STA 360: Homework 5

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9/20/21

3.12

a.

$$I(\theta) = -E[\frac{\partial^2(\log p(Y|\theta))}{\partial\theta^2}|\theta]$$

$$p(Y|\theta) = \binom{n}{Y}\theta^Y(1-\theta)^{n-Y}$$

$$\log(\binom{n}{Y}\theta^Y(1-\theta)^{n-Y}) = \log(\binom{n}{Y}) + Y\log\theta + (n-Y)\log(1-\theta)$$

$$\frac{\partial(\log(\binom{n}{Y}) + Y\log\theta + (n-Y)\log(1-\theta))}{\partial\theta} = \frac{Y}{\theta} - \frac{n-Y}{1-\theta}$$

$$\frac{\partial(\frac{Y}{\theta} - \frac{n-Y}{1-\theta})}{\partial\theta} = -\frac{Y}{\theta^2} - \frac{n-Y}{(1-\theta)^2}$$

$$I(\theta) = -E[-\frac{Y}{\theta^2} - \frac{n-Y}{(1-\theta)^2}|\theta] = \frac{1}{\theta^2}E[Y] + \frac{n}{(1-\theta)^2} - \frac{1}{(1-\theta)^2}E[Y] = \frac{n\theta}{\theta^2} + \frac{n}{(1-\theta)^2} - \frac{n\theta}{(1-\theta)^2} = \frac{n}{\theta} + \frac{n(1-\theta)}{(1-\theta)^2} = \frac{n}{\theta} + \frac{n}{1-\theta}$$

$$p_J(\theta) \propto \frac{1}{\sqrt{\theta(1-\theta)}}$$

b.

$$\log(\binom{n}{Y}e^{\psi Y}(1+e^\psi)^{-n}) = \log(\binom{n}{Y}) + \psi Y - n\log(1+e^\psi)$$

$$\frac{\partial(\log(\binom{n}{Y}) + \psi Y - n\log(1+e^\psi))}{\partial\psi} = Y - \frac{ne^\psi}{1+e^\psi}$$

$$\frac{\partial(Y - \frac{ne^\psi}{1+e^\psi})}{\partial\psi} = \frac{ne^{2\psi}}{(1+e^\psi)^2} - \frac{ne^\psi}{1+e^\psi} = \frac{ne^{2\psi} - ne^\psi(1+e^\psi)}{(1+e^\psi)^2} = -\frac{ne^\psi}{(1+e^\psi)^2}$$

$$I(\psi) = -E[-\frac{ne^\psi}{(1+e^\psi)^2}|\psi] = \frac{ne^\psi}{(1+e^\psi)^2}$$

$$p_J(\psi) \propto \frac{e^{\psi/2}}{1+e^\psi}$$

c.

$$\psi = log(\frac{\theta}{1-\theta}) \rightarrow \theta = \frac{e^{\psi}}{1+e^{\psi}}$$

$$|\frac{dh}{d\psi}| = \frac{e^{\psi}}{(1+e^{\psi})^2}$$

$$p_{\psi}(\psi) = \frac{1}{\sqrt{\frac{e^{\psi}}{1+e^{\psi}}(1-\frac{e^{\psi}}{1+e^{\psi}})}} \times \frac{e^{\psi}}{(1+e^{\psi})^2} = (\frac{1+e^{\psi}}{e^{\psi}})^{1/2}(1+e^{\psi})^{1/2}\frac{e^{\psi}}{(1+e^{\psi})^2} = \frac{e^{\psi/2}}{1+e^{\psi}}$$

