Lab 5: Rejection Sampling

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Agenda

We can often end up with posterior distributions that we only know up to a normalizing constant. For example, in practice, we may derive

$$p(\theta \mid x) \propto p(x \mid \theta)p(\theta)$$

and find that the normalizing constant $p(x)$ is very difficult to evaluate. Such examples occur when we start building non-conjugate models in Bayesian statistics.

Given such a posterior, how can we appropriate it's density? One way is using rejection sampling. As an example, let's suppose our resulting posterior distribution is

$$f(x) \propto sin^2(\pi x), x \in [0, 1].$$

In order to understand how to approximate the density (normalized) of f, we will investigate the following tasks:

1. Plot the densities of $f(x)$ and the Unif(0,1) on the same plot. According to the rejection sampling approach sample from $f(x)$ using the Unif(0,1) pdf as an enveloping function.
2. Plot a histogram of the points that fall in the acceptance region. Do this for a simulation size of 10^2 and 10^5 and report your acceptance ratio. Compare the ratios and histograms.
3. Repeat Tasks 1 - 3 for Beta(2,2) as an enveloping function.
4. Provide the four histograms from Tasks 2 and 3 using the Uniform(0,1) and the Beta(2,2) enveloping proposals. Provide the acceptance ratios. Provide commentary.
5. Do you recommend the Uniform or the Beta(2,2) as a better enveloping function (or are they about the same)? If you were to try and find an enveloping function that had a high acceptance ratio, which one would you try and why?

Task 1

Plot the densities of $f(x)$ and the Unif(0,1) on the same plot.

Let's first create a sequence of points from 0 to 1, so that we can have a grid of points for plotting both of the proposed functions.

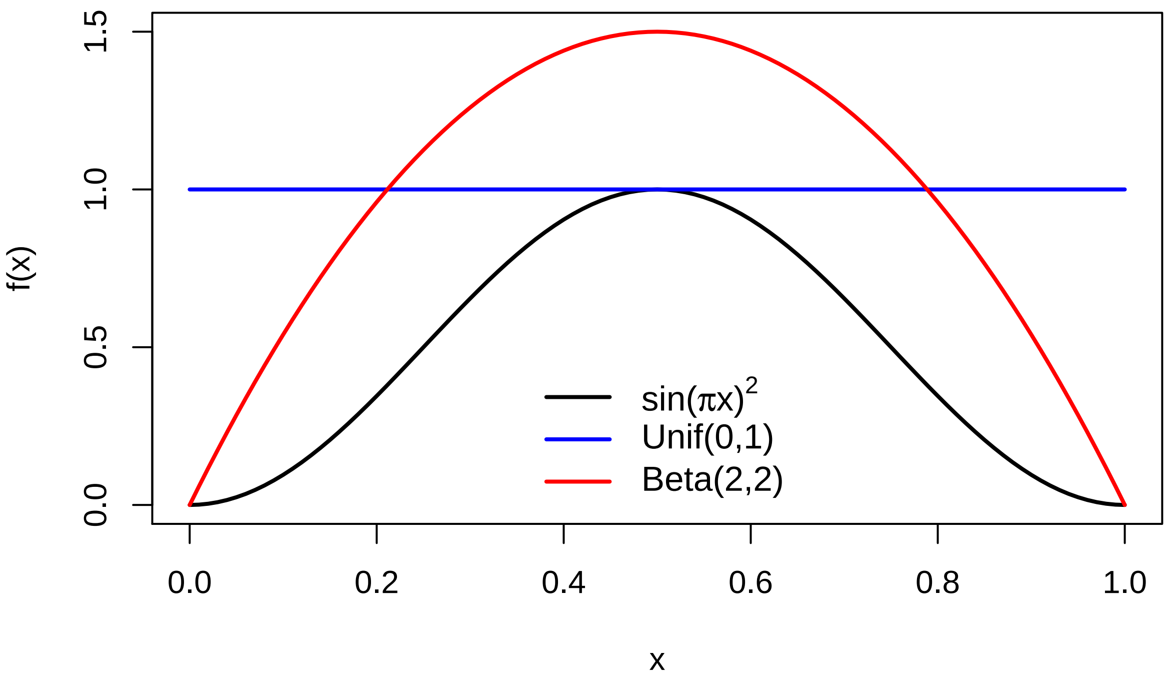
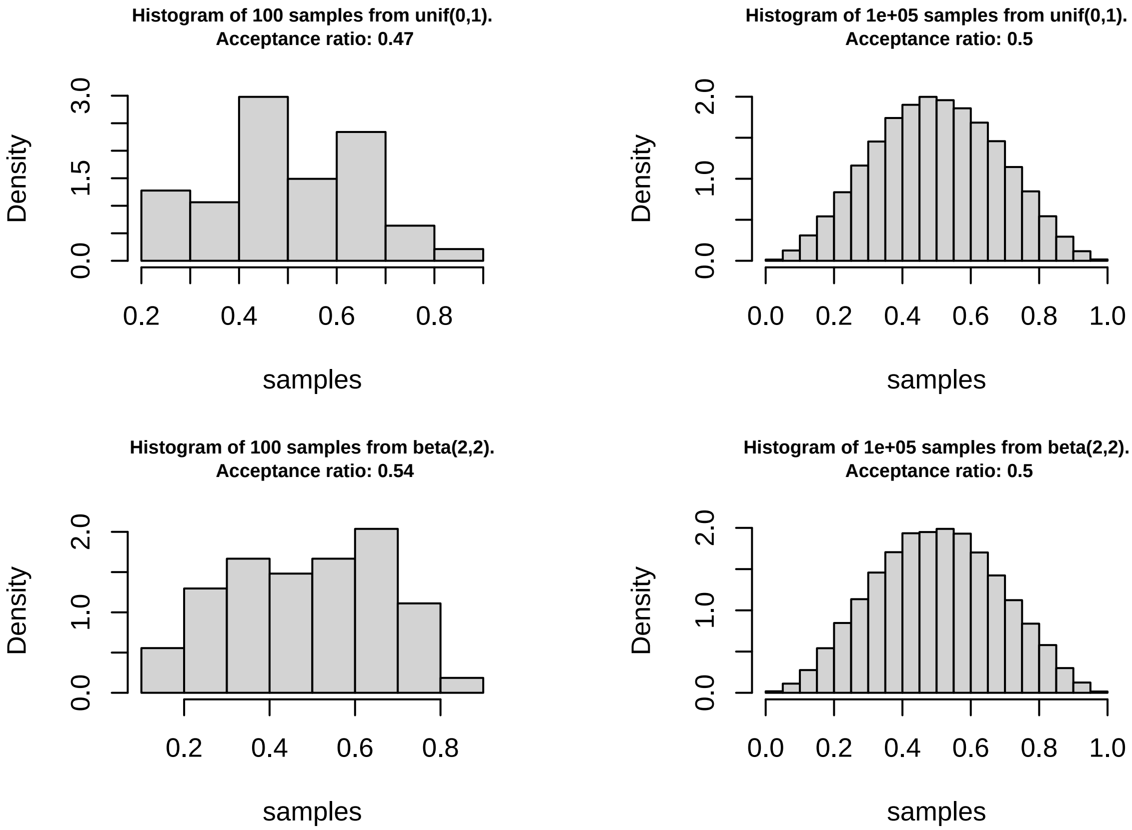


