IHS Math Seminar: January 16th, 2025

1 Rings

1. Is the set of even integers a ring? Why or why not?
2. Is the set of odd integers a ring? Why or why not?
3. Are the natural numbers (integers that are greater than or equal to zero) a ring? Why or why not?
4. Find a ring where the additive identity element is the equal to the multiplicative identity element.

2 Fields

1. Explain why the ring of integers, denoted $\mathbb{Z}$, is not a field.
2. Are the natural numbers a field? Why or why not?
3. The complex numbers are defined as numbers of the form $a + bi$ where a and b are integers and $i^2 = -1$. Find the multiplicative inverse for any complex number.
4. Consider the following fields: the real numbers, complex numbers and rational numbers. Which of these fields are contained in each other?

3 Polynomial Rings in Sage

1. Here are some example of rings that Sage recognizes.

Sage Notation	Math Notation	Definition
`ZZ`	$\mathbb{Z}$	The ring of **integers**
`QQ`	$\mathbb{Q}$	The ring of **rational numbers**
`RR`	$\mathbb{R}$	The ring of **real numbers**
`CC`	$\mathbb{C}$	The ring of **complex numbers**
`ZZ[sqrt(2)]`	$\mathbb{Z}[\sqrt{2}]$	The ring of numbers of the form $a + b\sqrt{2}$ where a and b are integers

Try factoring the polynomial $12x^6 - 28x^5 - 33x^4 + 21x^3 - 63x^2 + 189x + 162$ over various rings. What are some observations you can make?

2. The following code explore polynomial rings in Sage. Play with it and add some comments. What does each line do? What interesting things do you notice?

```
R.<x> = PolynomialRing( QQ )                # Sets up the ring of polynomials
                                            # with rational coefficients. (Also
                                            # registers x as a variable in this
                                            # ring.)

f = x^5 - 4*x^3 + 2*x^2 - 8
g = x^3 - 8

f(0) + g(1)

f / g
f // g
f % g


gcd(f,g)
lcm(f,g)

f*g / gcd(f,g)

factor(x^2 + 1)

R.<x> = PolynomialRing( QQ[i] )             # These coefficients are numbers
                                            # of the form a+bi, where a and b
                                            # are rational.

factor(x^2 + 1)
```

4 If you have time, go back and try some exercises from part 4 of worksheet 1!