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Higher order Bézier circles

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In the paper, it is proved that the order of the Bézier curves forming a full circle must be at least 6 for the curves to have all positive weights. Cubic Bézier curves that are circular arcs are also studied, and the control points for all such curves are derived. Some geometric properties of these curves are discussed.

Keywords: rational Bézier curves, circles, circular arcs

One of the most (if not the most) important reasons for using rational Bézier curves and B-splines for shape representation is to represent circles exactly. A quadratic curve has the lowest degree that is necessary to represent a circle with rational B-spline curves, and, indeed, it is the best representation for most applications. To represent a complete circle or an arc greater than 180°, the current practice is to piece multiple quadratic Bézier curves together. These Bézier curves are only position continuous in the homogeneous space, which creates difficulties in some applications.

We are also interested in methods to identify the analytical form of a B-spline curve, given just the control points and knot values. Such methods are important in the area of data exchange between systems. Without the correct analytical form, the receiving system may not be able to perform certain operations. For example, in a parametric design system, parametric information cannot be attached to the centre unless the system receives a circular arc. If the sending system supplies data only in B-spline form, the receiving system has to identify circular arcs from B-splines.

An excellent review of B-spline circles can be found in Reference 1. Other more recent work on this subject can be found in References 2 and 3. All the literature implicitly assumes that the curve segments are planar in the homogeneous space and are quadratic, except for one degree-elevated cubic curve in Reference 1. In order to search for a better B-spline representation of circles and to identify circular arcs in B-spline form, higher-order Bézier arcs have to be studied.

In this paper, we first look at the case in which one Bézier curve makes a full circle. The existence of such curves is interesting to us. We prove that, for the curve to have all positive weights, the curve must be at least quintic. We also point out that all such curves are 'improperly parameterized'[4].

In the latter half of this paper, we investigate all the cubic Bézier curves that are circular arcs. We show that there are no 'true' cubic Bézier circular arcs and we derive the explicit form of all these arcs. Finally, we prove that the maximum angle of a cubic Bézier circular arc without negative weights is 240°.

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PRELIMINARIES

A rational Bézier curve in 2D is a vector valued rational function obtained by a perspective projection of a nonrational Bézier curve in the homogeneous space (3D). We let H denote a perspective map from the origin, in 3D, onto the plane $w=1$. We have

$$\mathbf{C}(t)=(X(t),\ Y(t))=H\{\tilde{\mathbf{C}}(t)\}=H\{\tilde{X}(t),\ \tilde{Y}(t),\ W(t)\} \quad (1)$$

$$=H\left\{\sum_{i=0}^{n} B_i^n(t)\tilde{\mathbf{P}}_i\right\} \quad (2)$$

$$=\sum_{i=0}^{n} B_i^n(t)w_i\frac{\mathbf{P}_i}{\sum_{j=0}^{n} B_j^n(t)w_j} \quad (3)$$

where $\tilde{\mathbf{P}}_i=(\tilde{x}_i,\ \tilde{y}_i,\ w_i)=(w_ix_i,\ w_iy_i,\ w_i)$, $i=0\ldots,n$, are the homogeneous control points, $B_i^n(t)$ are the nth degree Bernstein functions, w_i are the weights, and $\mathbf{P}_i=H\{\tilde{\mathbf{P}}_i\}=(x_i,\ y_i)$ are the 2D control points.

It is easier to discuss the rational curves in the homogeneous space. For a circle, the nonrational homogeneous curves $\tilde{\mathbf{C}}(t)$ lie on a cone in the homogeneous space (see *Figure 1*). This can easily be seen from the circle equation (Equation 4):

$$\left(\frac{\tilde{X}(t)}{W(t)}\right)^2+\left(\frac{\tilde{Y}(t)}{W(t)}\right)^2=1 \quad (4)$$

$$\Rightarrow(\tilde{X}(t))^2+(\tilde{Y}(t))^2-(W(t))^2=0 \quad (5)$$

Equations 4 and 5 are those of a circle centred at the origin and with unit radius. Equation 5 is also that of a curve on a cone. The perspective map H projects the curve onto the base circle of the cone, forming the circle $\mathbf{C}(t)$.

B-spline circles can be constructed by piecing together Bézier curves going around the cone. To investigate

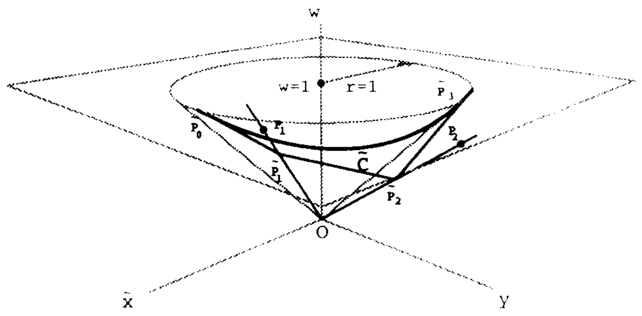

Figure 1 Bézier curves for arcs lie on cone in homogeneous space

B-spline circles of higher orders, we need to study the possible Bézier curves on the cone. We assume our circles/circular arcs are centred at the origin and have unit radii. This confines the cone of our discussion to the one shown in *Figure 1*, but does not sacrifice the generality of our results.

