Stable Menus of Public Goods

EC 2025

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Motivation

Decision makers often select "public goods" to provide to unit-demand agents:

- Which electives should an online school offer?
- Where should a vending machine company locate identical machines?
- When should I schedule my office hours?

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- View as matching problem: agents match to favorite available good.
- Each good needs minimum usage to justify existence.
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This talk:

- How to define stability for a matching in this setting?
- Existence of stable outcomes? Strategic considerations?

Related Work

Public projects (e.g. Papadimitriou + Schapira + Singer '08)

Committee selection (e.g. Aziz et al. '14, Jiang + Mungala + Wang '20)

Facility location (e.g. Hotelling '29, Procaccia + Tennenholtz '13)

Matching (e.g. Hatfield + Kojima '08, Dubins + Freedman '81, Roth '82)

Cooperative game theory / NTU games (e.g. Scarf '67)

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This work: no cardinal valuations

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Matching (e.g. Hatfield + Kojima '08, Dubins + Freedman '81, Roth '82)
This work: no capacity constraints, yes complementarities

Cooperative game theory / NTU games (e.g. Scarf '67)
This work: tighter bounds than those yielded by balancedness

Model

- n **agents**, denoted $N = \{1, \dots, n\}$.
- g **public goods**, denoted $G = \{1, \dots, g\}$.
- Each agent $i \in \{1, \dots, n\}$ has **complete preferences** $\succ_i$ over G.
- A **menu** $M \subseteq G$ induces a **matching**: agent i uses their favorite good in M

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Preliminary Defn. A menu $M \subseteq G$ of public goods is t-**stable** if:

- t-**feasibility**: each provided public good $\gamma \in M$ is used by $\geq t$ agents.
- t-**uncontestability**: there do not exist t "unhappy" agents, and an unprovided public good $\gamma \in G \setminus M$, such that each of these agents prefers γ over all provided public goods in M.

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- **Menu selection problem** = (agents, public goods, preferences).

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Example

$t = 4$

$n = 9$ agents

$g = 3$ goods

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No t-stable menu exists!

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When u/t sufficiently large, (t, u)-stable menus exist.

Overview

We are interested in questions of:

Existence

$\hookrightarrow$ For which t, u do (t, u)-stable menus exist for all menu selection problems?

Strategyproofness

$\hookrightarrow$ When existence guaranteed, for which g, t, u is there a SP mechanism?

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- $g = 2$: simple anonymous stable SP mechanism
- $g = 3, 4, 5, 6$: impossibility result (no anonymous stable SP mechanism)

Existence

Simple bounds

Lower bound

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Theorem. Let $g \geq 3$ and $u \leq 2t-2$. Then there exists a menu selection problem with no (t, u)-stable menu.

Upper bound

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Proof sketch. One can check these agents have no (t, u)-stable menu:

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Proposition. Let $g \geq 2$ and $u > g(t-1)$. Then for all menu selection problems, there exists a (t, u)-stable menu.

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Proposition. Let $g \geq 2$ and $u > g(t-1)$. Then for all menu selection problems, there exists a (t, u)-stable menu.

Proof sketch. Let $M := \{\gamma \in G : \exists t \text{ agents with favorite good } \gamma\}$.
One can check M is (t, u)-stable.

Tight characterization for $g \leq 6$

To guarantee existence of stable menus, simple bounds say:

$$\left.\begin{array}{l} \bullet\ \text{Necessary: } u \geq 2t-1. \\ \bullet\ \text{Sufficient: } u \geq g(t-1)+1. \end{array}\right\} \text{gap of factor of } \sim g$$

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This simple lower bound is tight for $g = 3, 4, 5, 6$:

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Proof sketch.

- $g = 3, 4$: analyze **greedy algorithm**. Analyzing cycle reveals stable menu.
- $g = 5, 6$: solve **computationally**. Using structural insights, reduce to polyhedra covering problem $\hookrightarrow$ 1 week on Harvard cluster using SMT solver.

Sidenote: reduction to polyhedra covering problem

- Encode menu selection problem as

$$x \in \mathbb{R}^{g!}.$$

Example

Menu selection problem:

- $2 \times 1 \succ 2 \succ 3$
- $3 \times 2 \succ 1 \succ 3$

$\hookrightarrow$ gives vector $x = (2, 0, 0, 3, 0, 0)$.

Sidenote: reduction to polyhedra covering problem

- Encode menu selection problem as

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- Construct polyhedron $P_M^{t,u,g}$ s.t.

