Dolbeault Cohomology of Vector Bundles

Sara Stephens

March 2, 2023

These are notes written for a talk as a part of Math 7670: Hodge Theory. They are based on Chapter 4 of Voisin's *Hodge Theory and Complex Algebraic Geometry I*, with a few supplementary sources.

1 Introduction

Our goal for today is to prove Dolbeault's theorem for vector bundles. Specifically, let X be a complex manifold, E a holomorphic vector bundle over X, with a corresponding sheaf of holomorphic sections $\mathcal{E}$. Then the Dolbeault complex computes the sheaf cohomology of $\mathcal{E}$. We have

$$H^i(X, \mathcal{E}) \cong H^i_{\mathrm{Dolb}}(X, \mathcal{E}) = \frac{\ker(\overline{\partial}_E : A^{0,i}_X(E) \to A^{0,i+1}_X(E))}{\mathrm{im}(\overline{\partial}_E : A^{0,i-1}_X(E) \to A^{0,i}_X(E))}$$

Recall that $A^{p,q}_X(E)$ is $C^\infty(X, \Omega^{p,q}_X \otimes_{\mathbb{C}} E)$, the C^∞ sections of (p,q) forms valued in E. This theorem is the complex analogue to a theorem of de Rham's, saying that the sheaf cohomology of $\underline{\mathbb{R}}$ (the sheaf of locally constant real valued functions) for smooth manifolds is isomorphic to it's de Rham cohomology.

2 Sheaves

Let's start with some preliminaries about sheaves.

Definition. A *sheaf* $\mathcal{F}$ of abelian groups on a topological space X consists of the data of an abelian group $\mathcal{F}(U)$ for every open set $U \subseteq X$ and restriction maps $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for open sets $V \subseteq U$ such that

1. $\mathcal{F}(\emptyset) = 0$
2. $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map
3. if $W \subseteq V \subseteq U$ are open sets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$
4. *(locality)* if $\{U_i\}_{i \in I}$ is an open cover of U and $s, t \in \mathcal{F}(U)$ are sections, then if $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$

5. *(gluing)* if $\{U_i\}_{i \in I}$ is an open cover of U and $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$

Note that items 1-3 constitute a presheaf and 4-5 are the sheaf axioms. We have the following proposition.

Proposition. Every presheaf gives rise to a unique sheaf. This process is called sheafification.

Proof. We defer to Voisin Lemma 4.4 or Hartshorne II Proposition 1.2.

We now consider an important sheaf for complex manifolds.

Definition. If X is a complex manifold, the *structure sheaf* $\mathcal{O}_X$ is a sheaf assigning holomorphic functions to every open set $U \subseteq X$:

$$\mathcal{O}_X(U) = \{f : U \to \mathbb{C} \text{ holomorphic}\}$$

We will also need the following definition which helps us capture the behaviour of a sheaf around a given point.

Definition. The *stalk* of a sheaf $\mathcal{F}$ on a topological space X at a point $x \in X$ is the direct limit $\lim_{U \ni x} \mathcal{F}(U)$ over open sets $U \subseteq X$.

Now that we've defined a sheaf, we introduct the notion of a sheaf of modules, which we will need to study the sheaf cohomology of holomorphic vector bundles.

Definition. Let $\mathcal{A}$ be a sheaf on a topological space X. A *sheaf of $\mathcal{A}$-modules* over X is a sheaf $\mathcal{F}$ such that $\mathcal{F}(U)$ has an $\mathcal{A}$-module structure, and the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are compatible with restriction maps $\mathcal{A}(U) \to \mathcal{A}(V)$. Specifically, the restriction of $fg \in \mathcal{F}(U)$ is the restriction of f times the restriction of g for any $f \in \mathcal{A}(U), g \in \mathcal{F}(U)$.

Throughout, we will keep the following important example of a sheaf of modules in mind.

Example. If E is a holomorphic vector bundle over a complex manifold X, the holomorphic sections of E form a sheaf of $\mathcal{O}_X$ modules, denoted $\mathcal{E}$.

We need one more definition of characterizing a sheaf of modules that is locally isomorphic to the direct sum of several copies of the structure sheaf.

Definition. A sheaf $\mathcal{F}$ of $\mathcal{A}$-modules is *locally free* if for $U \subseteq X$ open, we have $\mathcal{F}|_U \cong \mathcal{A}^n|_U$ as a sheaf of $\mathcal{A}$-modules. The integer n is called the *rank* of $\mathcal{F}$.

We are now able to characterize the correspondance between vector bundles and locally free sheaves.

Proposition. We have a bijection (equivalence of categories) between locally free sheaves and vector bundles.