Given a circular arc, there are an infinite number of homogeneous curves $\tilde{\mathbf{C}}(t)$ that can be mapped to the arc. For example, given the Bézier control points $\mathbf{P}_i$ of an arc, all the homogeneous curves that have the same shape invariance (obtained via rational linear reparameterization)[5] and that are a result of degree elevation/reduction[6] are mapped to the same arc. As an example, the shape invariance formula for cubic curves is[6]

$$\frac{w_1^2}{w_0 w_2} = c_1$$

$$\frac{w_2^2}{w_1 w_3} = c_2 \tag{6}$$

Two cubic curves with the same c_1 and c_2 are of the same shape and are mapped to the same circular arc. Geometrically, the effect of adjusting the weights is to move the control points $\tilde{\mathbf{P}}_i$ along the line connecting $\mathbf{P}_i$ and the origin (see *Figure 1*). Through the shape invariance, we can fix the weights of the control points at the two ends to unity. In 3D, this means the end control points are on the base circle of the cone. Given a curve, after we have fixed the end weights, the positions of the control points $\tilde{\mathbf{P}}_i$ are fixed. Hereafter, we assume that the first and the last control points stay on the base circle of the cone.

BÉZIER CURVE AS COMPLETE CIRCLE

In this section, we study the Bézier curves that make a full circle. In particular, we are interested in those with all positive weights. We examine the Bézier curves with increasing higher polynomial orders, beginning with quadratic curves.

The fact that a rational quadratic Bézier curve cannot represent a full circle (even with negative weights) is evident from the literature. The following is a simple proof, included for completeness.

All nonrational quadratic Bézier curves are parabolae, and all the parabolae on a cone are created by cutting the cone with planes parallel to the rulings. In *Figure 2*, we can push the cutting plane as close to the ruling (R in *Figure 2*) as possible, and hence make $\mathbf{C}(t)$ as close to a full circle as possible, but the cutting plane cannot contain the ruling. When that happens, $\tilde{\mathbf{C}}(t)$ becomes a double line and $\mathbf{C}(t)$ a point. All the quadratic Bézier arcs have the following control points:

$$\tilde{\mathbf{P}}_0 = (\cos\theta, -\sin\theta, 1)$$

$$\tilde{\mathbf{P}}_1 = (1, 0, \cos\theta)$$

$$\tilde{\mathbf{P}}_2 = (\cos\theta, \sin\theta, 1) \tag{7}$$

where θ is one-half of the sweeping angle of the arc. If no negative weights are allowed ($\cos\theta > 0$), $\mathbf{C}(t)$ is less than 180°. When the middle weight is zero ($\theta = 90°$), $\mathbf{C}(t)$ is exactly 180°.

The next natural question is 'how about cubic curves?'. If quadratic curves cannot represent a full circle, can cubic curves do it? Unfortunately, the answer is negative. We provide the following simple proof.

For the cubic curve $H(\tilde{\mathbf{C}}(t))$ to be a circle, $\tilde{\mathbf{P}}_3$ must be on the line $\overline{\tilde{\mathbf{P}}_0\mathbf{O}}$, and we can assume that both $\tilde{\mathbf{P}}_0$ and $\tilde{\mathbf{P}}_3$ are at the same point in 3D. Since $\tilde{\mathbf{C}}(t)$ is on the cone, the tangents of $\tilde{\mathbf{C}}(t)$ at $\tilde{\mathbf{P}}_0$ and at $\tilde{\mathbf{P}}_3$ must lie on the tangent plane of the cone at $\tilde{\mathbf{P}}_0$. By the tangent property of Bézier curves, $\tilde{\mathbf{P}}_1$ is on the tangent line of the curve at $\tilde{\mathbf{P}}_0$, and $\tilde{\mathbf{P}}_2$ is on the tangent line of the curve at $\tilde{\mathbf{P}}_3$. From the above discussion, all the four control points are on the tangent plane of the cone at $\tilde{\mathbf{P}}_0$; hence, by the convex hull property of Bézier curves, the curve $\tilde{\mathbf{C}}(t)$ is on the tangent plane. We know that the tangent plane only intersects the cone at a line. $\mathbf{C}(t)$ is a point.