Task 4

According to the rejection sampling approach sample from f(x) using the Unif(0,1) pdf as an enveloping function. In order to do this, we write a general rejection sampling function that also allows us to plot the historams for any simulation size. Finally, our function also allows us to look at task 4 quite easily.

```
set.seed(123)
```

```
x <- seq(0, 1, 10^-2)
fx <- function(x) sin(pi * x)^2
sim_fun <- function(f, envelope = "unif", par1 = 0, par2 = 1, n = 10^2, plot = TRUE){

  r_envelope <- match.fun(paste0("r", envelope))
  d_envelope <- match.fun(paste0("d", envelope))
  proposal <- r_envelope(n, par1, par2)
  density_ratio <- f(proposal) / d_envelope(proposal, par1, par2)
  samples <- proposal[runif(n) < density_ratio]
  acceptance_ratio <- length(samples) / n
  if (plot) {
    hist(samples, probability = TRUE,
         main = paste0("Histogram of ",
                       n, " samples from ",
                       envelope, "(", par1, ",", par2,
                       ").\n Acceptance ratio: ",
                       round(acceptance_ratio,2)),
                       cex.main = 0.75)
  }
  list(x = samples, acceptance_ratio = acceptance_ratio)
}
```

```
par(mfrow = c(2,2), mar = rep(4, 4))
unif_1 <- sim_fun(fx, envelope = "unif", par1 = 0, par2 = 1, n = 10^2)
unif_2 <- sim_fun(fx, envelope = "unif", par1 = 0, par2 = 1, n = 10^5)
beta_1 <- sim_fun(fx, envelope = "beta", par1 = 2, par2 = 2, n = 10^2)
beta_2 <- sim_fun(fx, envelope = "beta", par1 = 2, par2 = 2, n = 10^5)
```

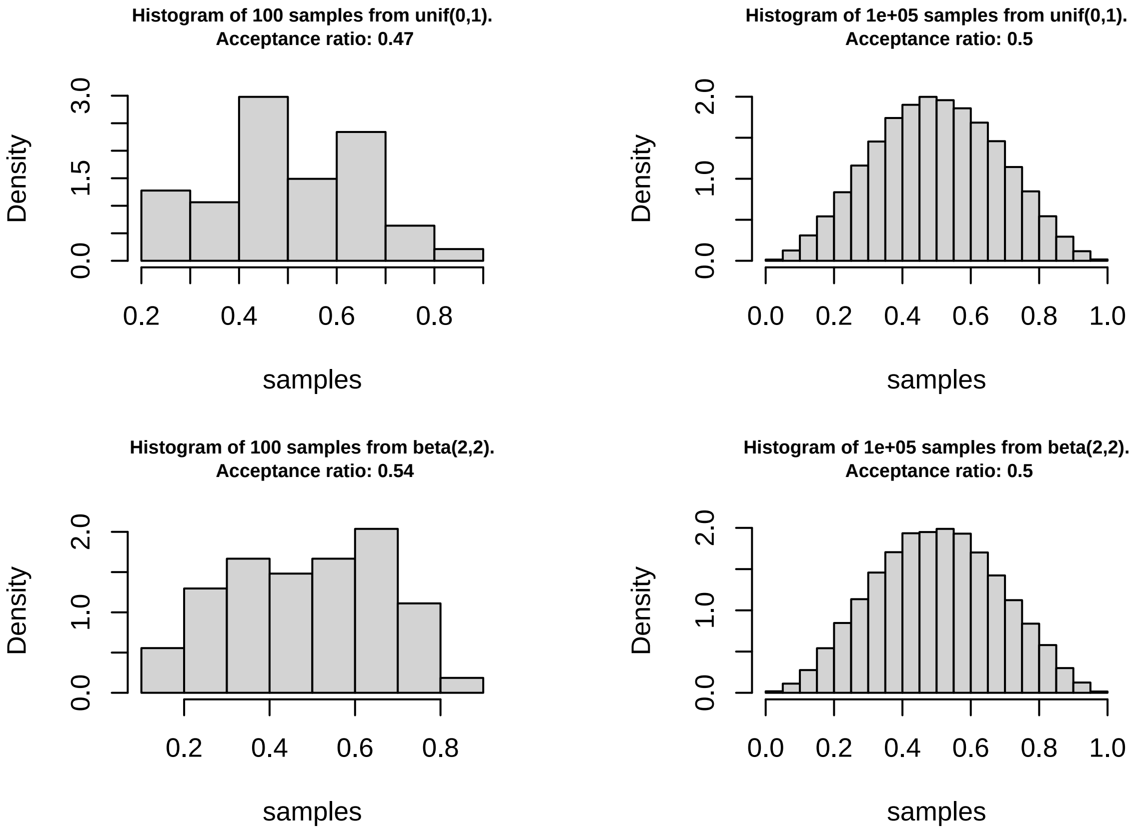


Figure 1: Comparision of the output of the rejection sampling for 100 versus 100,000 simulations with uniform and beta distributions as envelope functions.

