4.7 Change of bases

* Ex: Assume $B_1 = \{u_1, u_2\}$ $B = \{v_1, v_2\}$ are two bases of some vector space V.
let w be in V. then, there are unique c_1, c_2 scalars st
$w = c_1 u_1 + c_2 u_2$ $\quad [W]_{B_1} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

we want to find $[W]_{B_2}$

$$[W]_{B_2} = c_1 [u_1]_{B_2} + c_2 [u_2]_{B_2}$$

$$\begin{bmatrix} [u_1]_{B_2} & [u_2]_{B_2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} [u_1]_{B_2} & [u_2]_{B_2} \end{bmatrix}}_{} [W]_{B_1}$$

transition matrix from B_1 to B_2
and is denoted $P_{B_1 \to B_2}$

$P_{B_1 \to B_2}$ acts on coordinate vector relative to B_1 and gives as output the coordinate vector of the same vector relative to B_2

Important
if $B_1 = \{u_1, u_2 \to u_n\}$ $B_2 = \{v_1, v_2 \to v_n\}$ then $P_{B_1 \to B_2} = \begin{bmatrix} [u_1]_{B_2} & [u_2]_{B_2} & \dots & [u_n]_{B_2} \end{bmatrix}$

old basis new basis

Ex: $B_1 = \{u_1, u_2, u_3\}$ $B_2 = \{v_1, v_2, v_3\}$

* $P_{B_1 \to B_2}$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad [u_3]_{B_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Warning
$P_{B_1 \to B_2}$ is always invertiable and its inverse

* $P^{-1}_{B_1 \to B_2} = P_{B_2 \to B_1}$

Algorithm to find transition matrix from old basis to new: $P_{old \to new}$

* $\left[\text{new basis} \,\middle|\, \text{old basis} \right]$

$\quad \downarrow$

invertiable

columns are linearly indep

* we apply elementary row operation to the matrix on the left side until we reach its RREF $\longrightarrow$ identity matrix

* Ex: $V = \mathbb{R}^3$ Find $P_{B_1 \to B_2}$

$$B_1 = \left\{ u_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} \qquad B_2 = \left\{ v_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$[v_1 \; v_2 \; v_3][u_1 \; u_2 \; u_3]$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right] \quad R_2 \leftrightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 2 \\ 3 & 1 & -1 & 2 & 2 & 1 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 5R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 2 \\ 0 & -2 & -1 & -1 & 5 & -5 \\ 0 & 2 & 2 & 6 & -4 & 11 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 2 \\ 0 & -2 & -1 & -1 & 5 & -5 \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right] \quad R_2 \to R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 2 \\ 0 & -2 & 0 & 4 & 6 & 1 \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right] \quad R_2 \to \frac{-1}{2} R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & \frac{5}{2} \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$$

Hence,

$$P_{B_1 \to B_2} = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}$$

* Ex: $V = P_1$ Find $P_{B_1 \to B_2}$

$B_1 = \{p_1 = 6 + 3x \quad p_2 = 10 + 2x\}$, $B_2 = \{q_1 = 2 \quad q_2 = 3 + 2x\}$

* The standard basis of P_1 is $S = \{1, x\}$

$[p_1]_S = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ $\quad [p_2]_S = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$

$[q_1]_S = \begin{matrix} 2 \\ 0 \end{matrix}$ $\quad [q_2]_S = \begin{matrix} 3 \\ 2 \end{matrix}$

$[[q_1]_S \; [q_2]_S] \; | \; [p_1]_S \; [p_2]_S]$

$\left[\begin{array}{cc|cc} 2 & 3 & 6 & 10 \\ 0 & 2 & 3 & 2 \end{array}\right]$ $\quad R_2 \to R_2 - \frac{1}{2} R_2$

$\begin{array}{cc|cc} 2 & 3 & 6 & 10 \\ 0 & 1 & \frac{3}{2} & 1 \end{array}$ $\quad R_1 \to R_1 -$

$\begin{array}{cc|cc} 2 & 0 & \frac{3}{2} & 7 \\ 0 & 1 & \frac{3}{2} & 1 \end{array}$ $\quad R_1 \to \frac{1}{2} R_1$

$\begin{array}{cc|cc} 1 & 0 & \frac{3}{4} & \frac{7}{2} \\ 0 & 1 & \frac{3}{2} & 1 \end{array}$

Hence, $P_{B_1 \to B_2} \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$

IMP

* Let $p = 1 + x$. Use the first part to find $[p]_{B_2}$

$[p]_{B_2} = P_{B_1 \to B_2} [p]_{B_1}$

$\begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{9} \\ \frac{-1}{6} \end{bmatrix} = \begin{bmatrix} \frac{-1}{4} \\ \frac{1}{2} \end{bmatrix}$

↓

solved in prev qs.