How about quartic curves? What are the quartic curves that are circles? To find all the quartic circles, we substitute the equation for the homogeneous Bézier curves into the equation of the cone (Equation 5). After equating the coefficients of $B_i^8(t)$, $i = 0, \ldots, 8$, to zero, we obtain nine equations.

Without loss of generality, we assume that $\tilde{\mathbf{P}}_0 = \tilde{\mathbf{P}}_4 = (1, 0, 1)$. Since $\tilde{\mathbf{P}}_1$ and $\tilde{\mathbf{P}}_3$ are on the tangent plane at $\tilde{\mathbf{P}}_0$, we also have $\tilde{x}_1 = w_1$ and $\tilde{x}_3 = w_3$. With these known conditions, the nine equations are reduced to five. With some simple algebraic manipulation, we have the following equations:

$$\tilde{y}_3 = -\tilde{y}_1 \tag{8}$$

$$\tilde{x}_3 = \tilde{x}_1 \tag{9}$$

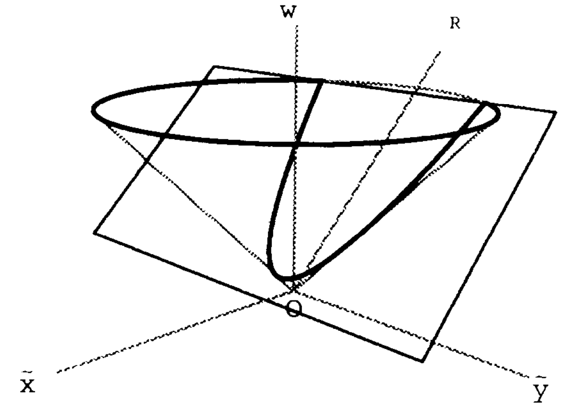

Figure 2 Single quadratic Bézier curve cannot be full circle

$$3\tilde{x}_2+4\tilde{y}_1^2-3w_2=0 \tag{10}$$

$$\tilde{x}_1\tilde{x}_2+\tilde{y}_1\tilde{y}_2-\tilde{x}_1w_2=0 \tag{11}$$

$$9\tilde{x}_2^2-8\tilde{y}_1^2+9\tilde{y}_2^2-9w_2^2=0 \tag{12}$$

Equations 10–12 have two sets of nontrivial solutions:

$$\tilde{y}_1=\alpha$$

$$\tilde{x}_2=-\frac{3w_2-4\tilde{x}_1^2+2}{3}$$

$$\tilde{y}_2=\frac{4}{3}\tilde{x}_1\alpha \tag{13}$$

and

$$\tilde{y}_1=-\alpha$$

$$\tilde{x}_2=-\frac{3w_2-4\tilde{x}_1^2+2}{3}$$

$$\tilde{y}_2=-\frac{4}{3}\tilde{x}_1\alpha \tag{14}$$

where $\alpha=(3w_2/2-\tilde{x}_1^2+1/2)^{1/2}$. We have the following control points for the quartic circles:

$$\tilde{\mathbf{P}}_0=(1,\ 0,\ 1) \tag{15}$$

$$\tilde{\mathbf{P}}_1=(\tilde{x}_1,\ \pm\alpha,\ \tilde{x}_1) \tag{16}$$

$$\tilde{\mathbf{P}}_2=\left(-\frac{3w_2-4\tilde{x}_1^2+2}{3},\ \pm\frac{4}{3}\tilde{x}_1\alpha,\ w_2\right) \tag{17}$$

$$\tilde{\mathbf{P}}_3=(-\tilde{x}_1,\ \mp\alpha,\ -\tilde{x}_1) \tag{18}$$

$$\tilde{\mathbf{P}}_4=(1,\ 0,\ 1) \tag{19}$$

In order for α to be a real number, we must have

$$w_2>-\frac{1}{3} \tag{20}$$

and

$$-\left(\frac{3w_2+1}{2}\right)^{1/2}<\tilde{x}_1<\left(\frac{3w_2+1}{2}\right)^{1/2} \tag{21}$$

Note that, when $w_2=-1/3$ or $\tilde{x}_1=\pm((3w_2+1)/2)^{1/2}$, $\mathbf{C}(t)$ is not a circle.

With this set of control points, it is easy to check that $W(t)>0$, for $t\in[0,1]$. That is, points on the circle can be computed safely. However, from Equations 16 and 18, $w_1=-w_3$; it is impossible to have a quartic Bézier circle with all positive weights.

