

k-colorings



Hardcore model

Independent sets

HOWEVER . . .

Three-colorings:

The local chain is fast for 3-colorings in 2-d [LRS]

but slow for 3-colorings in sufficiently high dimension.

[Galvin, Kahn, R, Sorkin], [Galvin, R]

Independent Sets

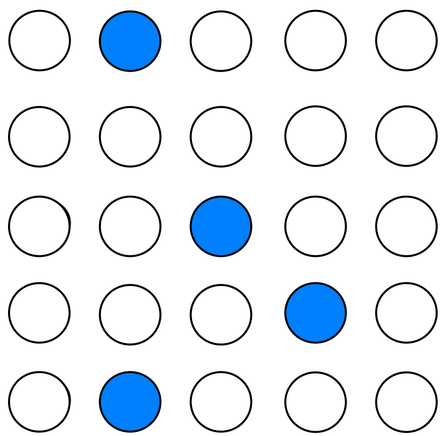
The local chain is fast for sparse Ind Sets in 2-d

[Luby, Vigoda],…, [Weitz]

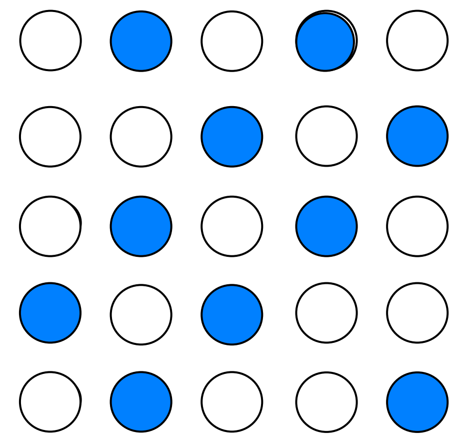
but slow for dense Ind Sets. [R.]

