Advanced Algorithms: Homework 9

Due on April 25, 2024 at 11:59pm EST

Professor Dana Randall Spring 2024

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course. EC problems are all-or-nothing. Furthermore, you must complete EC problems alone (i.e. without external resources or collaborators)

Exercise 1 (EC)

How many domino tilings are there on a 6 x 8 lattice region?

Exercise 2 (EC)

We are given λ and a graph $G = (V, E)$ and we would like to sample independent sets $\mathcal{I}$ on G. The weight of an independent set I is $\lambda^{|I|}/Z$ where $Z = \sum_{J \in \mathcal{I}} \lambda^{|J|}$ is the normalizing constant to make this into a probability distribution.

The Markov chain at each step picks a vertex $v \in V$ at random and a bit $b \in \{0, 1\}$ and tries to add v to the current independent set, if possible, and if $b = 0$ it tries to remove the vertex to move to a new independent set. It does these moves with transition probabilities given by:

$$P(I, I') = \begin{cases} \frac{1}{2n} \min\left(1, \lambda^{|I'|-|I|}\right), & \text{if } I \oplus I' = 1, \\ 1 - \sum_{J \sim I} P(I, J), & \text{if } I = I', \\ 0, & \text{otherwise,} \end{cases}$$

(where $I \oplus I'$ is the number of vertices that is one independent set but not the other). Convince yourself this chain will converge to the stationary distribution π. Give a coupling proof that shows that when λ is small enough, the chain is rapidly mixing. What is the best value of λ you can show this for?

Extra Practice

These problems are ungraded.

Exercise 3

All parts of this question pertain to a standard random walk on some graph.

(a) What is the hitting time h_{uv} for two adjacent vertices u, v on an n-vertex cycle?

(b) What is the hitting time h_{uv} for two (adjacent) vertices u, v in an n-vertex clique?

(c) Consider a random walk on an n-vertex clique starting at vertex 1. So $p^{(0)} = (1, 0, ..., 0)$. What is $p^{(1)}$ (the distribution after one step of the Markov chain) and how far is it in L_1 distance from the stationary distribution π? What is $p^{(2)}$ (the distribution after two steps of the Markov chain) and how far is it in L_1 distance from the stationary distribution π? (Note that this shows that the mixing time is much less than the hitting time for the clique)

Exercise 4

Say we have a Markov chain with states $1, \ldots, n$, and $P_{i,j}$ is the probability that you move into state j from state i. We put these in a matrix $\mathcal{P} = [P_{i,j}]$ in which the entry in the i^{th} row and j^{th} column is $P_{i,j}$.
For each state i, the probability that the Markov chain is in state i at time 0 is $\pi_0[i]$. We put these in a probability vector $\pi_0 = [\pi_0[1], \pi_0[2], \ldots, \pi_0[n]]$,

(a) Prove that after one time step, the distribution is $\pi_1 = \pi_0\mathcal{P}$. (Hint: Think conditional probability.)

(b) What is the distribution after n time steps?

(c) What equation does any stationary distribution π satisfy?

(d) Say there was a unique stationary distribution π, and the row sums of the matrix $\forall j \sum_i \mathcal{P}_{ij} = 1$. Such matrices are called *doubly stochastic*. For these, find the stationary distribution (in simple terms).

Exercise 5

Say the weather one day depends on the weather on the previous day (and nothing before that), and the weather is either sunny (S) or rainy (R). If it's sunny, the probability that it will be sunny tomorrow is .8, and if it's rainy, the probability that it will be rainy tomorrow is .6. We view this as a Markov chain.

(a) What is the transition matrix?

(b) If its sunny on monday, what is the probability that it is sunny on thursday?

(c) What is the stationary distribution? This can be thought of as the long run percentage of time that it is sunny or rainy.

(d) Say the weather depended on the previous two days instead. If it's sunny both today and yesterday, the probability that it's sunny today is .8, if its rainy today and yesterday, the probability that it's rainy today is .6, and otherwise there's a .5 chance that it's sunny today. How can we view this as a Markov chain, and what is the transition matrix?

Exercise 6

Say we place k balls on the vertices of a graph with n vertices so that each vertex has at most one ball. At each step, we pick a ball at random, and pick one of its neighbors at random. We then try to move the ball to its neighbor. If the neighbor is unoccupied, move the ball to this neighbor, otherwise do nothing. Assume the graph G is connected, and has at least k vertices.

(a) What is the state space Ω? What is $|\Omega|$?

(b) Show that this state space is connected by moves of this Markov chain.

(c) What is the stationary distribution, if we allowed multiple balls to occupy the same vertex? Give as simple an expression as you can, and express the distribution in terms of properties of the graph.

(Recall from class that when there is only one ball, this reduces to a simple random walk on the graph, which has stationary distribution $\pi[v] = \frac{deg(v)}{2|E|}$. This was proven to be the stationary distribution by showing that it satisfied the time reversibility equations.)

(d) Give an expression for the stationary distribution when we don't allow balls to occupy the same vertex.