If we require the curve to be symmetric with respect to the plane $y=0$, we have $\tilde{\mathbf{P}}_0=\tilde{\mathbf{P}}_4=(1,0,1)$ and

$$\tilde{\mathbf{P}}_1=(0,\ \pm\beta,\ 0) \tag{22}$$

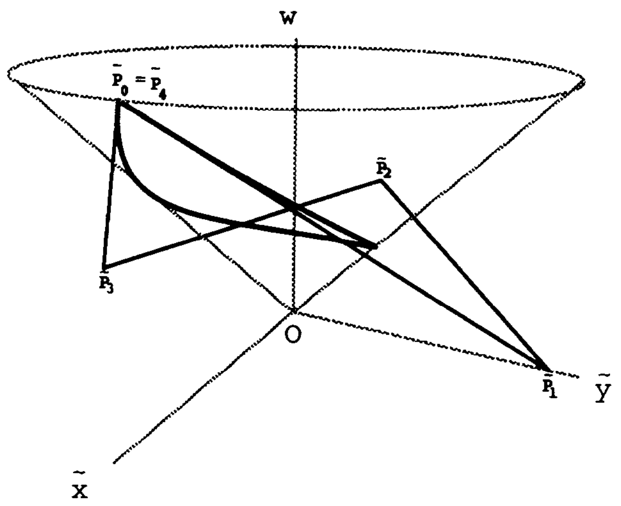

Figure 3 Quartic Bézier curve as circle ($w_1=w_3=0$)

$$\tilde{\mathbf{P}}_2=\left(-\frac{2}{3}-w_2,\ 0,\ w_2\right) \tag{23}$$

$$\tilde{\mathbf{P}}_3=(0,\ \mp\beta,\ 0) \tag{24}$$

where $\beta=(1/2+(3/2)w_2)^{1/2}$. In particular, when $w_2=1/3$, we have

$$\tilde{\mathbf{P}}_0=(1,\ 0,\ 1)$$

$$\tilde{\mathbf{P}}_1=(0,\ 1,\ 0)$$

$$\tilde{\mathbf{P}}_2=(-1,\ 0,\ 1/3)$$

$$\tilde{\mathbf{P}}_3=(0,\ -1,\ 0)$$

$$\tilde{\mathbf{P}}_4=(1,\ 0,\ 1) \tag{25}$$

Figure 3 shows this curve. This curve can also be obtained by squaring the quadratic semicircle with the control points*

$$\tilde{\mathbf{P}}_0=(1,\ 0,\ 1)$$

$$\tilde{\mathbf{P}}_1=(0,\ 1,\ 0)$$

$$\tilde{\mathbf{P}}_2=(-1,\ 0,\ 1) \tag{26}$$

Now we have a Bézier curve (a rational polynomial piece) as a circle. Unfortunately, all the quartic Bézier circles have zero or negative weights. This is highly undesirable in many applications. The next natural question is 'is it possible to have a quintic Bézier circle with all positive weights?'.

This question can be answered easily by elevating the degree of the curves in Equations 15–19. The formula for degree elevating Bézier curves can be found in Reference 6. The conditions for the degree-elevated curves

*This fact was first pointed out to the author by Tim Strotman at SDRC.

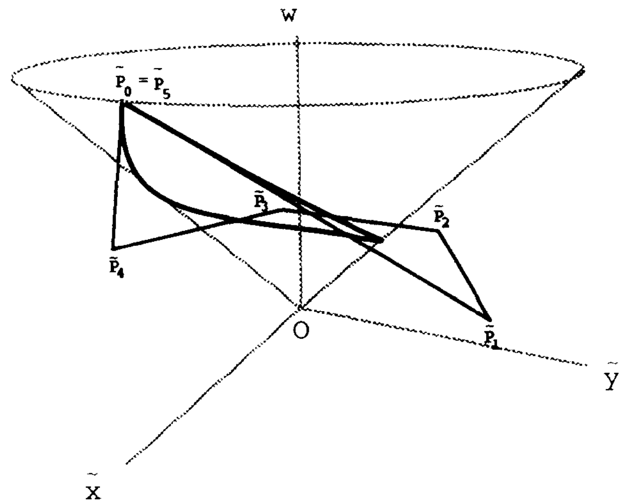

Figure 4 Quintic Bézier curve as circle [All weights are positive.]

to have all positive weights are

$$-\frac{1}{4} < \tilde{x}_1 < \frac{1}{4}$$

and

$$-\frac{3}{2} w_2 < \tilde{x}_1 < \frac{3}{2} w_2 \tag{27}$$

For example, the curve in Equations 25 satisfies these conditions, and all the weights of the degree-elevated curve are positive. This degree-elevated curve is shown in *Figure 4*. Its control points are