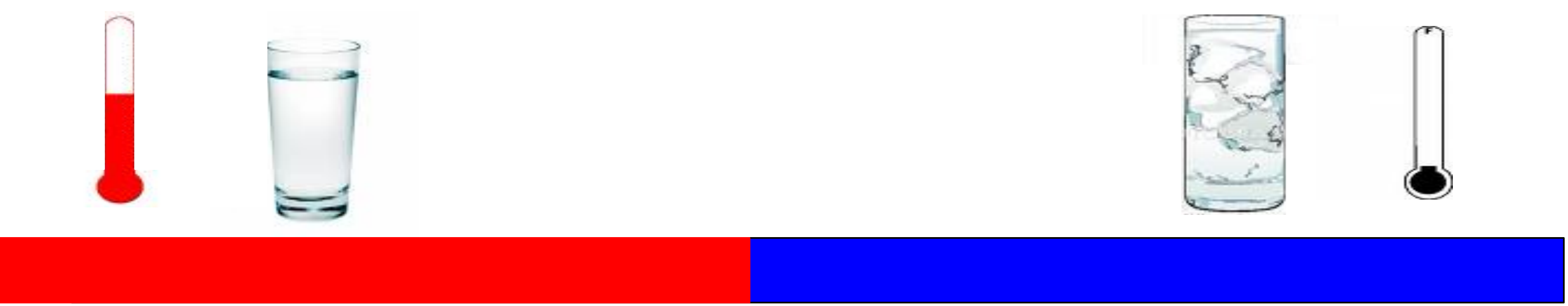
Weighted Independent Sets

Sparse



Dense



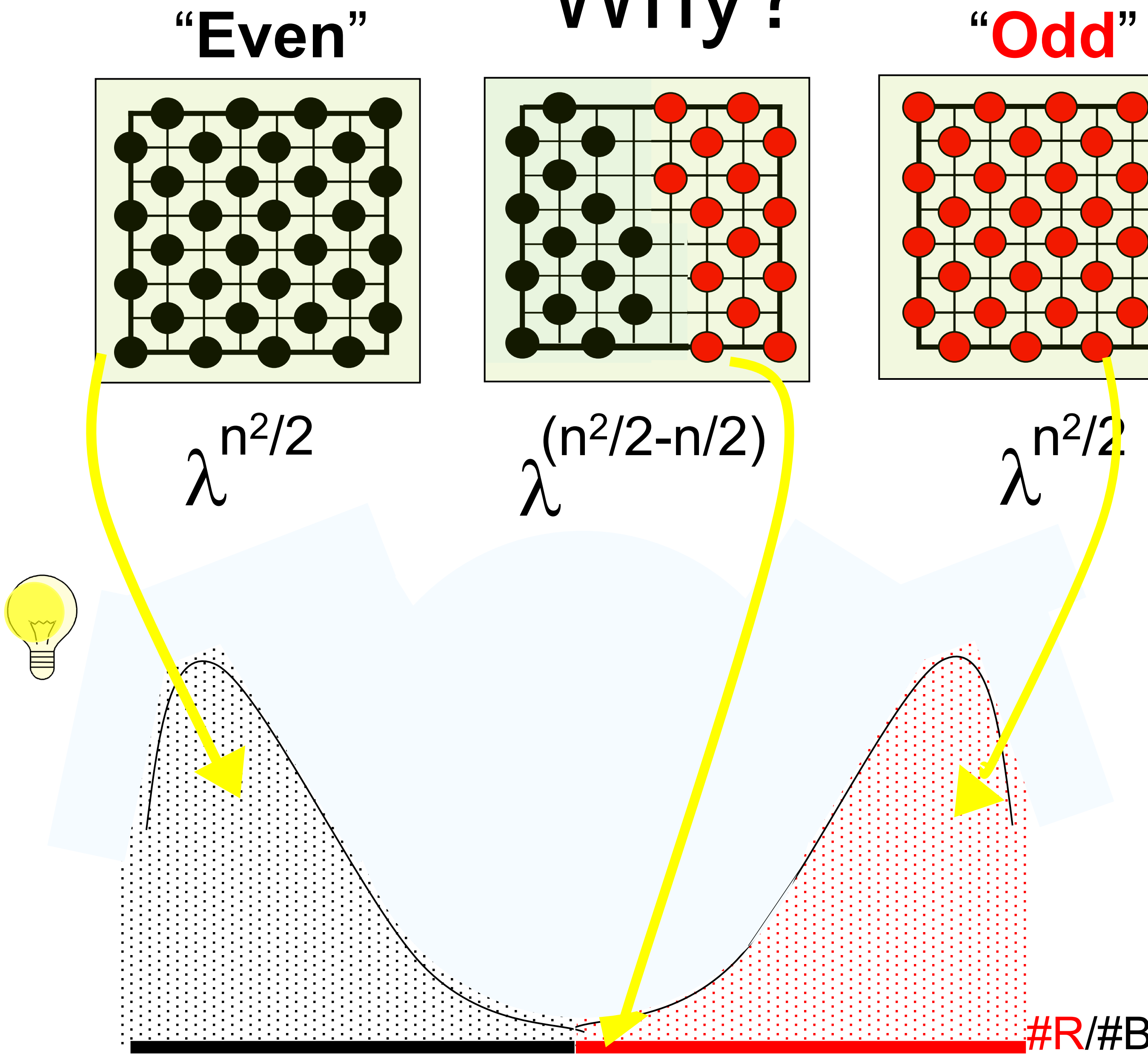


Fast

Phase Transition

Slow

Why?