Construction. We start by constructing a sheaf from a vector bundle $E \xrightarrow{\pi} X$. For open sets $U_i \subset X$, assignt to U_i the moprphisms $\{s : U_i \to E \mid \pi \circ s = id_{u_i}\}$. The open sets we look are U_i such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{C}^n$. We have $U_i \xrightarrow{s} U_i \times \mathbb{C}^n$ such that $\pi \circ s = id_{U_i}$. This is the same as looking at morphisms $U_i \to \mathbb{C}^n = \mathbb{C} \times \ldots \times \mathbb{C}$, ie. n regular functions on U_i, giving us a sheaf of sections $\mathcal{E}$.
We now construct the inverse correspondance from locally free sheaves $\mathcal{E}$ to vector bundles $E \xrightarrow{\pi} X$. If $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_X$ modules, there is a covering of X by open sets $\{U_i\}$ such that $\mathcal{E}|_{U_i} \cong \mathcal{O}_X^n|_{U_i}$. Denote the isomorphism as $\tau_{U_i} : \mathcal{E}|_{U_i} \xrightarrow{\sim} \mathcal{O}_X^n|_{U_i}$. On the intersection $U_i \cap U_j$ we have the isomorphism

$$\tau_{U_j} \circ \tau_{U_i}^{-1} : \mathcal{O}_X^n|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{O}_X^n|_{U_i \cap U_j}$$

The isomorphism is given by an $n \times n$ matrix M_{ij} where entries consisting of elements $\mathcal{O}_X|_{U_i \cap U_j}$. We can obtain the vector bundle E via gluing $U_i \cap \mathbb{C}^n$ via the identification

$$\begin{tikzcd}
U_i \times \mathbb{C}^n & & U_j \times \mathbb{C}^n \\
(U_i \cap U_j) \times \mathbb{C}^n \arrow[u, hook] \arrow[rr, bend left, "id \times M_{ij}"] & & (U_i \cap U_j) \times \mathbb{C}^n \arrow[u, hook]
\end{tikzcd}$$

Note our transition matrices satisfy the cocycle condition $M_{ij}M_{jk}M_{ki} = id$, giving us a vector bundle.

3 Sheaf Cohomology

We consider the category of sheaves of abelian groups over a topological space X. Let Γ be the global sections function $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$. Broadly, sheaf cohomology tells us how badly the function Γ fails to be exact. Note that Γ is always left exact.

In order to start taking about sheaf cohomology, we need to be able to construct resolutions of sheaves.

Proposition. The category of sheaves of abelian groups has sufficiently many injectives, meaning every such sheaf $\mathcal{F}$ admits an injective resolution $0 \to \mathcal{F} \to I_0 \to I_1 \to \ldots$.

Proof. We defer to Hartshorne III Proposition 2.2/2.3.

In practice, injective resolutions are difficult to work with, so we introduce the weaker notion of being acyclic.

Definition. An object M is *acyclic* for a functor F if $R^iF(M) = 0$ for all $i > 0$.

We are now able to state the following homological result allowing us to compute the cohomology using acyclic resolutions.

Proposition. If A has an F-acyclic resolution $0 \to A \to M^0 \to M^1 \to \ldots$, then $R^iF(A) = H^i(F(M^\cdot))$.

Proof. We defer to Voisin Proposition 4.32 or Hartshorne III Proposition 1.2A.

In order to use this result to determine our sheaf cohomology groups, we first need the following definition.

Definition. A sheaf $\mathcal{F}$ in a topological space is *flasque* if for all $V \subseteq U$ open, the map $\mathcal{F}(U) \twoheadrightarrow \mathcal{F}(V)$ is surjective.

Proposition. Flasque sheaves are acyclic for the global sections functor Γ.

Proof. We defer to Voisin Proposition 4.34 or Hartshorne III Proposition 2.5.

This allows us to define the sheaf cohomology groups as

$$H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$$

This is sometimes called the de Rham-Weil theorem.

4 Dolbeault's Theorem

We now turn out focus back to showing the Dolbeault complex does indeed compute sheaf cohomology for the sheaf of sections of a holomorphic vector bundle. Let X be a complex manifold, E a holomorphic vector bundle over X, with $\mathcal{E}$ it's associated locally free sheaf of $\mathcal{O}_X$ modules. Let $\mathcal{A}^{0,q}(E)$ be the sheaf of C^∞ sections of $\Omega^{0,q} \otimes_{\mathbb{C}} E$. Recall our definition of the Dolbeault operator

$$\overline{\partial}_E : \mathcal{A}^{0,q}(E) \to \mathcal{A}^{0,q+1}(E)$$

This operator allows us to resolve $\mathcal{E}$.

Proposition. The following complex resolves $\mathcal{E}$:

$$0 \to \mathcal{A}^{0,0}(E) \xrightarrow{\overline{\partial}_E} \mathcal{A}^{0,1}(E) \xrightarrow{\overline{\partial}_E} \dots \xrightarrow{\overline{\partial}_E} \mathcal{A}^{0,n}(E) \to 0$$

where $n = \dim_{\mathbb{C}} X$.