$$\tilde{\mathbf{P}}_0 = (1, 0, 1)$$

$$\tilde{\mathbf{P}}_1 = \left(\frac{1}{5}, \frac{4}{5}, \frac{1}{5}\right)$$

$$\tilde{\mathbf{P}}_2 = \left(-\frac{3}{5}, \frac{2}{5}, \frac{1}{5}\right)$$

$$\tilde{\mathbf{P}}_3 = \left(-\frac{3}{5}, -\frac{2}{5}, \frac{1}{5}\right)$$

$$\tilde{\mathbf{P}}_4 = \left(\frac{1}{5}, -\frac{4}{5}, \frac{1}{5}\right)$$

$$\tilde{\mathbf{P}}_5 = (1, 0, 1) \tag{28}$$

Finally, we point out that any Bézier curves in a full circle are improperly parameterized[4]. This is true regardless of the order of the Bézier curve. Improperly parameterized curves are multiple valued, with multiple parameter values corresponding to a curve point when the parameter of the curve goes from minus to positive infinity. An improperly parameterized curve $\mathbf{C}(t)$ can be written as $\mathbf{F}(s(t))$, where $\mathbf{F}(s)$ has a lower order than $\mathbf{C}(t)$, and $s(t)$ is a rational polynomial of t. In order to see that all Bézier full circles are multiple valued, we note that, if the Bézier curve $\mathbf{C}(t)$ is a full circle, all the points $\mathbf{C}(t)$ for $t \in [-\infty, \infty]$ are on the circle. Since $\mathbf{C}(t)$ with $t \in [0, 1]$ completes a circle, $\mathbf{C}(t)$ with $t \in [-\infty, \infty]$ must be multiple valued. Reference 4 provides algorithms to reduce the orders of improperly parameterized curves. However, even though the orders of the curves, as parametric functions, can always be reduced, the results may not be Bézier curves for a full circle any more.

The consequence of improper parameterization of Bézier full circles is that the Bézier circles can always be obtained by reparameterizing a lower-order Bézier arc using a rational polynomial parameter substitution. For example, the curve in Equations 25 can also be obtained by substituting $t = (u^2 - u)/(u^2 - 1/2)$ into the semicircle in Equations 26.

GENERAL CUBIC BÉZIER CIRCULAR ARCS

In this section, we find all cubic Bézier circular arcs. We can find all cubic Bézier arcs by substituting the equation for the homogeneous Bézier curves into that of the cone (Equation 5), as we did above for quartic full circles. After equating the coefficients of $B_i^6(t)$, $i = 0, \ldots, 6$, to zero, we obtain seven equations. For example, we can assume that $\tilde{\mathbf{P}}_0 = (\cos\theta, -\sin\theta, 1)$ and $\tilde{\mathbf{P}}_3 = (\cos\theta, \sin\theta, 1)$, where θ is one-half of the sweeping angle of the arc. The seven equations become five:

$$w_1 = (\cos\theta)\tilde{x}_1 - (\sin\theta)\tilde{y}_1 \tag{29}$$

$$w_2 = (\cos\theta)\tilde{x}_2 + (\sin\theta)\tilde{y}_2 \tag{30}$$

$$3(\sin^2\theta)\tilde{x}_1^2 + 3(\cos^2\theta)\tilde{y}_1^2 - 4(\sin\theta)\tilde{y}_2 + 6(\sin\theta)(\cos\theta)\tilde{x}_1\tilde{y}_1 = 0 \tag{31}$$

$$9(\sin^2\theta)\tilde{x}_1\tilde{x}_2 + 9(\cos\theta)(\sin\theta)\tilde{y}_1\tilde{x}_2 - 9(\cos\theta)(\sin\theta)\tilde{x}_1\tilde{y}_2 + 9(1 + \sin^2\theta)\tilde{y}_1\tilde{y}_2 - 2\sin^2\theta = 0 \tag{32}$$

$$3(\sin^2\theta)\tilde{x}_2^2 + 3(\cos^2\theta)\tilde{y}_2^2 + 4(\sin\theta)\tilde{y}_1 - 6(\sin\theta)(\cos\theta)\tilde{x}_2\tilde{y}_2 = 0 \tag{33}$$

As an example, we can set $\theta = \pi/2$. From Equations 29–33, we can obtain the following solution:

$$\tilde{\mathbf{P}}_0 = (0, -1, 1)$$

$$\tilde{\mathbf{P}}_1 = \left(\frac{2\alpha}{3}, -\frac{1}{3\alpha^2}, \frac{1}{3\alpha^2}\right)$$

$$\tilde{\mathbf{P}}_2 = \left(\frac{2}{3\alpha}, \frac{\alpha^2}{3}, \frac{\alpha^2}{3}\right)$$

$$\tilde{\mathbf{P}}_3 = (0, 1, 1) \tag{34}$$

where $\alpha = 3\tilde{x}_1/2$. The 2D control points are $\mathbf{P}_0 = (0, -1)$, $\mathbf{P}_1 = (2\alpha^3, -1)$, $\mathbf{P}_2 = (2/\alpha^3, 1)$ and $\mathbf{P}_3 = (0, 1)$. Several interesting properties of these curves are discussed below.