Summary

Where

How

What

When

Why

Summary

Where — Algebra, Combinatorics, Algorithms

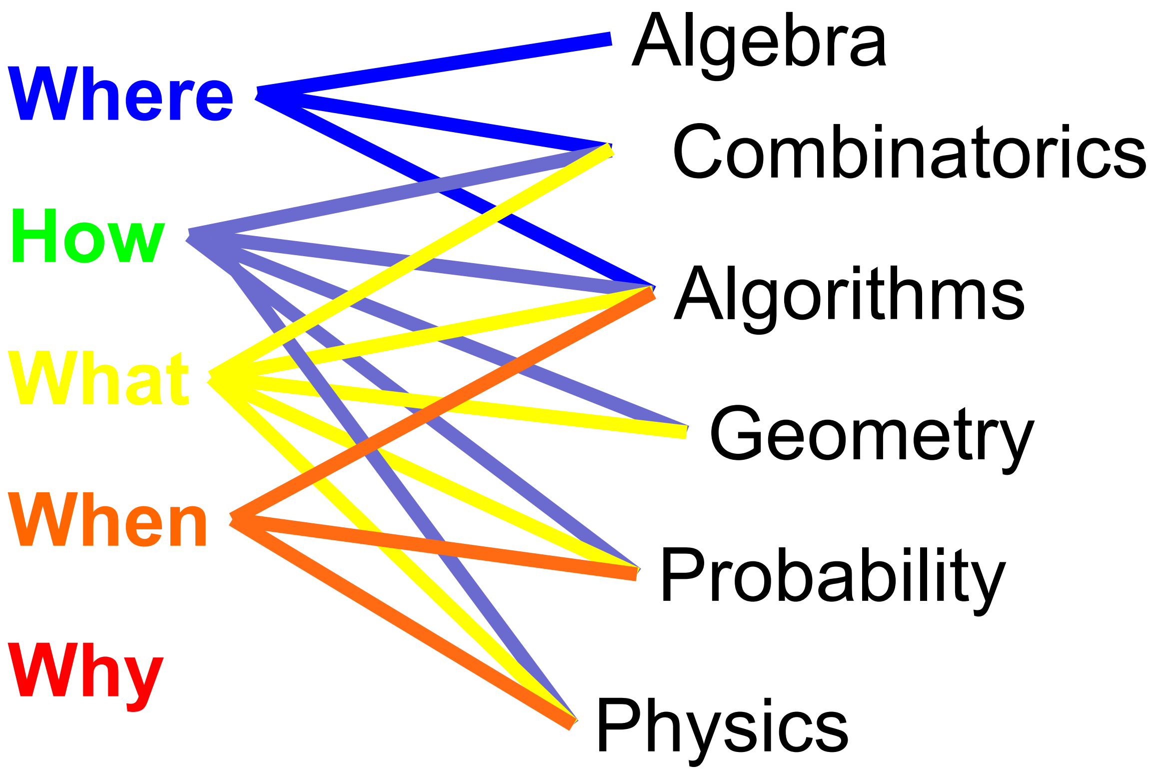
How

What

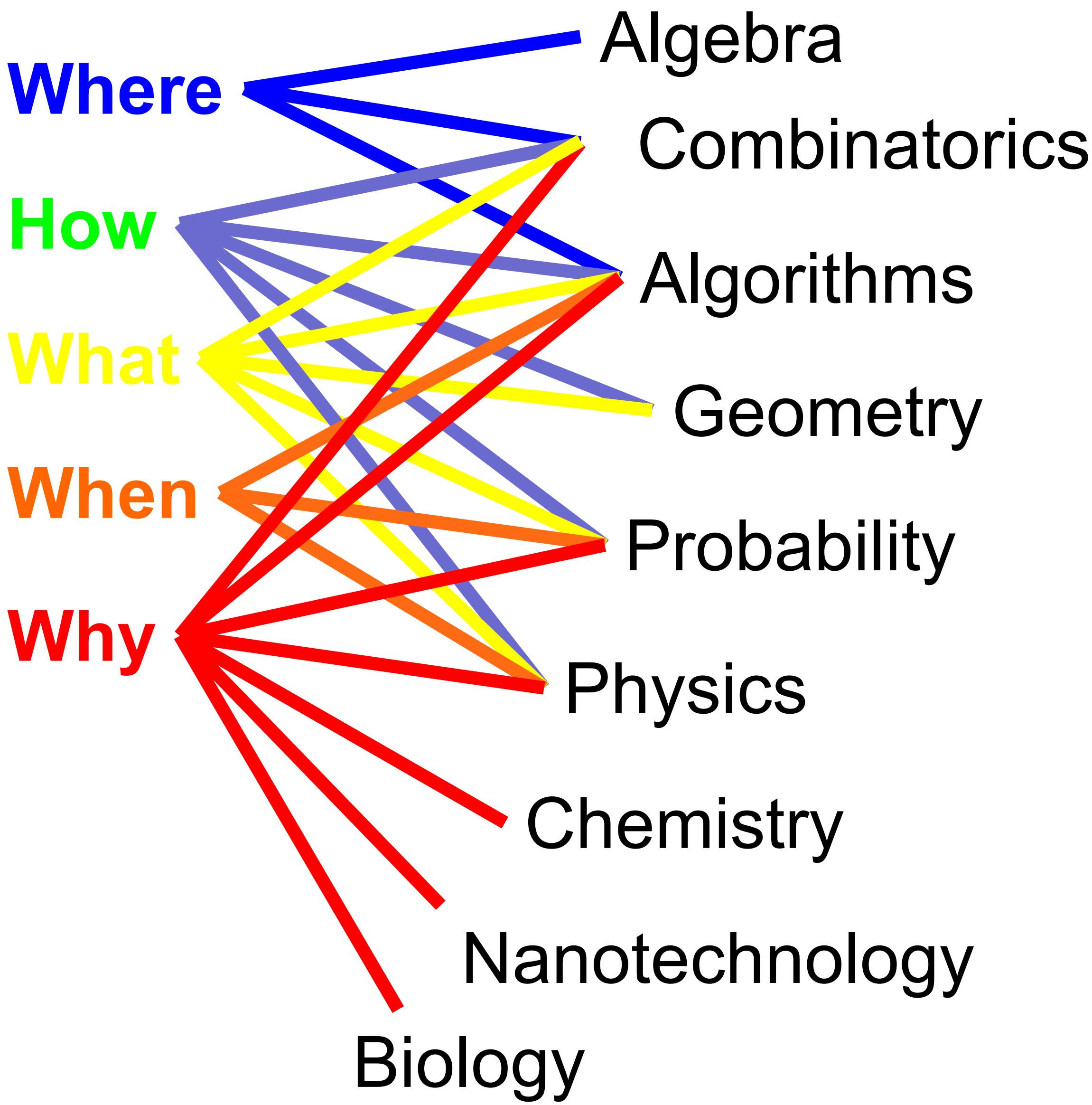
When

Why

Summary


Where
How
What
When
Why
Algebra
Combinatorics
Algorithms
Geometry
Probability
Physics