Proof. The kernel of $\overline{\partial}_E : \mathcal{A}^{0,0}(E) \to \mathcal{A}^{0,1}(E)$ is equal to the sheaf of holomorphic sections of E, ie. to $\mathcal{E}$. This is because holomorphic sections are given by n-tuples of holomorphic functions in local holomorphic trivializations. Since $\overline{\partial}_E$ acts like $\overline{\partial}$ on these tuples, the functions vanishing under $\overline{\partial}_E$ are the holomorphic functions (recall $\overline{\partial}$ 'detects holomorphicity').

We also have exactness since a section of $\mathcal{A}^{0,q}(E)$ is $\overline{\partial}_E$ closed if and only if it is locally $\overline{\partial}_E$- exact. This is a result know as the Dolbeault-Grothendieck Lemma, or the $\overline{\partial}_E$- Poincaré Lemma. $\square$

Lemma. (Dolbeault-Grothendieck) Let $\alpha \in \mathcal{A}^{0,q}(E), q > 0$. If $\overline{\partial}_E\alpha = 0$, then locally on X there exists a form $\beta \in \mathcal{A}^{0,q-1}(E)$ such that $\alpha = \overline{\partial}_E\beta$.

Proof. Recall that for an open set $U \subseteq X$ we have $\overline{\partial}_E\alpha|_U = \partial_U\alpha|_U$ for $\alpha \in \mathcal{A}^{p,q}(E)$. Therefore, it is enough to show that $\overline{\partial}$ is locally exact.
For the sake of time I will prove this lemma for the case for the case $n = 1$ where we have

$$0 \to \mathcal{A}^{0,0}(E) \xrightarrow{\partial} \mathcal{A}^{0,1}(E) \to 0$$

To prove it in full generality, apply this strategy variable by variable and induct.

Let $f : U \to \mathbb{C}$ be a C^∞ function where $U \subseteq X$ is open. We show for every $(0,1)$-form $f(z)d\overline{z}$ on U that we have an open cover $U = \cup_{i\in I}U_i$ such that $f(z)d\overline{z}|_{U_i} = \overline{\partial}g_i$, where $g_i : U_i \to \mathbb{C}$ is a C^∞ function.

For every point $z_0 \in U$, choose an ϵ neighbourhood such that $\overline{U_\epsilon}(z_0) \subset U$. After possibly multplying f by a suitable bump function, we can assume that for $|z - z_0| \geq \epsilon/2$ we have $f(z) = 0$. Note that this does not restrict the generality since we only want to find such a g locally around each $z_0 \in U$ such that $f = \overline{\partial}g$. Note, we can now consider f as a C^∞ function on all of $\mathbb{C}$, instead of just U, because it's some sort of bump function itself.

We define for $z \in U_\epsilon(z_0)$:

$$g(z) := \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(w)}{w-z}dw \wedge d\overline{w}$$

Substituting $w = w' + z$ gives

$$g(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(w'+z)}{w'}dw' \wedge d\overline{w'}$$

One can verify that $g(z)$ has class C^∞. Now we can use a generalized version of Cauchy's integral formula to obtain

$$2\pi i\frac{\partial}{\partial \overline{z}}g(z) = \int_{\mathbb{C}} \frac{\partial f}{\partial \overline{w}}(w'+z)\frac{1}{w'}dw' \wedge d\overline{w'} = \int_{\mathbb{C}} \frac{\partial f}{\partial \overline{w}}(w)\frac{1}{w-z}dw \wedge d\overline{w} = 2\pi i f(z)$$

Thus $\overline{\partial}g = f$ as required.
Note: We used the following general version of Cauchy's Integral formula, known as the Cauchy-Pompeiu Integral Formula:
Let $U \subset \mathbb{C}$ open, and $f : U \to \mathbb{C}$ a C^∞ function. Then for every $z_0 \in U$ and $\epsilon > 0$ such that $\overline{U_\epsilon}(z_0) \subset U$, we have

$$2\pi i f(z) = \int_{|w-z_0|=\epsilon} \frac{f(w)}{w-z}dw + \int_{\overline{U_\epsilon(z_0)}} \frac{\partial f}{\partial \overline{w}}(w)\frac{dw \wedge d\overline{w}}{w-z}$$

where $z \in U_\epsilon(z_0)$. $\square$

Now we've shown that the dolbeault complex resolves the sheaf $\mathcal{E}$, where $n = \dim_{\mathbb{C}}X$:

$$0 \to \mathcal{A}^{0,0}(E) \xrightarrow{\overline{\partial}_E} \mathcal{A}^{0,1}(E) \xrightarrow{\overline{\partial}_E} \dots \xrightarrow{\overline{\partial}_E} \mathcal{A}^{0,n}(E) \to 0$$

In order to show that this computes sheaf cohomology, we need the following definition.