When θ is taken as a variable, it is very difficult to obtain an explicit solution for Equations 29–33. In the following, we take a different approach. First, we show that there are no true cubic Bézier arcs. An order n Bézier curve is considered as a *true* order n Bézier arc only if it is properly parameterized and has no common factors in the numerators and denominators of the rational curve functions.

Theorem 1: If

$$\mathbf{C}(t)=\frac{(\tilde{X}(t),\ \tilde{Y}(t))}{W(t)} \tag{35}$$

is bounded for $t\in[-\infty,\infty]$, and $(\tilde{X}(t), \tilde{Y}(t), W(t))$ are polynomials and have no common dividers, none of $\tilde{X}(t)$, $\tilde{Y}(t)$ and $W(t)$ can have an odd degree. That is, if we write any of these polynomials as

$$a_n t^n + a_{n-1}t^{n-1} + \ldots + a_0 \tag{36}$$

with $a_n \neq 0$, n must be even. There is one exception to Theorem 1 when $\mathbf{C}(t)$ is at special positions, in which $\tilde{X}(t)$ and $\tilde{Y}(t)$ may be of odd degrees. This exception is discussed below.

Proof: We first argue that $W(t)$ cannot have an odd degree. If $W(t)$ is odd, $W(t)$ has a real root, say $\bar{t}$. When $t\to\bar{t}$, $\mathbf{C}(t)$ becomes unbound, which contradicts the fact that $\mathbf{C}(t)$ is a bounded curve. For $\tilde{X}(t)$ and $\tilde{Y}(t)$, if the degree of either of them is odd and is higher than that of $W(t)$, $\mathbf{C}(t)$ becomes unbound as t approaches infinity. This contradicts our assumption. If the degree of $\tilde{X}(t)$ is odd and is lower than that of $W(t)$, the x component of $\mathbf{C}(\infty)$ is zero. A simple affine transformation can move $\mathbf{C}(t)$ out of this special position. This transformation makes $\tilde{X}(t)$ of an even degree. The same argument applies when the degree of $\tilde{Y}(t)$ is odd and is lower than that of $W(t)$. □

If a Bézier curve $\mathbf{C}(t)$ is an arc, the whole $\mathbf{C}(t)$, for $t\in[-\infty,\infty]$, is on the circle and is bounded. This theorem says that a true cubic Bézier arc does not exist. That is, even though $\tilde{X}(t)$, $\tilde{Y}(t)$ and $W(t)$ are cubic, the x and y components of $\mathbf{C}(t)$ are always quadratic.

Note that the exception of the theorem does not occur for cubic Bézier arcs, since $W(t)$ must be quadratic, and $\tilde{X}(t)$ and $\tilde{Y}(t)$ cannot be linear.

From the theorem, we also know that the only possible cubic Bézier arcs are of the form

$$\mathbf{C}(t)=\frac{(\tilde{X}(t),\ \tilde{Y}(t))(aB_0^1(t)+bB_1^1(t))}{W(t)(aB_0^1(t)+bB_1^1(t))} \tag{37}$$

where

$$\mathbf{D}(t)=\frac{(\tilde{X}(t),\ \tilde{Y}(t))}{W(t)} \tag{38}$$

is a quadratic circular arc. Let the quadratic Bézier arc $\mathbf{D}(t)$ be

$$\tilde{\mathbf{P}}_0=(\cos\theta,\ -\sin\theta,\ 1) \tag{39}$$

$$\tilde{\mathbf{P}}_1=({w_2}^{1/2},\ 0,\ {w_2}^{1/2}\cos\theta) \tag{40}$$

$$\tilde{\mathbf{P}}_2=(w_2\cos\theta,\ w_2\sin\theta,\ w_2) \tag{41}$$

where θ is one-half of the sweeping angle of the arc. These control points can be obtained from Equations 7 by letting w_0 be 1, w_2 be a variable, and, at the same time, keeping the shape invariance

$$w_1^2/(w_0w_2)=\cos^2\theta \tag{42}$$

as a constant.

The denominator of the cubic curve in Equation 37 becomes

$$\begin{aligned}W(t)(aB_0^1(t)+bB_1^1(t)) &= aB_0^3(t)+\left(\frac{2}{3}a(\cos\theta)({w_2}^{1/2})+\frac{b}{3}\right)B_1^3(t)\\ &\quad+\left(\frac{aw_2}{3}+\frac{2}{3}b(\cos\theta)({w_2}^{1/2})\right)B_2^3(t)+bw_2B_3^3(t)\end{aligned} \tag{43}$$