Summary


Where
How
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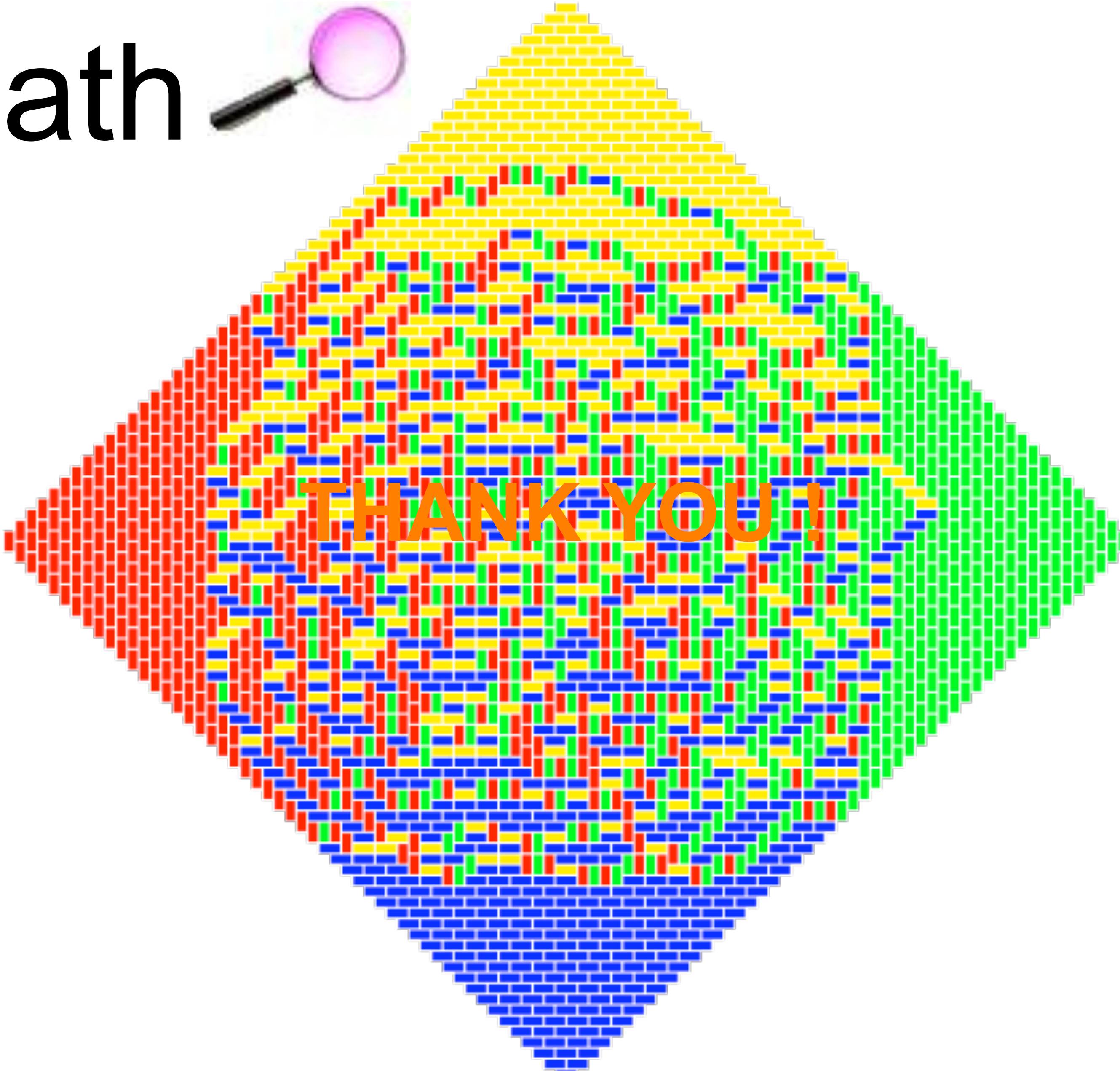
