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Cramer's Rule

1 Gabriel Cramer: Historical Brief

In this section, we will present the life story of Gabriel Cramer based on the articles "*Gabriel Cramer*" [1], [2], [5].

GAbriel Cramer is a renowned Swiss mathematician recognized for his book covering treatise on curves, *Introduction à l'analyse des lignes courbes algébriques*. His findings include Cramer's Rule, an algorithm that can solve a system of linear equations using determinants. Yet, his most significant contributions emerged from editing and proofreading other mathematical theories.

Born on July 31st, 1704, in Geneva, Cramer came into a family of three brothers and a physician as a dad, Jean Cramer. Growing up in an overachieving family, Gabriel graduated with a doctorate degree focused on the theory of sound at 18 years old. He was later appointed co-chair of the mathematical department with Giovanni Calandrini at Académie de Clavin, Geneva. Part of the position's requirements was consistent traveling and communicating with other lead mathematicians of the time, including Johann and Daniel Bernoulli, Euler, Halley, de Moiré, and Stirling. Visiting London, Leiden, and Paris, Cramer's discussions ignited the spark for his most influential theories.

In 1730, Gabriel Cramer was awarded second place by the Paris Academy for his work, *"Quelle est la cause de la figure elliptique des planètes et de la* *mobilité de leurs aphélies,"* which corresponds to identifying the causes of elliptical planets and the mobility of their aphelia. Following the prize, Cramer was re-directed to chair of the mathematical department in 1735. In addition to his teaching and connections, Gabriel published articles on geometry studies, the history of mathematics, and philosophy. Though his work was impeccable in mathematics, he and his siblings had a shared interest in local politics, where he served as a member of the Council of Two Hundred and another at the Council of Seventy in 1734 and 1749, respectively. Then, he used his knowledge in physics and math to perform artillery and fortification tasks, as well as reconstruction of buildings, and excavation techniques. With his political position, Cramer aided in spreading mathematical theories and findings of his fellow logicians. A decade later, he was directed to his initial work preference as a professor of philosophy at Académie de Clavin, during which he published the four-volume book, *Introduction à l'analyse des lignes courbes algébriques*. The book mainly covered Cramer's Rule, Cramer's Paradox, and fundamental principles of the concept of utility.

In 1751, the accumulation of his duties, work, and articles, as well as a fall off his carriage, had caused Cramer's health to deteriorate. Doctors had suggested he rests in the south of France for two months. However, Cramer died a year later, on January 4th, on his route to Bagnoli.

2 Cramer's Rule

Definitions 2.1 - 2.7 have been taken from the textbook Elementary Linear Algebra, International Metric Edition [3].

Definition 2.1. A *square matrix* is a matrix which has equal number of columns and rows.

Definition 2.2. A *diagonal* matrix is a matrix whose non-zero elements lie on the principal diagonal only.

Definition 2.3. The *identity* matrix I is a square diagonal matrix whose diagonal elements are all 1's. *Identity matrices* can be of any size. I_n is an identity matrix of size nxn.

Definition 2.4. The *determinant* of a square matrix A is a real number which can be associated with A. The determinant of A is denoted as $|A|$ or $det(A)$. Note that determinant can *only* be defined on square matrices.

To find the determinant of a square matrix A, it is convenient to use the *minors* and *cofactors* of entries in the matrix.

Definition 2.5. For a matrix A, where the element in row i and column j is denoted by a_{ij}, the *minor* M_{ij} of an entry a_{ij} is equal to the determinant of A after deleting row i and column j from A. The *cofactor* C_{ij} of a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Definition 2.6. To find $|A|$ for a square matrix A of size nxn ($n \geq 2$) using cofactors, we can apply a method called *cofactor expansion.* In *cofactor* *expansion*, we have that for any row i of A,

$$|A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + ... + a_{in}C_{in}.$$

Alternatively, we can apply *cofactor expansion* across any column j. Similarly, we get that

$$|A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + ... + a_{nj}C_{nj}.$$

Definition 2.7. The *inverse* of a matrix A is a unique matrix A^{-1} which, when multiplied with A, yields the identity matrix. That is, $AA^{-1} = A^{-1}A = I$. Not all square matrices have an *inverse.* For a matrix to have an *inverse* (to be invertible), its determinant must not equal to zero. That is to say, A is invertible and A^{-1} exists when $|A| \neq 0$.

Theorem 2.8 (Cramer's Rule)**.** *Let $Ax = b$ be a system of n linear equations in n variables, and A_k represent the matrix obtained by replacing the k-th column in A with b. Given that A is invertible, then for the unique solution x, we have that $x_i = \frac{|A_i|}{|A|}$.*

Proof. Suppose A is an invertible nxn matrix. First, we want to see that $Ax = b$ has a unique solution. Let t be any arbitrary solution to $Ax = b$. This means that $At = b$. Thus,

$$t = (A^{-1}A)t = A^{-1}(At) = A^{-1}b,$$

so we can see that $x = A^{-1}b$ is the only solution. Now, we aim to find the solution in a systematic way for each element of x. We will proceed using the proposed method in "Proof of Cramer's Rule" by Milson, R [4].