To keep the end weights of the cubic curve to 1, we have

$$a=1$$

$$w_2=1/b \tag{44}$$

The cubic Bézier arcs have control points

$$\tilde{\mathbf{P}}_0=(\cos\theta,\ -\sin\theta,\ 1) \tag{45}$$

$$\tilde{\mathbf{P}}_1=\left(\left(\frac{2}{3(b^{1/2})}+\frac{b\cos\theta}{3}\right),\ -\frac{b\sin\theta}{3},\ \left(\frac{2\cos\theta}{3(b^{1/2})}+\frac{b}{3}\right)\right) \tag{46}$$

$$\tilde{\mathbf{P}}_2=\left(\left(\frac{\cos\theta}{3b}+\frac{2(b^{1/2})}{3}\right),\ \frac{\sin\theta}{3b},\ \left(\frac{2(\cos\theta)(b^{1/2})}{3}+\frac{1}{3b}\right)\right) \tag{47}$$

$$\tilde{\mathbf{P}}_3=(\cos\theta,\ \sin\theta,\ 1) \tag{48}$$

When $\theta=\pi/2$ and $b=1/\alpha^2$, Equations 45–48 are reduced to those of the cubic semicircles in Equations 34.

One possible reason for using cubic Bézier arcs instead of quadratic arcs is to represent a circle or circular arc with fewer pieces of Bézier curves, and hence reduce the amount of data. Whether data size reduction is possible depends on the maximum angle that a cubic Bézier arc with all positive weights can achieve. We know that the maximum angle achievable by a quadratic Bézier arc without negative weights is 180°. In the following, we prove that a cubic Bézier arc can achieve an angle of 240°.

To have all nonnegative weights, we must have (from Equations 46 and 47)

$$\frac{2\cos\theta}{3(b^{1/2})}+\frac{b}{3}\geqslant 0 \tag{49}$$

and

$$\frac{2(\cos\theta)(b^{1/2})}{3}+\frac{1}{3b}\geqslant 0 \tag{50}$$

That is,

$$\cos\theta\geqslant-\beta/2 \tag{51}$$

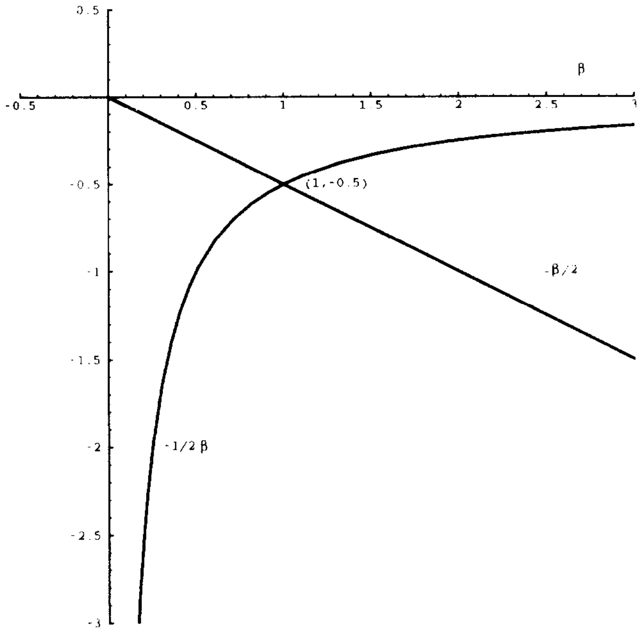

Figure 5 Graphs of $-\beta/2$ and $-1/(2\beta)$ as function of β

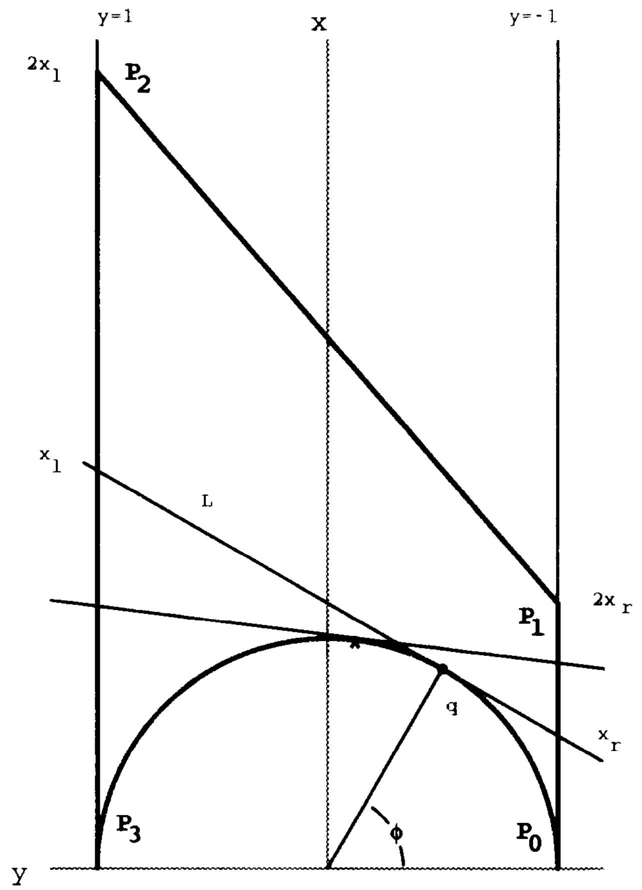

Figure 6 Geometric construction of cubic Bézier semicircle

and

$$\cos\theta \geqslant -1/(2\beta) \tag{52}$$

where $\beta = b(b^{1/2})$.