To check your answer

$[p]_{B_2} = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \end{bmatrix}$ This means that $p = \frac{-1}{4} q_1 + \frac{1}{2} q_2$

$\frac{-1}{4}(2) + \frac{1}{2}(3 + 2x) = \frac{-1}{2} + \frac{3}{2} + x = 1 + x$

Important

* $V = \mathbb{R}^3$

$$B_1 = \left\{ u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \; u_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

S the standard basis for $\mathbb{R}^3$

Find $P_{B \to S}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & u_1 \; u_2 \; u_3 \\ 0 & 0 & 1 & \end{array}\right]$$

Hence:

$$P_{B \to S} = [u_1 \; u_2 \; u_3] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

* Ex: $V = P_1$

$B = \{P_1 = 6 + 3x, \; P_2 = 10 + 2x\}$ Find $P_{B \to S}$

$$\left[\begin{array}{cc|cc} [1]_S & [x]_S & [P_1]_S & [P_2]_S \end{array}\right]$$

$$\left[\begin{array}{cc|cc} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} & & [P_1]_S & [P_2]_S \end{array}\right]$$

$$P_{B \to S} = \left[[P_1]_S \; [P_2]_S \right] = \begin{bmatrix} 6 & 10 \\ 3 & 2 \end{bmatrix}$$

* B_1, B_2, B_3 bases of R^2

$P_{B1\to B2} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ $\quad P_{B2\to B3} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix}$ $\quad$ Find $P_{B1\to B3}$

$$\begin{tikzcd} B1 \arrow[r, "P_{B1\to B2}"] \arrow[rr, bend right, "P_{B1\to B3}"'] & B2 \arrow[r, "P_{B2\to B3}"] & B3 \end{tikzcd}$$

$P_{B1\to B3}[u]_{B_1} = [u]_{B3}$

$P_{B2\to B3} \underbrace{P_{B1\to B2}[u]_{B1}}_{[u]_{B2}} = [u]_{B3}$ $\quad$ or $\quad$ $P_{B2\to B3}\, P_{B1\to B2}\, [u]_{B_1} = [u]_{B3}$

$P_{B1\to B3} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 31 & 11 \\ -1 & 2 \end{bmatrix}$

4.8 - 4.9

Let A be $m \times n$ matrix

i) The row space of A $R(A)$ is the subspace of $\mathbb{R}^n$ spanned by the rows of A (every row of A is in $\mathbb{R}^n$)

ii) The column space of $Col(A)$ is the subspace of $\mathbb{R}^m$

* $A = \begin{matrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{matrix}$ $\quad R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 7R_1$

$\begin{matrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{matrix}$ $\quad R_3 \to R_3 - R_2$

$\begin{matrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{matrix}$ in REF

$R(A) = span\{[1 \ -1 \ 3], [0 \ 1 \ -19]$

$Col(A) = Span\left\{\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -6 \end{bmatrix}\right\}$

* $A = \begin{matrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{matrix}$ $\quad R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$\begin{matrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{matrix}$ $\quad R_3 \to R_3 + R_2$

$\begin{matrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{matrix}$ in REF

$R(A) = span\{[1 \ 4 \ 5 \ 2], [0, -7, -7 \ -4]\}$

$Col(A) = span\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}\right\}$

Very important

* if A is row equivalent to B, then $R(A) = R(B)$ → elementary row operation doesn't affect row space

* elementary row operation might affect the column space
if A is row equivalent to B, then any relations satisfied by the columns of B must also be satisfied by the corresponding columns of A