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How to test if $\{1,2\}$ is (t,u)-stable?

t-**feasibility**:

- $\langle \underbrace{(1,1,0,0,1,0)}_{\text{types using 1}}, x\rangle \geq t$
- $\langle \underbrace{(0,0,1,1,0,1)}_{\text{types using 2}}, x\rangle \geq t$

u-**defendability**:

- $\langle \underbrace{(0,0,0,0,1,1)}_{\text{types demanding 3}}, x\rangle < u.$

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$$M \text{ is } (t,u)\text{-stable} \Leftrightarrow x \in P_M^{t,u,g}.$$

- There exists a stable menu M for a menu selection problem x if and only if

$$x \in \bigcup_{M \subseteq G} P_M^{t,u,g}.$$

- *All* menu selection problems have stable menus if and only if

$$\mathbb{Z}_{\geq 0}^{g!} \subseteq \bigcup_{M \subseteq G} P_M^{t,u,g}.$$

Example

Menu selection problem:

- $2 \times 1 \succ 2 \succ 3$
- $3 \times 2 \succ 1 \succ 3$

$\hookrightarrow$ gives vector $x = (2, 0, 0, 3, 0, 0)$.

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}}_{A} x \geq \underbrace{\begin{pmatrix} t \\ t \\ -u+1 \end{pmatrix}}_{b}$$

$P_{\{1,2\}}^{t,u,g} := \{v : \ Av \geq b\}$ encodes for which menu selection problems $\{1,2\}$ is stable.

Beyond $g \geq 7$

Theorem. Let $g \geq 7$ and $u \leq 23\lfloor \frac{t-1}{11} \rfloor$. Then there exists a menu selection problem with no (t, u)-stable menu. (cf. $u \geq 2t-1 \Leftrightarrow$ existence when $g \leq 6$)

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Set $x := \lfloor \frac{t-1}{11} \rfloor$. Then the following $70x$ agents have no (t,u)-stable menu:

#	Preference	#	Preference	#	Preference	#	Preference
$5x$	$1 \succ 2 \succ 3$	$3x$	$1 \succ 2 \succ 4 \succ 5$	x	$1 \succ 4 \succ 2 \succ 5$	x	$1 \succ 6 \succ 4 \succ 2$
$5x$	$2 \succ 3 \succ 4$	$3x$	$2 \succ 3 \succ 5 \succ 6$	x	$2 \succ 5 \succ 3 \succ 6$	x	$2 \succ 7 \succ 5 \succ 3$
$5x$	$3 \succ 4 \succ 5$	$3x$	$3 \succ 4 \succ 6 \succ 7$	x	$3 \succ 6 \succ 4 \succ 7$	x	$3 \succ 1 \succ 6 \succ 4$
$5x$	$4 \succ 5 \succ 6$	$3x$	$4 \succ 5 \succ 7 \succ 1$	x	$4 \succ 7 \succ 5 \succ 1$	x	$4 \succ 2 \succ 7 \succ 5$
$5x$	$5 \succ 6 \succ 7$	$3x$	$5 \succ 6 \succ 1 \succ 2$	x	$5 \succ 1 \succ 6 \succ 2$	x	$5 \succ 3 \succ 1 \succ 6$
$5x$	$6 \succ 7 \succ 1$	$3x$	$6 \succ 7 \succ 2 \succ 3$	x	$6 \succ 2 \succ 7 \succ 3$	x	$6 \succ 4 \succ 2 \succ 7$
$5x$	$7 \succ 1 \succ 2$	$3x$	$7 \succ 1 \succ 3 \succ 4$	x	$7 \succ 3 \succ 1 \succ 4$	x	$7 \succ 5 \succ 3 \succ 1$

- Simplified and cleaned from counterexample found by SMT solver.

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$5x$	$3 \succ 4 \succ 5$	$3x$	$3 \succ 4 \succ 6 \succ 7$	x	$3 \succ 6 \succ 4 \succ 7$	x	$3 \succ 1 \succ 6 \succ 4$
$5x$	$4 \succ 5 \succ 6$	$3x$	$4 \succ 5 \succ 7 \succ 1$	x	$4 \succ 7 \succ 5 \succ 1$	x	$4 \succ 2 \succ 7 \succ 5$
$5x$	$5 \succ 6 \succ 7$	$3x$	$5 \succ 6 \succ 1 \succ 2$	x	$5 \succ 1 \succ 6 \succ 2$	x	$5 \succ 3 \succ 1 \succ 6$
$5x$	$6 \succ 7 \succ 1$	$3x$	$6 \succ 7 \succ 2 \succ 3$	x	$6 \succ 2 \succ 7 \succ 3$	x	$6 \succ 4 \succ 2 \succ 7$
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- Also have somewhat improved upper bound: $u \geq (g-2)(t-1)+1$.