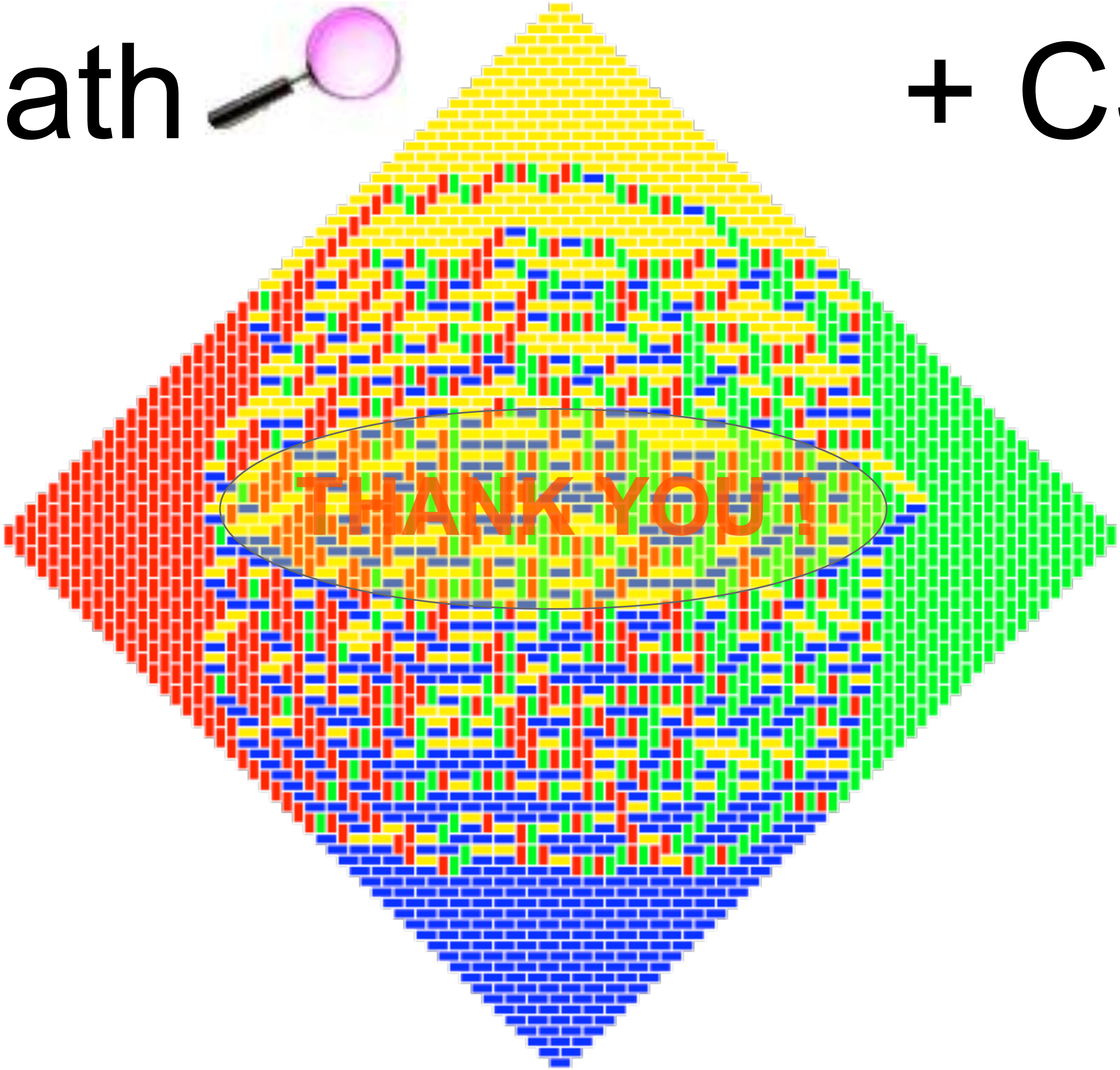
Algebra
Combinatorics
Algorithms
Geometry
Probability
Physics
Chemistry
Nanotechnology
Biology