* From the previous example:

$$\begin{bmatrix} 3 \\ -19 \\ 0 \end{bmatrix} = -16 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 19 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} = -16 \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} - 19 \begin{bmatrix} -1 \\ -4 \\ -6 \end{bmatrix}$$

hence, same elementary operations between rows and columns

$A = [A_1 \ A_2 \ A_3]$

$$col(A) = Span\{A_1 \ A_2 \ \underbrace{A_3}_{16A_1 - 19A_2}\} = span\{A_1 \ A_2\}$$

* The rank of matrix A rank(A)

$rank(A) = dim\ R(A) = dim\ Col(A)$

always same dimension

* $A = \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\left\{ \begin{bmatrix} 1 & -3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \right\}$ is a basis for $R(A)$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for $col(A)$

hence, $rank(A) = 2$

$*$ Let A be $m \times n$ matrix, the null space of A $N(A)$ is the subspace of solutions to the homogeneous system $\underset{m\times n}{A}\overset{n\times 1}{X} = 0_{\mathbb{R}^n}$

$N(A)$ is a subspace of $\mathbb{R}^n$

$*$ Find $N(A)$ where $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ consider the system $AX = 0_{\mathbb{R}^3}$

augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so if $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution to $AX = 0_{\mathbb{R}^3}$ then: $y = 2x$, x and z are free

$$X = \begin{bmatrix} x \\ 2x \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad x, z \text{ in } \mathbb{R}$$

Hence, $N(A) = \text{span} \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{linearly independent}} \right\}$ $\longrightarrow$ forms a basis for $N(A)$

$*$ The nullity of matrix A $\text{nullity}(A)$ is the dimension of $N(A)$

$\text{nullity}(A) = \dim N(A)$

* Find a basis for the subspace of $\mathbb{R}^4$

spanned by $v_1 = \begin{bmatrix} 1 \\ 1 \\ -4 \\ -3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -2 \end{bmatrix}$ $v_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}$

span $\{v_1, v_2, v_3\}$ = col([v_1 v_2 v_3])

$$[v_1\ v_2\ v_3] = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & -1 \\ -4 & 2 & 3 \\ -3 & -2 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 4R_1 \\ R_4 \to R_4 + 3R_1 \end{array}$$

$$\begin{matrix} 1 & 2 & 2 \\ 0 & -2 & -3 \\ 0 & 10 & 11 \\ 0 & 4 & 8 \end{matrix} \quad \begin{array}{l} R_3 \to R_3 + 5R_2 \\ R_4 \to R_4 + 2R_2 \end{array}$$

$$\begin{matrix} 1 & 2 & 2 \\ 0 & -2 & -3 \\ 0 & 0 & -4 \\ 0 & 0 & 2 \end{matrix} \quad R_4 \to R_4 + \frac{1}{2}R_3$$

$$\begin{matrix} 1 & 2 & 2 \\ 0 & -2 & -3 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{matrix} \quad \text{in REF}$$

hence $\{v_1, v_2, v_3\}$ is a basis for span $\{v_1\ v_2\ v_3\}$

v_1, v_2, v_3 is linearly independent

* $W = \text{span}\left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 3 \\ 7 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 3 \\ 9 \\ 3 \end{bmatrix}, v_4 = \begin{bmatrix} -5 \\ 3 \\ 5 \\ -1 \end{bmatrix} \right\}$

note: if $\{v_1\ v_2\ v_3\ v_4\}$ is not a basis for w, then inside this set some vectors are linear combinations of others.

$W = \text{col}([v_1\ v_2\ v_3\ v_4])$

$$[v_1\ v_2\ v_3\ v_4] = \begin{matrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{matrix} \qquad \begin{matrix} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\begin{matrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 0 & 10 & 10 & 10 \\ 0 & 4 & 4 & 4 \end{matrix} \qquad \begin{matrix} R_3 \to R_3 - \frac{10}{3} R_2 \\ R_4 \to R_4 - \frac{4}{3} R_2 \end{matrix}$$

$$\begin{matrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad \text{in REF}$$

v_1, v_2 are linearly independent and so $\{v_1\ v_2\}$ is a basis for w
moreover, v_3 and v_4 are linear combinations of v_1 and v_2.