Figure 1: Comparision of the target function and the Unif(0,1) and the Beta(2,2) densities on the same plot.

Tasks 2 – 4

According to the rejection sampling approach sample from f(x) using the Unif(0,1) pdf as an enveloping function. In order to do this, we write a general rejection sampling function that also allows us to plot the historams for any simulation size. Finally, our function also allows us to look at task 4 quite easily.

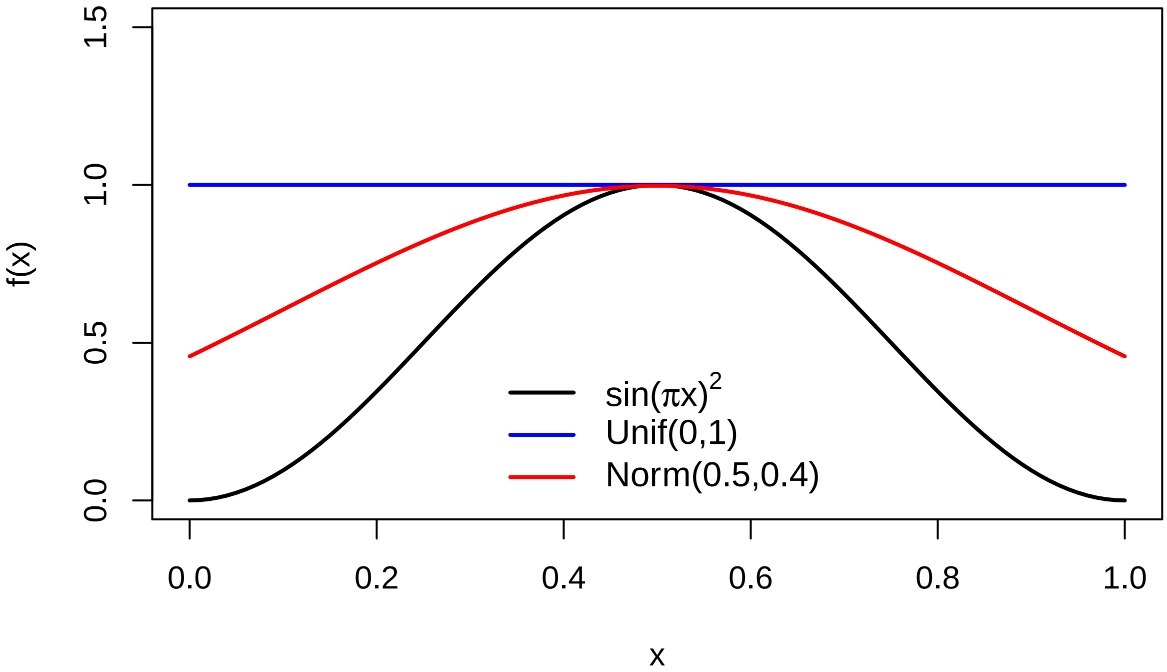


The lowest acceptance ratio results from taking 100 samples from the Uniform(0,1) distribution, whereas the highest acceptance ratio actually comes taking 100 samples from the Beta(2,2) distribution. The histogram for Beta(2,2) 100 samples is slightly more normal than the histogram for the Uniform(0,1) distribution. However, both the histogram shape and acceptance ratios for the smaller samples are most likely not consistent across samples, due to sampling variation. Both are nowhere near as representative of a normal curve as the two 10ˆ6 sample graphs. The two 10ˆ6 sample graphs look pretty similar to each other and yield the same acceptance ratio as well (0.5). This is unlikely to vary much based on the Law of Large Numbers.

Figure 2: Comparision of the output of the rejection sampling for 100 versus 100,000 simulations with uniform and beta distributions as envelope functions.

Task 5

It seems that the Uniform(0,1) and Beta(2,2) enveloping functions are about the same in terms of their acceptance ratios. To get a higher acceptance ratio, I might try a normal distribution, because the shape of f(x) looks kind of like a normal distribution.



f(x)
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
x
sin(πx)^2
Unif(0,1)
Norm(0.5,0.4)