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$5x$	$4 \succ 5 \succ 6$	$3x$	$4 \succ 5 \succ 7 \succ 1$	x	$4 \succ 7 \succ 5 \succ 1$	x	$4 \succ 2 \succ 7 \succ 5$
$5x$	$5 \succ 6 \succ 7$	$3x$	$5 \succ 6 \succ 1 \succ 2$	x	$5 \succ 1 \succ 6 \succ 2$	x	$5 \succ 3 \succ 1 \succ 6$
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- Also have somewhat improved upper bound: $u \geq (g-2)(t-1)+1$.
- When $g \geq 7$, existence question open for $\underbrace{23\left\lfloor \frac{t-1}{11} \right\rfloor < u \leq (g-2)(t-1)}_{\sim g \text{ gap}}$.

Improved Upper Bound

Theorem. Let $g \geq 7$ and $u \geq (g-2)(t-1)+1$. Then every menu selection problem has a (t,u)-stable menu.

Proof relies on lemma:

Lemma. Fix $g \geq 2$ and $u \geq 2t-1$. Then at least one is true:

(1) $\exists M \subseteq G$ with $|M| = 1$ such that M is u-uncontestable.

(2) $\exists M \subseteq G$ with $|M| = 2$ such that M is t-feasible.

Proof sketch of lemma. Let x_{ij} denote the number of agents who prefer i over j. Assume both (1) and (2) are false.

- $\neg(1)$ says $\forall i \in G$ that $\bigvee_{j \neq i}(x_{ji} \geq u)$.
- $\neg(2)$ says $\forall \{i,j\} \subset G$ that $(x_{ij} < t) \vee (x_{ji} < t)$.

Solve this "2SAT instance" by hand, using repeatedly the fact

$$\forall i,j,k \in G,\ x_{ij} \geq u \wedge x_{kj} < t \implies x_{ik} \geq t.$$

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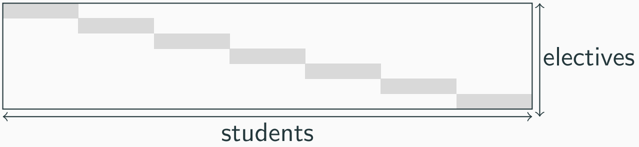
Proof sketch of theorem.

- If (1), then either some such M is also t-feasible, or $\emptyset$ is stable.
- Otherwise, if (2), then the maximum possible lobby size is $(g-2)(t-1) < u$. (Using other machinery, suffices to assume that each good is top-ranked by $\leq t-1$ agents.)

What about large numbers of goods?

When $g \geq 7$, existence question open for $\underbrace{23\left\lfloor \frac{t-1}{11} \right\rfloor < u \leq (g-2)(t-1)}_{\sim g \text{ gap}}$.

- Consider online university with 600 possible electives.
- Absent further assumptions, we only guarantee a $(11, 5981)$-stable menu.
- But: suppose there are 100 departments, each with 6 possible electives. And assume each student only likes electives in their department.
- Then, due to "block structure" of students' preferences, we guarantee a (11,21)-stable menu.



Strategyproofness

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Does there exist an anonymous strategyproof mechanism

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Theorem. For $g = 3, 4, 5, 6$, there is no anonymous stable SP mechanism.

Proof sketch. Given voting problem, carefully transform into menu selection problem and invoke Gibbard–Statterthwaite (transform so that unanimity implied by stability). Challenge: menu selection problem should only have singletons as stable menus.

Strategyproofness: technicalities

Formally, we consider mechanisms that work on subsets of *stability parameters*.

A set of stability parameters $\mathcal{S} \subseteq \mathbb{N}^2$ *guarantees existence* if $\forall (t, u) \in \mathcal{S}$, every menu selection problem has a (t, u)-stable solution.

E.g. when $g \in \{3, 4, 5, 6\}$, then $\mathcal{S} = \{(t, u) : \; u \geq 2t - 1\}$ guarantees existence.

We're actually looking for anonymous stable strategyproof mechanisms with respect to the set of stability parameters $\mathcal{S}$:

$$\mathcal{M} : \; (\text{menu selection problem}) \times (t, u) \mapsto (t, u)\text{-stable menu?}$$

Theorem (again). For $g = 3, 4, 5, 6$, and $\mathcal{S} = \{(t, u) : \; u \geq 2t - 1\}$, there is no anonymous stable SP mechanism.

Future work:

- For $g \in \{3, 4, 5, 6\}$, can we get positive results by restricting to smaller $\mathcal{S}$?
- What happens for $g \geq 7$?
- What about incomplete preference lists?

Takeaways

- We introduce a new model for a matching market, with complementarities and no capacity constraints.
- For $g \leq 6$, we provide a tight characterization for when stable menus exist.
- For $g \geq 7$, we provide lower and upper bounds for when stable menus exist.
- For $3 \leq g \leq 6$, there are fundamental barriers for strategyproofness.

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Thank you! Questions?