The graphs of $-\beta/2$ and $-1/(2\beta)$ are shown in *Figure* 5. It is easy to see from *Figure 5* that the maximum angle is achieved when $\beta = b = 1$, $w_1 = w_2 = 0$, and $\theta = 120°$. Since θ is one-half of the angle, the maximum angle of a cubic Bézier arc without negative weights is 240°.

CUBIC SEMICIRCLES

In this section, we revisit cubic Bézier semicircles and give some of the geometric properties of the curves. The control points of the semicircles are given in Equations 34.

First, we provide a geometric method for constructing an arbitrary rational cubic Bézier arc forming a semicircle. As shown in *Figure 6*, we draw two lines $y = -1$ and $y = 1$. After that, we draw an arbitrary tangent line L to the semicircle. Let the tangent point be q. L intersects the vertical lines at x_r and x_l. We find the points $(2x_r, -1)$ and $(2x_l, 1)$ on the lines. These two points, along with $(0, -1)$ and $(0, 1)$, are the 2D control points of the cubic semicircle. The weights are

$$w_1 = \frac{1}{3(x_r)^{2/3}}$$

$$w_2 = \frac{1}{3(x_l)^{2/3}} \tag{53}$$

The fact that $x_l x_r = 1$ for any line tangent to the semicircle can be proved by simple algebra. The rest of the construction follows directly from Equations 34 with $\alpha = x_r^{1/3}$. The angle ϕ (see *Figure 6*) at the tangent point is related to α by $\tan\phi = (2\alpha^3)/(1-\alpha^6)$.

We check the parameterization of the semicircles by examining the points at $t = 1/2$. It is easy to calculate that $\mathbf{C}(1/2) = (2\alpha/(1+\alpha^2), (\alpha^2-1)/(1+\alpha^2))$. This point is shown in *Figure 6* (by a cross), along with the line tangent to the semicircle at this point. This tangent line intersects $y = -1$ and $y = 1$ at $x = \alpha$ and $x = 1/\alpha$, respectively. From the construction, for $\alpha < 1$, $\mathbf{C}(1/2)$ lies between point $(1, 0)$ and q. As q moves towards $(0, -1)$ (with decreasing α), $\mathbf{C}(1/2)$ follows, producing an increasingly skewed parameterization. An interesting case is when q is at $(1, 0)$. In this case, $\alpha = 1$, and the curve is symmetric with respect to the $y = 0$ line. The control points of this curve are $\tilde{\mathbf{P}}_0 = (0, -1, 1)$, $\tilde{\mathbf{P}}_1 = (2/3, -1/3, 1/3)$, $\tilde{\mathbf{P}}_2 = (2/3, 1/3, 1/3)$ and $\tilde{\mathbf{P}}_3 = (0, 1, 1)$; these can also be obtained by elevating the degree of the quadratic semicircle[1].

CONCLUSIONS

We have investigated some interesting properties of Bézier curves as circles/circular arcs. To the author's knowledge, the representation of a full circle with a single piece of Bézier curve has not previously been presented in the literature. It is probably to theoreticians' dismay that all these curves are improperly parameterized.

An in-depth study was carried out on cubic Bézier arcs. Although they may be of practical use, we have proved that all these curves can be obtained from quadratic arcs. The equations for the control points of these curves may be useful in devising algorithms for arc identification. As to the representation power of cubic curves, it is quite disappointing to learn that the maximum angle of a cubic Bézier arc (with nonnegative weights) is not much larger than that of a quadratic. The geometric construction of cubic semicircles is interesting. However, use of the knowledge obtained about cubic arcs to construct better B-spline circles remains an exercise. Whether the exercise will be futile is a concern, given the limited flexibility of cubic arcs as revealed by this paper. Is it worth going to quartic arcs in search of better B-spline circles? Or, more fundamentally, do true quartic Bézier arcs exist? An even broader question is that of the existence of true Bézier arcs other than quadratic arcs.

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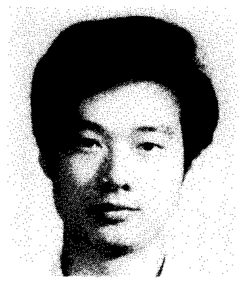

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