Math

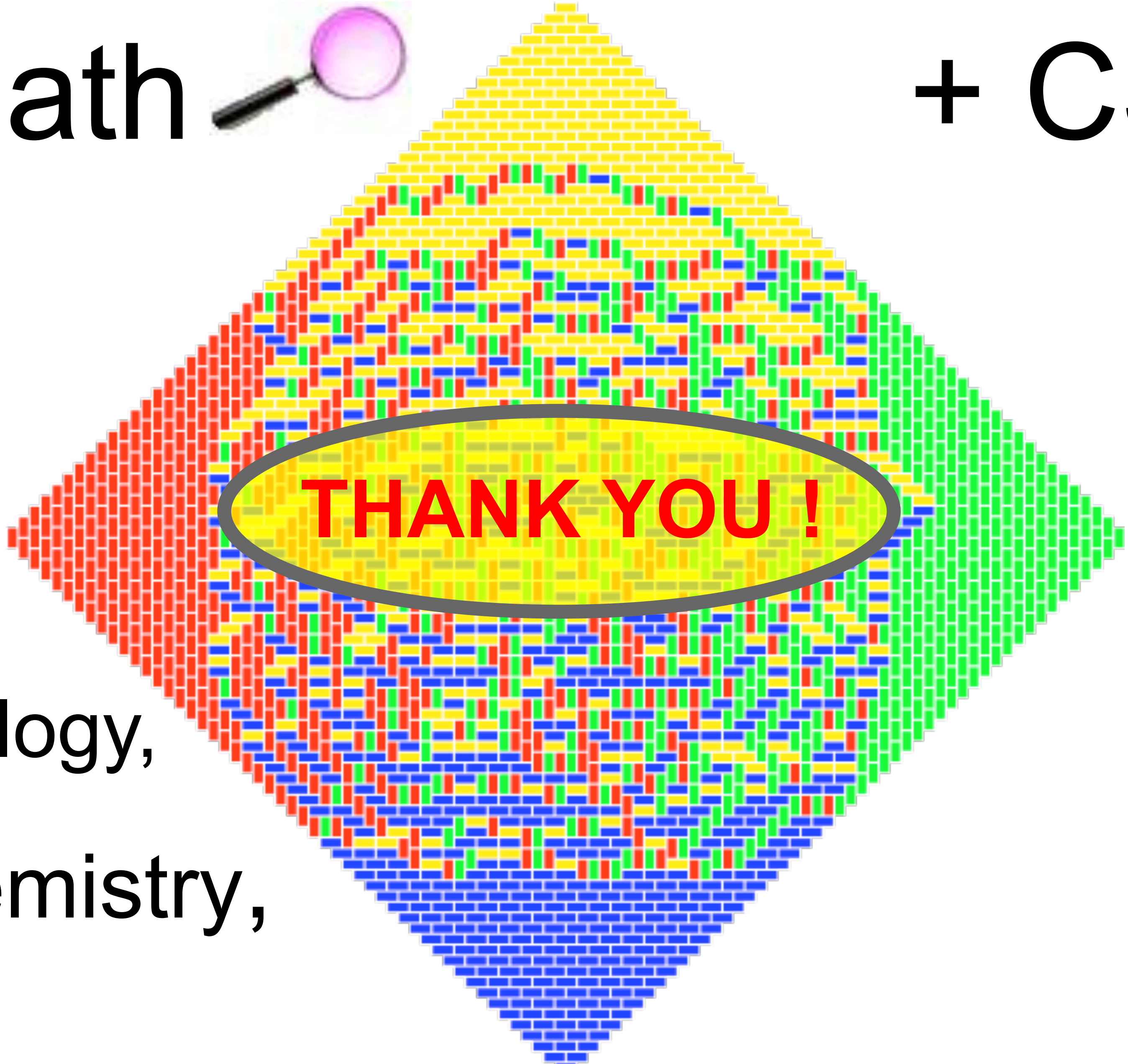


THANK YOU !

