CS 6515: Homework 8

Due on Sunday, November 19, 11:59pm via Gradescope.
Late submission with 10% penalty until Monday, November 20, 11:59pm.

Professor Brito

CS 6515

Suggested reading.

See `Markov Chains and mixing times` by Levin, Peres, and Wilmer, **chapters 1 and 2** for Introduction and basic properties; **chapter 3** for Metropolis and Glauber dynamics; **chapter 4** for mixing times; and **chapter 5** for coupling.

Problem 1

(Reversible random walks) The Metropolis chain was motivated by the desire of building a walk that converges to a given distribution π. In this problem we check this property.
Let J be a transition matrix with all entries positive, and let π be a probability distribution with all its entries positive.

(a) Let P be the transition matrix of a random walk and let μ be a probability satisfying

$$\mu(x)P(x,y) = \mu(y)P(y,x) \quad \text{for all } x, y \in \Omega.$$

Show that μ is stationary, this is, check that $\mu = \mu P$.

(b) Show that the Metropolis random walk is reversible with respect to π, the given probability distribution.

Problem 2

(Optimization over graphs) Let $G = (V, E)$ be a regular graph, and let $f : V \to \mathbb{R}$ be a real valued function on the set of vertices of G. Our goal is to find a vertex v for which $f(v) = f^*$ is maximal. We run the Metropolis chain with matrix J equal to the transition matrix of the random walk on G and probability π proportional to $\lambda^{f(v)}$ (note that, for a large graph, an exhaustive search is inefficient, as well as calculating the normalizing constant!).

(a) Compute the transition probabilities $P(v, w)$ for $v, w \in V$. Your answer may depend on λ.

(b) When $\lambda > 1$, the probability π puts more mass on the maximum values of f. In fact, taking large values of λ yields more guarantees of finding one such optimal argument. Confirm this statement by computing

$$\lim_{\lambda \to \infty} \pi(v).$$

Problem 3

(Random Walk on the hypercube) The hypercube of dimension n is the graph with vertex set $\{0,1\}^n$ and an edge connecting two vertices x and y if the ℓ_1 distance is exactly 1 (i.e.: x and y differ in exactly one coordinate). Consider the lazy random walk on the hypercube: at each step, we choose a random neighbor and move there with probability 1/2, and otherwise stay in our current state.

(a) Check that the lazy random walk in the hypercube is irreducible. Justify your answer.

(b) Check that the lazy random walk in the hypercube is aperiodic. Justify your answer.

(c) Find the stationary distribution. Justify your answer.

(d) Note that one can simulate the lazy random walk as a Glauber Dynamic: when at state x, we pick a random coordinate i and flip a fair coin, independently of our choice. If the coin lands heads we set $x_i = 1$, otherwise we set $x_i = 0$. Let τ be the random variable defined by the first time all coordinates were chosen at least once. Compute $\mathbb{E}(\tau)$.
Hint: we did this in class!

(ϟ) (*Extra credit*) Find the distribution of the random walk at time τ. Justify your answer.