Definition. A *fine sheaf* $\mathcal{F}$ on a topological space X is a sheaf of $\mathcal{A}$-modules where $\mathcal{A}$ is a sheaf of rings on X such that for every open cover $\{U_i\}$ of X, there is a partition of unity $\{f_i\}, \sum_i f_i = 1$ subordinate to the cover $(\mathrm{supp}(f_i) \subseteq U_i)$.

The following is an important example of a fine sheaf.

Example. If X is a complex manifold, and E a holomorphic vector bundle, the sheaf of (p, q) forms with values in E, denoted $\mathcal{A}^{p,q}(E)$ is fine. This comes from the fact that we have a C^∞ partition of unity subordinate to a locally finite open cover of X.

Recall, we had that for a Γ-acyclic resolution, sheaf cohomology can be computed using the derived functors of the resoluion. In order to get our desired result, Dolbeault's theorem for bundles, we need to show that fine sheaves are Γ-acyclic.

Proposition. If $\mathcal{F}$ is a fine sheaf, $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Proof. We resolve $\mathcal{F}$ with the Godement resolution. For an open set $U \subseteq X$, defined the sheaf $\mathcal{F}_{God}$ by

$$U \mapsto \mathcal{F}_{God}(U) := \bigoplus_{x \in U} \mathcal{F}_X$$

This sheaf is flasque. We get a flasque resolution

$$0 \hookrightarrow \mathcal{F} \overset{d_0}{\hookrightarrow} \bigoplus_{x \in X} \mathcal{F}_X \overset{d_1}{\hookrightarrow} \bigoplus_{x \in X} \mathrm{coker}(d_0)_x \overset{d_2}{\hookrightarrow} \bigoplus_{x \in X} \mathrm{coker}(d_1)_x \overset{d_3}{\hookrightarrow} \dots$$

Denote this flasque resolution by $I^\cdot$, with $\mathcal{F} \subseteq I^0$. Since flasque sheaves are Γ-acyclic, we have

$$H^k(X, \mathcal{F}) = \frac{\ker(\Gamma(I^k) \to \Gamma(I^{k+1}))}{\mathrm{im}(\Gamma(I^{k-1}) \to \Gamma(I^k))}$$

We want to show if α is a k-cocyle then it is also a coboundary. Let $\alpha \in \ker(\Gamma(I^k) \to \Gamma(I^{k+1}))$. Since the complex $I^\cdot$ is exact, α locally comes from I^{k-1}. There exists an open cover $\{U_i\}$ of X such that $\alpha|_{U_i} = d\beta_i$ where $\beta_i \in I^{k-1}(U_i)$. Let $\{f_i\}$ be a partition of unity subordinate to the cover $\{U_i\}$, ie. $\mathrm{supp}(f_i) \subseteq U_i$ where $f_i \in \mathcal{A}(X)$. Let $\beta = \sum_i f_i \beta_i$ where the sum is locally finite.

Note that $f_i\beta_i$ is a section of I^{k-1} on X with value in I^{k-1}_X being 0 for x outside U_i, and $f_i\beta_i|_{U_i}$ for $x \in U_i$. The sum is well defined since it is locally finite.

Since $\alpha = \sum_i f_i \alpha|_{U_i}$, we get $d\beta = \alpha$. □

We are at last ready to state our main result.

Theorem. (Dolbeault Cohomology for Vector Bundles) Let X be a complex manifold, E a holomorphic vector bundle over X, with a corresponding sheaf of holomorphic sections $\mathcal{E}$. Then the Dolbeault complex $(\mathcal{A}^{0,\star}, \overline{\partial}_E)$ computes the sheaf cohomology of $\mathcal{E}$. We have

$$H^i(X, \mathcal{E}) \cong H^i_{\text{Dolb}}(X, \mathcal{E}) = \frac{\ker(\overline{\partial}_E : A_X^{0,i}(E) \to A_X^{0,i+1}(E))}{\operatorname{im}(\overline{\partial}_E : A_X^{0,i-1}(E) \to A_X^{0,i}(E))}$$

Proof. We use the Dolbeault resolution of $\mathcal{E}$:

$$0 \to \mathcal{A}^{0,0}(E) \xrightarrow{\overline{\partial}_E} \mathcal{A}^{0,1}(E) \xrightarrow{\overline{\partial}_E} \dots \xrightarrow{\overline{\partial}_E} \mathcal{A}^{0,n}(E) \to 0$$

Since the sheaves $\mathcal{A}^{0,\star}(E)$ are fine, this resolution is Γ-acyclic. Thus, since we have an acyclic resolution, we can use it to compute sheaf cohomology $H^i(X, \mathcal{E})$. $\square$