Math + CS



Math + CS

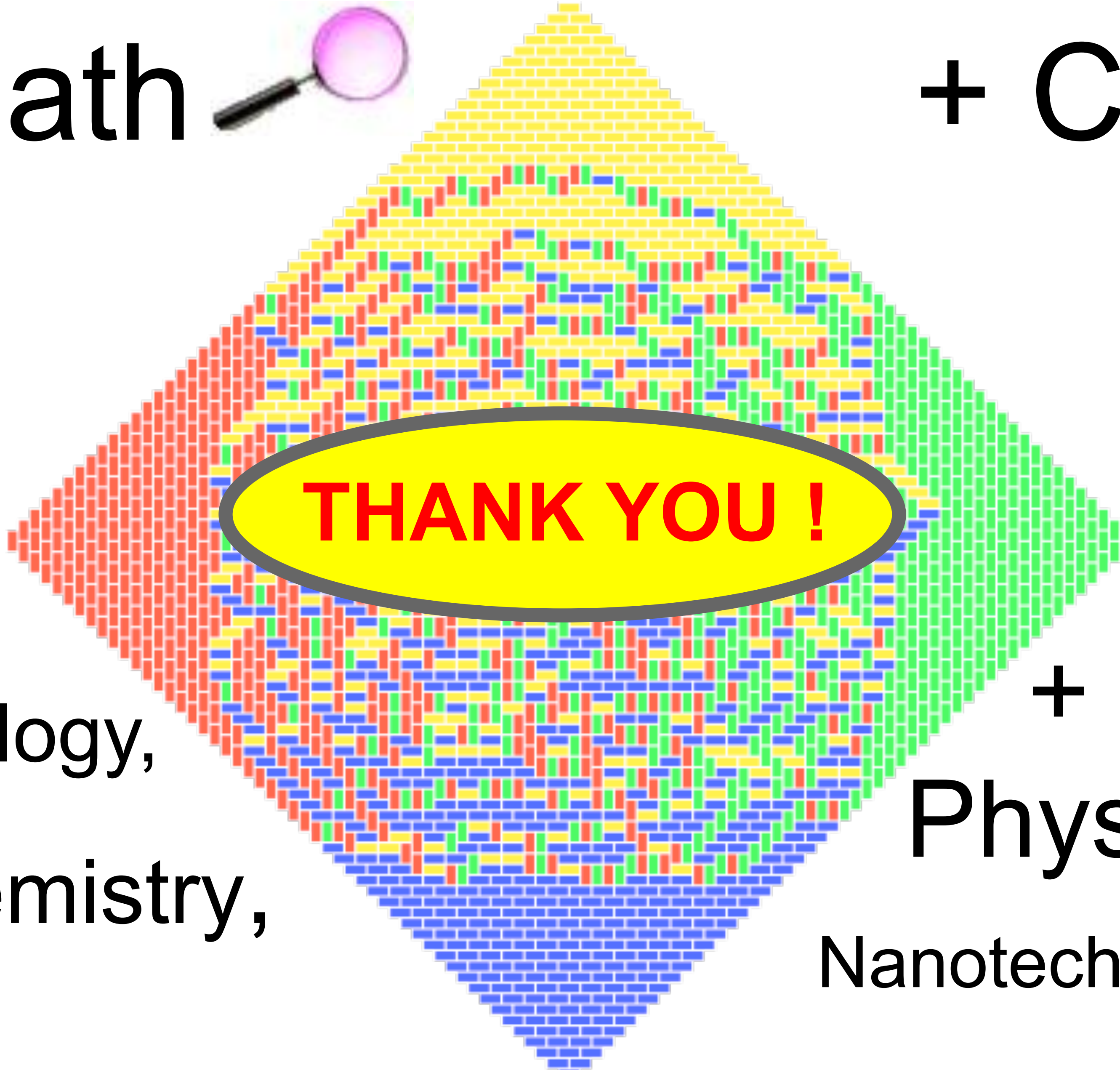


\+

Biology,

Chemistry,

Math + CS



THANK YOU !

+ Biology, Chemistry,

+ Physics, Nanotechnology,

. . .

THANK YOU !