$R_2 \to \frac{1}{3} R_2$

$$\begin{matrix} 1 & -3 & -1 & -5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad R_1 \to R_1 + 3R_2$$

$$\begin{matrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad \text{in RREF}$$

$\longrightarrow c_4 = 2c_1 + c_2$

$c_3 = 2c_1 + c_2$

hence, $v_3 = 2v_1 + v_2$
$v_4 = -2v_1 + v_2$

Applications to linear system

* $(S)\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$

$S \Leftrightarrow AX = b$

$$A = \begin{array}{ccc} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \\ \downarrow & \downarrow & \downarrow \\ A_1 & A_2 & A_3 \end{array} \qquad X = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \qquad b = \begin{array}{c} 5 \\ -2 \\ 3 \end{array}$$

$$[A_1 \ A_2 \ A_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$$

The system is consistent if there are scalars x_1, x_2, x_3 st $[A_1 \ A_2 \ A_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$

$b = \underbrace{x_1 A_1 + x_2 A_2 + x_3 A_3}_{\text{Linear combinations of } A_1, A_2, A_3}$

Thus, b is in span of the columns of A i.e b is in col(A)

Hence, (S) is consistent iff $b \in \text{col}(A)$

* Find rank(A), N(A):

$$A = \begin{matrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{matrix} \qquad \begin{matrix} R_3 \to R_3 - 2R_1 \\ R_2 \to R_2 + 3R_1 \end{matrix}$$

$$\begin{matrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{matrix} \qquad R_3 \to R_3 - \frac{1}{5} R_2$$

$$\begin{matrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \text{in REF}$$

$\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \right\}$ is a basis for col(A)

hence, rank(A) = dim col(A) = 2

$R_2 \to \frac{1}{5} R_2$

$$\begin{matrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad R_1 \to R_1 - 2R_2$$

$$\begin{matrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \text{in RREF}$$

if $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ is a solution to $AX = 0_{\mathbb{R}^3}$ then:

$x_3 = -2x_4 + 2x_5$

$x_1 = 2x_2 + x_4 - 3x_5$

x_2, x_4 are free.

Cont.

Then $X = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{v_2} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{v_3}}_{\text{linear combinations of } v_1, v_2, v_3}$ with x_2, x_4, x_5 in $\mathbb{R}$

Hence $N(A) = \text{Span}\{v_1\ v_2\ v_3\}$ and $\{v_1\ v_2\ v_3\}$ is a basis for $N(A)$

$\text{nullity}(A) = \dim N(A) = 3$

Warning

* Let A be mxn matrix Then:

$\text{rank}(A) + \text{nullity}(A) = \underset{\substack{\downarrow \\ \text{number of columns of } A}}{n}$

Very important

* Let A be $m \times n$ matrix :

(1) $R(A)$ is a subspace of $\mathbb{R}^n$, $\dim R(A) \leq n$ or $\text{rank}(A) \leq n$

(2) $\text{col}(A)$ is a subspace of $\mathbb{R}^m$, $\dim \text{col}(A) \leq m$ or $\text{rank}(A) \leq m$

(1) and (2) imply $\text{rank } A \leq \min\{m, n\}$

* $\text{rank}(A) = 0$ iff $\longrightarrow$ null matrix

* $0\underset{\text{any vector in } \mathbb{R}^n}{X} = 0_{\mathbb{R}^m}$ $\qquad N(A) = \mathbb{R}^n$

Warning

* Since the rows of A are the columns of A^T and the columns of A are rows of A^T

$col(A) = R(A^T)$

$R(A) = col(A^T)$

$rank(A) = rank(A^T)$

* Let A be a $m \times n$ matrix

Then: A^T is $n \times m$ matrix

So $rank(A^T) + nullity(A^T) = m$ or $rank(A) + nullity(A^T) = m$

Ex: Let A be 5×7 matrix st $rank(A) = 4$ Find $nullity(A^T)$

$rank(A) = rank(A^T) = 4$

$rank(A^T) + nullity(A^T) = 5$

$nullity(A^T) = 5 - 4 = 1$

* Let A be $n \times n$ matrix

Then: A is invertiable iff $det(A) \neq 0$ iff $col(A)$ are linearly independent

iff $col(A) = \mathbb{R}^n$

iff $r(A)$ are linearly independent

iff $R(A) = \mathbb{R}^n$

iff $rank(A) = n$

iff $nullity(A) = 0$

iff $AX = 0_{\mathbb{R}^n}$ has only the trivial solution

iff $AX = b$ for every b in $\mathbb{R}^n$ has a unique solution