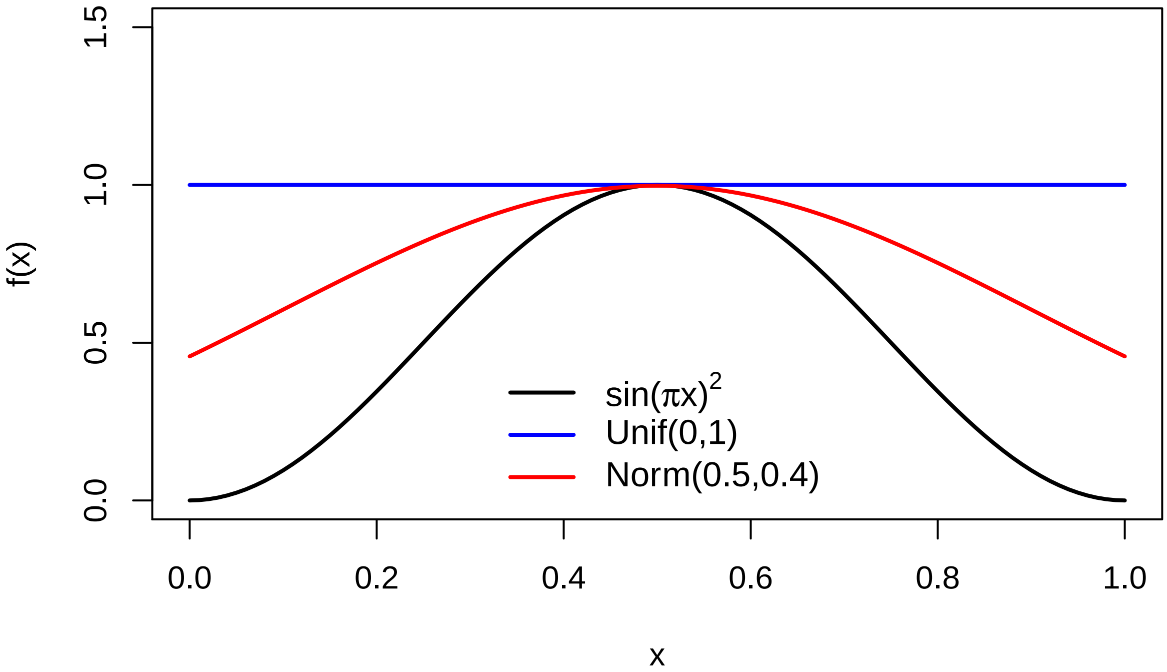
The lowest acceptance ratio results from taking 100 samples from the Uniform(0,1) distribution, whereas the highest acceptance ratio actually comes taking 100 samples from the Beta(2,2) distribution. The histograms for 100 samples somewhat resemble f(x), but not clearly and are slightly skewed in different directions. Both the histogram shape and acceptance ratios for the smaller samples are most likely not consistent across samples, due to sampling variation. The two 10ˆ6 sample graphs look very similar to f(x) and to each other, and yield the same acceptance ratio as well (0.5).

Task 5

It seems that the Uniform(0,1) and Beta(2,2) enveloping functions are about the same in terms of their acceptance ratios. To get a higher acceptance ratio, I might try a normal distribution, because the shape of f(x) looks kind of like a normal distribution.

```r
# grid of points
x <- seq(0, 1, 10^-2)
```

```r
fx <- function(x) sin(pi * x)^2
plot(fx, xlim = c(0,1), ylim = c(0,1.5), ylab = "f(x)", lwd = 2)
curve(dunif, add = TRUE, col = "blue", lwd = 2)
curve(dnorm(x,0.5,0.4), add = TRUE, col = "red", lwd = 2)
legend("bottom", legend = c(expression(paste("sin(",pi,"x)"^"2")),"Unif(0,1)",
"Norm(0.5,0.4)"), col = c("black", "blue", "red"), lty = c(1,1,1), bty = "n", cex = 1.1, lwd = 2)
```



1.5
1.0
0.5
0.0
f(x)
sin(πx)^2
Unif(0,1)
Norm(0.5,0.4)
0.0
0.2
0.4
0.6
0.8
1.0
x