Let e_i represent the i-th column of the identity matrix I_n, let a_i represent the i-th column of A, and let X_i denote the matrix formed by replacing the i-th column of I_n with the column matrix x. We know that for any two matrices, say C and D, the k-th column of the product CD is simply the product of C and the k-th column of D. Thus, we have that

$$
\begin{aligned}
AX_i &= A(e_1, ..., e_{i-1}, x, e_{i+1}, ..., e_n) && (1)\\
&= (Ae_1, ..., Ae_{i-1}, Ax, Ae_{i+1}, ..., Ae_n) && (2)\\
&= (a_1, ..., a_{i-1}, b, a_{i+1}, ..., a_n) && (3)\\
&= A_i && (4)
\end{aligned}
$$

where A_i is the matrix obtained by replacing the i-th column in A with b. We now have to get $\det(X_i)$- we can use the fact that X_i is equal to the identity matrix I_n, with column i replaced with x. Using co-factor expansion, we get that

$$det(X_i) = (-1)^{i+i} \cdot x_i \cdot det(I_{n-1}) = 1 \cdot x_i \cdot 1 = x_i$$

Using the properties of determinants,

$$|A_i| = det(A_i) = det(AX_i) = det(A) \cdot det(X_i) = |A| \cdot x_i.$$

Finally, using simple isolation of variables, we have that

$$x_i = \frac{|A_i|}{|A|}.$$

$\blacksquare$

The following is an example to solve a system of linear equations using Cramer's rule:

Let us to try and solve the following system of linear equations:

$$x + y - z = 6$$

$$3x - 2y + z = -5$$

$$x + 3y - 2z = 14$$

We can rewrite this system using the matrix representation: $A\ \underline{x} = b$. So, the coefficient matrix, A would be:

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{bmatrix}$$

And b would be:

$$\begin{bmatrix} 6 \\ -5 \\ 14 \end{bmatrix}$$

Next, we need to find A_1, A_2, A_3 by replacing the first, second, or third column of each of the three matrices respectively with the constant column vector b. Doing so we will obtain:

$$A_1 = \begin{bmatrix} 6 & 1 & -1 \\ -5 & -2 & 1 \\ 14 & 3 & -2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 6 & -1 \\ 3 & -5 & 1 \\ 1 & 14 & -2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 6 \\ 3 & -2 & -5 \\ 1 & 3 & 14 \end{bmatrix}$$

Next, we will find the determinant of each of the above matrices using the cofactors method or the diagonals method.

$\det(A) =$ -3, $\det(A_1) =$ -3, $\det(A_2) =$ -9, $\det(A_3) = 6$

To construct the variable matrix x, we will use Cramer's rule:

$$x_i = \frac{|A_i|}{|A|}$$

.

$$x = \frac{|A_1|}{|A|} = \frac{-3}{-3} = 1$$

$$y = \frac{|A_2|}{|A|} = \frac{-9}{-3} = 3$$

$$z = \frac{|A_3|}{|A|} = \frac{6}{-3} = -2$$

Therefore,

$$b = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

3 Exercise 27: Page 142

This exercise is taken from the textbook Elementary Linear Algebra, International Metric Edition [3].

Use Cramer's Rule to solve the system of linear equations for x and y.

$$kx + (1-k)y = 1$$

$$(1-k)x + ky = 3$$

For what value(s) of k will the system be inconsistent?

We need to represent the system of linear equations in the form of

$A\ \underline{x} = b$. We will start with constructing the coefficient matrix of the system of linear equations:

$$\begin{bmatrix} k & 1-k \\ 1-k & k \end{bmatrix}$$

The constant column vector $\underline{b}$ will be written as:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Now we will compute the determinant of the matrix A.

$$det(A) = k^2 - (1-k)^2$$

$$det(A) = k^2 - (1 - 2k + k^2) = k^2 - 1 + 2k - k^2$$

$$det(A) = 2k - 1$$

Since the matrix A has 2 columns then we need to find A_1 and A_2.

A_1 is the matrix A but with the first column substituted with the constant column vector b.

$$\begin{bmatrix} 1 & 1-k \\ 3 & k \end{bmatrix}$$

Now we will compute the determinant of the matrix A_1.

$$det(A_1) = k - 3(1-k)$$

$$det(A_1) = k - 3 + 3k = 4k - 3$$

A_2 is the matrix A but with the second column substituted with the constant column vector b.

$$\begin{bmatrix} k & 1 \\ 1-k & 3 \end{bmatrix}$$

Now we will compute the determinant of the matrix A_2.

$$det(A_2) = 3k - (1)(1-k)$$

$$det(A_2) = 3k - 1 + k = 4k - 1$$

Now, we are ready to find the variable column vector:

$$\begin{bmatrix} x \\ \\ y \end{bmatrix}$$

Where

$$x = \frac{|A_1|}{|A|}$$

and

$$y = \frac{|A_2|}{|A|}$$

Therefore,

$$x = \frac{4k - 3}{2k - 1}, y = \frac{4k - 1}{2k - 1}$$

The system is inconsistent if there are no solutions to the system of linear equations. This can occur if the coefficient matrix is singular (not invertible) and at least one of the matrices A_i is invertible bearing in mind that if all matrices of A_i are singular, then the system has infinitely many solutions. Equate the determinant of matrix A to zero to find the values of k.

$$det(A) = 2k - 1 = 0$$

$$k = \frac{1}{2}$$

For this value of k, we can tell that the determinant of A_1 does not equal to zero.

$$det(A) = 4k - 3 = 4(\frac{1}{2}) - 3 = 2 - 3 = -1$$

Thus, when $k = 0.5$, the system is inconsistent.

References

[1] *Gabriel Cramer*. Scolab, 2009

[2] J J O'Connor and E F Robertson. *Gabriel Cramer*. University of St. Andrews, 2000

[3] Larson, Ron. *Elementary Linear Algebra, International Metric Edition.* Available from: VitalSource Bookshelf, (8th Edition). Cengage Learning EMEA, 2017.

[4] Milson, R. *Proof of Cramer's Rule*. planetmath, 2013.

[5] Vav Petic. *Gabriel Cramer*. 2017

Link to presentation:

https://drive.google.com/file/d/1ls6pJUQ5Pt5OM-4KsYNm36L067QeIoeL/view?usp=sharing

Link to code (on github repo):

https://github.com/saraa-mohamedd/S22-LA-Project